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Research article

General and optimal decay rates for a viscoelastic wave equation with strong damping

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Abstract: This work is devoted to investigating the decay properties for a nonlinear viscoelastic wave equation with strong damping. Under certain class of relaxation functions and initial data and using the perturbed energy method, we obtain general and optimal decay results.

Keywords: viscoelastic wave equation; strong damping; Lyapunov function; decay
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1. Introduction

In this article, we study the following nonlinear viscoelastic problem

$$\begin{cases} |u_t|^\rho u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_t = u|u|^{p-2}, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of R^n $(n \geq 1)$ with a smooth boundary $\partial\Omega$, g denote the kernel of memory term and the index number p and ρ satisfy the following conditions

$$\begin{cases} 2 < p < +\infty, \text{ if } n = 1,2, & 2 < p \leq \frac{2(n-1)}{n-2}, \text{ if } n \geq 3, \\ 0 < \rho < +\infty, \text{ if } n = 1,2, & 0 < \rho \leq \frac{2}{n-2}, \text{ if } n \geq 3. \end{cases} \tag{1.2}$$

This model of equation in (1.1) arise in the theory of viscoelasticity and physics and represent the propagation of several materials which possess a capacity to storage and dissipate mechanical energy. In the last half century, the existence and stability properties of solutions have been considered by many mathematicians, to motivate our work, we recall some literature related to our work.

In the absence of viscoelastic term and $\rho = 0$, Xu and Lian [18] studied the following initial boundary value problem at three different initial energy levels,

$$u_{tt} - \triangle u - \omega\triangle u_t + \mu u_t = u \ln |u|, (x, t) \in \Omega \times (0, \infty),$$

they proved the local existence of weak solution, and in the framework of potential well, they showed the global existence, energy decay of the solution with sub-critical initial energy, then by scaling technique, parallelly extended all the results for the subcritical case to the critical case. A similar result was also obtained by [2, 20].

In the case when $\rho = 0$ and the strong damping term $-\triangle u_t$ is replaced by the damping mechanism $u_t|u_t|^{m-2}$, the form of the classical equation as follows,

$$u_{tt} - \triangle u + \int_0^t g(t - s)\triangle u(s)ds + au_t|u_t|^{m-2} = bu|u|^{p-2}, \text{ in } \Omega \times (0, \infty), \tag{1.3}$$

where a, b are two positive constant, the index number $m \geq 1, p \geq 2$, Ω is a bounded domain. Messaoudi [17] discussed the interaction between the damping term and the source term, which was first considered by Levine [8, 10] for $m = 1$. He obtained, under suitable conditions on g and initial data, that the solutions exist globally for any initial data if $m \geq p$ and blow up in finite time with negative initial energy if $p > m$. On the other hand, Messaoudi [15, 16] considered equation (1.3) for $a = b = 0$ or $a = 0$, by using the perturbed energy method and under the supposition that $g'(t) \leq -\xi(t)g(t)$, proved that the solution energy is general decay not necessarily of exponential or polynomial type.

In the case when $\rho \neq 0$, as the following class of quasilinear viscoelastic equations,

$$|u_t|^\rho u_{tt} - \triangle u - \triangle u_{tt} + \int_0^t g(t - s)\triangle u(s)ds - \gamma\triangle u_t + au_t|u_t|^{m-2} = bu|u|^{p-2}, \tag{1.4}$$

defined in a bounded domain and the index number $m \geq 1, p > 2$. This equation can model some materials whose density rely on the velocity u_t. The asymptotic behavior of solutions for equation (1.4) has been studied by many authors. For example, when $a = b = 0$, Cavalcanti et al. [1] proved that there exist the global results for $\gamma \geq 0$ and exponential decay for $\gamma > 0$. When $a = 0$, by using the potential well method, Messaoudi et al. [14] obtained a global existence and an exponential decay result. When $a = \gamma = 0$, Liu [11], by constructing a suitable Lyapunov function and using the perturbed energy method, proved that the solution energy is general decay. Furthermore, when $a = b = \gamma = 0$, Messaoudi and Al-Khulaifi [13] establish a general and optimal decay of solution energy with the relaxation function satisfies $g'(t) \leq -\xi(t)g^\theta(t)$, $1 \leq \theta < \frac{3}{2}$. Also, in [12], under the condition $g'(t) \leq -\varepsilon(t)\chi(g(t))$, where χ is increasing and convex without any additional constraints, Mustafa established energy decay results that address both the optimality and generality by using the multiplier method and some properties of convex functions. For results of the same nature, we refer the readers to [4, 6, 7, 19] and the references therein.

