Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 18253–18279.
DOI: 10.3934/math.20221004
Received: 10 June 2022
Revised: 14 July 2022
Accepted: 14 July 2022
Published: 11 August 2022

Research article

A study on a special case of the Sturm-Liouville equation using the Mittag-Leffler function and a new type of contraction

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Abstract: One of the most famous equations that are widely used in various branches of physics, mathematics, financial markets, etc. is the Langevin equation. In this work, we investigate the existence of the solution for two generalized fractional hybrid Langevin equations under different boundary conditions. For this purpose, the problem of the existence of a solution will become the problem of finding a fixed point for an operator defined in the Banach space. To achieve the result, one of the recent fixed point techniques, namely the α-ψ-contraction technique, will be used. We provide sufficient conditions to use this type of contraction in our main theorems. In the calculations of the auxiliary lemmas that we present, the Mittag-Leffler function plays a fundamental role. The fractional derivative operators used are of the Caputo type. Two examples are provided to demonstrate the validity of the obtained theorems. Also, some figures and a table are presented to illustrate the results.

Keywords: α-ψ-Contraction; fixed point; fractional differential equation; the Mittag-Leffler function; the Sturm-Liouville equation
Mathematics Subject Classification: 34A40, 34C10

1. Introduction and preliminaries

Needless to say, many researchers today are working on various aspects of fractional derivatives. Perhaps the reason for all this attention is the high ability of fractional calculus to model natural phenomena. The most important advantage of the fractional derivative over the integer-order derivative is its ability in memory, which gives it a special ability to model. With the success of researchers in this field, we are witnessing the growth and application of fractional differential equations in mathematics, physics, engineering and biology every day. In bio-math, for example, various teams around the world using fractional differential equations have developed models for the human liver [1], mumps virus [2], Corona virus in China [3], heroin addiction [4] and hepatitis B [5]. Other researchers have developed models for thermostats [6–9]. The stability of fractional dynamic systems was also examined by others [10–13]. Attempts have also been made to find numerical and analytical solutions to the fractional cases of important equations such as Sturm-Liouville and others [14–30].

As we know, in the history of mathematics and physics, many famous equations have been of interest to various researchers. One of them is the Sturm-Liouville equation. It should be noted that some other important equations such as the Hermite, Jacobi, Bessel and Legendre equations can also be written as Sturm-Liouville [31]. However, what we want to address in this study is one of the most important special cases of Sturm-Liouville, called the Langevin equation. As we know, the beginning of the 20th century saw many of great scientific developments, and 1905 is one of the most important years in the history of science. It was in this year that Einstein published four important papers on the theory of special relativity, the photoelectric effect and Brownian motion. A year later, the young Polish physicist Marian Smoluchowski gave an independent explanation of Brownian motion [32]. Finally, in 1908, the French physicist Paul Langevin untied the 80-year-old conundrum with the publication of an article entitled "On The Theory of Brownian Motion" [33]. In doing so, he formulated an important equation known as the Langevin equation. Langevin's research had a significant impact on the research fields of mathematics and physics and is still often cited. The Langevin equation is an important tool in mathematical physics, and it can be used to describe physical phenomena concerning the temporal evolution of Brownian motion in oscillating environments [34–36]. The most important applications of the Langevin equation are describing as anomalous diffusion [37], modeling gait variability [38] and modeling of memory in financial markets [39]. Numerous contributions have been made to the numerical and analytical solutions of the Langevin equation in recent years. For example, in 2012 [40], Bhrawy et al. proposed the Jacobi-Gauss-Lobatto collocation method for solving numerically the nonlinear fractional Langevin equation. In the same year, Ahmad et al. studied a nonlinear Langevin equation involving two fractional orders using Krasnoselskii's fixed point theorem [41]. In 2017 [42], Wang et al. examined the existence of local center stable manifolds of Langevin differential equations. In 2018 upper-solution and lower-solution methods were presented for Langevin equations with two fractional orders [43]. Qualitative properties of the implicit impulsive Langevin equation involving mixed order derivatives were investigated by Zada et al. [44]. In 2020 the stochastic resonance (SR) of the fractional order Langevin equation was reviewed by periodic modulated noise with mass fluctuation [45]. For more contributions about the fractional Langevin equation we refer to [46–52].

In addition to the above, the study of hybrid differential equations has recently received much attention from new researchers. Nonlinear differential equations that have quadratic perturbations are called hybrids in the literature [53]. Using fixed point theory, researchers examined the necessary and sufficient conditions for the existence of such equations [53–55]. In 2010 [56], Dhage and Lakshmikantham developed the theory of existence for hybrid nonlinear differential equations under mixed Lipschitz and Caratheodory conditions. The reader, for more information about this manner, can refer to [57–61].

To the best of our knowledge, there is no study of the hybrid version of the fractional Langevin equation using the Mittag-Leffler function. Therefore, motivated by the aforesaid topics presented in this work, we intend to investigate the existence of the solution to the fractional Langevin equation with the hybrid boundary condition using the α-ψ-contraction technique and the Mittag-Leffler function.

In 2011, Zhao et al. [59] investigated the following fractional hybrid differential equation:

$$\begin{cases} \mathcal{D}^{\iota}\Big(\dfrac{\vartheta(\mathbf{t})}{\mathbf{g}(\mathbf{t},\vartheta(\mathbf{t}))}\Big) = \mathbf{f}(\mathbf{t},\vartheta(\mathbf{t})), \\ \\ \vartheta(0) = 0, \end{cases} \tag{1.1}$$

where $0 < \iota < 1$, $\mathbf{g} \in C([0,1]\times\mathbb{R}, \mathbb{R}-\{0\})$, and $f \in C([0,1]\times\mathbb{R},\mathbb{R})$.

In 2020 [52], Fazli et al. investigated the following fractional Langevin equation involving two fractional orders:

$$\begin{cases} \mathcal{D}_c^{\iota_1}(\mathcal{D}_c^{\iota_2}+\lambda)\vartheta(\mathbf{t}) = \mathbb{F}(\mathbf{t},\vartheta(\mathbf{t})), & 0 < \mathbf{t} < 1, \\ \vartheta^{(i)}(0) = \mu_i, & 0 \le i < l, \\ \vartheta^{(i+p)}(0) = \nu_i, & 0 \le i < n, \end{cases}$$

where $m-1 < \iota_2 \le m$, $n-1 < \iota_1 \le n$, $l = \max\{m,n\}$, $m,n \in \mathbb{N}$, $\mathcal{D}_c^{\iota_1}$, and $\mathcal{D}_c^{\iota_2}$ denote Caputo fractional derivatives and $\mathbb{F} : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a given function.

In this work motivated by the above, at first, we investigate the generalized fractional hybrid Langevin equation which reads as follows:

$$\begin{cases} \mathcal{D}_c^{\zeta_1}\Big((\mathcal{D}_c^{\zeta_2}+\lambda)(\dfrac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}) - \mathfrak{I}(\mathbf{t},\vartheta(\mathbf{t}))\Big) = \mathbb{A}(\mathbf{t},\vartheta(\mathbf{t})), \\ \Big[(\mathcal{D}_c^{\zeta_2}+\lambda)(\dfrac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))})\Big]_{\mathbf{t}=0} = \mathfrak{I}(0,\vartheta(0)), \\ \displaystyle\sum_{\mathbf{i}=0}^{\mathbf{n}} \frac{\kappa_{\mathbf{i}}\vartheta(\mu_{\mathbf{i}})}{\wp(\mu_{\mathbf{i}},\vartheta(\mu_{\mathbf{i}}))} + \sum_{\mathbf{j}=0}^{\mathbf{m}} \frac{\tau_{\mathbf{j}}\vartheta(\nu_{\mathbf{j}})}{\wp(\nu_{\mathbf{j}},\vartheta(\nu_{\mathbf{j}}))} = \mathbf{a}_{\mathbf{n},\mathbf{m}}, \end{cases} \tag{1.2}$$

where $\zeta_1,\zeta_2 \in (0,1]$, $\mathbb{I} = [0,T]$, such that $T > 0$. Moreover, $\mathcal{D}_c^{\zeta_1}$ and $\mathcal{D}_c^{\zeta_2}$ denote Caputo fractional derivatives, $\mathbb{A},\mathfrak{I} \in C(\mathbb{I}\times\mathbb{R},\mathbb{R})$, $\wp \in C(\mathbb{I}\times\mathbb{R},\mathbb{R}-\{0\})$, $\mathbf{a}_{\mathbf{n},\mathbf{m}} : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ is a sequence, $\kappa_{\mathbf{i}},\tau_{\mathbf{j}} \in \mathbb{R}$, and $\mu_{\mathbf{i}},\nu_{\mathbf{j}} \in [0,1]$.

Next, we investigate the generalized fractional hybrid Langevin equation via the integral boundary condition which reads as

$$\begin{cases} \mathcal{D}_c^{\zeta_1}(\mathcal{D}_c^{\zeta_2}+\lambda)\Big(\dfrac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),\ldots,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big) = \mathbb{B}(\mathbf{t},\vartheta(\mathbf{t})), \\ \vartheta(0) = 0, \\ \Big[(\mathcal{D}_c^{\zeta_2}+\lambda)\Big(\dfrac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),\ldots,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)\Big]_{\mathbf{t}=0} = 0, \end{cases} \tag{1.3}$$

where $\zeta_1, \zeta_2, \varsigma_1, \varsigma_2, ..., \varsigma_k \in (0, 1]$, $\mathcal{D}_c^{\zeta_1}$ and $\mathcal{D}_c^{\zeta_2}$ denote Caputo fractional derivatives, $\mathcal{I}^{\varsigma_1}, \mathcal{I}^{\varsigma_2}, ..., \mathcal{I}^{\varsigma_1}$ denote Riemann−Liouville fractional integrals, $\mathbb{A} \in C(\mathbb{I} \times \mathbb{R}, \mathbb{R})$, and $\Upsilon \in C(\mathbb{I} \times \mathbb{R}^{2k+1}, \mathbb{R} - \{0\})$.

It is well known that the Riemann−Liouville fractional integral of order ι of a function $\mathbf{f}$ is defined by $I^\iota \mathbf{f}(\mathbf{t}) = \frac{1}{\Gamma(\iota)} \int_0^t (t-s)^{\iota-1} f(s) ds (\iota > 0)$, and the Caputo derivative of order ι for a function $\mathbf{f}$ is defined by

$$\mathcal{D}^\iota \mathbf{f}(\mathbf{t}) = \frac{1}{\Gamma(n-\iota)} \int_0^{\mathbf{t}} \frac{\mathbf{f}^{(n)}(s)}{(\mathbf{t}-s)^{\iota-n+1}} ds,$$

where $n = [\iota] + 1$. For more details on the Riemann−Liouville fractional integral and Caputo derivative, see [62–64].

