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Research article

Meromorphic solutions of $f^n + P_d(f) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z} + p_3e^{\alpha_3 z}$

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Abstract: By using Nevanlinna of the value distribution of meromorphic functions, we investigate the transcendental meromorphic solutions of the non-linear differential equation

$$f^n + P_d(f) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z} + p_3e^{\alpha_3 z},$$

where $P_d(f)$ is a differential polynomial in f of degree $d(0 \leq d \leq n-3)$ with small meromorphic coefficients and $p_i, \alpha_i (i = 1, 2, 3)$ are nonzero constants. We show that the solutions of this type equation are exponential sums and they are in $\Gamma_0 \cup \Gamma_1 \cup \Gamma_3$ which will be given in Section 1. Moreover, we give some examples to illustrate our results.

Keywords: Nevanlinna theory; complex differential equations; exponential sums; meromorphic solutions
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1. Introduction and main results

It is an important and quite difficult problem to prove the existence of solutions of complex differential equations. Recently, more and more people investigate the solutions of complex differential equations by using Nevanlinna theory. Throughout this paper, we assume that the reader is familiar with the standard notations and fundamental results in Nevanlinna theory, such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $S(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevanlinna theory. For more details, see [3, 4, 14].

First of all, we give the following notations. For $\lambda \in \mathbb{C} \setminus \{0\}$, denote

$\Gamma_0 = \{\lambda_0 e^{\lambda z}\colon\ \lambda_0 \text{ is a nonzero constant}\}$,

$\Gamma_1 = \{\lambda_1 e^{\lambda z} + \lambda_0\colon\ \lambda_0 \text{ and } \lambda_1 \text{ are nonzero constants}\}$,

$\Gamma_2 = \{\lambda_1 e^{\lambda z} + \lambda_2 e^{-\lambda z}\colon\ \lambda_1 \text{ and } \lambda_2 \text{ are nonzero constants}\}$,

$\Gamma_3 = \{\lambda_1 e^{\lambda z} + \lambda_2 e^{-2\lambda z}\colon\ \lambda_1 \text{ and } \lambda_2 \text{ are nonzero constants}\}$.

In [13], the authors show that the differential equation $4f^3 + 3f'' = -\sin 3z$ has exactly three nonconstant entire solutions, namely $f_1(z) = \sin z$, $f_2(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$ and $f_3(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. Since $\sin 3z$ is a linear combination of e^{3iz} and e^{-3iz}, this has attracted many scholars to study the following more general differential equation

$$f^n(z) + P_d(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.1}$$

where $p_d(f)$ is a differential polynomial in f of degree d, see [5–10, 13, 15, 16]. Now, we recall the following classic result due to Li [6].

Theorem A. [6] Let $n \geq 2$ be an integer and $P_d(f)$ be a differential polynomial in f of degree at most $n-2$, and $p_i, \alpha_i (i = 1, 2)$ be nonzero constants and $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental meromorphic solution of (1.1) and satisfying $N(r, f) = S(r, f)$, then one of the following holds:

(1) $f(z) = c_0 + c_1 e^{\alpha_1 z/n}$;
(2) $f(z) = c_0 + c_2 e^{\alpha_2 z/n}$;
(3) $f(z) = c_1 e^{\alpha_1 z/n} + c_2 e^{\alpha_2 z/n}$, and $\alpha_1 + \alpha_2 = 0$,

where c_0 is a small function of $f(z)$ and $c_1,\ c_2$ are constants satisfying $c_1^n = p_1, c_2^n = p_2$.

If $d \leq n-2$, Theorem A shows that the entire solutions of (1.1) are exponential sums and they are in $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. Firstly, we consider the existence of entire solutions of (1.1) under the condition $d = n-1$. We get the following result.

Theorem 1.1. *Let $n \geq 2$ be an integer and $P_d(f)$ be a differential polynomial in f of degree $n-1$, and $p_i,\ \alpha_i (i = 1, 2)$ be nonzero constants and $\alpha_1 \neq \alpha_2$. If f is a transcendental entire solution of* (1.1)*, then $f \notin \Gamma_3$.*

Note that the right-hand side of (1.1) is a linear combination of $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$, it is natural and interesting to replace $p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$ with $h(z) := p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + \cdots + p_k e^{\alpha_k z}$. In other words, how to find the solutions of the following equation

$$f^n + P_d(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + \cdots + p_k e^{\alpha_k z}, \tag{1.2}$$

where $k \geq 3$ is an integer, $\alpha_i (i = 1, 2, \cdots, k)$ are distinct nonzero constants, $P_d(f)$ is a differential polynomial in f with degree d and $p_i, \alpha_i (i = 1, 2, \cdots, k)$ are nonzero constants?

In this perspective, Xue [12] investigated the entire solutions of (1.2) for $k = 3$ and he proved the following result.

Theorem B. [12] Let $n \geq 2$ and $P_d(f)$ be a differential polynomial in f of degree $d \leq n-1$. Suppose that $p_i,\ \alpha_i (i = 1, 2, 3)$ are nonzero constants and $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3|$. If $f(z)$ is a transcendental entire solution of of (1.2), then $f(z) = a_1 e^{\alpha_1 z/n}$, where a_1 is a non-zero constant such that $a_1^n = p_1$, and α_j are in one line for $j = 1, 2, 3$.

In fact, Examples 2–4 in this section show that (1.2) has some solutions different from $a_1 e^{\alpha_1 z/n}$ under the condition $k = 3$. In this paper, we further consider the existence of the transcendental meromorphic solutions of (1.2) under the conditions $d \leq n-3$ and $k = 3$. Without loss of generality, we assume that $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3|$. We obtain the following result.