As far as we know, the decay property for equation (1.1) has not been considered. In this article, we consider the decay property of solution energy for problem (1.1), we obtain the following result: under certain class of relaxation functions and initial data, by using some inequalities and constructing a suitable Lyapunov function, we establish a general decay result for problem (1.1). Moreover, without restrictive conditions, we also obtain the optimal polynomial decay which seldom appear in previous literature.

This article is organized as follows. In section 2, we present some material needed for our work. In section 3, we show the global existence of solution and establish the general decay result.

2. Preliminaries

In this part, we give some theorems and lemmas needed in the proof of our results. Firstly, we make the following assumptions.

(A1) $g(t) : [0, +\infty) \to (0, +\infty)$ is a non-increasing C^1 function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = l > 0.$$

(A2) There exists a positive differentiable function $\xi(t) : [0, +\infty) \to (0, +\infty)$ such that

$$g'(t) \le -\xi(t)g^r(t), \quad t \ge 0, \quad 1 \le r < \frac{3}{2},$$

and $\xi(t)$ satisfies

$$\xi'(t) \le 0, \quad \int_0^{+\infty} \xi(t)dt = +\infty, \quad \forall\ t > 0.$$

For our work, we introduce the following functionals:

$$\begin{aligned} J(t) &= \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u(t)\|_p^p, \\ I(t) &= (1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 + (g \circ \nabla u)(t) - \|u(t)\|_p^p, \\ E(t) &= \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u(t)\|_p^p, \end{aligned}$$

where

$$(g \circ \nabla v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2 d\tau.$$

Then, we state a local existence theorem to the problem (1.1) that can be proved by combining arguments of [1,3,9].

Theorem 2.1 ([5], Theorem 2.1)**.** Suppose that (1.2) and $(A1)$ hold and initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ is given. Then problem (1.1) has a unique local solution

$$u \in C([0, T]); H_0^1(\Omega)), \quad u_t \in C([0, T]; H_0^1(\Omega)).$$

Lemma 2.2. Assume the assumption (1.2) and $(A1)$ hold. Let u be a solution of (1.1). Then $E(t)$ is non-increasing. In addition, we get the following energy inequality

$$\frac{d}{dt}E(t) = -\|\nabla u_t\|_2^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le 0. \tag{2.1}$$

Proof. Multiplying (1.1) by u_t and integrating over Ω, we can get

$$\frac{d}{dt}\Big\{\frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx\Big\}$$

$$-\int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\nabla u(s)dxds = -\int_\Omega |\nabla u_t|^2 dx. \tag{2.2}$$

Now, we estimate the last term in the left-hand side of equation (2.2) as follows

$$\begin{aligned}
&\int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\nabla u(s)dxds \\
&= \int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\Big[\nabla u(s)-\nabla u(t)\Big]dxd\tau + \int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\nabla u(t)dxds \\
&= -\frac{1}{2}\int_0^t g(t-s)\Big(\frac{d}{dt}\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dx\Big)ds + \int_0^t g(s)\Big(\frac{d}{dt}\frac{1}{2}\int_\Omega |\nabla u(t)|^2 dx\Big)ds \\
&= -\frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds\Big] + \frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(s)\int_\Omega |\nabla u(t)|^2 dxds\Big] \\
&\quad + \frac{1}{2}\int_0^t g'(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2 dx.
\end{aligned} \tag{2.3}$$

Insert (2.3) into (2.2), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big\{\frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx\Big\} \\
&\quad - \frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(s)ds\|\nabla u(t)\|_2^2\Big] + \frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds\Big] \\
&= -\int_\Omega |\nabla u_t|^2 dx + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le 0.
\end{aligned} \tag{2.4}$$

This complete the proof. □

The following Lemma is used for studying the decay of solution.

Lemma 2.3 ([13], Corollary 2.1)**.** Assume that $g(t)$ satisfies $(A1)$ and $(A2)$ and u is the solution of (1.1) then we have

$$\xi(t)(g\circ\nabla u)(t) \le C[-E'(t)]^{\frac{1}{2r-1}}.$$

3. The decay result

In this section, we state and prove the decay result for global solutions. Firstly, we establish the global existence theorem.

