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Research article

New results on the divisibility of power GCD and power LCM matrices

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Abstract: Let a, b and n be positive integers and let S be a set consisting of n distinct positive integers $x_1, ..., x_{n-1}$ and x_n. Let (S^a) (resp. $[S^a]$) denote the $n \times n$ matrix having $\gcd(x_i, x_j)^a$ (resp. $\mathrm{lcm}(x_i, x_j)^a$) as its (i, j)-entry. For any integer $x \in S$, if $(y < x, y|z|x$ and $y, z \in S) \Rightarrow z \in \{y, x\}$, then y is called a greatest-type divisor of x in S. In this paper, we establish some results about the divisibility between (S^a) and (S^b), between (S^a) and $[S^b]$ and between $[S^a]$ and $[S^b]$ when $a|b$, S is gcd closed (i.e., $\gcd(x_i, x_j) \in S$ for all $1 \le i, j \le n$), and $\max_{x \in S}\{|\{y \in S : y \text{ is a greatest-type divisor of } x \text{ in } S\}|\} = 2$.

Keywords: divisibility; power matrix; greatest-type divisor; gcd-closed set
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1. Introduction

For arbitrary integers x and y, we denote by (x, y) (resp. $[x, y]$) the greatest common divisor (resp. least common multiple) of integers x and y. Let a, b and n be positive integers. Let S be a set consisting of n distinct positive integers $x_1, ..., x_{n-1}$ and x_n. Let (S^a) (resp. $[S^a]$) stand for the $n \times n$ matrix with $(x_i, x_j)^a$ (resp. $[x_i, x_j]^a$) as its (i, j)-entry, which is called *ath power GCD matrix* (resp. *ath power LCM matrix*). In 1875, Smith [19] proved that

$$\det((i, j))_{1 \le i, j \le n} = \prod_{k=1}^{n} \varphi(k), \tag{1.1}$$

where φ is the Euler's phi function. After that, many generalizations of Smith's determinant (1.1) were published (see, for instance, [1–18] and [20–27]).

The set S is said to be *factor closed (FC)* if $(x \in S, d|x) \Rightarrow d \in S$. We say that S is *gcd closed* if S contains $(x_i, x_j) \in S$ for all integers i and j with $1 \le i, j \le n$. Obviously, an FC set must be gcd closed, but the converse is not true. As usual, let $\mathbf{Z}$ and $|S|$ denote the ring of integers and the cardinality of the set S, respectively. In 1999, Hong [10] introduced the concept of a greatest-type divisor when he [10] solved completely the renowned Bourque-Ligh conjecture [2]. For any integer $x \in S$, if

$$(y < x, y|z|x \text{ and } y, z \in S) \Rightarrow z \in \{y, x\},$$

then y is called a *greatest-type divisor* of x. One defines a subset $G_S(x)$ of S as follows:

$$G_S(x) := \{y \in S : y \text{ is a greatest-type divisor of } x \text{ in } S\}.$$

Let $M_n(\mathbf{Z})$ stand for the ring of $n \times n$ matrices over the integers. Bourque and Ligh [2] proved that (S) divides $[S]$ in the ring $M_n(\mathbf{Z})$ if S is FC. Namely, $\exists\, B \in M_n(\mathbf{Z})$ such that $[S] = B(S)$ or $[S] = (S)B$. Hong [12] showed that such a factorization is no longer true in general when S is gcd closed and $\max_{x \in S}\{|G_S(x)|\} = 2$. The results of Bourque-Ligh and Hong are extended by Korkee and Haukkanen [17] and by Chen, Hong and Zhao [5]. Feng, Hong and Zhao [6], Zhao [22], Altinisik, Yildiz and Keskin [1] and Zhao, Chen and Hong [23] used the greatest-type divisor to make important progress on an open problem of Hong raised in [12].

On the other hand, Hong [15] initially studied the divisibility among power GCD matrices and among power LCM matrices. It was proved in [15] that $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$ if $a|b$ and S is a divisor chain (that is, $x_{\sigma(1)}|...|x_{\sigma(n)}$ for a permutation σ of $\{1, ..., n\}$), and such factorizations are no longer true if $a \nmid b$ and $|S| \geq 2$. Evidently, a divisor chain is gcd closed but not conversely. Recently, Zhu [24] and Zhu and Li [27] confirmed three conjectures of Hong raised in [15] stating that if $a|b$ and S is a gcd-closed set with $\max_{x \in S}\{|G_S(x)|\} = 1$, then the bth power GCD matrix (S^b) (resp. the bth power LCM matrix $[S^b]$) is divisible by the ath power GCD matrix (S^a), and the bth power LCM matrix $[S^b]$ is divisible by the ath power LCM matrix (S^a). One naturally asks the following question: If $a|b$ and S is gcd closed and $\max_{x \in S}\{|G_S(x)|\} = 2$, then is it true that $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$ hold in $M_n(\mathbf{Z})$? In particular, the following interesting question arises.