Theorem 1.2. *Let $n \geq 3$ be an integer, $k = 3$ and $P_d(f)$ be a differential polynomial in f of degree $d \leq n-3$. If f is a transcendental meromorphic solution of* (1.2) *satisfying $N(r, f) = S(r, f)$, then one of the following holds:*

(1) $f(z) = c_1 e^{\alpha_1 z/n}$;
(2) $f(z) = c_1 e^{\alpha z} + c_0$ *and* $\alpha_j = \frac{n-j}{n}\alpha_1 (j = 2, 3)$;
(3) $f(z) = c_1 e^{2\alpha z} + c_3 e^{-\alpha z}$ *and* $\alpha_2 = \frac{2n-3}{2n}\alpha_1$, $\alpha_3 = -\frac{1}{2}\alpha_1$,

where $\alpha = \frac{(\alpha_1+\alpha_2+\alpha_3)}{3(n-1)}$ and c_0, c_1, c_3 are nonzero constants satisfying $c_1^n = p_1$, $c_3^n = p_3$. Furthermore, if $d < n-3$, then $f(z) = c_1 e^{\alpha_1 z/n}$.

Remark. By the result due to Steinmetz [11], we have $T(r, h) = K(1 + o(1))\frac{r}{2\pi}(r \to \infty)$, where K denote the perimeter of convex polygon which is formed by $\{0, \alpha_1, \alpha_2, \cdots, \alpha_k\}$ (if $0, \alpha_1, \alpha_2, \cdots, \alpha_k$ are collinear, then $K = 2\max_{i,j\in\Lambda} |\alpha_i - \alpha_j|$, where $\Lambda = \{0, 1, 2, \cdots, k\}$ and $\alpha_0 = 0$). This implies (1.2) has no rational solution.

By Theorem A and Theorem 1.1, we see that the entire solutions of (1.1) are not in Γ_3. Since the entire solutions of (1.2) are in $\Gamma_0 \cup \Gamma_1 \cup \Gamma_3$ under the conditions $k = 3$ and $d \leq n-3$, this implies that the solutions of (1.2) are different from (1.1). Below, we give some examples to show the existence of the solutions of (1.2).

Example 1. *For any $n \geq 5$, $f^n + ff'' + f' = e^{nz} + e^{2z} + e^z$ has exactly one entire solution $f(z) = e^z$.*

Example 2. *For $n = 5$ and $d = 2$, $f^5 - 10ff' + 5f' - 1 = 10e^{3z} + 5e^{4z} + e^{5z}$ has exactly one entire solution $f(z) = e^z + 1$.*

Example 3. *For $n = 5$ and $d = 2$, $f^5 - \frac{50}{9}f^2 - \frac{5}{9}f'f'' = e^{-5z} + 5e^{7z} + e^{10z}$ has exactly one entire solution $f(z) = e^{-z} + e^{2z}$.*

Example 4. *Let $n = 3$ and $d = 0$, then*

$$f^3 - 1 = e^{3z} + 3e^{2z} + 3e^z$$

has three entire solutions $f_j(z) = \omega_j(e^z + 1)$, where $\omega_j = e^{\frac{2j\pi i}{3}} (j = 1, 2, 3)$.

And

$$f^3 - 3 = e^{-3z} + 3e^{3z} + e^{6z}$$

has three solutions $f_j(z) = \omega_j(e^{-z} + e^{2z})$, where $\omega_j = e^{\frac{2j\pi i}{3}} (j = 1, 2, 3)$.

By Theorem 1.2, we can prove the following two corollaries.

Corollary 1.1. *Let $b_j (j = 0, 1, \cdots, k)$ be constants and $\{\lambda, p_1, p_2, p_3\} \subset \mathbb{C} \setminus \{0\}$. If f is a transcendental merommorphic solution of the following nonlinear differential equation*

$$f^4 + b_0 f + b_1 f' + b_2 f'' + \cdots + b_k f^{(k)} = p_1 e^{4\lambda z} + p_2 e^{3\lambda z} + p_3 e^{2\lambda z} \tag{1.3}$$

with $N(r, f) = S(r, f)$. Then λ is a root of polynomials $-3b_0 + b_1 z + b_2 z^2 + \cdots + b_k z^k$. In particularly, if $b_0 = 0$, then (1.3) *has no transcendental merommorphic solution satisfying $N(r, f) = S(r, f)$.*

Let $b_0 = 1, b_1 = 2, b_2 = 1$, then $-3b_0 + b_1 z + b_2 z^2$ has two roots 1 and -3. If $\lambda = 1$ or $\lambda = -3$, it can be verified that equation $f^4 + f + 2f' + f'' = e^{4\lambda z} + 4e^{3\lambda z} + 6e^{2\lambda z}$ has a solution $f(z) = -e^{\lambda z} - 1$.

Corollary 1.2. *Let d, $b_j (j = 0, 1, \cdots, k)$ be constants and $\{\lambda, p_1, p_2, p_3\} \subset \mathbb{C} \setminus \{0\}$. If f is a transcendental meromorphic solution of the following nonlinear differential equation*

$$f^4 + b_0 f + b_1 f' + b_2 f'' + \cdots + b_k f^{(k)} + d = p_1 e^{-8\lambda z} + p_2 e^{-5\lambda z} + p_3 e^{4\lambda z} \tag{1.4}$$

with $N(r, f) = S(r, f)$. Then $d = 0$ and λ is a root of polynomials $b_0 + 7b_1 z - 5b_2 z^2 + \cdots + \left(3 + (-2)^{k+1}\right) b_k z^k$.