Lemma 3.1. Assume the assumption $(A1)$, $(A2)$ and (1.2) hold and the initial data $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ such that

$$\vartheta := \frac{C^p}{l}\Big(\frac{2p}{l(p-2)}E(0)\Big)^{(p-2)/2} < 1, \quad I(u_0) > 0, \tag{3.1}$$

then $I(u(t)) > 0$ for $\forall\ t > 0$.

Proof. Due to $I(u_0) > 0$, then there exists a time $T_m < T$ such that $I(u(t)) \ge 0$ for $\forall t \in [0, T_m)$. So, we have

$$J(t) = \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{p}\|u(t)\|_p^p$$

$$= \frac{p-2}{2p}(1-\int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + \frac{p-2}{2p}(g\circ\nabla u)(t) + \frac{1}{p}I(u(t))$$
$$\geq \frac{p-2}{2p}\Big\{(1-\int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t)\Big\},\ \forall\ t\in[0,T_m). \tag{3.2}$$

By applying $(A1)$, (2.1) and (3.2), we deduce

$$\begin{aligned} l\|\nabla u(t)\|_2^2 &\leq (1-\int_0^t g(s)ds)\|\nabla u(t)\|_2^2 \leq \frac{2p}{p-2}J(t) \\ &\leq \frac{2p}{p-2}E(t) \\ &\leq \frac{2p}{p-2}E(0), \quad \forall\ t\in[0,T_m). \end{aligned} \tag{3.3}$$

Then, combine $(A1)$, (3.1) and (3.3), we arrive at

$$\begin{aligned} \|u(t)\|_p^p &\leq C^p\|\nabla u(t)\|_2^p \\ &\leq \frac{C^p}{l}\|\nabla u(t)\|_2^{p-2} l\|\nabla u(t)\|_2^2 \\ &\leq \frac{C^p}{l}\Big[\frac{2p}{l(p-2)}E(0)\Big]^{(p-2)/2} l\|\nabla u(t)\|_2^2 \\ &\leq \vartheta l\|\nabla u(t)\|_2^2 \\ &< (1-\int_0^t g(\tau)d\tau)\|\nabla u(t)\|_2^2. \end{aligned} \tag{3.4}$$

Therefore,

$$I(t) = (1-\int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t) - \|u(t)\|_p^p > 0,$$

for $\forall t\in[0,T_m)$. Repeating the process and using the fact that

$$\lim_{t\to T_m}\frac{C^p}{l}\Big(\frac{2p}{l(p-2)}E(0)\Big)^{(p-2)/2} \leq \vartheta < 1,$$

$T_m$is extended to T. □

Theorem 3.2. Assume that $(A1)$, $(A2)$ and (1.2) hold, and the initial data $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ and satisfies (3.1), then the solution is global and bounded.

Proof. Our objective is to prove $\|\nabla u(t)\|_2^2 + \|u_t(t)\|_{\rho+2}^{\rho+2}$ is bounded independently of t. For the aim, we apply the Lemma 3.1 to obtain

$$\begin{aligned} E(0) \geq E(t) &= J(t) + \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} \\ &\geq \frac{p-2}{2p}\Big((1-\int_0^t g(s)ds)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t)\Big) + \frac{1}{p}I(u(t)) + \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} \end{aligned}$$

$$\geq \frac{p-2}{2p}\Big((1-\int_0^t g(s)ds)\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)\Big)+\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2},$$

for $I(u(t))\geq 0$ and $(g\circ\nabla u)(t)$ is positive. Therefore, we have

$$\|\nabla u(t)\|_2^2+\|u_t(t)\|_{\rho+2}^{\rho+2}\leq CE(0),\quad 0\leq t<\infty.$$

Then we complete the proof. □

For obtaining the general decay rate estimate, let us consider the following functionals

$$L(t):=ME(t)+\varepsilon\chi(t)+\zeta(t), \tag{3.5}$$

where M and ε are positive constants and

$$\chi(t):=\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx,$$
$$\zeta(t):=-\int_\Omega \frac{|u_t|^\rho u_t}{\rho+1}\int_0^t g(t-s)(u(t)-u(s))dsdx.$$

Lemma 3.3. For M large enough while ε is small enough, then we have the following relation

$$\alpha_1 L(t)\leq E(t)\leq \alpha_2 L(t) \tag{3.6}$$

holds, where α_1 and α_2 are two positive constants.