Problem 1.1. *Let S be a gcd-closed set with* $\max_{x \in S}\{|G_S(x)|\} = 2$. *Is it true that $(S)|(S^b)$, $(S)|[S^b]$ and $[S]|[S^b]$ hold in $M_n(\mathbf{Z})$?*

In this paper, our main goal is to study Problem 1.1. To state our main result, we need the following concept, also due to Hong.

Definition 1.2. ([9, 14]) *Let S be a finite set of distinct positive integers, and let r be an integer with $1 \leq r \leq |S| - 1$. The set S is called 0-fold gcd closed if S is gcd closed. The set S is called r-fold gcd closed if there is a divisor chain $R \subset S$ with $|R| = r$ such that* $\max(R)|\min(S \setminus R)$, *and the set $S \setminus R$ is gcd closed.*

Clearly, any r-fold gcd-closed set must be an $(r-1)$-fold gcd-closed set, and the converse is not true. We can now state the main result of this paper.

Theorem 1.3 *Let a, b and n be positive integers. Then, each of the following is true:*

(i). *If $a|b$ and $n \leq 3$, then for any gcd-closed set S with $|S| = n$, one has $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$.*

(ii). *If $a|b$ and $n \geq 4$, then for any $(n-3)$-fold gcd-closed set S with $|S| = n$, one has $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$.*

(iii). *Let $n \geq 4$ and $b \geq 2$. If $36 \nmid b$, then there exists an $(n-4)$-fold gcd-closed set S_1 with $|S_1| = n$ and* $\max_{x \in S_1}\{|G_{S_1}(x)|\} = 2$ *such that $(S_1) \nmid (S_1^b)$. If $b \not\equiv 0, 35 \pmod{36}$, then there exists an $(n-4)$-fold* *gcd-closed set* S_2 *with* $|S_2| = n$ *and* $\max_{x\in S_2}\{|G_{S_2}(x)|\} = 2$ *such that* $(S_2) \nmid [S_2^b]$. *If* $b \not\equiv 0, 11, 100 \pmod{110}$*, then there exists an* $(n-4)$*-fold gcd-closed set* S_3 *with* $|S_3| = n$ *and* $\max_{x\in S_3}\{|G_{S_3}(x)|\} = 2$ *such that* $[S_3] \nmid [S_3^b]$.

It is obvious that

$$(S^a)|(S^b) \iff (S_\sigma^a)|(S_\sigma^b), (S^a)|[S^b] \iff (S_\sigma^a)|[S_\sigma^b], \text{ and } [S^a]|[S^b] \iff [S_\sigma^a]|[S_\sigma^b]$$

for any permutation σ on the set $\{1, ..., n\}$, where $S_\sigma := \{x_{\sigma(1)}, ..., x_{\sigma(n)}\}$. Thus, without loss of generality, we may let $x_1 < \cdots < x_n$ in what follows.

This paper is organized as follows. In Section 2, we supply some preliminary results that are needed in the proof of Theorem 1.3. Then, in Section 3, we present the proof of Theorem 1.3.

2. Preliminary lemmas

At first, for any arithmetic function f, we define the reciprocal arithmetic function $\frac{1}{f}$ for any positive integer m by

$$\frac{1}{f}(m) := \begin{cases} 0 & \text{if } f(m) = 0, \\ \dfrac{1}{f(m)} & \text{otherwise.} \end{cases}$$

We need two known results which give the formulas for the determinants of the power LCM matrix and power GCD matrix on gcd-closed sets.

Lemma 2.1. [13, Lemma 2.1] *If* S *is gcd closed, then*

$$\det[S^a] = \prod_{k=1}^{n} x_k^{2a}\alpha_{a,k}, \tag{2.1}$$

where

$$\alpha_{a,k} := \sum_{\substack{d|x_k \\ d\nmid x_t, x_t<x_k}} \left(\frac{1}{\xi_a} * \mu\right)(d), \tag{2.2}$$

where μ *is the Möbius function,* ξ_a *is defined by* $\xi_a(x) = x^a$*, and* $\frac{1}{\xi_a} * \mu$ *is the Dirichlet product of* $\frac{1}{\xi_a}$ *and* μ.