Definition 1.1. *[65] Let $\alpha, \beta > 0$, $\lambda \in \mathbb{R}$ and $\sigma \in \mathcal{L}_1([0, 1])$. The Prabhakar integral is formulated as*

$$\mathbb{E}(\alpha, \beta, \lambda)\sigma(\mathbf{t}) = \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\beta-1} \mathbf{E}_{\alpha,\beta} \lambda(\mathbf{t}-\mathbf{s})^\alpha d\mathbf{s},$$

where $\mathbf{E}_{\alpha,\beta}(.)$ is called the two parameter Mittag-Leffler function, such that it is defined by

$$\mathbf{E}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}.$$

Lemma 1.2. *[65] Assume that $\alpha, \beta, \gamma > 0$, $\lambda \in \mathbb{R}$, and $\sigma \in \mathcal{L}_1([0, 1])$. Then, we have*

$$\mathcal{I}^\gamma \mathbb{E}(\alpha, \beta, \lambda)\sigma(\mathbf{t}) = \mathbb{E}(\alpha, \beta, \lambda)\mathcal{I}^\gamma \sigma(\mathbf{t}) = \mathbb{E}(\alpha, \beta+\gamma, \lambda)\sigma(\mathbf{t}).$$

Samet et al. [66] introduced the following new contraction in fixed point theory, which will play a key role in this paper. As we recall from [51], throughout our main theorems, we will utilize the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$, with Ψ, such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ. Let $T : \mathcal{Z} \to \mathcal{Z}$ be a self map and $\alpha : \mathcal{Z} \times \mathcal{Z} \to [0, +\infty)$ be a function. We say that T is α-admissible whenever $\alpha(x, z) \geq 1$ implies that $\alpha(Tx, Tz) \geq 1$. Furthermore, assume that $\psi \in \Psi$. Then, a self-map $T : \mathcal{Z} \to \mathcal{Z}$ is called an α-ψ-contraction whenever $\forall x, z \in \mathcal{Z}$ and we have $\alpha(x, z)d(Tx, Tz) \leq \psi(d(x, z))$. The following lemma is widely used in the sequel.

Lemma 1.3. *[66] Let $(\mathcal{Z}, d)$ be a complete metric space and $T : \mathcal{Z} \to \mathcal{Z}$ an α-ψ-contraction and α-admissible map. Assume that there exists $z_0 \in \mathcal{Z}$ such that $\alpha(z_0, Tz_0) \geq 1$. Furthermore, if $\{z_n\}$ is a sequence in $\mathcal{Z}$ with $z_n \to z$ and $\alpha(z_n, z_{n+1}) \geq 1$ for all $n \geq 1$, we have $\alpha(z_n, z) \geq 1$, $\forall n \geq 1$. Then, T has a fixed point.*

2. Main results

Lemma 2.1. *Assume that $\sigma \in \mathcal{L}^1([0, 1])$. Consider the following equation:*

$$\mathcal{D}_c^{\zeta_1}\Big((\mathcal{D}_c^{\zeta_2} + \lambda)(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t}, \vartheta(\mathbf{t}))}) - \mathfrak{I}(\mathbf{t}, \vartheta(\mathbf{t}))\Big) = \sigma(\mathbf{t}), \tag{2.1}$$

with boundary conditions

$$\begin{cases} \Big[(\mathcal{D}_c^{\zeta_2} + \lambda)(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t}, \vartheta(\mathbf{t}))})\Big]_{\mathbf{t}=0} = \mathfrak{I}(0, \vartheta(0)), \\ \sum_{\mathbf{i}=0}^{\mathbf{n}} \frac{\kappa_{\mathbf{i}} \vartheta(\mu_{\mathbf{i}})}{\wp(\mu_{\mathbf{i}}, \vartheta(\mu_{\mathbf{i}}))} + \sum_{\mathbf{j}=0}^{\mathbf{m}} \frac{\tau_{\mathbf{j}} \vartheta(\nu_{\mathbf{j}})}{\wp(\nu_{\mathbf{j}}, \vartheta(\nu_{\mathbf{j}}))} = \mathbf{a}_{\mathbf{n},\mathbf{m}}. \end{cases} \tag{2.2}$$

Then, the function $\vartheta \in C([0,1])$ *is a solution of the problems* (2.1) *and* (2.2) *whenever*

$$\begin{aligned}
\vartheta(\mathbf{t}) = {} & \wp(\mathbf{t},\vartheta(\mathbf{t}))\Big[\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\sigma(\mathbf{s})d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})\sigma(\mathbf{s})d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})\sigma(\mathbf{s})d\mathbf{s}\Big] \\
& + \wp(\mathbf{t},\vartheta(\mathbf{t}))\Big[\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\Im(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})\Im(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})\Im(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s} \\
& + \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\mathbf{a}_{\mathbf{n},\mathbf{m}}\Big],
\end{aligned} \tag{2.3}$$

where

$$\Phi = \frac{1}{\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\mathbf{E}_{\zeta_2,1}(-\lambda\mu_{\mathbf{i}}^{\zeta_2}) + \sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\mathbf{E}_{\zeta_2,1}(-\lambda\nu_{\mathbf{j}}^{\zeta_2})}$$

and

$$\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\mathbf{E}_{\zeta_2,1}(-\lambda\mu_{\mathbf{i}}^{\zeta_2}) + \sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\mathbf{E}_{\zeta_2,1}(-\lambda\nu_{\mathbf{j}}^{\zeta_2}) \neq 0.$$

Proof. The following equation is another version of Eq (2.1):

$$\mathcal{I}^{1-\zeta_1}\Big[\frac{d}{d\mathbf{t}}\Big((\mathcal{D}_c^{\zeta_2}+\lambda)(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}) - \Im(\mathbf{t},\vartheta(\mathbf{t}))\Big)\Big] = \sigma(\mathbf{t}).$$

By applying the operator $\mathcal{I}^{\zeta_1}$ on both sides, we deduce that

$$\mathcal{I}^{1}\Big[\frac{d}{d\mathbf{t}}\Big((\mathcal{D}_c^{\zeta_2}+\lambda)(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}) - \Im(\mathbf{t},\vartheta(\mathbf{t}))\Big)\Big] = \mathcal{I}^{\zeta_1}\sigma(\mathbf{t}),$$

which implies that

$$(\mathcal{D}_c^{\zeta_2}+\lambda)(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}) - \Im(\mathbf{t},\vartheta(\mathbf{t})) - \Big[(\mathcal{D}_c^{\zeta_2}+\lambda)(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))})\Big]_{\mathbf{t}=0} + \Im(0,\vartheta(0)) = \mathcal{I}^{\zeta_1}\sigma(\mathbf{t}).$$

Since

$$\Big[(\mathcal{D}_c^{\zeta_2}+\lambda)(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))})\Big]_{\mathbf{t}=0} = \Im(0,\vartheta(0)),$$

then we get

$$(\mathcal{D}_c^{\zeta_2}+\lambda)(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}) - \Im(\mathbf{t},\vartheta(\mathbf{t})) = \mathcal{I}^{\zeta_1}\sigma(\mathbf{t}).$$

Equivalently,

$$\mathcal{D}_c^{\zeta_2}\Big(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}\Big)+\lambda\Big(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}\Big)=\mathcal{I}^{\zeta_1}\sigma(\mathbf{t})+\mathfrak{I}(\mathbf{t},\vartheta(\mathbf{t})).$$

The Laplace transform of the above equation implies that

$$\mathbf{s}^{\zeta_2}\mathfrak{L}_{\mathbf{s}}\Big(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}\Big)-\ell_0\mathbf{s}^{\zeta_2-1}+\lambda\mathfrak{L}_{\mathbf{s}}\Big(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}\Big)=\mathfrak{L}_{\mathbf{s}}(\mathcal{I}^{\zeta_1}\sigma(\mathbf{t}))+\mathfrak{L}_{\mathbf{s}}(\mathfrak{I}(\mathbf{t},\vartheta(\mathbf{t}))),$$

where $\ell_0=\big(\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}\big)_{\mathbf{t}=0}$. So, by utilizing the inverse Laplace transform, we get

$$\begin{aligned}\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}&=\mathfrak{L}_{\mathbf{s}}^{-1}\Big(\frac{\mathfrak{L}_{\mathbf{s}}\mathcal{I}^{\zeta_1}\sigma(\mathbf{t}))}{\mathbf{s}^{\zeta_2}+\lambda}\Big)+\mathfrak{L}_{\mathbf{s}}^{-1}\Big(\frac{\mathfrak{L}_{\mathbf{s}}(\mathfrak{I}(\mathbf{t},\vartheta(\mathbf{t})))}{\mathbf{s}^{\zeta_2}+\lambda}\Big)+\ell_0\mathfrak{L}_{\mathbf{s}}^{-1}\Big(\frac{\mathbf{s}^{\zeta_2-1}}{\mathbf{s}^{\zeta_2}+\lambda}\Big)\\&=\mathbf{t}^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda\mathbf{t}^{\zeta_2})\star\mathcal{I}^{\zeta_1}\sigma(\mathbf{t})+\mathbf{t}^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda\mathbf{t}^{\zeta_2})\star\mathfrak{I}(\mathbf{t},\vartheta(\mathbf{t}))+\ell_0\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\\&=\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathcal{I}^{\zeta_1}\sigma(\mathbf{s})d\mathbf{s}\\&+\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s}+\ell_0\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2}).\end{aligned}$$

Since

$$\begin{aligned}\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathcal{I}^{\zeta_1}\sigma(\mathbf{s})d\mathbf{s}=\mathbb{E}(\zeta_2,\zeta_2,-\lambda)\mathcal{I}^{\zeta_1}\sigma(\mathbf{t})&=\mathbb{E}(\zeta_2,\zeta_1+\zeta_2,-\lambda)\sigma(\mathbf{t})\\&=\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\sigma(\mathbf{s})d\mathbf{s},\end{aligned}$$

we get

$$\begin{aligned}\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))}&=\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\sigma(\mathbf{s})d\mathbf{s}\\&+\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s}+\ell_0\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2}).\end{aligned}\tag{2.4}$$

For simplicity, let

$$\Xi_1(\mathbf{t})=\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\sigma(\mathbf{s})d\mathbf{s}$$

and

$$\Xi_2(\mathbf{t})=\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s}.$$

So,

$$\sum_{\mathbf{i}=0}^{\mathbf{n}}\frac{\kappa_{\mathbf{i}}\vartheta(\mu_{\mathbf{i}})}{\wp(\mu_{\mathbf{i}},\vartheta(\mu_{\mathbf{i}}))}=\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\Xi_1(\mu_{\mathbf{i}})+\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\Xi_2(\mu_{\mathbf{i}})+\ell_0\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\mathbf{E}_{\zeta_2,1}(-\lambda\mu_{\mathbf{i}}{}^{\zeta_2}),$$

and

$$\sum_{\mathbf{j}=0}^{\mathbf{m}} \frac{\tau_{\mathbf{j}}\vartheta(\nu_{\mathbf{j}})}{\wp(\nu_{\mathbf{j}},\vartheta(\nu_{\mathbf{j}}))} = \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}\Xi_1(\nu_{\mathbf{j}}) + \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}\Xi_2(\nu_{\mathbf{j}}) + \ell_0 \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}\mathbf{E}_{\zeta_2,1}(-\lambda\nu_{\mathbf{j}}{}^{\zeta_2}).$$

Thus,

$$\ell_0 = \Phi[\mathbf{a}_{\mathbf{n},\mathbf{m}} - \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}\Xi_1(\mu_{\mathbf{i}}) - \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}\Xi_2(\mu_{\mathbf{i}}) - \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}\Xi_1(\nu_{\mathbf{j}}) - \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}\Xi_2(\nu_{\mathbf{j}})],$$

where

$$\Phi = \frac{1}{\sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}\mathbf{E}_{\zeta_2,1}(-\lambda\mu_{\mathbf{i}}{}^{\zeta_2}) + \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}\mathbf{E}_{\zeta_2,1}(-\lambda\nu_{\mathbf{j}}{}^{\zeta_2})}.$$

Now, by substituting the value of ℓ_0 in (2.4), we obtain

$$\begin{aligned}\frac{\vartheta(\mathbf{t})}{\wp(\mathbf{t},\vartheta(\mathbf{t}))} &= \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\sigma(\mathbf{s})d\mathbf{s} \\ &+ \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\Im(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s} + \mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\Phi[\mathbf{a}_{\mathbf{n},\mathbf{m}} \\ &- \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}\Xi_1(\mu_{\mathbf{i}}) - \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}\Xi_2(\mu_{\mathbf{i}}) - \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}\Xi_1(\nu_{\mathbf{j}}) - \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}\Xi_2(\nu_{\mathbf{j}})].\end{aligned} \tag{2.5}$$

After doing some simple computations and substitutions, we get (2.3). □

Now, we are going to establish and prove our first main theorem.