Proof. Applying Hölder inequality, Young inequality, Sobolev embedding theorem and (3.3) we have

$$\begin{aligned}\Big|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx\Big| &\leq \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+1}\|u\|_{\rho+2}\\ &\leq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{(\rho+1)(\rho+2)}\|u\|_{\rho+2}^{\rho+2}\\ &\leq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{C^{\rho+2}}{(\rho+1)(\rho+2)}\|\nabla u\|_2^{\rho+2}\\ &\leq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{C^{\rho+2}}{(\rho+1)(\rho+2)}\Big(\frac{2pE(0)}{l(p-2)}\Big)^{\frac{\rho}{2}}\|\nabla u\|_2^2,\end{aligned}$$

and

$$\begin{aligned}&\Big|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g(t-s)((u(t)-u(s))dsdx\Big|\\ &\leq \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+1}\Big(\int_\Omega\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^{\rho+2}dx\Big)^{\frac{1}{\rho+2}}\\ &\leq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{(\rho+1)(\rho+2)}\int_\Omega\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^{\rho+2}dx\\ &\leq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{C^{\rho+2}(1-l)^{\rho+1}}{(\rho+1)(\rho+2)}\Big(\frac{4pE(0)}{l(p-2)}\Big)^{\frac{\rho}{2}}(g\circ\nabla u)(t)\end{aligned}$$

When M is large enough and ε is small enough, we arrive at

$$\begin{aligned}L(t) &\leq ME(t) + \frac{\varepsilon+1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{C^{\rho+2}\varepsilon}{(\rho+1)(\rho+2)}\Big(\frac{2pE(0)}{l(p-2)}\Big)^{\frac{\rho}{2}}\|\nabla u\|_2^2\\ &+ \frac{C^{\rho+2}(1-l)^{\rho+1}}{(\rho+1)(\rho+2)}\Big(\frac{4pE(0)}{l(p-2)}\Big)^{\frac{\rho}{2}}(g\circ\nabla u)(t)\\ &\leq \frac{\varepsilon+1+M}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \Big[\frac{M}{2} + \frac{C^{\rho+2}(1-l)^{\rho+1}}{(\rho+1)(\rho+2)}\Big(\frac{4pE(0)}{l(p-2)}\Big)^{\frac{\rho}{2}}\Big](g\circ\nabla u)(t)\\ &+ \Big(\frac{M}{2}(1-\int_0^t g(\tau)d\tau) + \frac{C^{\rho+2}\varepsilon}{(\rho+1)(\rho+2)}\Big(\frac{2pE(0)}{l(p-2)}\Big)^{\frac{\rho}{2}}\Big)\|\nabla u\|_2^2\\ &\leq \frac{1}{\alpha_1}E(t).\end{aligned}$$

Analogously, we have

$$\begin{aligned}L(t) &\geq \frac{M-\varepsilon-1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \Big[\frac{M}{2} - \frac{C^{\rho+2}(1-l)^{\rho+1}}{(\rho+1)(\rho+2)}\Big(\frac{4pE(0)}{l(p-2)}\Big)^{\frac{\rho}{2}}\Big](g\circ\nabla u)(t)\\ &+ \Big(\frac{M}{2}(1-\int_0^t g(s)ds) - \frac{C^{\rho+2}\varepsilon}{(\rho+1)(\rho+2)}\Big(\frac{2pE(0)}{l(p-2)}\Big)^{\frac{\rho}{2}}\Big)\|\nabla u\|_2^2\\ &\geq \frac{1}{\alpha_2}E(t).\end{aligned}$$

□

Lemma 3.4. Under the assumption $(A1)$ and $(A2)$, let u be the solution of (1.1), then the functional

$$\chi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx, \tag{3.7}$$

satisfies

$$\begin{aligned}\chi'(t) &\leq \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - (\frac{l}{2} - \frac{1}{4\eta_2})\|\nabla u\|_2^2 + \frac{1-l}{2l}(g\circ\nabla u)(t)\\ &+ \eta_2\|\nabla u_t\|_2^2 + \|u\|_p^p.\end{aligned} \tag{3.8}$$