Let $b_0 = -70, b_1 = 5, b_2 = -7$, then $b_0 + 7b_1 z - 5b_2 z^2$ has two roots 1 and -2. It can be verified that $f^4 - 70f + 5f' - 7f'' = 16e^{-8z} + 96e^{-5z} + 81e^{4z}$ has a solution $f(z) = 2e^{-2z} + 3e^{z}$ and $f^4 - 70f + 5f' - 7f'' = 81e^{16z} + 324e^{10z} + 81e^{-8z}$ has a solution $f(z) = 3e^{-4z} + 3e^{2z}$.

The paper is devoted to investigate the solutions of (1.2) under the conditions $k = 3$ and $d \leq n - 3$ and we obtain some new results. There are still many problems to be solved. For further study, we arise the following questions.

Question 1. How to find the solutions of (1.2) under the condition $k > 3$?

Question 2. How to find the solutions of (1.2) under the conditions $k = 3$ and $d \leq n - 2$?

2. Lemmas

To prove the Theorems, we need the following lemmas.

Lemma 2.1. *[1, 2] Suppose that $f(z)$ is meromorphic and transcendental in the plane and that*

$$f^n P(f) = Q(f),$$

where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f)$ is at most n. Then

$$m(r, P(f)) = S(r, f).$$

Lemma 2.2. *If f is a transcendental meromorphic solution of the following equation*

$$af^3 + bf^2 f' + cf(f')^2 + d(f')^3 = E, \tag{2.1}$$

where a, b, c, d, E are constants and $adE \neq 0$, then f satisfies one of the following cases:

(i) $f(z) = Ae^{\frac{-c}{3d}z} + B$;
(ii) $f(z) = \frac{Dd}{c} e^{\frac{c}{3d}z} - \frac{E}{D^2 c} e^{-\frac{2c}{3d}z}$,

where A, B, D are nonzero constants satisfying $B^3 = E$.

Proof. Suppose that f is a transcendental meromorphic solution of (2.1). Since $E \neq 0$, it's obviously that f has no pole and all zeros of f with multiplicity 1. By (2.1), we have

$$a + bf'/f + c(f'/f)^2 + d(f'/f)^3 = E/f^3. \tag{2.2}$$

By the lemma of logarithmic derivative and (2.2), we get $m(r, 1/f) = S(r, f)$. This implies

$$N(r, 1/f) = T(r, f) + S(r, f). \tag{2.3}$$

Differentiating (2.1), we obtain

$$3af^2 f' + bf^2 f'' + 2bf(f')^2 + 2cff'f'' + 3d(f')^2 f'' + c(f')^3 = 0. \tag{2.4}$$

Let $\omega(z) = \frac{cf'+3df''}{f}$, by a similar analysis as in Lemma 6 in [6], we can deduce $\omega(z)$ has no pole. So we have $T(r, \omega) = m(r, \omega) = S(r, f)$. Let

$$f'' = \frac{\omega}{3d} f - \frac{c}{3d} f'. \tag{2.5}$$

Substituting (2.5) into (2.4), we obtain

$$\frac{b\omega}{3d} f^3 + \left(3a - \frac{bc}{3d} + \frac{2c\omega}{3d}\right) f^2 f' + \left(2b - \frac{2c^2}{3d} + \omega\right) f(f')^2 = 0. \tag{2.6}$$

By combining (2.3) and (2.6), we conclude that

$$\begin{cases} \dfrac{b\omega}{3d} = 0, \\ 3a - \dfrac{bc}{3d} + \dfrac{2c\omega}{3d} = 0, \\ 2b - \dfrac{2c^2}{3d} + \omega = 0. \end{cases} \tag{2.7}$$

Now, we distinguish the following two cases to discuss.

Case 1. If $\omega = 0$. By (2.5), we have $f' = c_0 e^{\frac{-c}{3d}z}$, this implies $f(z) = Ae^{\frac{-c}{3d}z} + B$ with $A, B \in \mathbb{C}$. By substituting $f(z) = Ae^{\frac{-c}{3d}z} + B$ into (2.1), we have $B^3 = E$. This gives (i).

Case 2. If $\omega \neq 0$. By (2.7), it is easy to see that $b = 0$ and $4c^3 = -27ad^2$. Therefore (2.1) is equivalent to the following equation.

$$\left(f' + \frac{2c}{3d} f\right)^2 \left(df' - \frac{c}{3} f\right) = E. \tag{2.8}$$

Since f has no pole, we obtain by (2.8) that

$$\begin{cases} f' + \dfrac{2c}{3d} f = Ce^{\alpha(z)}, \\ df' - \dfrac{c}{3} f = \dfrac{E}{C^2} e^{-2\alpha(z)}, \end{cases} \tag{2.9}$$

where $\alpha(z)$ is an entire function and $C \in \mathbb{C}\backslash\{0\}$. By (2.9), we can deduce

$$\begin{cases} f = \dfrac{Cd}{c} e^{\alpha(z)} - \dfrac{E}{C^2 c} e^{-2\alpha(z)}, \\ f' = \dfrac{C}{3} e^{\alpha(z)} + \dfrac{2E}{3C^2 d} e^{-2\alpha(z)}. \end{cases} \tag{2.10}$$

It follows from (2.10) that $\alpha(z) = \frac{3c}{d} z + c_1$, where $c_1 \in \mathbb{C}$. Hence $f = \frac{Dd}{c} e^{\frac{c}{3d}z} - \frac{E}{D^2 c} e^{-\frac{2c}{3d}z}$ with $D \in \mathbb{C}\backslash\{0\}$. This gives (ii).

□

Remark. If f is a transcendental meromorphic solution of (2.7), it follows from Lemma 2.2 that $f \in \Gamma_1 \cup \Gamma_3$.