Theorem 2.2. *Let the following conditions hold:*

($\mathcal{P}_1$) *There exist* $\chi_1, \chi_2, \chi_3 \in C(\mathcal{I}, \mathbb{R}^+)$ *such that the following inequalities are satisfied,*

$$\begin{cases} |\mathbb{A}(\mathbf{t},\mathbf{r}_{11}) - \mathbb{A}(\mathbf{t},\mathbf{r}_{12})| \le \chi_1(\mathbf{t})|\mathbf{r}_{12}-\mathbf{r}_{11}|, \quad \forall(\mathbf{t},\mathbf{r}_{11},\mathbf{r}_{12}) \in \mathcal{I}\times\mathbb{R}\times\mathbb{R}, \\ |\Im(\mathbf{t},\mathbf{r}_{21}) - \Im(\mathbf{t},\mathbf{r}_{22})| \le \chi_2(\mathbf{t})|\mathbf{r}_{22}-\mathbf{r}_{21}|, \quad \forall(\mathbf{t},\mathbf{r}_{21},\mathbf{r}_{22}) \in \mathcal{I}\times\mathbb{R}\times\mathbb{R}, \\ |\wp(\mathbf{t},\mathbf{r}_{31}) - \wp(\mathbf{t},\mathbf{r}_{32})| \le \chi_3(\mathbf{t})|\mathbf{r}_{32}-\mathbf{r}_{31}|, \forall(\mathbf{t},\mathbf{r}_{31},\mathbf{r}_{32}) \in \mathcal{I}\times\mathbb{R}\times\mathbb{R}. \end{cases}$$

($\mathcal{P}_2$) *There exists* $\zeta > 0$*, such that*

$$(\mho_1\zeta + \mho_2)(\chi_3^*\zeta + \wp_0^*) \le \zeta \text{ and } (2\chi_3^* + \wp_0^*)\mho_1 + \chi_3^*\mho_2 < 1,$$

where

$$\mho_1 = \Big(\frac{\chi_1^*}{\zeta_1+\zeta_2} + \frac{\chi_2^*}{\zeta_2}\Big)\Big(\Pi_1^* + |\Phi|\Pi_1^*\Pi_2^*(\sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}} + \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}})\Big)$$

and

$$\mho_2 = \Big(\frac{\mathbb{A}_0^*}{\zeta_1+\zeta_2} + \frac{\Im_0^*}{\zeta_2}\Big)\Big(\Pi_1^* + |\Phi|\Pi_1^*\Pi_2^*(\sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}} + \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}})\Big) + |\Phi|\Pi_1^*|\mathbf{a}_{\mathbf{n},\mathbf{m}}|,$$

in which χ_1^*, χ_2^*, χ_3^*, $\wp_0^*$, $\Im_0^*$, $\mathbb{A}_0^*$, Π_1^*, Π_2^* *indicate the supremum of* $\chi_1(\mathbf{t})$, $\chi_2(\mathbf{t})$, $\chi_3(\mathbf{t})$, $\wp(\mathbf{t},0)$, $\Im(\mathbf{t},0)$, $\mathbb{A}(\mathbf{t},0)$, $|\mathbf{E}_{\zeta_2,\zeta_2}(-\mathbf{t}^{\zeta_2})|$, *and* $|\mathbf{E}_{\zeta_2,1}(-\mathbf{t}^{\zeta_2})|$, *respectively.*

Then, the problem formulated in (1.2) *has at least one solution.*

Proof. Let $\mathbf{Z}^\star = C(\mathcal{I})$. Define $\alpha : \mathbf{Z}^\star \times \mathbf{Z}^\star \to [0,\infty)$ by

$$\alpha(\vartheta, \tilde{\vartheta}) = \begin{cases} 1 \text{ if } \vartheta(\mathbf{t}) \leq \zeta \text{ for all } \mathbf{t} \in \mathcal{I} \\ 0 \text{ otherwise.} \end{cases}$$

Furthermore, we define the operator $\hat{\mathcal{K}} : \bar{V}_\zeta(0) \to \mathbf{Z}^\star$ by

$$\begin{aligned}
(\hat{\mathcal{K}}\vartheta)(t) = {} & \wp(\mathbf{t}, \vartheta(\mathbf{t}))\Big[\int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s}\Big] \\
& + \wp(\mathbf{t}, \vartheta(\mathbf{t}))\Big[\int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s} \\
& + \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\mathbf{a}_{\mathbf{n},\mathbf{m}}\Big].
\end{aligned}$$

Define two single-valued mappings $\hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2 : \bar{V}_\zeta(0) \to \mathbf{Z}^\star$ by

$$\begin{aligned}
(\hat{\mathcal{B}}_1\vartheta)(t) = {} & \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s} \\
& + \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s} \\
& - \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s} \\
& + \Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\mathbf{a}_{\mathbf{n},\mathbf{m}}
\end{aligned}$$

and

$$(\hat{\mathcal{B}}_1\vartheta)(t) = \wp(\mathbf{t}, \vartheta(\mathbf{t})).$$

Also, note that $\hat{\mathcal{K}}(\vartheta) = \hat{\mathcal{B}}_1\vartheta\hat{\mathcal{B}}_2\vartheta$ for all $\vartheta \in \bar{V}_\zeta(0)$. In view of Lemma 2.1, $(\hat{\mathcal{K}}\vartheta)$ is the integral version of problem 1.2. By using Lemma (1.3), σ_0 solves problem 1.2 if and only if σ_0 is a fixed point of $(\hat{\mathcal{K}}\vartheta)$. Now, we are going to prove that all conditions of Lemma 1.3 hold for $\hat{\mathcal{K}}$. Let $\alpha(\vartheta, \tilde{\vartheta}) \geq 1$. Then, $\vartheta(\mathbf{t}) \leq \zeta$ and $\tilde{\vartheta}(\mathbf{t}) \leq \zeta$ for $\mathbf{t} \in \mathcal{I}$. Thus,

$$|\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))| = |\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t})) - \mathbb{A}(\mathbf{t}, 0) + \mathbb{A}(\mathbf{t}, 0)| \leq \chi_1(\mathbf{t})|\vartheta(\mathbf{t})| + \mathbb{A}(\mathbf{t}, 0) \leq \chi_1^*\zeta + \mathbb{A}_0^*,$$

and

$$|\mathfrak{I}(\mathbf{t}, \vartheta(\mathbf{t}))| = |\mathfrak{I}(\mathbf{t}, \vartheta(\mathbf{t})) - \mathfrak{I}(\mathbf{t}, 0) + \mathfrak{I}(\mathbf{t}, 0)| \leq \chi_2(\mathbf{t})|\vartheta(\mathbf{t})| + \mathfrak{I}(\mathbf{t}, 0) \leq \chi_2^*\zeta + \mathfrak{I}_0^*.$$

On the other hand, we have the following inequalities:

$$\begin{aligned} |\int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t} - \mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s}| &\leq \Pi_1^*(\chi_1^*\zeta + \mathbb{A}_0^*) \int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\zeta_1+\zeta_2-1} d\mathbf{s} \\ &\leq \frac{\Pi_1^*\chi_1^*}{\zeta_1 + \zeta_2}\zeta + \frac{\Pi_1^*\mathbb{A}_0^*}{\zeta_1 + \zeta_2}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &|\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2}) \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}} \int_0^{\mu_{\mathbf{i}}} (\mu_{\mathbf{i}} - \mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}} - \mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s}| \\ &\leq \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_1^* \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}}{\zeta_1 + \zeta_2}\zeta + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathbb{A}_0^* \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}}{\zeta_1 + \zeta_2}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &|\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2}) \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}} \int_0^{\nu_{\mathbf{j}}} (\nu_{\mathbf{j}} - \mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}} - \mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))d\mathbf{s}| \\ &\leq \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_1^* \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}}{\zeta_1 + \zeta_2}\zeta + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathbb{A}_0^* \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}}{\zeta_1 + \zeta_2} \end{aligned} \tag{2.8}$$

and

$$|\int_0^{\mathbf{t}} (\mathbf{t} - \mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t} - \mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s}| \leq \frac{\Pi_1^*\chi_2^*}{\zeta_2}\zeta + \frac{\Pi_1^*\mathfrak{I}_0^*}{\zeta_2}, \tag{2.9}$$

$$\begin{aligned} &|\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2}) \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}} \int_0^{\mu_{\mathbf{i}}} (\mu_{\mathbf{i}} - \mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}} - \mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s}| \\ &\leq \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_2^* \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}}{\zeta_2}\zeta + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathfrak{I}_0^* \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}}}{\zeta_2}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} &|\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2}) \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}} \int_0^{\nu_{\mathbf{j}}} (\nu_{\mathbf{j}} - \mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}} - \mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s}, \vartheta(\mathbf{s}))d\mathbf{s}| \\ &\leq \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_2^* \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}}{\zeta_2}\zeta + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathfrak{I}_0^* \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}}}{\zeta_2}. \end{aligned} \tag{2.11}$$

Now, since

$$
\begin{aligned}
|(\hat{\mathscr{B}}_1\vartheta)(t)| \leq & \,|\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t}))d\mathbf{s}| \\
& + |\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t}))d\mathbf{s}| \\
& + |\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_1+\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t}))d\mathbf{s} \\
& + \int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s}| \\
& + |\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s}| \\
& + |\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})\mathfrak{I}(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s}| \\
& + |\Phi\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})\mathbf{a}_{\mathbf{n},\mathbf{m}}|,
\end{aligned}
$$