Proof. Taking a time derivative of (3.7) and applying equation (1.1), we can deduce

$$\begin{aligned}\chi'(t) &= \int_\Omega |u_t|^\rho u_{tt} u dx + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2}\\ &= \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - \|\nabla u(t)\|_2^2 + \int_\Omega\int_0^t g(t-s)\nabla u(s)ds\nabla u(t)dsdx\\ &- \int_\Omega \nabla u_t \nabla u dx + \|u\|_p^p.\end{aligned} \tag{3.9}$$

We now estimate the third term on the right-hand side of (3.9), yields

$$\begin{aligned}
&\int_\Omega \nabla u(t) \int_0^t g(t-s)\nabla u(s)dsdx \\
&\quad \leq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega \Big(\int_0^t g(t-s)\nabla u(s)ds\Big)^2 dx \\
&\quad \leq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega \Big(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)| + |\nabla u(t)|)ds\Big)^2 dx, \qquad (3.10)
\end{aligned}$$

at present, estimate the second term in the right-hand side of (3.10), for $\forall\ \eta_1 > 0$, we can arrive at

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)| + |\nabla u(t)|)ds\Big)^2 dx \\
&\leq \int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2 dx + \int_\Omega \Big(\int_0^t g(t-s)|\nabla u(t)|ds\Big)^2 dx \\
&\quad + 2\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)\Big(\int_0^t g(t-s)|\nabla u(t)|ds\Big)dx \\
&\leq (1+\frac{1}{\eta_1})\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2 dx \\
&\quad + (1+\eta_1)\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(t)|ds\Big)^2 dx \\
&\leq (1+\frac{1}{\eta_1})(1-l)(g\circ\nabla u)(t) + (1+\eta_1)(1-l)^2\|\nabla u\|_2^2. \qquad (3.11)
\end{aligned}$$

Inserting (3.11) into (3.10), we get

$$\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx \\
&\quad \leq \frac{1}{2}(1+(1+\eta_1)(1-l)^2)\|\nabla u\|_2^2 + \frac{1}{2}(1+\frac{1}{\eta_1})(1-l)(g\circ\nabla u)(t). \qquad (3.12)
\end{aligned}$$

For the forth term in the right-hand side of (3.9), for $\forall\ \eta_2 > 0$, we can get

$$\begin{aligned}
&\int_\Omega \nabla u_t \nabla u dx \\
&\quad \leq \eta_2\|\nabla u_t\|_2^2 + \frac{1}{4\eta_2}\|\nabla u\|_2^2. \qquad (3.13)
\end{aligned}$$

Inserting (3.12) and (3.13) into (3.9), and choosing $\eta_1 = l/(1-l)$, we can deduce

$$\begin{aligned}
\chi'(t) \leq& \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - (\frac{l}{2} - \frac{1}{4\eta_2})\|\nabla u\|_2^2 + \frac{1-l}{2l}(g\circ\nabla u)(t) \\
&+ \eta_2\|\nabla u_t\|_2^2 + \|u\|_p^p. \qquad (3.14)
\end{aligned}$$

□

Lemma 3.5. Under the assumption $(A1)$ and $(A2)$, and let u be the solution of (1.1), then the functional

$$\zeta(t) = -\int_\Omega \frac{|u_t|^\rho u_t}{\rho+1}\int_0^t g(t-s)(u(t)-u(s))dsdx, \tag{3.15}$$

satisfies

$$\begin{aligned}\zeta'(t) \le\ & \delta\Big(1+2(1-l)^2+C^{2(p-1)}\Big(\frac{2pE(0)}{(p-2)l}\Big)^{p-2}\Big)\|\nabla u\|_2^2 \\ & +(1-l)\Big(2\delta+\frac{3}{4\delta}+\frac{C^2}{4\delta}\Big)(g\circ\nabla u)(t)-\frac{g^{\rho+1}(0)C^{\rho+2}}{(\rho+1)(\rho+2)}(g'\circ\nabla u)(t) \\ & -\Big(\frac{1}{\rho+1}(\int_0^t g(s)ds)-\frac{1}{\rho+2}\Big)\|u_t\|_{\rho+2}^{\rho+2}+\delta\|\nabla u_t\|_2^2.\end{aligned} \tag{3.16}$$