- If $f \in \Gamma_1$, then a, b, c, d satisfy $bc = 9ad$ and $3bd = c^2$ by (2.7). Suppose that $b = c = 3$ and $a = d = 1$. Then for any $A \in \mathbb{C}\backslash\{0\}$ and $B^3 = E$, it can be verified that $f(z) = Ae^{-z} + B$ is a solution of
$$f^3 + 3f^2 f' + 3f(f')^2 + (f')^3 = E.$$
- If $f \in \Gamma_3$, then a, b, c, d satisfy $b = 0$ and $4c^3 + 27ad = 0$ by (2.7). Let $a = -4$, $b = 0$, $c = 3$ and $d = 1$, then $f(z) = \frac{D}{3}e^z - \frac{E}{3D^2}e^{-2z}$ with $D \in \mathbb{C}\backslash\{0\}$ is a solution of
$$-4f^3 + 3f(f')^2 + (f')^3 = E.$$

Lemma 2.3. *Let $a_i (i = 1, 2, 3, 5)$ be constants, $a_4 = -n$, $a_6 = -3n(n-1)$, $a_7 = -n(n-1)(n-2)$ with $n \geq 3$ and $\psi \not\equiv 0$ is a small function of e^z. If f is a transcendental meromorphic solution of the following equation*

$$a_1 f^3 + a_2 f^2 f' + a_3 f^2 f'' + a_4 f^2 f''' + a_5 f(f')^2 + a_6 f f' f'' + a_7 (f')^3 = \psi \tag{2.11}$$

with $\rho(f) \geq 1$, then one of the following holds:

(i) $f(z) = c_1 e^{\alpha z} + c_2$;
(ii) $f(z) = \frac{c_3}{3\alpha} e^{-\alpha z} - \frac{(n-1)\psi}{a_5(n-2)c_3^2} e^{2\alpha z}$,

where $\alpha = \frac{a_5}{3n(n-1)^2}$ and c_1, c_2, c_3 are nonzero constants satisfying $c_2^3 = \psi$.

Proof. If f is a transcendental meromorphic solution of (2.11), we can deduce $N(r, f) = N(r, \psi) = S(r, f)$ and
$$N_{(2}(r, 1/f) \leq N\left(r, \frac{1}{\psi}\right) = S(r, f). \tag{2.12}$$

Differentiating (2.11) and it yields

$$\begin{aligned} \psi' = &\left(3a_1 f' + a_2 f'' + a_3 f^{(3)} + a_4 f^{(4)}\right) f^2 + a_5 (f')^3 + (a_6 + 3a_7)(f')^2 f'' \\ &+ \left[2a_2 (f')^2 + 2(a_3 + a_5) f' f'' + a_6 (f'')^2 + (2a_4 + a_6) f' f^{(3)}\right] f. \end{aligned} \tag{2.13}$$

Let
$$\varphi = \frac{(a_5\psi - a_7\psi')f' + (a_6 + 3a_7)\psi f''}{f}. \tag{2.14}$$

It follows from (2.11)–(2.14) that $N(r, \varphi) \leq N_{(2}\left(r, \frac{1}{f}\right) + N(r, f) + N(r, \psi) = S(r, f)$. This implies $T(r, \varphi) = S(r, f)$. Now we rewrite (2.14) as the form

$$f'' = A(z) f' + B(z) f, \tag{2.15}$$

where
$$A(z) = \frac{a_7\psi' - a_5\psi}{(a_6 + 3a_7)\psi}, \quad B(z) = \frac{\varphi}{(a_6 + 3a_7)\psi}. \tag{2.16}$$

Differentiating (2.15), we get

$$f''' = (A' + A^2 + B)f' + (B' + AB)f \tag{2.17}$$

and

$$f^{(4)} = (A'' + 3AA' + A^3 + 2AB + 2B')f' + (2A'B + AB' + A^2B + B'' + B^2)f. \tag{2.18}$$

By substituting (2.15) and (2.17) into (2.11), we have

$$\begin{aligned}(a_1 + a_3B + a_4(B' + AB))f^3 + &(a_2 + a_3A + a_4(A' + A^2 + B) + a_6B)f^2 f' \\ &+ (a_5 + a_6A)f(f')^2 + a_7(f')^3 = \psi.\end{aligned} \tag{2.19}$$

By substituting (2.15), (2.17) and (2.18) into (2.13), we obtain

$$b_1f^3 + b_2f^2f' + b_3f(f')^2 + a_7\frac{\psi'}{\psi}(f')^3 = \psi', \tag{2.20}$$

where

$$b_1 = a_2B + a_3(B' + AB) + a_4(2A'B + AB' + A^2B + B^2 + B'') + a_6B^2,$$

$$\begin{aligned}b_2 = 3a_1 + a_2A + a_3(A' + A^2 + 3B) + a_4(A^3 + 4AB + A'' + 3AA' + 4B') \\ + 2a_5B + a_6(3AB + B'),\end{aligned}$$

and

$$b_3 = 2a_2 + 2a_3A + 2a_4(A^2 + B + A') + 2a_5A + a_6(2A^2 + 2B + A') + 3a_7B.$$

By Combining (2.19) and (2.20), we have

$$\begin{aligned}&\left(b_1 - (a_1 + a_3B + a_4(B' + AB))\frac{\psi'}{\psi}\right) f^3 + \left(b_3 - (a_5 + a_6A)\frac{\psi'}{\psi}\right) f(f')^2 \\ &+ \left(b_2 - (a_2 + a_3A + a_4(A' + A^2 + B) + a_6B)\frac{\psi'}{\psi}\right) f^2 f' = 0.\end{aligned} \tag{2.21}$$

Applying the lemma of logarithmic derivative to (2.11), we get $m\left(r, \frac{1}{f}\right) = S(r, f)$. This implies $N\left(r, \frac{1}{f}\right) = T(r, f) + S(r, f)$. We then obtain by (2.21) that

$$\begin{cases}\Gamma(z) := b_1 - (a_1 + (a_3 + a_4)B + a_4B')\dfrac{\psi'}{\psi} \equiv 0, \\ \Delta(z) := b_2 - \left(a_2 + a_3A + a_4(A' + A^2) + (a_4 + a_6)B\right) \dfrac{\psi'}{\psi} \equiv 0, \\ \Lambda(z) := b_3 - (a_5 + a_6A)\dfrac{\psi'}{\psi} \equiv 0.\end{cases} \tag{2.22}$$

Now, we claim that ψ has no zero and pole.