by using (2.6)–(2.11), we conclude that

$$
\begin{aligned}
|(\hat{\mathscr{B}}_1\vartheta)(t)| \leq & \frac{\Pi_1^*\chi_1^*}{\zeta_1+\zeta_2}\zeta + \frac{\Pi_1^*\mathbb{A}_0^*}{\zeta_1+\zeta_2} + \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_1^*\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}}{\zeta_1+\zeta_2}\zeta + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathbb{A}_0^*\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}}{\zeta_1+\zeta_2} \\
& + \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_1^*\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}}{\zeta_1+\zeta_2}\zeta + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathbb{A}_0^*\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}}{\zeta_1+\zeta_2} + \frac{\Pi_1^*\chi_2^*}{\zeta_2}\zeta + \frac{\Pi_1^*\mathfrak{I}_0^*}{\zeta_2} \\
& + \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_2^*\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}}{\zeta_2}\zeta + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathfrak{I}_0^*\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}}{\zeta_2} + \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_2^*\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}}{\zeta_2}\zeta + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathfrak{I}_0^*\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}}{\zeta_2} \\
& + |\Phi|\Pi_1^*|\mathbf{a}_{\mathbf{n},\mathbf{m}}| = \Big[\frac{\Pi_1^*\chi_1^*}{\zeta_1+\zeta_2} + \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_1^*\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}}{\zeta_1+\zeta_2} + \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_1^*\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}}{\zeta_1+\zeta_2} + \frac{\Pi_1^*\chi_2^*}{\zeta_2} \\
& + \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_2^*\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}}{\zeta_2} + \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_2^*\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}}{\zeta_2}\Big]\zeta + \frac{\Pi_1^*\mathbb{A}_0^*}{\zeta_1+\zeta_2} + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathbb{A}_0^*\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}}{\zeta_1+\zeta_2} \\
& + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathbb{A}_0^*\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}}{\zeta_1+\zeta_2} + \frac{\Pi_1^*\mathfrak{I}_0^*}{\zeta_2} + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathfrak{I}_0^*\sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}}}{\zeta_2} + \frac{|\Phi|\Pi_1^*\Pi_2^*\mathfrak{I}_0^*\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}}}{\zeta_2} + |\Phi|\Pi_1^*|\mathbf{a}_{\mathbf{n},\mathbf{m}}| \\
& = \Big(\frac{\chi_1^*}{\zeta_1+\zeta_2} + \frac{\chi_2^*}{\zeta_2}\Big)\Big(\Pi_1^* + |\Phi|\Pi_1^*\Pi_2^*(\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}} + \sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}})\Big)\zeta + \Big(\frac{\mathbb{A}_0^*}{\zeta_1+\zeta_2} + \frac{\mathfrak{I}_0^*}{\zeta_2}\Big)\Big(\Pi_1^* \\
& + |\Phi|\Pi_1^*\Pi_2^*(\sum_{\mathbf{j}=0}^{\mathbf{m}}\tau_{\mathbf{j}} + \sum_{\mathbf{i}=0}^{\mathbf{n}}\kappa_{\mathbf{i}})\Big) + |\Phi|\Pi_1^*|\mathbf{a}_{\mathbf{n},\mathbf{m}}| = \mho_1\zeta + \mho_2,
\end{aligned}
$$

and this yields

$$|(\hat{\mathscr{B}}_1\vartheta)(t)| \leq \mho_1\zeta + \mho_2. \tag{2.12}$$

Also, we have

$$|\hat{\mathscr{B}}_2\vartheta| \leq |\wp(\mathbf{t},\vartheta(\mathbf{t}))| \leq \chi_3^*\zeta + \wp_0^*. \tag{2.13}$$

Hence, by using (2.12) and (2.13), we get

$$|(\hat{\mathcal{K}}\vartheta)(t)| \leq (\mho_1\zeta + \mho_2)(\chi_3^*\zeta + \wp_0^*) \leq \zeta.$$

Similarly, we obtain

$$|(\hat{\mathcal{K}}\tilde{\vartheta})(t)| \leq \zeta,$$

that is, $\alpha(\hat{\mathcal{K}}\vartheta, \hat{\mathcal{K}}\tilde{\vartheta}) \geq 1$. So, $\hat{\mathcal{K}}$ is an α-admissible mapping. Furthermore, we can obtain

$$\int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})||\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t}))-\mathbb{A}(\mathbf{t},\tilde{\vartheta}(\mathbf{t}))|d\mathbf{s} \leq \frac{\Pi_1^*\chi_1^*}{\zeta_1+\zeta_2}\|\vartheta-\tilde{\vartheta}\|, \tag{2.14}$$

and

$$\begin{aligned}&|\Phi||\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})|\sum_{\mathbf{i}=0}^{\mathbf{n}}|\kappa_{\mathbf{i}}|\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_1+\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})||\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t}))\\&-\mathbb{A}(\mathbf{t},\tilde{\vartheta}(\mathbf{t}))|d\mathbf{s} \leq \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_1^*\sum_{\mathbf{i}=0}^{\mathbf{n}}|\kappa_{\mathbf{i}}|}{\zeta_1+\zeta_2}\|\vartheta-\tilde{\vartheta}\|.\end{aligned} \tag{2.15}$$

Also,

$$\begin{aligned}&|\Phi||\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})|\sum_{\mathbf{j}=0}^{\mathbf{m}}|\tau_{\mathbf{j}}| \in t_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_1+\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})||\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t}))\\&-\mathbb{A}(\mathbf{t},\tilde{\vartheta}(\mathbf{t}))|d\mathbf{s} \leq \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_1^*\sum_{\mathbf{j}=0}^{\mathbf{m}}|\tau_{\mathbf{j}}|}{\zeta_1+\zeta_2}\|\vartheta-\tilde{\vartheta}\|.\end{aligned} \tag{2.16}$$

Then,

$$\int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})||\Im(\mathbf{s},\vartheta(\mathbf{s}))-\Im(\mathbf{s},\tilde{\vartheta}(\mathbf{s}))|d\mathbf{s}| \leq \frac{\Pi_1^*\chi_2^*}{\zeta_2}\|\vartheta-\tilde{\vartheta}\|, \tag{2.17}$$

and

$$\begin{aligned}&|\Phi||\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})|\sum_{\mathbf{i}=0}^{\mathbf{n}}|\kappa_{\mathbf{i}}|\int_0^{\mu_{\mathbf{i}}}(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})||\Im(\mathbf{s},\vartheta(\mathbf{s}))\\&-\Im(\mathbf{s},\tilde{\vartheta}(\mathbf{s}))|d\mathbf{s} \leq \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_2^*\sum_{\mathbf{i}=0}^{\mathbf{n}}|\kappa_{\mathbf{i}}|}{\zeta_2}\|\vartheta-\tilde{\vartheta}\|,\end{aligned} \tag{2.18}$$

so

$$\begin{aligned}&|\Phi||\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})|\sum_{\mathbf{j}=0}^{\mathbf{m}}|\tau_{\mathbf{j}}|\int_0^{\nu_{\mathbf{j}}}(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})||\Im(\mathbf{s},\vartheta(\mathbf{s}))\\&-\Im(\mathbf{s},\tilde{\vartheta}(\mathbf{s}))|d\mathbf{s} \leq \frac{|\Phi|\Pi_1^*\Pi_2^*\chi_2^*\sum_{\mathbf{j}=0}^{\mathbf{m}}|\tau_{\mathbf{j}}|}{\zeta_2}\|\vartheta-\tilde{\vartheta}\|.\end{aligned} \tag{2.19}$$

Now,

$$
\begin{aligned}
|(\hat{\mathcal{B}}_1\vartheta)(t) - (\hat{\mathcal{B}}_1\tilde{\vartheta})(t)| \leq & \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_1+\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})||\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t})) - \mathbb{A}(\mathbf{t},\tilde{\vartheta}(\mathbf{t}))|d\mathbf{s} \\
& + |\Phi||\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})| \sum_{\mathbf{i}=0}^{\mathbf{n}} |\kappa_{\mathbf{i}}| \int_0^{\mu_{\mathbf{i}}} (\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_1+\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})||\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t})) \\
& - \mathbb{A}(\mathbf{t},\tilde{\vartheta}(\mathbf{t}))|d\mathbf{s} \\
& + |\Phi||\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})| \sum_{\mathbf{j}=0}^{\mathbf{m}} |\tau_{\mathbf{j}}| \int_0^{\nu_{\mathbf{j}}} (\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_1+\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\nu_{\mathbf{j}}-\mathbf{s})^{\zeta_2})||\mathbb{A}(\mathbf{t},\vartheta(\mathbf{t})) \\
& - \mathbb{A}(\mathbf{t},\tilde{\vartheta}(\mathbf{t}))|d\mathbf{s} \\
& + \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})||\Im(\mathbf{s},\vartheta(\mathbf{s})) - \Im(\mathbf{s},\tilde{\vartheta}(\mathbf{s}))|d\mathbf{s} \\
& + \int_0^{\mathbf{t}} (\mathbf{t}-\mathbf{s})^{\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})||\Im(\mathbf{s},\vartheta(\mathbf{s})) - \Im(\mathbf{s},\tilde{\vartheta}(\mathbf{s}))|d\mathbf{s} \\
& + |\Phi||\mathbf{E}_{\zeta_2,1}(-\lambda\mathbf{t}^{\zeta_2})| \sum_{\mathbf{i}=0}^{\mathbf{n}} |\kappa_{\mathbf{i}}| \int_0^{\mu_{\mathbf{i}}} (\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mu_{\mathbf{i}}-\mathbf{s})^{\zeta_2})||\Im(\mathbf{s},\vartheta(\mathbf{s})) \\
& - \Im(\mathbf{s},\tilde{\vartheta}(\mathbf{s}))|d\mathbf{s}.
\end{aligned}
$$

Hence, we get

$$|(\hat{\mathcal{B}}_1\vartheta)(t) - (\hat{\mathcal{B}}_1\tilde{\vartheta})(t)| \leq \mho_1\|\vartheta - \tilde{\vartheta}\|.$$

Also, we have

$$|\hat{\mathcal{B}}_2\vartheta - \hat{\mathcal{B}}_2\tilde{\vartheta}| \leq \chi_3(\mathbf{t})|\wp(\mathbf{t},\vartheta(\mathbf{t})) - \wp(\mathbf{t},\tilde{\vartheta}(\mathbf{t}))| \leq \chi_3^*\|\vartheta - \tilde{\vartheta}\|. \tag{2.20}$$

On the other hand, $\hat{\mathcal{K}}(\vartheta) = \hat{\mathcal{B}}_1\vartheta\hat{\mathcal{B}}_2\vartheta$ for all $\vartheta \in \bar{V}_\zeta(0)$, so

$$
\begin{aligned}
|\hat{\mathcal{K}}(\vartheta) - \hat{\mathcal{K}}(\tilde{\vartheta})| &= |\hat{\mathcal{B}}_1\vartheta\hat{\mathcal{B}}_2\vartheta - \hat{\mathcal{B}}_1\tilde{\vartheta}\hat{\mathcal{B}}_2\tilde{\vartheta}| = |\hat{\mathcal{B}}_1\vartheta\hat{\mathcal{B}}_2\vartheta - \hat{\mathcal{B}}_1\vartheta\hat{\mathcal{B}}_2\tilde{\vartheta} + \hat{\mathcal{B}}_1\vartheta\hat{\mathcal{B}}_2\tilde{\vartheta} - \hat{\mathcal{B}}_1\tilde{\vartheta}\hat{\mathcal{B}}_2\tilde{\vartheta}| \\
&= |\hat{\mathcal{B}}_1\vartheta(\hat{\mathcal{B}}_2\vartheta - \hat{\mathcal{B}}_1\vartheta\hat{\mathcal{B}}_2\tilde{\vartheta}) + \hat{\mathcal{B}}_2\tilde{\vartheta}(\hat{\mathcal{B}}_1\vartheta - \hat{\mathcal{B}}_1\tilde{\vartheta})| \\
&\leq |\hat{\mathcal{B}}_1\vartheta||\hat{\mathcal{B}}_2\vartheta - \hat{\mathcal{B}}_2\tilde{\vartheta}| + |\hat{\mathcal{B}}_2\tilde{\vartheta}||\hat{\mathcal{B}}_1\vartheta - \hat{\mathcal{B}}_1\tilde{\vartheta}| \\
&\leq \chi_3^*(\mho_1\zeta + \mho_2)\|\vartheta - \tilde{\vartheta}\| + \mho_1(\chi_3^*\zeta + \wp_0^*)\|\vartheta - \tilde{\vartheta}\| \\
&= \Big((2\chi_3^* + \wp_0^*)\mho_1 + \chi_3^*\mho_2\Big)\|\vartheta - \tilde{\vartheta}\|
\end{aligned}
$$

and hence

$$\|\hat{\mathcal{K}}(\vartheta) - \hat{\mathcal{K}}(\tilde{\vartheta})\| \leq \Big((2\chi_3^* + \wp_0^*)\mho_1 + \chi_3^*\mho_2\Big)\|\vartheta - \tilde{\vartheta}\|.$$

Let $\Psi(\mathbf{t}) = \ell\mathbf{t}$, where $\ell = (2\chi_3^* + \wp_0^*)\mho_1 + \chi_3^*\mho_2 < 1$. Thus,

$$\|\hat{\mathcal{K}}(\vartheta) - \hat{\mathcal{K}}(\tilde{\vartheta})\| \leq \Psi(\|\vartheta - \tilde{\vartheta}\|).$$

Therefore, as also stated in [51], $\hat{\mathcal{K}}$ is an $\alpha-\psi$-contractive mapping. Assume that $\{\vartheta_n\}$ is a sequence in $\mathbf{Z}$ where $\alpha(\vartheta_{n-1}, \vartheta_n) \geq 1$ for all $n \geq 1$, and $\vartheta_n \to \vartheta \in \mathbf{Z}$. Thus, $\vartheta_n(\vartheta) \leq \xi$. Then, $\lim_{n\to\infty} \vartheta_n(\vartheta) = \vartheta \leq \varsigma$. So, $\alpha(\vartheta_n, \vartheta) \geq 1$ for all $n \geq 1$. Also, evidently, there exists $\vartheta_0 \in \mathbf{Z}$ such that $\alpha(\vartheta_0, \hat{\mathcal{K}}\vartheta_0) \geq 1$. Then, all conditions of Lemma 1.3 are valid, and $\hat{\mathcal{K}}$ has a fixed point in $\mathbf{Z}$ which is a solution for the problem (1.2). □

We further need the following lemma to prove our second main result.