Proof. Taking a derivative of $\zeta(t)$, we have

$$\begin{aligned}\zeta'(t) =& -\int_\Omega |u_t|^\rho u_{tt}\int_0^t g(t-s)(u(t)-u(s))dsdx \\ & -\int_\Omega \frac{|u_t|^\rho u_t}{\rho+1}\int_0^t g'(t-s)(u(t)-u(s))dsdx \\ & -\frac{1}{\rho+1}\Big(\int_0^t g(s)ds\Big)\|u_t\|_{\rho+2}^{\rho+2}.\end{aligned} \tag{3.17}$$

Inserting equation (1.1) into (3.17), we get

$$\begin{aligned}\zeta'(t) =& -\int_\Omega \triangle u\int_0^t g(t-s)(u(t)-u(s))dsdx \\ & +\int_\Omega\Big(\int_0^t g(t-s)\Delta u(s)ds\Big)\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)dx \\ & -\int_\Omega \triangle u_t\int_0^t g(t-s)(u(t)-u(s))dsdx \\ & -\int_\Omega u|u|^{p-2}\int_0^t g(t-s)(u(t)-u(s))dsdx \\ & -\int_\Omega \frac{|u_t|^\rho u_t}{\rho+1}\int_0^t g'(t-s)(u(t)-u(s))dsdx \\ & -\frac{1}{\rho+1}\Big(\int_0^t g(s)ds\Big)\|u_t\|_{\rho+2}^{\rho+2}.\end{aligned} \tag{3.18}$$

We now estimate the first term in the right-hand side of (3.18), for $\forall\ \delta>0$, we can deduce

$$\begin{aligned}& -\int_\Omega \Delta u\int_0^t g(t-s)(u(t)-u(s))dsdx \\ & \le \int_\Omega \nabla u\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\ & \le \delta\|\nabla u\|_2^2+\frac{1-l}{4\delta}(g\circ\nabla u)(t).\end{aligned} \tag{3.19}$$

For the second term

$$
\begin{aligned}
&\int_\Omega \Big(\int_0^t g(t-s)\Delta u(s)ds\Big)\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)dx\\
&\le -\int_\Omega \Big(\int_0^t g(t-s)\nabla u(s)ds\Big)\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s)ds\Big)dx\\
&\le \delta\int_\Omega \Big(\int_0^t g(t-s)\nabla u(s)ds\Big)^2dx+\frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big)^2dx\\
&\le \delta\int_\Omega \Big(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|)ds\Big)^2dx\\
&\quad+\frac{1}{4\delta}\int_0^t g(t-s)ds\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2dsdx\\
&\le 2\delta\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\Big)^2dx\\
&\quad+2\delta(1-l)^2\int_\Omega|\nabla u|^2dx+\frac{1}{4\delta}(1-l)(g\circ\nabla u)(t)\\
&\le (2\delta+\frac{1}{4\delta})(1-l)(g\circ\nabla u)(t)+2\delta(1-l)^2\|\nabla u\|_2^2.
\end{aligned}
\tag{3.20}
$$

Similarly, we have

$$
\begin{aligned}
&-\int_\Omega \triangle u_t\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&\quad\le \int_\Omega \nabla u_t\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\\
&\quad\le \delta\|\nabla u_t\|_2^2+\frac{1-l}{4\delta}(g\circ\nabla u)(t),
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&-\int_\Omega u|u|^{p-2}\int_0^t g(t-s)(u(t)-u(s))dsdx\\
&\quad\le \delta\|u\|_{2(p-1)}^{2(p-1)}+\frac{1-l}{4\delta}C^2(g\circ\nabla u)(t)\\
&\quad\le \delta C^{2(p-1)}\Big(\frac{2pE(0)}{(p-2)l}\Big)^{p-2}\|\nabla u\|_2^2+\frac{1-l}{4\delta}C^2(g\circ\nabla u)(t),
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
&-\int_\Omega \frac{|u_t|^\rho u_t}{\rho+1}\int_0^t g'(t-s)(u(t)-u(s))dsdx\\
&\le \frac{1}{\rho+1}\Big[\|u_t\|_{\rho+2}^{\rho+1}\int_\Omega\Big(\int_0^t g'(t-s)(u(t)-u(s))ds\Big)^{\rho+2}dx\Big]\\
&\le \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{(\rho+1)(\rho+2)}\int_\Omega\Big(\int_0^t g'(t-s)(u(t)-u(s))ds\Big)^{\rho+2}dx\\
&\le \frac{1}{(\rho+1)(\rho+2)}\int_\Omega\Big(\int_0^t g'(t-s)ds\Big)^{\rho+1}\Big(\int_0^t g'(t-s)|u(t)-u(s)|^{\rho+2}ds\Big)dx
\end{aligned}
$$

$$+\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}$$
$$\leq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} - \frac{g^{\rho+1}(0)C^{\rho+2}}{(\rho+1)(\rho+2)}(g' \circ \nabla u)(t). \tag{3.23}$$