If not, firstly, we assume that ψ has a pole and denote it by z_0. Obviously, z_0 is also a pole of f by (2.11). Hence, there is a sufficiently small neighborhood $U(z_0)$ of z_0, for any z in $U(z_0)$, by (2.11), we have

$$\psi = \mu(z - z_0)^{-3m-3} + O((z - z_0)^{-3m-2}), \quad f = \nu(z - z_0)^{-m} + O((z - z_0)^{-m+1}), \tag{2.23}$$

where $\mu\nu \neq 0$ and $m \in \mathbb{N}^+$. Note that $a_6 = -3n(n-1)$, $a_7 = -n(n-1)(n-2)$.

By substituting (2.23) into (2.14) and (2.16), we can deduce

$$A(z) = A_1(z-z_0)^{-1} + O(1), \quad B(z) = B_1(z-z_0)^{-2} + O((z-z_0)^{-1}), \tag{2.24}$$

where $A_1 = -\frac{(n-2)(m+1)}{n-1}$ and $B_1 = \frac{m(m+1)}{n-1}$. By substituting (2.23) and (2.24) into (2.22), we have

$$\Gamma(z) = a_4 B_1(K_1 + B_1 + 6)(z-z_0)^{-4} + O((z-z_0)^{-3})(z \to z_0), \tag{2.25}$$

where $K_1 = \left((\frac{n-2}{n-1})^2(m+1) + 4\frac{n-2}{n-1} + 3m - 6\right)(m+1)$. If $n \geq 4$, it is easy to see that $K_1 \geq \frac{5}{9} > 0$. By (2.22) and (2.25), we can get a contradiction. So $n = 3$. This shows that

$$\Lambda(z) = -\frac{27}{2}(m+1)(m-3)(z-z_0)^{-4} + O((z-z_0)^{-1})(z \to z_0). \tag{2.26}$$

Note that $K_1 > 0$ if $m = n = 3$. We can easily get a contradiction by (2.22), (2.25) and (2.26). This means ψ has no pole.

Secondly, if ψ has a zero and denote it by z_1. It is easy to see that z_1 is not a zero of f with multiplicity 1 by (2.11). Next, we distinguish the following two cases to discuss.

Case i). Suppose that z_1 is a zero of f with multiplicity $t(t \geq 2)$. Hence, there is a sufficiently small neighborhood $U(z_1)$ of z_1, and for any z in $U(z_1)$, by (2.11), we have

$$\psi = \mu_1(z-z_1)^s + O((z-z_1)^{s-1}), \quad f = \nu_1(z-z_1)^t + O((z-z_1)^{t-1}), \tag{2.27}$$

where $\mu_1\nu_1 \neq 0$. Furthermore, we can deduce $s = 3t - 3$.

By substituting (2.27) into (2.14) and (2.16), we get

$$A(z) = A_2(z-z_1)^{-1} + O(1), \quad B(z) = B_2(z-z_1)^{-2} + O((z-z_1)^{-1}), \tag{2.28}$$

where $A_2 = \frac{(n-2)(t-1)}{n-1}$ and $B_2 = \frac{t(t-1)}{n-1}$. By substituting (2.27) and (2.28) into (2.22), we have

$$\Gamma(z) = a_4 B_2(K_2 + A_2^2 + B_2 + 6)(z-z_1)^{-4} + O((z-z_1)^{-3})(z \to z_1),$$

where $K_2 = \left(3t + 6 - \frac{4(n-2)}{n-1}\right)(t-1) > 0$ for $n \geq 3$. This implies $\Gamma(z) \not\equiv 0$, which contradicts to (2.22). Hence z_1 is not a zero of f with multiplicity ≥ 2.

Case ii). Suppose that z_1 is not a zero of f. Note that z_1 is a pole of $A(z)$ with multiplicity 1 by (2.16), therefore z_1 is a pole of $B(z)$ with multiplicity 1 or it is not a pole of B by (2.17). Suppose that ψ satisfies (2.27), we then deduce $A(z) = \frac{(n-2)s}{3(n-1)}(z-z_1)^{-1} + O(1)$. By a calculation, we have

$$\Lambda(z) = a_4 A_3 s\left([(6n-4)A_3 - 3(n-1)]s - 3n - 1\right)(z-z_1)^{-2} + O((z-z_1)^{-1})(z \to z_1),$$

where $A_3 = \frac{n-2}{3(n-1)}$. It's easy to see that $(6n-4)A_3 - 3(n-1) < 0$ for $n \geq 3$. So $\Lambda(z) \not\equiv 0$, which also contradicts to (2.22).