Lemma 2.3. *Consider the following equation:*

$$\mathcal{D}_c^{\zeta_1}(\mathcal{D}_c^{\zeta_2}+\lambda)\left(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\right)=\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t})), \tag{2.21}$$

with boundary conditions

$$\begin{cases}\vartheta(0)=0,\\ \left[(\mathcal{D}_c^{\zeta_2}+\lambda)\left(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\right)\right]_{\mathbf{t}=0}=0.\end{cases} \tag{2.22}$$

Then, the function $\vartheta \in C([0,1])$ is a solution of problems (2.21) *and* (2.22) *if*

$$\vartheta(\mathbf{t})=\overline{\Upsilon}(\mathbf{t})\int_0^{\mathbf{t}}\int_0^{\mathbf{s}}(\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathbb{B}(\tau,\vartheta(\tau))d\tau d\mathbf{s},$$

where

$$\overline{\Upsilon}(\mathbf{t})=\frac{1}{\Gamma(\zeta_1)}\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t})).$$

Proof. Equation (2.21) can be written as

$$\mathcal{I}_c^{1-\zeta_1}\left[\frac{d}{d\mathbf{t}}(\mathcal{D}_c^{\zeta_2}+\lambda)\left(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\right)\right]=\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t})).$$

By applying the operator I^{ζ_1} on both sides, we get

$$\mathcal{I}_c^{1}\left[\frac{d}{d\mathbf{t}}(\mathcal{D}_c^{\zeta_2}+\lambda)\left(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\right)\right]=\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t})).$$

Consequently,

$$\begin{aligned}&(\mathcal{D}_c^{\zeta_2}+\lambda)\left(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\right)\\&-\left[(\mathcal{D}_c^{\zeta_2}+\lambda)\left(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\right)\right]_{\mathbf{t}=0}=\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t})).\end{aligned}$$

Now, since

$$\left[(\mathcal{D}_c^{\zeta_2}+\lambda)\left(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\right)\right]_{\mathbf{t}=0}=0,$$

we then have

$$(\mathcal{D}_c^{\zeta_2}+\lambda)\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)=\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t})).$$

Equivalently,

$$\begin{aligned}&\mathcal{D}_c^{\zeta_2}\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)\\&+\lambda\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)=\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t})).\end{aligned}\tag{2.23}$$

The Laplace transform of the above equation implies that

$$\begin{aligned}&\mathbf{s}^{\zeta_2}\mathfrak{L}_\mathbf{s}\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)+\mathbf{s}^{\zeta_2-1}\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)_{\mathbf{t}=0}\\&+\lambda\mathfrak{L}_\mathbf{s}\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)=\mathfrak{L}_\mathbf{s}\Big(\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t}))\Big).\end{aligned}$$

Since

$$\mathbf{s}^{\zeta_2-1}\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)_{\mathbf{t}=0}=0,$$

we get

$$\begin{aligned}&\mathbf{s}^{\zeta_2}\mathfrak{L}_\mathbf{s}\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)\\&+\lambda\mathfrak{L}_\mathbf{s}\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)=\mathfrak{L}_\mathbf{s}\Big(\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t}))\Big),\end{aligned}$$

that is,

$$\mathfrak{L}_\mathbf{s}\Big(\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\Big)=\frac{1}{\mathbf{s}^{\zeta_2}+\lambda}\mathfrak{L}_\mathbf{s}\Big(\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t}))\Big).$$

Now, by utilizing the inverse Laplace transform, we obtain

$$\begin{aligned}&\frac{\vartheta(\mathbf{t})}{\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))}\\&=\mathfrak{L}_\mathbf{s}^{-1}\Big(\frac{1}{\mathbf{s}^{\zeta_2}+\lambda}\mathfrak{L}_\mathbf{s}\Big(\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t}))\Big)\Big)\\&=\mathbf{t}^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda\mathbf{t}^{\zeta_2})\star\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{t},\vartheta(\mathbf{t}))\\&=\int_0^\mathbf{t}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathcal{I}_c^{\zeta_1}\mathbb{B}(\mathbf{s},\vartheta(\mathbf{s}))d\mathbf{s}\\&=\frac{1}{\Gamma(\zeta_1)}\int_0^\mathbf{t}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\int_0^\mathbf{s}(\mathbf{s}-\tau)^{\zeta_1-1}\mathbb{B}(\tau,\vartheta(\tau))d\tau d\mathbf{s}\\&=\frac{1}{\Gamma(\zeta_1)}\int_0^\mathbf{t}\int_0^\mathbf{s}(\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathbb{B}(\tau,\vartheta(\tau))d\tau d\mathbf{s},\end{aligned}$$

which implies that

$$\vartheta(\mathbf{t}) = \overline{\Upsilon}(\mathbf{t}) \int_0^{\mathbf{t}} \int_0^{\mathbf{s}} (\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathbb{B}(\tau,\vartheta(\tau))d\tau d\mathbf{s},$$

where

$$\overline{\Upsilon}(\mathbf{t}) = \frac{1}{\Gamma(\zeta_1)}\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t})).$$

This complete the proof. □

Theorem 2.4. *Suppose that the following conditions hold:*

($\mathcal{P}_1$) *There exist $\beta, \chi_3 \in C(\mathcal{I}, \mathbb{R}^+)$ such that the following inequalities are satisfied:*

$$\begin{cases} |\mathbb{B}(\mathbf{t},\mathbf{r}_1) - \mathbb{B}(\mathbf{t},\mathbf{r}_2)| \le \chi_3(\mathbf{t})|\mathbf{r}_2 - \mathbf{r}_1|, \\ |\Upsilon(\mathbf{t}, r_0, r_1, ..., r_\kappa) - \Upsilon(\mathbf{t}, s_0, s_1, ..., s_\kappa)| \le \beta(\mathbf{t}) \sum_{m=0}^{\kappa} |r_m - s_m|, \end{cases}$$

for all $(\mathbf{t},\mathbf{r}_1,\mathbf{r}_2) \in \mathcal{I}\times\mathbb{R}\times\mathbb{R}$ and $(\mathbf{t}, r_0, r_1, ..., r_\kappa, s_0, s_1, ..., s_\kappa) \in \mathcal{I}\times\mathbb{R}^{2\kappa+2}$.

($\mathcal{P}_2$) *There exists $\xi > 0$ such that*

$$(\beta^*\xi \sum_{m=0}^{\kappa} \frac{1}{\Gamma(\varsigma_m+1)} + \Upsilon^*)(\frac{\Gamma(\zeta_2)\aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*)}{\Gamma(\zeta_1+\zeta_2+1)}) \le \xi$$

and

$$\frac{\Gamma(\zeta_2)\aleph_1^\star}{\Gamma(\zeta_1+\zeta_2+1)}\Big((2\chi_3^*\beta^*\xi + \mathbb{B}_0^*)\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)} + \Upsilon^*\chi_3^*\Big) < 1,$$

where β^*, χ_3^*, $\mathbb{B}_0^*$, $\aleph_1^\star$, Υ^* indicate the suprema of $\beta(\mathbf{t})$, $\chi_3(\mathbf{t})$, $\mathbb{B}(\mathbf{t},0)$, $\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda\mathbf{t}^{\zeta_2})$ and $\Upsilon(\mathbf{t},0,0,0,...,0,0)$, respectively.

Then, the problem mentioned in (1.3) *has at least one solution.*

Proof. Let $\mathbf{Z}^\star = C(\mathcal{I})$. Define $\alpha : \mathbf{Z}^\star \times \mathbf{Z}^\star \to [0,\infty)$ by

$$\alpha(\vartheta,\tilde{\vartheta}) = \begin{cases} 1 & \text{if} \quad \vartheta(\mathbf{t}) \le \xi, \quad \forall \mathbf{t} \in \mathcal{I}, \\ 0, & \quad \text{otherwise.} \end{cases}$$

Furthermore, we define the operator $\hat{\mathcal{K}} : \bar{V}_\zeta(0) \to \mathbf{Z}^\star$ by

$$(\hat{\mathcal{K}}\vartheta)(\mathbf{t}) = \overline{\Upsilon}(\mathbf{t}) \int_0^{\mathbf{t}} \int_0^{\mathbf{s}} (\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathbb{B}(\tau,\vartheta(\tau))d\tau d\mathbf{s},$$

where

$$\overline{\Upsilon}(\mathbf{t}) = \frac{1}{\Gamma(\zeta_1)}\Upsilon(\mathbf{t},\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}),\mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}),...,\mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t})).$$

Define two single-valued mappings $\hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2 : \bar{V}_\zeta(0) \to \mathcal{X}$ by

$$(\hat{\mathscr{B}}_1\vartheta)(t) = \overline{\Upsilon}(\mathbf{t}) = \Upsilon(\mathbf{t}, \vartheta(\mathbf{t}), \mathcal{I}^{\varsigma_1}\vartheta(\mathbf{t}), \mathcal{I}^{\varsigma_2}\vartheta(\mathbf{t}), ..., \mathcal{I}^{\varsigma_k}\vartheta(\mathbf{t}))$$

and

$$(\hat{\mathscr{B}}_2\vartheta)(t) = \frac{1}{\Gamma(\zeta_1)}\int_0^{\mathbf{t}}\int_0^{\mathbf{s}}(\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\mathbb{B}(\tau, \vartheta(\tau))d\tau d\mathbf{s}.$$