Lemma 2.2. *If* S *is gcd closed, then*

$$\det(S^a) = \prod_{k=1}^{n} \eta_{a,k}, \tag{2.3}$$

where

$$\eta_{a,k} := \sum_{\substack{d|x_k \\ d\nmid x_t, x_t<x_k}} (\xi_a * \mu)(d). \tag{2.4}$$

Proof. This follows immediately from [3, Theorem 2] applied to $f = \xi_a$. □

Lemma 2.3. *Let u be a positive integer. Then,*

$$\sum_{d|u}(\xi_a * \mu)(d) = u^a,$$

and

$$\sum_{d|u}\left(\frac{1}{\xi_a} * \mu\right)(d) = u^{-a}.$$

Proof. The results follow immediately from [11, Lemma 7] applied to $f = \xi_a$ and $f = \dfrac{1}{\xi_a}$ respectively. □

Lemma 2.4. *Let $\alpha_{a,k}$ and $\eta_{a,k}$ be defined as in* (2.2) *and* (2.4)*, respectively. Then, $\alpha_{a,1} = x_1^{-a}$ and $\eta_{a,1} = x_1^a$.*

Proof. Lemma 2.4 follows immediately from Lemma 2.3. □

Lemma 2.5. [24, Theorem 1.3] *Let S be gcd closed and* $\max_{x\in S} |G_S(x)| = 1$ *and let a and b be positive integers with a|b. Then, in the ring $M_n(\mathbf{Z})$, we have $(S^a)|(S^b)$ and $(S^a)|[S^b]$.*

Lemma 2.6. [6, Lemma 2.2] *Let S be a gcd-closed set with* $\max_{x\in S}\{|G_S(x)|\} = 2$. *Let $\alpha_{a,k}$ and $\eta_{a,k}$ be defined as in* (2.2) *and* (2.4)*, respectively. Then, for any $2 \le k \le n$, we have*

$$\alpha_{a,k} = \begin{cases} \dfrac{1}{x_k^a} - \dfrac{1}{x_{k_0}^a}, & \text{if } G_S(x_k) = \{x_{k_0}\}, \\ \dfrac{1}{x_k^a} - \dfrac{1}{x_{k_1}^a} - \dfrac{1}{x_{k_2}^a} + \dfrac{1}{x_{k_3}^a}, & \text{if } G_S(x_k) = \{x_{k_1}, x_{k_2}\} \text{ and } (x_{k_1}, x_{k_2}) = x_{k_3}, \end{cases} \tag{2.5}$$

and

$$\eta_{a,k} = \begin{cases} x_k^a - x_{k_0}^a, & \text{if } G_S(x_k) = \{x_{k_0}\}, \\ x_k^a - x_{k_1}^a - x_{k_2}^a + x_{k_3}^a, & \text{if } G_S(x_k) = \{x_{k_1}, x_{k_2}\} \text{ and } (x_{k_1}, x_{k_2}) = x_{k_3}. \end{cases} \tag{2.6}$$

Lemma 2.7. [27, Theorem 1.1] *Let S be gcd closed and* $\max_{x\in S} |G_S(x)| = 1$ *and let a and b be positive integers with a|b. Then, in the ring $M_n(\mathbf{Z})$, one has $[S^a]|[S^b]$.*

3. Proof of Theorem 1.3

In this section, we use the lemmas presented in the previous section to give the proof of Theorem 1.3.

Proof of Theorem 1.3. First, we prove part (i). The conditions $n \le 3$ and S being gcd closed imply that S satisfies $\max_{x\in S}\{|G_S(x)|\} = 1$. It then follows immediately from Lemmas 2.5 and 2.7 that part (i) is true. Part (i) is proved.

Subsequently, we prove part (ii). First of all, any $(n-3)$-fold gcd-closed set S must satisfy either

$$x_1|x_2|...|x_{n-3}|x_{n-2}|x_{n-1}|x_n,$$

or

$$x_1|x_2|...|x_{n-3}|x_{n-2} \text{ and } (x_n, x_{n-1}) = x_{n-2}.$$

So, any $(n-3)$-fold gcd-closed set S also satisfies $\max_{x\in S}\{|G_S(x)|\} = 1$. Part (ii) follows immediately from Lemmas 2.5 and 2.7. Part (ii) is proved.

Finally, we show part (iii). To do so, it is sufficient to prove that there exist $(n-4)$-fold gcd-closed sets S_1, S_2 and S_3 with $|S_i| = n$ and $\max_{x\in S_i}\{|G_{S_i}(x)|\} = 2$ $(1 \le i \le 3)$ such that

$$\det(S_1) \nmid \det(S_1^b), \det(S_2) \nmid \det[S_2^b], \det[S_3] \nmid \det[S_3^b].$$

Let us continue the proof of part (iii) of Theorem 1.3, which is divided into the following cases.