By the above discussion, we conclude that ψ has no zero and pole. It follows from $T(r,\psi) = S(r,e^z)$ that ψ is a constant. We then obtain by (2.22) that $b_1 = b_2 = b_3 \equiv 0$. Furthermore, it is easy to see that $A(z) \equiv -\frac{a_5}{a_6+3a_7}$ by (2.16), we then deduce $B(z)$ is a constant by the fact $b_3 \equiv 0$. Now (2.20) becomes

$$(a_1 + a_3B + a_4AB)f^3 + (a_2 + a_3A + a_4(A^2+B) + a_6B)f^2f' + (a_5 + a_6A)f(f')^2 + a_7(f')^3 = \psi. \tag{2.29}$$

By Lemma 2.2, we have $f(z) = c_1e^{\alpha z} + c_2$ or $f(z) = \frac{c_3}{3\alpha}e^{-\alpha z} - \frac{(n-1)\psi}{a_5(n-2)c_3^2}e^{2\alpha z}$, where $\alpha = \frac{a_5}{3n(n-1)^2}$ and c_1, c_2, c_3 are nonzero constants satisfying $c_2^3 = \psi$.

□

Lemma 2.4. *[14, Corollary of Theorem 1.52] If* $f_j(z)$ $(j = 1, 2, \cdots, n+1)(n \geq 2)$, $g_j(z)(j = 1, 2, \cdots, n)(n \geq 1)$ *are entire functions satisfying the following two conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$;

(ii) When $1 \leq j \leq n+1, 1 \leq k \leq n$, *the order of* f_j *is less than that of* $e^{g_k(z)}$. *When* $n \geq 2$ *and* $1 \leq j \leq n+1, 1 \leq h \leq k \leq n$, *the orders of* f_j *are also less than that of* $e^{g_h(z)-g_k(z)}$.

Then $f_j(z) \equiv 0 (j = 1, 2, \cdots, n+1)$.

By Lemma 2.4. we have

Lemma 2.5. *Let* $a_0, \cdots, a_n$ *be constant, and let* $b_1, \cdots, b_n \in \mathbb{C}\backslash\{0\}$ *be distinct constants. Then*

$$\sum_{j=1}^{n} a_j e^{b_j z} \equiv a_0$$

holds only when $a_0 = a_1 = \cdots = a_n = 0$.

3. Proof of theorems

Proof of Theorem 1.1. Suppose that $f = \lambda_1 e^{\lambda z} + \lambda_2 e^{-2\lambda z}$ is a solution of (1.1). By substituting f into (1.1), we have

$$\lambda_1^n e^{n\lambda z} + \lambda_1 \lambda_2^{n-1} e^{(-2n+3)\lambda z} + \lambda_2^n e^{-2n\lambda z} + \sum_{j \notin \Xi} \tau_j e^{j\lambda z} = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{3.1}$$

where $\Xi = \{n, -2n+3, -2n\}$. By Lemma 2.5 and (3.1), we can get a contradiction. Hence $f \notin \Gamma_3$. This completes the proof of Theorem 1.1.

□

Proof of Theorem 1.2. Let f be a transcendental meromorphic solution of (1.2) satisfying $N(r, f) = S(r, f)$. By differentiating (1.2), we have

$$nf^{n-1}f' + (P_d(f))' = q_1 e^{\alpha_1 z} + q_2 e^{\alpha_2 z} + q_3 e^{\alpha_3 z}, \tag{3.2}$$

and

$$nf^{n-1}f'' + n(n-1)f^{n-2}(f')^2 + (P_d(f))'' = r_1 e^{\alpha_1 z} + r_2 e^{\alpha_2 z} + r_3 e^{\alpha_3 z}, \tag{3.3}$$

where $q_i = p_i \alpha_i$ and $r_i = p_i \alpha_i^2 (i = 1, 2, 3)$. Notice that (1.2), (3.2), (3.3) can be written as inhomogeneous linear systems of equations form as $AX = b$, where

$$A = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}, \ X = \begin{pmatrix} e^{\alpha_1 z} \\ e^{\alpha_2 z} \\ e^{\alpha_3 z} \end{pmatrix}, \ B = \begin{pmatrix} f^{n-1} + P_d(f) \\ nf^{n-1}f' + (P_d(f))' \\ nf^{n-1}f'' + n(n-1)f^{n-2}(f')^2 + (P_d(f))'' \end{pmatrix}.$$

By a calculation, $D = |A| = p_1 p_2 p_3 (\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1) \neq 0$. So, there is a unique solution $X^T = (e^{\alpha_1 z}, e^{\alpha_2 z}, e^{\alpha_3 z})$ of $AX = b$. It can be verified that $e^{\alpha_1 z}$ satisfies

$$\begin{aligned} p_1(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)e^{\alpha_1 z} = {} & \alpha_2 \alpha_3 f^n - (\alpha_2 + \alpha_3) nf^{n-1}f' \\ & + nf^{n-1}f'' + n(n-1)f^{n-2}(f')^2 + E_d(f), \end{aligned} \tag{3.4}$$

where

$$E_d(f) = \alpha_2\alpha_3 P_d(f) - (\alpha_2 + \alpha_3)(P_d(f))' + (P_d(f))''. \tag{3.5}$$

By differentiating (3.4) and eliminate $e^{\alpha_1 z}$, we can deduce

$$p_2 p_3(\alpha_3 - \alpha_2) f^{n-3}\Psi = R_d(f), \tag{3.6}$$

where

$$R_d(f) = (E_d(f))' - \alpha_1 E_d(f), \tag{3.7}$$

which is a differential polynomial in f with degree $\leq n-3$ and

$$\begin{aligned}\Psi(z) = {} & \alpha_1\alpha_2\alpha_3 f^3 - (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) n f^2 f' + (\alpha_1 + \alpha_2 + \alpha_3) n f^2 f'' - n f^2 f''' \\ & + (\alpha_1 + \alpha_2 + \alpha_3) n(n-1) f (f')^2 - 3n(n-1) f f' f'' - n(n-1)(n-2)(f')^3.\end{aligned} \tag{3.8}$$

By (3.6) and Lemma 2.1, we have $T(r, \Psi) = m(r, \Psi) + S(r, f) = S(r, f)$. Below, we distinguish two cases to discuss.