According to Lemma 2.3, $(\hat{\mathcal{K}}\vartheta)$ is the integral version of problem 1.3. By using Lemma (1.3), σ_0 solves the problem 1.3 if and only if σ_0 is a fixed point of $(\hat{\mathcal{K}}\vartheta)$. Now, we are going to prove that all conditions of Lemma 1.3 hold true for $\mathbb{H}$. At first, assume that $\alpha(\vartheta, \tilde{\vartheta}) \geq 1$. Then, $\vartheta(\mathbf{t}) \leq \zeta$, and $\tilde{\vartheta}(\mathbf{t}) \leq \zeta$ for $\mathbf{t} \in \mathcal{I}$. Hence,

$$|\mathbb{B}(\mathbf{t}, \vartheta(\mathbf{t}))| = |\mathbb{B}(\mathbf{t}, \vartheta(\mathbf{t})) - \mathbb{B}(\mathbf{t}, 0) + \mathbb{B}(\mathbf{t}, 0)| \leq \chi_3(\mathbf{t})|\vartheta(\mathbf{t})| + \mathbb{B}(\mathbf{t}, 0) \leq \chi_3^*\xi + \mathbb{B}_0^*.$$

Then, we deduce that

$$\begin{aligned}
|(\hat{\mathscr{B}}_2\vartheta)(\mathbf{t})| &\leq \frac{1}{\Gamma(\zeta_1)}\int_0^{\mathbf{t}}\int_0^{\mathbf{s}}(\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}|\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})||\mathbb{B}(\tau, \vartheta(\tau))|d\tau d\mathbf{s} \\
&\leq \aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*) \times \frac{1}{\Gamma(\zeta_1)}\int_0^{\mathbf{t}}\int_0^{\mathbf{s}}(\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}d\tau d\mathbf{s} \\
&\leq \aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*) \times \frac{1}{\Gamma(\zeta_1+1)}\int_0^1 \mathbf{s}^{\zeta_1}(1-\mathbf{s})^{\zeta_2-1} \\
&= \aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*) \times \frac{1}{\Gamma(\zeta_1+1)}\mathbf{B}(\zeta_1+1, \zeta_2) \text{ (where } \mathbf{B} \text{ is the beta function)} \\
&= \aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*) \times \frac{1}{\Gamma(\zeta_1+1)} \times \frac{\Gamma(\zeta_1+1)\Gamma(\zeta_2)}{\Gamma(\zeta_1+\zeta_1+1)} = \frac{\Gamma(\zeta_2)\aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*)}{\Gamma(\zeta_1+\zeta_1+1)},
\end{aligned}$$

which yields

$$|(\hat{\mathscr{B}}_2\vartheta)(\mathbf{t})| \leq \frac{\Gamma(\zeta_2)\aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*)}{\Gamma(\zeta_1+\zeta_1+1)}. \tag{2.24}$$

Furthermore, we can conclude that

$$\begin{aligned}
\left|\Upsilon(\mathbf{t}, \vartheta(\mathbf{t}), \mathcal{I}_c^{\varsigma_1}\vartheta(\mathbf{t}), \mathcal{I}_c^{\varsigma_2}\vartheta(\mathbf{t}), ..., \mathcal{I}_c^{\varsigma_\kappa}\vartheta(\mathbf{t}))\right| &= \Big|\Upsilon(\mathbf{t}, \vartheta(\mathbf{t}), \mathcal{I}_c^{\varsigma_1}\vartheta(\mathbf{t}), \mathcal{I}_c^{\varsigma_2}\vartheta(\mathbf{t}), ..., \mathcal{I}_c^{\varsigma_\kappa}\vartheta(\mathbf{t})) \\
&\quad - \Upsilon(\mathbf{t}, 0, 0, 0, ..., 0, 0) + \Upsilon(\mathbf{t}, 0, 0, 0, ..., 0, 0)\Big| \\
&\leq \beta(\mathbf{t})\sum_{m=0}^{\kappa}|\mathcal{I}_c^{\varsigma_m}\vartheta(\mathbf{t})| + |\Upsilon(\mathbf{t}, 0, 0, 0, ..., 0, 0)| \\
&\leq \beta^*\xi\sum_{m=0}^{\kappa}\frac{1}{\Gamma(m_\kappa+1)} + \Upsilon^*,
\end{aligned}$$

which implies that

$$|(\hat{\mathscr{B}}_1\vartheta)(\mathbf{t})| \leq \beta^*\xi\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)} + \Upsilon^*. \tag{2.25}$$

By applying (2.24) and (2.25), we have

$$(\hat{\mathcal{K}}\vartheta)(\mathbf{t}) \leq (\beta^*\xi \sum_{m=0}^{\kappa} \frac{1}{\Gamma(\varsigma_m+1)} + \Upsilon^*)(\frac{\Gamma(\zeta_2)\aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*)}{\Gamma(\zeta_1+\zeta_1+1)}) \leq \xi,$$

and so $(\hat{\mathcal{K}}\vartheta)(\mathbf{t}) \leq \xi$. Similarly, we can prove that $(\hat{\mathcal{K}}\tilde{\vartheta})(\mathbf{t}) \leq \xi$. Hence, $\alpha(\hat{\mathcal{K}}\vartheta, \hat{\mathcal{K}}\tilde{\vartheta}) \geq 1$, that is, $\hat{\mathcal{K}}$ is an α-admissible mapping. Next, we will prove that $\hat{\mathcal{K}}$ is an α-ψ-contractive mapping. Note that

$$\begin{aligned}
&|(\hat{\mathcal{B}}_2\vartheta)(\mathbf{t}) - (\hat{\mathcal{B}}_2\tilde{\vartheta})(\mathbf{t})| \\
&\leq \frac{1}{\Gamma(\zeta_1)} \int_0^{\mathbf{t}} \int_0^{\mathbf{s}} (\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})|\mathbb{B}(\tau,\vartheta(\tau)) - \mathbb{B}(\tau,\tilde{\vartheta}(\tau))|d\tau d\mathbf{s} \\
&\leq \frac{1}{\Gamma(\zeta_1)} \int_0^{\mathbf{t}} \int_0^{\mathbf{s}} (\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1}\mathbf{E}_{\zeta_2,\zeta_2}(-\lambda(\mathbf{t}-\mathbf{s})^{\zeta_2})\chi_3(\tau)|\vartheta(\tau) - \tilde{\vartheta}(\tau)|d\tau d\mathbf{s} \\
&\leq \frac{1}{\Gamma(\zeta_1)} \times \aleph_1^\star\chi_3^*\|\vartheta - \tilde{\vartheta}\| \times \int_0^{\mathbf{t}} \int_0^{\mathbf{s}} (\mathbf{s}-\tau)^{\zeta_1-1}(\mathbf{t}-\mathbf{s})^{\zeta_2-1} d\tau d\mathbf{s} \\
&\leq \frac{\Gamma(\zeta_2)\aleph_1^\star\chi_3^*}{\Gamma(\zeta_1+\zeta_1+1)}\|\vartheta - \tilde{\vartheta}\|.
\end{aligned}$$

Thus,

$$|(\hat{\mathcal{B}}_2\vartheta)(\mathbf{t}) - (\hat{\mathcal{B}}_2\tilde{\vartheta})(\mathbf{t})| \leq \frac{\Gamma(\zeta_2)\aleph_1^\star\chi_3^*}{\Gamma(\zeta_1+\zeta_1+1)}\|\vartheta - \tilde{\vartheta}\|. \tag{2.26}$$

Finally, we deduce that

$$\begin{aligned}
|(\hat{\mathcal{B}}_1\vartheta)(\mathbf{t}) - (\hat{\mathcal{B}}_1\tilde{\vartheta})(\mathbf{t})| &\leq \beta(\mathbf{t}) \sum_{m=0}^{\kappa} |\mathcal{I}_c^{\varsigma_m}\vartheta(\mathbf{t}) - \mathcal{I}_c^{\varsigma_m}\tilde{\vartheta}(\mathbf{t})| \\
&\leq \beta(\mathbf{t}) \sum_{m=0}^{\kappa} \mathcal{I}_c^{\varsigma_m}|\vartheta(\mathbf{t}) - \tilde{\vartheta}(\mathbf{t})| \\
&\leq \beta^* \sum_{m=0}^{\kappa} \frac{1}{\Gamma(\varsigma_m+1)}\|\vartheta - \tilde{\vartheta}\|.
\end{aligned}$$

So, we get

$$|(\hat{\mathcal{B}}_1\vartheta)(\mathbf{t}) - (\hat{\mathcal{B}}_1\tilde{\vartheta})(\mathbf{t})| \leq \beta^* \sum_{m=0}^{\kappa} \frac{1}{\Gamma(\varsigma_m+1)}\|\vartheta - \tilde{\vartheta}\|. \tag{2.27}$$

Therefore, by using (2.24)–(2.27), we obtain

$$\begin{aligned}
|(\hat{\mathcal{K}}\vartheta)(\mathbf{t}) - (\hat{\mathcal{K}}\tilde{\vartheta})(\mathbf{t})| &= |(\hat{\mathcal{B}}_1\vartheta)(\mathbf{t})(\hat{\mathcal{B}}_2\vartheta)(\mathbf{t}) - (\hat{\mathcal{B}}_1\tilde{\vartheta})(\mathbf{t})(\hat{\mathcal{B}}_2\tilde{\vartheta})(\mathbf{t})| \\
&= |(\hat{\mathcal{B}}_1\vartheta)(\mathbf{t})(\hat{\mathcal{B}}_2\vartheta)(\mathbf{t}) - (\hat{\mathcal{B}}_1\vartheta)(\mathbf{t})(\hat{\mathcal{B}}_2\tilde{\vartheta})(\mathbf{t}) + (\hat{\mathcal{B}}_1\vartheta)(\mathbf{t})(\hat{\mathcal{B}}_2\tilde{\vartheta})(\mathbf{t}) - (\hat{\mathcal{B}}_1\tilde{\vartheta})(\mathbf{t})(\hat{\mathcal{B}}_2\tilde{\vartheta})(\mathbf{t})| \\
&\leq |(\hat{\mathcal{B}}_1\vartheta)(\mathbf{t})||(\hat{\mathcal{B}}_2\vartheta)(\mathbf{t}) - (\hat{\mathcal{B}}_2\tilde{\vartheta})(\mathbf{t})| + |(\hat{\mathcal{B}}_2\vartheta)(\mathbf{t})||(\hat{\mathcal{B}}_1\vartheta)(\mathbf{t}) - (\hat{\mathcal{B}}_1\tilde{\vartheta})(\mathbf{t})| \\
&\leq (\beta^*\xi \sum_{m=0}^{\kappa} \frac{1}{\Gamma(\varsigma_m+1)} + \Upsilon^*) \times \frac{\Gamma(\zeta_2)\aleph_1^\star\chi_3^*}{\Gamma(\zeta_1+\zeta_1+1)}\|\vartheta - \tilde{\vartheta}\| \\
&\quad + \frac{\Gamma(\zeta_2)\aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*)}{\Gamma(\zeta_1+\zeta_1+1)} \times \beta^* \sum_{m=0}^{\kappa} \frac{1}{\Gamma(\varsigma_m+1)}\|\vartheta - \tilde{\vartheta}\|.
\end{aligned}$$

Now, we minimize the right side of the above inequality:

$$\begin{aligned}
&(\beta^*\xi\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)}+\Upsilon^*)\times\frac{\Gamma(\zeta_2)\aleph_1^{\star}\chi_3^*}{\Gamma(\zeta_1+\zeta_1+1)}\|\vartheta-\tilde{\vartheta}\|\\
&+\frac{\Gamma(\zeta_2)\aleph_1^{\star}(\chi_3^*\xi+\mathbb{B}_0^*)}{\Gamma(\zeta_1+\zeta_1+1)}\times\beta^*\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)}\|\vartheta-\tilde{\vartheta}\|\\
&=\frac{\Gamma(\zeta_2)\aleph_1^{\star}}{\Gamma(\zeta_1+\zeta_1+1)}\Big(\chi_3^*\beta^*\xi\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)}+\beta^*(\chi_3^*\xi+\mathbb{B}_0^*)\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)}+\Upsilon^*\chi_3^*\Big)\|\vartheta-\tilde{\vartheta}\|\\
&=\frac{\Gamma(\zeta_2)\aleph_1^{\star}}{\Gamma(\zeta_1+\zeta_1+1)}\Big((2\chi_3^*\beta^*\xi+\mathbb{B}_0^*)\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)}+\Upsilon^*\chi_3^*\Big)\|\vartheta-\tilde{\vartheta}\|.
\end{aligned}$$