Case 1-1. $b \equiv 2, 3 \pmod 4$. Let n be an integer with $n \ge 4$ and $S_1 = S(h) = \{x_1, ..., x_n\}$ with

$$x_k = h^{k-1},\ 1 \le k \le n-3,\ x_{n-2} = 2h^{n-4},\ x_{n-1} = 7h^{n-4},\ x_n = 28h^{n-4}$$

and $h \equiv 2, 3 \pmod 5$. By Definition 1.2, we know that S_1 is $(n-4)$-fold gcd closed. Since $G_{S(h)}(x_k) = \{h^{k-2}\}$ for all integers k with $2 \le k \le n-3$, $G_{S(h)}(x_{n-2}) = \{h^{n-4}\}$, $G_{S(h)}(x_{n-1}) = \{h^{n-4}\}$, $G_{S(h)}(x_n) = \{2h^{n-4}, 7h^{n-4}\}$, and $(2h^{n-4}, 7h^{n-4}) = h^{n-4}$, by Lemmas 2.2, 2.4 and (2.6), one has

$$\det(S(h)^b) = (2^b-1)(7^b-1)(28^b-2^b-7^b+1)h^{\frac{b(n-4)(n+1)}{2}}(h^b-1)^{n-4}.$$

So,

$$\det(S(h)) = 2^3 \times 3 \times 5h^{\frac{(n-4)(n+1)}{2}}(h-1)^{n-4}.$$

We claim that $5 \nmid \det(S(h)^b)$.

First, $b \equiv 2, 3 \pmod 4$ yields $2^b - 1 \not\equiv 0 \pmod 5$, $7^b - 1 \equiv 2^b - 1 \not\equiv 0 \pmod 5$ and

$$\begin{aligned} 28^b - 2^b - 7^b + 1 &\equiv 3^b - 2\cdot 2^b + 1 \equiv 3^2 - 2\times 2^2 + 1 \equiv 2 \not\equiv 0 \pmod 5 \\ &\text{or} \equiv 3^3 - 2\times 2^3 + 1 \equiv 2 \not\equiv 0 \pmod 5. \end{aligned}$$

Also, $h \equiv 2, 3 \pmod 5$ implies that h is a primitive root modulo 5. So $h^4 \equiv 1 \pmod 5$. Thus

$$\begin{aligned} h^b - 1 &\equiv h^2 - 1 \equiv 3 \not\equiv 0 \pmod 5 \\ \text{or } &\equiv h^3 - 1 \equiv 2 \text{ (or 1) } \not\equiv 0 \pmod 5. \end{aligned}$$

Hence, $\dfrac{\det(S_1^b)}{\det(S_1)} \notin \mathbf{Z}$ holds in this case.

Case 1-2. $b \equiv 0, 1 \pmod 4$ and $b \not\equiv 0 \pmod{36}$, namely,

$$b \equiv 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24, 25, 28, 29, 32, 33 \pmod{36}.$$

Let n be an integer with $n \ge 4$ and $S_1 = S(l) = \{x_1, ..., x_n\}$ with

$$x_k = l^{k-1},\ 1 \le k \le n-3,\ x_{n-2} = 2l^{n-4},\ x_{n-1} = 13l^{n-4},\ x_n = 52l^{n-4}$$

and $l \equiv 2, 3, 10, 13, 14, 15 \pmod{19}$. By Definition 1.2, one knows that S_1 is $(n-4)$-fold gcd closed. Since $G_{S(l)}(x_k) = \{l^{k-2}\}$ for all integers k with $2 \le k \le n-3$, $G_{S(l)}(x_{n-2}) = \{l^{n-4}\}$, $G_{S(l)}(x_{n-1}) = \{l^{n-4}\}$, $G_{S(l)}(x_n) = \{2l^{n-4}, 13l^{n-4}\}$, and $(2l^{n-4}, 13l^{n-4}) = l^{n-4}$, by Lemmas 2.2, 2.4 and (2.6), one derives that

$$\det(S(l)^b) = (2^b-1)(13^b-1)(52^b-2^b-13^b+1)l^{\frac{b(n-4)(n+1)}{2}}(l^b-1)^{n-4}.$$

So,

$$\det(S(l)) = 2^3 \times 3 \times 19 l^{\frac{(n-4)(n+1)}{2}} (l-1)^{n-4}.$$

We assert that $19 \nmid \det(S(l)^b)$.