Case 1. If $\Psi \equiv 0$. Then $R_d(f) \equiv 0$ by (3.6). We now claim that $E_d(f) \equiv 0$, if not, by (3.7), we have

$$E_d(f) = Ae^{\alpha_1 z}, \tag{3.9}$$

where A is a non-zero constant. Therefore, by (3.4), we get

$$\begin{aligned}(p_1(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1) - A)e^{\alpha_1 z} = {} & \alpha_2\alpha_3 f^n - (\alpha_2 + \alpha_3) n f^{n-1} f' \\ & + n f^{n-1} f'' + n(n-1) f^{n-2}(f')^2,\end{aligned} \tag{3.10}$$

If $A \neq p_1(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)$, it follows from (3.10) that f has no pole and zero. Therefore $f = B_1 e^{az}$. By substituting $f = B_1 e^{az}$ into (3.10), we can deduce $a = \frac{\alpha_1}{n}$. Note that $P_d(f)$ is an differential polynomial in f of degree $d \leq n-3$ and $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3|$, it can be verified that $E_d(f) \neq Ae^{\alpha_1 z}$ by (3.5). This contradicts (3.9).

So we have $A = p_1(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)$. By (3.10), we obtain

$$\alpha_2\alpha_3 f^2 - (\alpha_2 + \alpha_3) n f f' + n f f'' + n(n-1)(f')^2 = 0. \tag{3.11}$$

It can be verified that f also has no pole and zero by (3.11), this means $f = B_2 e^{bz}$. By substituting $f = B_2 e^{bz}$ into (3.10), we get $b = \frac{\alpha_2}{n}$ or $b = \frac{\alpha_3}{n}$. This implies that $E_d(f) \neq Ae^{\alpha_1 z}$ by $|\alpha_1| \geq |\alpha_j| (j = 2, 3)$ and (3.5).

Hence $E_d(f) \equiv 0$. It follows from (3.5) that

$$\begin{aligned}p_1(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)e^{\alpha_1 z} = {} & \alpha_2\alpha_3 f^n - (\alpha_2 + \alpha_3) n f^{n-1} f' \\ & + n f^{n-1} f'' + n(n-1) f^{n-2}(f')^2.\end{aligned} \tag{3.12}$$

By (3.12), we see that f has no pole and zero. By a similar discussion as above, we have $f^n = p_1 e^{\alpha_1 z}$. This is (i) of Theorem 1.2.

Case 2. If $\Psi \not\equiv 0$. Applying the lemma of logarithmic derivative to (3.8), we get $m(r, \frac{\psi}{f^3}) = \frac{1}{3} m(, \frac{\psi}{f^3}) + S(r, f) = S(r, f)$. Therefore,

$$m(r, P_d(f)) \leq m(r, \frac{P_d(f)}{f^d}) + m(r, f^d) + S(r, f) \leq dT(r, f) + S(r, f).$$

Since $N(r, f) = S(r, f)$, by the above equation, we have

$$(n-d)T(r, f) + S(r, f) \leq T(r, f^n + P_d(f)) \leq (n+d)T(r, f) + S(r, f). \tag{3.13}$$

By (1.2) and (3.13), we get $\rho(f) = \rho(p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + p_3 e^{\alpha_3 z}) = 1$. It follows from Lemma 2.3 that $f(z) = c_1 e^{\beta z} + c_0$ or $f(z) = c_1 e^{2\beta z} + c_3 e^{-\beta z}$, where $c_i (i = 0, 1, 2, 3)$ are nonzero constants and $\beta = \frac{\alpha_1+\alpha_2+\alpha_3}{3(n-1)}$.

If $f = c_1 e^{\beta z} + c_0$, by substituting f into (1.2), we have

$$\begin{aligned} c_1^n e^{n\beta z} + n c_1^{n-1} c_0 e^{(n-1)\beta z} + n(n-1) c_1^{n-2} c_0^2 e^{(n-2)\beta z} \\ - p_1 e^{\alpha_1 z} - p_2 e^{\alpha_2 z} - p_3 e^{\alpha_3 z} + \Sigma_{i=0}^{n-3} d_i e^{i\beta z} \equiv 0. \end{aligned} \tag{3.14}$$

Note that $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3|$ and $p_1 p_2 p_3 c_1 c_0 \neq 0$, then by (3.14) and Lemma 2.5, we get

$$\begin{cases} n\beta = \alpha_1 \quad \text{ and } \ c_1^n = p_1, \\ (n-1)\beta = \alpha_2 \ \text{ and } \ n c_1^{n-1} c_0 = p_2, \\ (n-2)\beta = \alpha_3 \ \text{ and } \ n(n-1) c_1^{n-2} c_0^2 = p_3. \end{cases}$$

By the above equations, we can deduce $\alpha_j = \frac{n-j}{n}\alpha_1 (j = 2, 3)$, which satisfies the (ii) of Theorem 1.2.