Hence, we get

$$|(\hat{\mathcal{K}}\vartheta)(\mathbf{t})-(\hat{\mathcal{K}}\tilde{\vartheta})(\mathbf{t})|\leq\frac{\Gamma(\zeta_2)\aleph_1^{\star}}{\Gamma(\zeta_1+\zeta_1+1)}\Big((2\chi_3^*\beta^*\xi+\mathbb{B}_0^*)\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)}+\Upsilon^*\chi_3^*\Big)\|\vartheta-\tilde{\vartheta}\|.$$

Now, we define the map $\psi:[0,\infty)\to[0,\infty)$ by

$$\psi(\mathbf{t})=\frac{\Gamma(\zeta_2)\aleph_1^{\star}}{\Gamma(\zeta_1+\zeta_1+1)}\Big((2\chi_3^*\beta^*\xi+\mathbb{B}_0^*)\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m+1)}+\Upsilon^*\chi_3^*\Big)\mathbf{t}.$$

Therefore, as also stated in [51], $\psi\in\Psi$, and $\|\hat{\mathcal{K}}\vartheta-\hat{\mathcal{K}}\tilde{\vartheta}\|\leq\psi(\|\sigma-\tilde{\sigma}\|)$. On the other hand, if $\alpha(\vartheta,\tilde{\vartheta})=0$, then $\alpha(\vartheta,\tilde{\vartheta})\|\hat{\mathcal{K}}\vartheta-\hat{\mathcal{K}}\tilde{\vartheta}\|=0\leq\psi(\|\vartheta-\tilde{\vartheta}\|)$. Thus, for all $\vartheta,\tilde{\vartheta}\in\mathbf{Z}$, we possess $\alpha(\vartheta,\tilde{\vartheta})\|\hat{\mathcal{K}}\vartheta-\hat{\mathcal{K}}\tilde{\vartheta}\|\leq\psi(\|\vartheta-\tilde{\vartheta}\|)$. Hence, we conclude that $\hat{\mathcal{K}}$ is α-ψ-contractive mapping. Suppose that $\{\vartheta_n\}$ is a sequence in $\mathbf{Z}$ such that $\alpha(\vartheta_{n-1},\vartheta_n)\geq1$ for all $n\geq1$, and $\vartheta_n\to\vartheta\in\mathbf{Z}$. So, $\vartheta_n(\vartheta)\leq\xi$. Then $\lim_{n\to\infty}\vartheta_n(\vartheta)=\vartheta\leq\varsigma$, i.e., $\alpha(\vartheta_n,\vartheta)\geq1$ for all $n\geq1$. Also, evidently, there exists $\vartheta_0\in\mathbf{Z}$ such that $\alpha(\vartheta_0,\hat{\mathcal{K}}\vartheta_0)\geq1$. Then, all conditions of Lemma 1.3 are valid, and $\hat{\mathcal{K}}$ has a fixed point in $\mathbf{Z}$ which is a solution for the problem (1.3). □

3. Examples

To illustrate the effectiveness of the proposed method in this work, two test examples are carried out in this section.

Example 3.1. *Consider the following fractional boundary value problem:*

$$\begin{cases}
\mathcal{D}_c^{\frac{10}{11}}\Big((\mathcal{D}_c^{\frac{1}{3}}+1)(\frac{\vartheta(\mathbf{t})}{\frac{t}{200}|\vartheta(\mathbf{t})|+\frac{2+\ln(1+\mathbf{t})}{1+\ln(1+\mathbf{t})}})-\frac{\cos^3(\mathbf{t})}{400(1+\mathbf{t}^2)}(\vartheta(\mathbf{t})+e^{-\mathbf{t}^2})\Big)=\frac{\sin^2\frac{\pi\mathbf{t}}{2}}{60}(\frac{1}{70}\vartheta(\mathbf{t})+\frac{1}{2}),\\
\mathcal{D}_c^{\frac{2}{3}}\Big(\frac{\vartheta(\mathbf{t})}{\frac{\mathbf{t}}{200}|\vartheta(\mathbf{t})|+\frac{2+\ln(1+\mathbf{t})}{1+\ln(1+\mathbf{t})}}\Big)_{\mathbf{t}=0}=\frac{1}{400}(\vartheta(0)+1),\\
\sum_{\mathbf{i}=0}^{2}\frac{100^{-\mathbf{i}}\vartheta(\pi^{-\mathbf{i}})}{\wp(\pi^{-\mathbf{i}},\vartheta(\pi^{-\mathbf{i}}))}+\sum_{\mathbf{j}=0}^{3}\frac{200^{-\mathbf{j}}\vartheta(\mathbf{e}^{-\mathbf{j}})}{\wp(\mathbf{e}^{-\mathbf{j}},\vartheta(\mathbf{e}^{-\mathbf{j}}))}=\frac{1}{100\pi+200e}.
\end{cases}\tag{3.1}$$

In this case, we take $\zeta_1 = \frac{10}{11}$, $\zeta_2 = \frac{1}{3}$, $\zeta = 1$, $\lambda = 0.1$, $\kappa_{\mathbf{i}} = 100^{-i}$ *(*$i = 0, 1, 2$*)*, $\tau_{\mathbf{j}} = 200^{-j}$ *(*$j = 0, 1, 2, 3$*)*, $\mathbf{a_{n,m}} = \frac{1}{n\pi + me}$, $\wp(\mathbf{t}, \vartheta(\mathbf{t})) = \frac{t}{200}|\vartheta(\mathbf{t})| + \frac{2+\ln(1+\mathbf{t})}{1+\ln(1+\mathbf{t})}$, $\Im(\mathbf{t}, \vartheta(\mathbf{t})) = \frac{\cos^3(\mathbf{t})}{400(1+\mathbf{t}^2)}(\vartheta(\mathbf{t}) + e^{-\mathbf{t}^2})$ *and* $\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t})) = \frac{\sin^2 \frac{\pi \mathbf{t}}{2}}{60}(\frac{1}{70}\vartheta(\mathbf{t}) + \frac{1}{3})$. *To better understand this example, graphs of some functions are presented in Figures 1–3.*

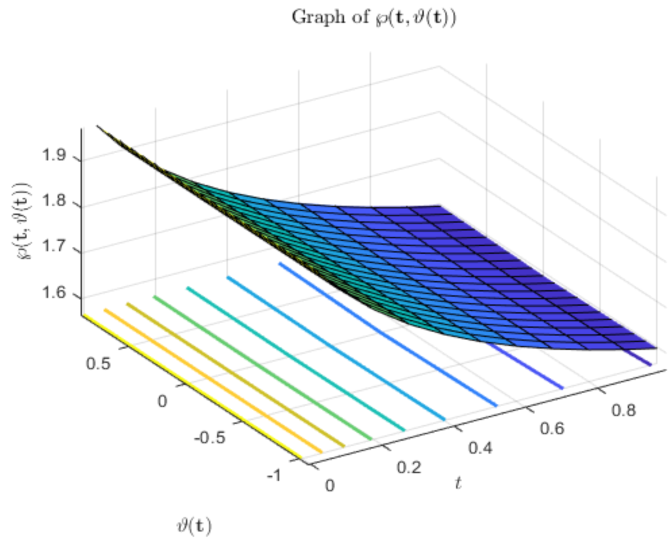

Figure 1. The graph of $\wp(\mathbf{t}, \vartheta(\mathbf{t}))$ in Example 3.1.

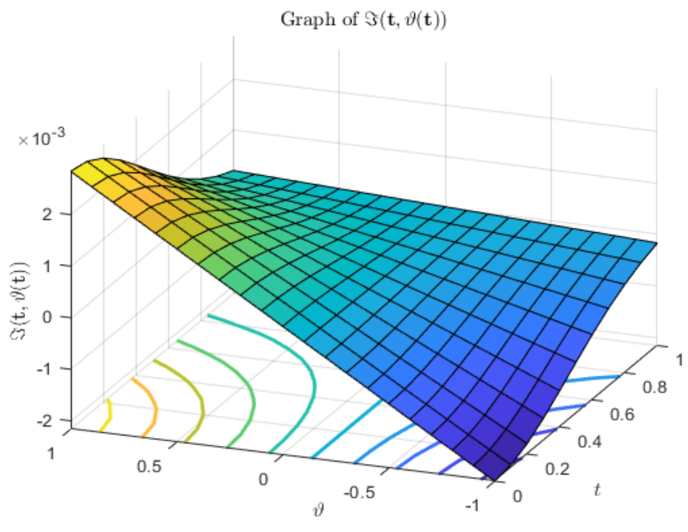

Figure 2. The graph of $\Im(\mathbf{t}, \vartheta(\mathbf{t}))$ in Example 3.1.

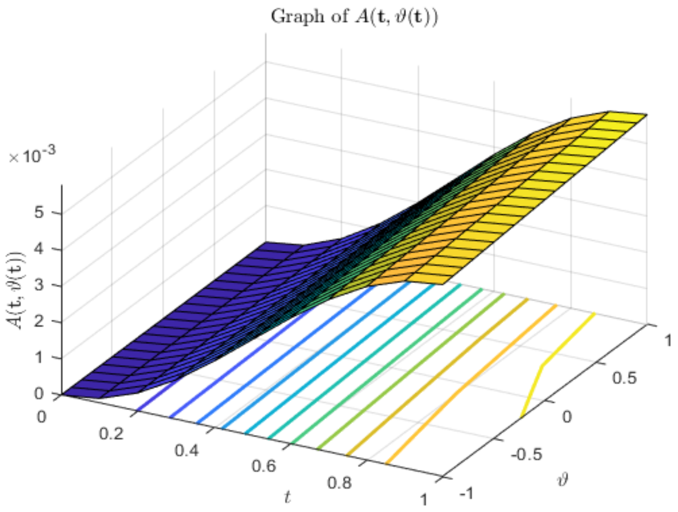

Figure 3. The graph of $\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t}))$ in Example 3.1.