Since

$$b \equiv 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24, 25, 28, 29, 32, 33 \pmod{36},$$

one deduces that $2^b - 1 \not\equiv 0 \pmod{19}$, $13^b - 1 \not\equiv 0 \pmod{19}$, and

$$\begin{aligned}
52^b - 2^b - 13^b + 1 \equiv 14^b - 2^b - 13^b + 1 &\equiv 14^4 - 2^4 - 13^4 + 1 \equiv 17 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^5 - 2^5 - 13^5 + 1 \equiv 3 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^8 - 2^8 - 13^8 + 1 \equiv 18 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^9 - 2^9 - 13^9 + 1 \equiv 2 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{12} - 2^{12} - 13^{12} + 1 \equiv 13 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{13} - 2^{13} - 13^{13} + 1 \equiv 4 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{16} - 2^{16} - 13^{16} + 1 \equiv 3 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{17} - 2^{17} - 13^{17} + 1 \equiv 3 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{20} - 2^{20} - 13^{20} + 1 \equiv 5 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{21} - 2^{21} - 13^{21} + 1 \equiv 8 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{24} - 2^{24} - 13^{24} + 1 \equiv 9 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{25} - 2^{25} - 13^{25} + 1 \equiv 18 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{28} - 2^{28} - 13^{28} + 1 \equiv 2 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{29} - 2^{29} - 13^{29} + 1 \equiv 16 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{32} - 2^{32} - 13^{32} + 1 \equiv 18 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv 14^{33} - 2^{33} - 13^{33} + 1 \equiv 12 \not\equiv 0 \pmod{19}.
\end{aligned}$$

The condition $l \equiv 2, 3, 10, 13, 14, 15 \pmod{19}$ implies that l is a primitive root modulo 19. So, $l^{18} \equiv 1 \pmod{19}$. Thus,

$$\begin{aligned}
l^b - 1 &\equiv l^4 - 1 \equiv 15, 4, 5, 3, 16, 8 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^5 - 1 \equiv 12, 14, 2, 13, 9, 1 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^8 - 1 \equiv 8, 5, 16, 15, 3, 4 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^9 - 1 \equiv 17 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^{12} - 1 \equiv 10, 6 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^{13} - 1 \equiv 2, 13, 12, 14, 1, 9 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^{16} - 1 \equiv 4, 16, 3, 8, 15, 5 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^{17} - 1 \equiv 9, 12, 1, 2, 14, 13 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^{20} - 1 \equiv 3, 8, 4, 16, 5, 15 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^{21} - 1 \equiv 7, 11 \not\equiv 0 \pmod{19}\\
&\text{or } \equiv l^{24} - 1 \equiv 6, 10 \not\equiv 0 \pmod{19}
\end{aligned}$$

$$\begin{aligned}
&\text{or} \equiv l^{25} - 1 \equiv 13, 1, 14, 9, 2, 12 \not\equiv 0 \pmod{19}\\
&\text{or} \equiv l^{28} - 1 \equiv 16, 15, 8, 5, 4, 3 \not\equiv 0 \pmod{19}\\
&\text{or} \equiv l^{29} - 1 \equiv 14, 9, 13, 1, 12, 2 \not\equiv 0 \pmod{19}\\
&\text{or} \equiv l^{32} - 1 \equiv 5, 3, 6, 4, 8, 16 \not\equiv 0 \pmod{19}\\
&\text{or} \equiv l^{33} - 1 \equiv 11, 12, 7 \not\equiv 0 \pmod{19}.
\end{aligned}$$

Hence, $\frac{\det(S_1^b)}{\det(S_1)} \notin \mathbf{Z}$ holds as expected in this case.

Case 2-1 $b \equiv 1, 2 \pmod 4$. Let $n \geq 4$ and $S_2 = S(r) = \{x_1, ..., x_n\}$ with

$$x_k = r^{k-1},\ 1 \leq k \leq n-3,\ x_{n-2} = 2r^{n-4},\ x_{n-1} = 17r^{n-4},\ x_n = 68r^{n-4}$$

and $r \equiv 2, 3 \pmod 5$. Since $G_{S(r)}(x_k) = \{r^{k-2}\}$ for all integers k with $2 \leq k \leq n-3$, $G_{S(r)}(x_{n-2}) = \{r^{n-4}\}$, $G_{S(r)}(x_{n-1}) = \{r^{n-4}\}$, $G_{S(r)}(x_n) = \{2r^{n-4}, 17r^{n-4}\}$, and $(2r^{n-4}, 17r^{n-4}) = r^{n-4}$, by Lemmas 2.1, 2.4 and (2.5), we have

$$\det[S(r)^b] = (-1)^{n-4}2^b \times 17^b \times 68^b(2^b - 1)(17^b - 1)(1 - 34^b - 4^b + 68^b)r^{\frac{b(n-4)(n+3)}{2}}(r^b - 1)^{n-4}.$$

From Lemmas 2.2, 2.4 and (2.6), one derives that

$$\det(S(r)^b) = (2^b - 1)(17^b - 1)(68^b - 2^b - 17^b + 1)r^{\frac{b(n-4)(n+1)}{2}}(r^b - 1)^{n-4}.$$

So,

$$\det(S(r)) = 2^5 \times 5^2 r^{\frac{(n-4)(n+1)}{2}}(r-1)^{n-4}.$$

One claims that $5 \nmid \det[S(r)^b]$.