Combining (3.18)–(3.23), we can deduce

$$\begin{aligned}\zeta'(t) \leq\ & \delta\Big(1 + 2(1-l)^2 + C^{2(p-1)}\Big(\frac{2pE(0)}{(p-2)l}\Big)^{p-2}\Big)\|\nabla u\|_2^2 \\ & + (1-l)\Big(2\delta + \frac{3}{4\delta} + \frac{C^2}{4\delta}\Big)(g \circ \nabla u)(t) - \frac{g^{\rho+1}(0)C^{\rho+2}}{(\rho+1)(\rho+2)}(g' \circ \nabla u)(t) \\ & - \Big(\frac{1}{\rho+1}(\int_0^t g(s)ds) - \frac{1}{\rho+2}\Big)\|u_t\|_{\rho+2}^{\rho+2} + \delta\|\nabla u_t\|_2^2. \end{aligned} \tag{3.24}$$

□

Theorem 3.6. Let initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given and satisfy (3.1). Assume that (1.2), (A1) and (A2) hold. Then, for each $t_0 > 0$, there exist strictly positive constants K and k such that the solution of (1.1) satisfies, for all $t \geq t_0$,

$$E(t) \leq Ke^{-k\int_{t_0}^t \xi(s)ds}, \quad r = 1, \tag{3.25}$$
$$E(t) \leq K\Big[\frac{1}{1 + \int_{t_0}^t \xi^{2r-1}(s)ds}\Big]^{\frac{1}{2r-2}}, \quad 1 < r < \frac{3}{2}. \tag{3.26}$$

Proof. Taking a derivative of (3.5), we can obtain

$$L'(t) = ME'(t) + \varepsilon\chi'(t) + \zeta'(t). \tag{3.27}$$

Since g is continuous and $g(0) > 0$, then there exists $t_0 > 0$ such that

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \quad \forall\ t \geq t_0. \tag{3.28}$$

By using (2.1), (3.8), (3.16) and (3.27), we arrive at

$$\begin{aligned} L'(t) \leq\ & -(M - \varepsilon\eta_2 - \delta)\|\nabla u_t\|_2^2 + \Big(\frac{M}{2} - \frac{g^{\rho+1}(0)C^{\rho+2}}{(\rho+1)(\rho+2)}\Big)(g' \circ \nabla u)(t) \\ & - \Big[\Big(\frac{l}{2} - \frac{1}{4\eta_2}\Big)\varepsilon - \delta\Big(1 + 2(1-l)^2 + C^{2(p-1)}\Big(\frac{2pE(0)}{(p-2)l}\Big)^{p-2}\Big)\Big]\|\nabla u\|_2^2 \\ & + (1-l)\Big(\frac{\varepsilon}{2l} + 2\delta + \frac{3}{4\delta} + \frac{C^2}{4\delta}\Big)(g \circ \nabla u)(t) + \|u\|_p^p \\ & - \Big(\frac{g_0 - \varepsilon}{\rho+1} - \frac{1}{\rho+2}\Big)\|u_t\|_{\rho+2}^{\rho+2}. \end{aligned} \tag{3.29}$$

At this point, we choose $\varepsilon < g_0$ such that

$$\frac{g_0 - \varepsilon}{\rho+1} - \frac{1}{\rho+2} > 0, \tag{3.30}$$

then take δ and η_2 small enough such that

$$\Big(\frac{l}{2}-\frac{1}{4\eta_2}\Big)\varepsilon-\delta\Big(1+2(1-l)^2+C^{2(p-1)}\Big(\frac{2pE(0)}{(p-2)l}\Big)^{p-2}\Big)>0. \tag{3.31}$$