If $f = c_2 e^{2\beta z} + c_3 e^{-\beta z}$, substitute f into (1.2), we have

$$c_2^n e^{2n\beta z} + c_3^n e^{-n\beta z} - p_1 e^{\alpha_1 z} - p_2 e^{\alpha_2 z} - p_3 e^{\alpha_3 z} + \Sigma_{i=1}^{n-1} d_i e^{(3j-n)\beta z} \equiv 0. \tag{3.15}$$

By (3.15) and Lemma 2.5, we have

$$\begin{cases} 2n\beta = \alpha_1 \quad \text{and} \ \ c_2^n = p_1, \\ -n\beta = \alpha_3 \ \ \text{and} \ \ c_3^n = p_3. \end{cases} \tag{3.16}$$

Since $\beta = \frac{\alpha_1+\alpha_2+\alpha_3}{3(n-1)}$, we can deduce $\alpha_2 = \frac{2n-3}{2n}\alpha_1$ and $\alpha_3 = -\frac{1}{2}\alpha_1$ by (3.16). So, we have (iii) of Theorem 1.2.

Furthermore, if $d < n-3$, then by (3.6) and lemma 2.1, we have $T(r, \Psi) = S(r, f) = T(r, f\Psi)$. This means $\Psi \equiv 0$. It follows that $f^n = p_1 e^{\alpha_1 z}$ by the above discussion of Case 1. Hence we complete the proof of Theorem 1.2. □

Proof of Corollary 1.1. Suppose that f is a transcendental meromorphic solution of (1.3) satisfying $N(r, f) = S(r, f)$. By Theorem 1.2, we have

$$f(z) = \lambda_1 e^{\lambda z} + \lambda_0, \tag{3.17}$$

where λ_0 and λ_1 are non-zero constant satisfying $\lambda_1^n = p_1$. By substituting (3.17) into (1.3), we can deduce

$$b_0 + \lambda_0^3 = 0 \tag{3.18}$$

$$-3b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_k\lambda^k = 0.$$

This means λ is a root of polynomials $-3b_0 + b_1 z + b_2 z^2 + \cdots + b_k z^k$.

In particularly, if $a_0 = 0$, it follows from (3.18) that $\lambda_0 = 0$. It is a contradiction. Hence, (1.3) has no transcendental merommorphic solution satisfying $N(r, f) = S(r, f)$. This completes the proof of Corollary 1.1.

□

Proof of Corollary 1.2. Suppose that f is a transcendental meromorphic solution of (1.4) satisfying $N(r, f) = S(r, f)$. By Theorem 1.2, we have

$$f(z) = \lambda_1 e^{-2\lambda z} + \lambda_3 e^{\lambda z}, \tag{3.19}$$

where λ_0 and λ_1 are non-zero constant satisfying $\lambda_j^n = p_j (j = 1, 3)$. By substituting (3.19) into (1.4), we can deduce that $d = 0$ and

$$b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_k\lambda^k + 4\lambda_1\lambda_3^2 = 0, \tag{3.20}$$

$$b_0 - 2b_1\lambda + 4b_2\lambda^2 + \cdots + (-2)^k b_k\lambda^k + 6\lambda_1\lambda_3^2 = 0. \tag{3.21}$$

By (3.20) and (3.21), we can deduce λ is a root of polynomials $b_0+7b_1z-5b_2z^2+\cdots+\left(3 + (-2)^{k+1}\right) b_k z^k$. We complete the proof of Corollary 1.2.

□

4. Conclusions

In this paper, we consider the meromorphic solutions of (1.2) with few poles under the conditions $k = 3$, $n \geq 3$ and $d \leq n-3$. We proved that all of the solutions of (1.2) are in $\Gamma_0 \cup \Gamma_1 \cup \Gamma_3$. In particular, for $n - 3 = d = 1$, if the following differential equation

$$f^4 + b_0 f + b_1 f' + b_2 f'' + \cdots + b_k f^{(k)} = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + p_3 e^{\alpha_3 z}$$

has a transcendental meromorphic solution f satisfying $N(r, f) = S(r, f)$, then $\{\alpha_1, \alpha_2, \alpha_3\} = \{4\lambda, 3\lambda, 2\lambda\}$ or $\{\alpha_1, \alpha_2, \alpha_3\} = \{-8\lambda, -5\lambda, 4\lambda\}$. Moreover

(1) λ is a root of polynomials $-3b_0 + b_1z + b_2z^2 + \cdots + b_kz^k$ if $\{\alpha_1, \alpha_2, \alpha_3\} = \{4\lambda, 3\lambda, 2\lambda\}$;
(2) λ is a root of polynomials $b_0 + 7b_1z - 5b_2z^2 + \cdots + \left(3 + (-2)^{k+1}\right) b_kz^k$ if $\{\alpha_1, \alpha_2, \alpha_3\} = \{-8\lambda, -5\lambda, 4\lambda\}$.

It is a natural idea to investigate the solutions of (1.2) for $k = 3$, $n \geq 2$ and $d \leq n - 2$. Are all entire solutions of (1.2) in $\Gamma_0 \cup \Gamma_1 \cup \Gamma_3$ under this conditions? In fact, the following examples show that (1.2) has other forms of entire solutions.

- Let $n = 2$ and $d = 1$, then $f(z) = e^z + 6e^{-z} + 6e^{-2z}$ is a solution of

$$f^2 + f' - f'' - 12 = e^{2z} + 72e^{-3z} + 36e^{-4z}.$$

- Let $n = 3, 4$ and $d = n - 2$, then $f(z) = e^{-z} + e^z$ is a solution of

$$f^3 - \frac{3}{2}(f + f') = e^{-3z} + 3e^{-z} + e^{3z}$$

and

$$f^4 - (f + f')^2 - 6 = e^{-4z} + 4e^{-2z} + e^{4z}.$$

- Let $n = 3$ and $d = 1$, then $f(z) = e^z + e^{3z}$ is a solution of

$$f^3 - \frac{1}{2}(f' - f) = 3e^{5z} + 3e^{7z} + e^{9z}.$$

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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