First, $b \equiv 1, 2 \pmod 4$ yields $2^b - 1 \not\equiv 0 \pmod 5$, $17^b - 1 \equiv 2^b - 1 \not\equiv 0 \pmod 5$ and

$$\begin{aligned}
68^b - 34^b - 4^b + 1 &\equiv 3^b - 2 \cdot 4^b + 1 \equiv 3 - 2 \times 2^2 + 1 \equiv 1 \not\equiv 0 \pmod 5\\
&\text{or} \equiv 3^2 - 2 \times 2^4 + 1 \equiv 3 \not\equiv 0 \pmod 5.
\end{aligned}$$

Also, $r \equiv 2, 3 \pmod 5$ implies that r is a primitive root modulo 5. So, $r^4 \equiv 1 \pmod 5$. Thus,

$$\begin{aligned}
r^b - 1 &\equiv r - 1 \equiv 1 \text{ (or 2)} \not\equiv 0 \pmod 5\\
\text{or } &\equiv r^2 - 1 \equiv 3 \not\equiv 0 \pmod 5.
\end{aligned}$$

Hence, $\frac{\det[S_2^b]}{\det(S_2)} \notin \mathbf{Z}$ holds as required in this case.

Case 2-2. $b \equiv 0, 3 \pmod 4$ and $b \not\equiv 0, 35 \pmod{36}$, namely,

$$b \equiv 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24, 27, 28, 31, 32 \pmod{36}.$$

Let n be an integer with $n \geq 4$ and $S_2 = S(l) = \{x_1, ..., x_n\}$ with

$$x_k = l^{k-1},\ 1 \leq k \leq n-3,\ x_{n-2} = 2l^{n-4},\ x_{n-1} = 13l^{n-4},\ x_n = 52l^{n-4}$$

and $l \equiv 2, 3, 10, 13, 14, 15 \pmod{19}$. Since $G_{S(l)}(x_k) = \{l^{k-2}\}$ for all integers k with $2 \le k \le n-3$, $G_{S(l)}(x_{n-2}) = \{l^{n-4}\}$, $G_{S(l)}(x_{n-1}) = \{l^{n-4}\}$, $G_{S(l)}(x_n) = \{2l^{n-4}, 13l^{n-4}\}$, and $(2l^{n-4}, 13l^{n-4}) = l^{n-4}$, by Lemmas 2.1, 2.4 and (2.5), one has

$$\det[S(l)^b] = (-1)^{n-4}2^b \times 13^b \times 52^b(2^b-1)(13^b-1)(1-26^b-4^b+52^b)l^{\frac{b(n-4)(n+3)}{2}}(l^b-1)^{n-4}.$$

By Lemmas 2.2, 2.4 and (2.6), one has

$$\det(S(l)^b) = (2^b-1)(13^b-1)(52^b-2^b-13^b+1)l^{\frac{b(n-4)(n+1)}{2}}(l^b-1)^{n-4}.$$

So,

$$\det(S(l)) = 2^3 \times 3 \times 19 l^{\frac{(n-4)(n+1)}{2}}(l-1)^{n-4}.$$

One asserts that $19 \nmid \det[S(l)^b]$.

Since

$$b \equiv 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24, 27, 28, 31, 32 \pmod{36},$$