Once δ and η_2 are fixed, we choose M sufficiently large such that

$$M-\varepsilon\eta_2-\delta>0, \tag{3.32}$$

and

$$\frac{M}{2}-\frac{g^{\rho+1}(0)C^{\rho+2}}{(\rho+1)(\rho+2)}>0. \tag{3.33}$$

Hence, there exist two positive constants k_1 and k_2 such that

$$L'(t)\le -k_1E(t)+k_2(g\circ\nabla u)(t),\quad \forall\ t\ge t_0. \tag{3.34}$$

Multiplying both sides of (3.34) by $\xi(t)$, we obtain

$$\xi(t)L'(t)\le -k_1\xi(t)E(t)+k_2\xi(t)(g\circ\nabla u)(t),\quad \forall\ t\ge t_0. \tag{3.35}$$

In the case when $r=1$, by using (A2) and (2.1), then from (3.35) we can infer that

$$\begin{aligned}\xi(t)L'(t)&\le -k_1\xi(t)E(t)+k_2(\xi g\circ\nabla u)(t)\\&\le -k_1\xi(t)E(t)-k_2(g'\circ\nabla u)(t)\\&\le -k_1\xi(t)E(t)-k_2E'(t),\quad \forall\ t\ge t_0.\end{aligned} \tag{3.36}$$

We let $F(t):=\xi(t)L(t)+k_2E(t)$, which is equivalent to $E(t)$,then from (3.36) we can arrive at

$$F'(t)\le -k\xi(t)F(t),\quad \forall\ t\ge t_0. \tag{3.37}$$

A simple integration of (3.37) leads to

$$F(t)\le F(t_0)e^{-k\int_{t_0}^{t}\xi(s)ds},\quad \forall\ t\ge t_0. \tag{3.38}$$

In the case when $1<r<\frac{3}{2}$, we again consider (3.35) and use Lemma 2.3 to get

$$\xi(t)L'(t)\le -k_1\xi(t)E(t)+k_2C[-E'(t)]^{\frac{1}{2r-1}},\quad \forall\ t\ge t_0. \tag{3.39}$$

Multiplying both sides of (3.39) by $\xi^{\nu}E^{\nu}(t)$, where $\nu=2r-2$, then applying Young inequality, we can infer that

$$\begin{aligned}\xi^{\nu+1}E^{\nu}(t)L'(t)&\le -k_1\xi^{\nu+1}(t)E^{\nu+1}(t)+k_2C(\xi E)^{\nu}[-E'(t)]^{\frac{1}{\nu+1}},\\&\le -k_1\xi^{\nu+1}(t)E^{\nu+1}(t)+k_2C(\varepsilon(\xi E)^{\nu+1}(t)-C_\varepsilon E'(t))\\&=-(k_1-k_2C\varepsilon)(\xi E)^{\nu+1}(t)-k_2CC_\varepsilon E'(t),\quad \forall t\ge t_0.\end{aligned} \tag{3.40}$$

We choose $\varepsilon<\frac{k_1}{k_2C}$ such that $k_3:=k_1-k_2C\varepsilon>0$ and thanks to $\xi'(t)\le 0$ and $E'(t)\le 0$, we can deduce that

$$(\xi^{\nu+1}E^{\nu}L)'(t)\le \xi^{\nu+1}(t)E^{\nu}(t)L'(t)\le -k_3(\xi E)^{\nu+1}(t)-k_2CC_\varepsilon E'(t),\quad \forall t\ge t_0. \tag{3.41}$$

Then we have

$$(\xi^{\nu+1}E^{\nu}L + k_2CC_{\varepsilon}E)'(t) \le -k_3(\xi E)^{\nu+1}(t).$$

Let $G := \xi^{\nu+1}E^{\nu}L + k_2CC_{\varepsilon}E \sim E$. Then we deduce

$$G'(t) \le -k_4(\xi G)^{\nu+1}(t) = -k_4\xi^{2r-1}G^{2r-1}, \quad \forall t \ge t_0,\ k_4 > 0.$$

By integrating over (t_0, t) and applying the condition that $G \sim E$, we arrive at

$$E(t) \le K\Big[\frac{1}{1+\int_{t_0}^{t}\xi^{2r-1}(s)ds}\Big]^{\frac{1}{2r-2}}, \quad \forall t \ge t_0. \tag{3.42}$$

This completes the proof. □

4. Conclusions

In this paper, we consider a viscoelastic wave equation with strong damping, by constructing a suitable Lyapunov function, we establish a general decay result, Moreover, without restrictive conditions, we also obtain the optimal polynomial decay result.

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Conflict of interest

The authors declare no conflict of interest.

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