Furthermore, in this case we have

$$|\wp(\mathbf{t}, \vartheta(\mathbf{t})) - \wp(\mathbf{t}, \tilde{\vartheta}(\mathbf{t}))| \le \frac{t}{200}|\vartheta(\mathbf{t}) - \tilde{\vartheta}(\mathbf{t})|,$$

and

$$|\mathfrak{I}(\mathbf{t}, \vartheta(\mathbf{t})) - \mathfrak{I}(\mathbf{t}, \tilde{\vartheta}(\mathbf{t}))| \le \frac{\cos^3(\mathbf{t})}{400(1 + \mathbf{t}^2)}|\vartheta(\mathbf{t}) - \tilde{\vartheta}(\mathbf{t})|.$$

Also,

$$|\mathbb{A}(\mathbf{t}, \vartheta(\mathbf{t})) - \mathbb{A}(\mathbf{t}, \tilde{\vartheta}(\mathbf{t}))| \le \frac{\sin^2 \frac{\pi \mathbf{t}}{2}}{4200}|\vartheta(\mathbf{t}) - \tilde{\vartheta}(\mathbf{t})|.$$

Now, we find $\chi_1^* \approx 0.0002380952$, $\chi_2^* \approx 0.0025$, $\chi_3^* \approx 0.005$, $\wp_0^* = 2$, $\mathfrak{I}_0^* \approx 0.0025$, $\mathbb{A}_0^* \approx 0.0055555556$. *Also,* $\mathbf{a}_{2,3} \approx 0.0011657517$, $\Pi_1^* \approx 0.9421485569355$, *and* $\Pi_2^* \approx 1$. *So,*

$$|\Phi| = \frac{1}{|\sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}} \mathbf{E}_{\zeta_2,1}(-\lambda \mu_{\mathbf{i}}{}^{\zeta_2}) + \sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}} \mathbf{E}_{\zeta_2,1}(-\lambda \nu_{\mathbf{j}}{}^{\zeta_2})|} \approx 0.4962471002,$$

and

$$\mho_1 = \left(\frac{\chi_1^*}{\zeta_1 + \zeta_2} + \frac{\chi_2^*}{\zeta_2}\right)\left(\Pi_1^* + |\Phi|\Pi_1^*\Pi_2^*(\sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}} + \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}})\right) \approx 0.0149828586729,$$

$$\mho_2 = \left(\frac{\mathbb{A}_0^*}{\zeta_1 + \zeta_2} + \frac{\mathfrak{I}_0^*}{\zeta_2}\right)\left(\Pi_1^* + |\Phi|\Pi_1^*\Pi_2^*(\sum_{\mathbf{j}=0}^{\mathbf{m}} \tau_{\mathbf{j}} + \sum_{\mathbf{i}=0}^{\mathbf{n}} \kappa_{\mathbf{i}})\right) + |\Phi|\Pi_1^*|\mathbf{a}_{\mathbf{n},\mathbf{m}}| \approx 0.0238649003463.$$

Thus, we get

$$(\mho_1\zeta + \mho_2)(\chi_3^*\zeta + \wp_0^*) \approx 0.077889756833496 \le 1 = \zeta$$

and

$$(2\chi_3^* + \wp_0^*)\mho_1 + \chi_3^*\mho_2 \approx 0.0002691530884605 < 1.$$

Hence, all conditions of Theorem 2.2 hold, and the problem (3.1) *has at least one solution.*

Example 3.2. *Consider the following fractional hybrid Langevin problem:*

$$\begin{cases} \mathcal{D}_c^{\frac{6}{7}}(\mathcal{D}_c^{\frac{1}{6}} + \frac{1}{2})\Big(\frac{\vartheta(\mathbf{t})}{\frac{\cos^2\pi\mathbf{t}}{1+e^{-\mathbf{t}^2}} + \frac{\vartheta(\mathbf{t})}{|\mathbf{t}|+e^{\pi e\mathbf{t}}} + \frac{1-\sin^6\pi^2 t}{(14+\pi t^2)\Gamma(\frac{\pi}{8})}\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{-\frac{8-\pi}{8}}\vartheta(\mathbf{s})d\vartheta(\mathbf{s})}\Big) = \frac{2^{-\frac{1}{1+t}}|\vartheta(\mathbf{t})|}{1+|\vartheta(\mathbf{t})|} + \frac{1}{1+3^{t+4}}e^{-\pi t}, \\ \vartheta(0) = 0, \\ \Big[(\mathcal{D}_c^{\frac{1}{6}} + \frac{1}{2})\Big(\frac{\vartheta(\mathbf{t})}{\frac{\cos^2\pi\mathbf{t}}{1+e^{-\mathbf{t}^2}} + \frac{\vartheta(\mathbf{t})}{|\mathbf{t}|+e^{\pi e\mathbf{t}}} + \frac{1-\sin^6\pi^2 t}{(14+\pi t^2)\Gamma(\frac{\pi}{8})}\int_0^{\mathbf{t}}(\mathbf{t}-\mathbf{s})^{-\frac{8-\pi}{8}}\vartheta(\mathbf{s})ds}\Big)\Big]_{\mathbf{t}=0} = 0. \end{cases} \tag{3.2}$$

In this case, we take $\xi = 0.1$, $\zeta_1 = \frac{6}{7}$, $\zeta_2 = \frac{1}{6}$, $\varsigma_0 = 0$, $\varsigma_1 = \frac{\pi}{8}$, $\lambda = \frac{1}{2}$,

$$\mathbb{B}(\mathbf{t}, \mathbf{x}) = \frac{2^{-\frac{1}{1+t}}|\mathbf{x}|}{1 + |\mathbf{x}|} + \frac{1}{1 + 3^{\mathbf{t}+4}}e^{-\pi\mathbf{t}},$$

and

$$\Upsilon(\mathbf{t}, \mathbf{y}, \mathbf{z}) = \frac{\cos^2\pi\mathbf{t}}{1 + e^{-\mathbf{t}^2}} + \frac{\mathbf{y}}{|\mathbf{t}| + e^{\pi e\mathbf{t}}} + \frac{1 - \sin^6\pi^2 t}{14 + \pi t^2}\mathbf{z}.$$

Then, $|\mathbb{B}(\mathbf{t}, \mathbf{x}_1) - \mathbb{B}(\mathbf{t}, \mathbf{x}_2)| \leq 2^{-\frac{1}{1+t}}|\mathbf{x}_1 - \mathbf{x}_2|$, *and*

$$|\Upsilon(\mathbf{t}, \mathbf{y}_1, \mathbf{z}_1) - \Upsilon(\mathbf{t}, \mathbf{y}_2, \mathbf{z}_2)| \leq \max\Big\{\frac{1}{|\mathbf{t}| + e^{\pi e\mathbf{t}}}, \frac{1 - \sin^6\pi^2 t}{14 - \pi t^2}\Big\}(|\mathbf{y}_1 - \mathbf{y}_2| + |\mathbf{z}_1 - \mathbf{z}_2|).$$

So, $\chi_3(\mathbf{t}) = 2^{-\frac{1}{1+t}}$ *and* $\beta(\mathbf{t}) = \max\{\frac{1}{|\mathbf{t}|+e^{\pi e\mathbf{t}}}, \frac{1-\sin^6\pi^2 t}{14+\pi t^2}\}$. *Now, we obtain* $\chi_3^* \approx 0.7071067811865$ *and* $\beta^* = 0.5$. *Furthermore,* $\mathbb{B}_0^* = 0.0121951219512$, $\Upsilon^* \approx 0.73105857863$, *and* $\aleph_1^\star \approx 0.1796520355079$. *Then we find*

$$(\beta^*\xi\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m + 1)} + \Upsilon^*)(\frac{\Gamma(\zeta_2)\aleph_1^\star(\chi_3^*\xi + \mathbb{B}_0^*)}{\Gamma(\zeta_1 + \zeta_1 + 1)}) \approx 0.0687162196187 \leq 0.1 = \xi,$$

and

$$\frac{\Gamma(\zeta_2)\aleph_1^\star}{\Gamma(\zeta_1 + \zeta_1 + 1)}\Big((2\chi_3^*\beta^*\xi + \mathbb{B}_0^*)\sum_{m=0}^{\kappa}\frac{1}{\Gamma(\varsigma_m + 1)} + \Upsilon^*\chi_3^*\Big) \approx 0.6861695496883 < 1.$$

Hence, all conditions of Theorem 2.4 hold, and the problem (3.2) *has at least one solution. In addition, to better understand this example, the graphs and numerical results of some functions are presented in Figures 4 and 5 and Table 1.*

Table 1. Numerical results for some functions in Example 3.2.

t	$\chi_3(\mathbf{t})$	$\frac{1}{\lvert\mathbf{t}\rvert+e^{\pi e\mathbf{t}}}$	$\frac{1-\sin^6\pi^2\mathbf{t}}{14+\pi\mathbf{t}^2}$	$\beta(\mathbf{t})$
0	0.5	1	0.0714	1
0.1	0.5325	0.4083	0.0472	0.4083
0.2	0.5612	0.1748	0.0279	0.1748
0.3	0.5867	0.0754	0.0700	0.0754
0.4	0.6095	0.0324	0.0592	0.0592
0.5	0.6299	0.0138	0.0094	0.0138
0.6	0.6484	0.0059	0.0659	0.0659
0.7	0.6651	0.0025	0.0617	0.0617
0.8	0.6803	0.0010	0.0003	0.0010
0.9	0.6943	0.0004	0.0593	0.0593
1	07071	0.0001	0.0579	0.0579

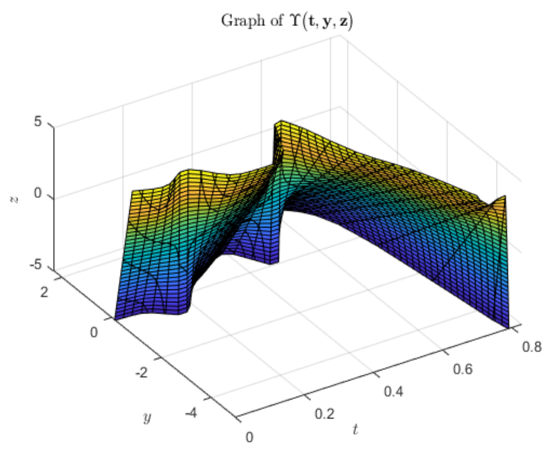

Figure 4. The graph of $\Upsilon(\mathbf{t}, \mathbf{y}, \mathbf{z})$ in Example 3.2.

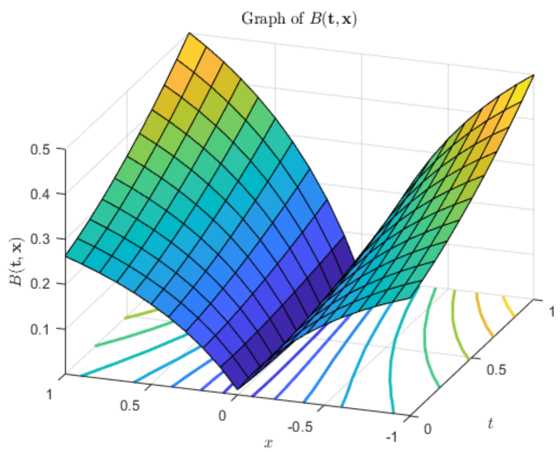

Figure 5. The graph of $\mathbb{B}(\mathbf{t}, \mathbf{x})$ in Example 3.2.

4. Conclusions

In this work, we have presented some sufficient conditions dealing with the existence of the solution for the generalized fractional hybrid Langevin equation with two different boundary conditions. First, in the form of two auxiliary lemmas, we solve a simplified form of our problem with the help of the Mittag-Leffler function, Laplace transforms and fractional calculus. Then, we design a fixed point problem in Banach space; in fact, we want to turn the problem of finding a solution into the problem of finding a fixed point for an operator. Once the desired operator is defined, the existence results are proved using the α-ψ-contraction theorem. Our proposed method is very simple and practical, so researchers can use our approach to solve equations that model natural phenomena. For our two main theorems, we have provided two numerical examples to test the efficiencies of our proposed method in numerical calculations. We obtained these results using the Caputo derivative, so other researchers can test our results with different fractional derivative operators, such as Hadamard, etc., to pave the way.

Acknowledgments

The third and fourth authors were supported by Azarbaijan Shahid Madani University. The authors express their gratitude to dear unknown referees for their helpful suggestions which improved the final version of this paper.

Conflict of interest

The authors declare that they have no competing interests.

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