we have $2^b - 1 \not\equiv 0 \pmod{19}, 13^b - 1 \not\equiv 0 \pmod{19}$ and

$$\begin{aligned}
52^b - 26^b - 4^b + 1 \equiv 14^b - 7^b - 4^b + 1 &\equiv 14^4 - 7^4 - 4^4 + 1 \equiv 2 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^7 - 7^7 - 4^7 + 1 \equiv 10 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^8 - 7^8 - 4^8 + 1 \equiv 8 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{11} - 7^{11} - 4^{11} + 1 \equiv 6 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{12} - 7^{12} - 4^{12} + 1 \equiv 4 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{15} - 7^{15} - 4^{15} + 1 \equiv 1 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{16} - 7^{16} - 4^{16} + 1 \equiv 4 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{19} - 7^{19} - 4^{19} + 1 \equiv 4 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{20} - 7^{20} - 4^{20} + 1 \equiv 18 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{23} - 7^{23} - 4^{23} + 1 \equiv 2 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{24} - 7^{24} - 4^{24} + 1 \equiv 15 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{27} - 7^{27} - 4^{27} + 1 \equiv 17 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{28} - 7^{28} - 4^{28} + 1 \equiv 14 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{31} - 7^{31} - 4^{31} + 1 \equiv 6 \not\equiv 0 \pmod{19}\\
\text{or } &\equiv 14^{32} - 7^{32} - 4^{32} + 1 \equiv 1 \not\equiv 0 \mod 19.
\end{aligned}$$

The condition $l \equiv 2, 3, 10, 13, 14, 15 \pmod{19}$ means that l is a primitive root modulo 19. So, $l^{18} \equiv 1 \pmod{19}$. Therefore,

$$
\begin{aligned}
l^b - 1 \equiv l^4 - 1 &\equiv 15, 4, 5, 3, 16, 8 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^7 - 1 &\equiv 13, 1, 14, 9, 2, 12 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^8 - 1 &\equiv 8, 5, 16, 15, 3, 4 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{11} - 1 &\equiv 14, 9, 13, 1, 12, 2 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{12} - 1 &\equiv 10, 6 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{15} - 1 &\equiv 11, 7 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{16} - 1 &\equiv 4, 16, 3, 8, 15, 5 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{19} - 1 &\equiv 1, 2, 9, 12, 13, 14 \not\equiv 0 \quad \text{mod } 19) \\
\text{or} \equiv l^{20} - 1 &\equiv 3, 8, 4, 16, 5, 15 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{23} - 1 &\equiv 12, 14, 2, 13, 9, 1 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{24} - 1 &\equiv 6, 10 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{27} - 1 &\equiv 17 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{28} - 1 &\equiv 16, 15, 8, 5, 4, 3 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{31} - 1 &\equiv 2, 13, 12, 14, 1, 9 \not\equiv 0 \pmod{19} \\
\text{or} \equiv l^{32} - 1 &\equiv 5, 3, 6, 4, 8, 16 \not\equiv 0 \pmod{19}.
\end{aligned}
$$

Hence, $\dfrac{\det[S_2^b]}{\det(S_2)} \notin \mathbf{Z}$ holds as desired in this case.

Case 3-1. $b \equiv 2, 3, 4, 5, 6, 7, 8, 9 \pmod{10}$. Let n be an integer with $n \geq 4$ and $S_3 = S(h) = \{x_1, ..., x_n\}$ with

$$x_k = h^{k-1},\ 1 \leq k \leq n-3,\ x_{n-2} = 2h^{n-4},\ x_{n-1} = 7h^{n-4},\ x_n = 28h^{n-4}$$

and $h \equiv 2, 6, 7, 8 \pmod{11}$. Since $G_{S(h)}(x_k) = \{h^{k-2}\}$ for all integers k with $2 \leq k \leq n-3$, $G_{S(h)}(x_{n-2}) = \{h^{n-4}\}, G_{S(h)}(x_{n-1}) = \{h^{n-4}\}, G_{S(h)}(x_n) = \{2h^{n-4}, 7h^{n-4}\}$, and $(2h^{n-4}, 7h^{n-4}) = h^{n-4}$, by Lemmas 2.1, 2.4 and (2.5), it can be derived that

$$\det[S(h)^b] = (-1)^{n-4} 2^b \times 7^b \times 28^b (2^b - 1)(7^b - 1)(1 - 14^b - 4^b + 28^b) h^{\frac{b(n-4)(n+3)}{2}} (h^b - 1)^{n-4}.$$

So,

$$\det[S(h)] = (-1)^{n-4} 2^4 \times 3 \times 7^2 \times 11 h^{\frac{(n-4)(n+3)}{2}} (h-1)^{n-4}.$$

We claim that $11 \nmid \det[S(h)^b]$.

First, we have $2^b - 1 \not\equiv 0 \pmod{11}$, $7^b - 1 \not\equiv 0 \pmod{11}$ and

$$
\begin{aligned}
28^b - 14^b - 4^b + 1 \equiv 6^b - 3^b - 4^b + 1 &\equiv 6^2 - 3^2 - 4^2 + 1 \equiv 1 \not\equiv 0 \pmod{11} \\
&\text{or} \equiv 6^3 - 3^3 - 4^3 + 1 \equiv 5 \not\equiv 0 \pmod{11} \\
&\text{or} \equiv 6^4 - 3^4 - 4^4 + 1 \equiv 3 \not\equiv 0 \pmod{11} \\
&\text{or} \equiv 6^5 - 3^5 - 4^5 + 1 \equiv 9 \not\equiv 0 \pmod{11} \\
&\text{or} \equiv 6^6 - 3^6 - 4^6 + 1 \equiv 10 \not\equiv 0 \pmod{11} \\
&\text{or} \equiv 6^7 - 3^7 - 4^7 + 1 \equiv 6 \not\equiv 0 \pmod{11} \\
&\text{or} \equiv 6^8 - 3^8 - 4^8 + 1 \equiv 2 \not\equiv 0 \pmod{11} \\
&\text{or} \equiv 6^9 - 3^9 - 4^9 + 1 \equiv 7 \not\equiv 0 \pmod{11},
\end{aligned}
$$

since $b \equiv 2,3,4,5,6,7,8,9 \pmod{10}$. Also, $h \equiv 2,6,7,8 \pmod{11}$ implies that h is a primitive root modulo 11. So, $h^{10} \equiv 1 \pmod{11}$. Thus,

$$\begin{aligned} h^b - 1 &\equiv h^2 - 1 \equiv 3,2,4,8 \not\equiv 0 \pmod{11} \\ \text{or } &\equiv h^3 - 1 \equiv 7,6,1,5 \not\equiv 0 \pmod{11} \\ \text{or } &\equiv h^4 - 1 \equiv 4,8,2,3 \not\equiv 0 \pmod{11} \\ \text{or } &\equiv h^5 - 1 \equiv 9 \not\equiv 0 \pmod{11} \\ \text{or } &\equiv h^6 - 1 \equiv 8,4,3,2 \not\equiv 0 \pmod{11} \\ \text{or } &\equiv h^7 - 1 \equiv 6,7,5,1 \not\equiv 0 \pmod{11} \\ \text{or } &\equiv h^8 - 1 \equiv 2,3,8,4 \not\equiv 0 \pmod{11} \\ \text{or } &\equiv h^9 - 1 \equiv 5,1,7,6 \not\equiv 0 \pmod{11}. \end{aligned}$$

Hence, $\frac{\det[S_3^b]}{\det[S_3]} \notin \mathbf{Z}$ holds in this case.

Case 3-2. $b \equiv 0,1 \pmod{10}$ and $b \not\equiv 0,11,100 \pmod{110}$, namely,

$$b \equiv 10,20,21,30,31,40,41,50,51,60,61,70,71,80,81,90,91,101 \pmod{110}.$$

Let $n \geq 4$ and $S_3 = S(l) = \{x_1, ..., x_n\}$ with

$$x_k = l^{k-1},\ 1 \leq k \leq n-3,\ x_{n-2} = 2l^{n-4},\ x_{n-1} = 13l^{n-4},\ x_n = 52l^{n-4}$$

and

$$l \equiv 2,3,4,5,7,9,10,11,13,14,15,16,17,18,19,20,21 \pmod{23}.$$

Since $G_{S(l)}(x_k) = \{l^{k-2}\}$ for all integers k with $2 \leq k \leq n-3$, $G_{S(l)}(x_{n-2}) = \{l^{n-4}\}$, $G_{S(l)}(x_{n-1}) = \{l^{n-4}\}$, $G_{S(l)}(x_n) = \{2l^{n-4}, 13l^{n-4}\}$, and $(2l^{n-4}, 13l^{n-4}) = l^{n-4}$, from Lemmas 2.1, 2.4 and (2.5), one has

$$\det[S(l)^b] = (-1)^{n-4} 2^b \times 13^b \times 52^b (2^b - 1)(13^b - 1)(1 - 26^b - 4^b + 52^b) l^{\frac{b(n-4)(n+3)}{2}} (l^b - 1)^{n-4}.$$

So,

$$\det[S(l)] = (-1)^{n-4} 2^5 \times 3 \times 13^2 \times 23 l^{\frac{(n-4)(n+3)}{2}} (l-1)^{n-4}.$$

We assert that $23 \nmid \det[S(l)^b]$.

The condition

$$b \equiv 10,20,21,30,31,40,41,50,51,60,61,70,71,80,81,90,91,101 \pmod{110}$$

yields $2^b - 1 \not\equiv 0 \pmod{110}$, $13^b - 1 \not\equiv 0 \pmod{110}$ and

$$\begin{aligned}52^b - 26^b - 4^b + 1 \equiv 6^b - 3^b - 4^b + 1 &\equiv 6^{10} - 3^{10} - 4^{10} + 1 \equiv 14 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{20} - 3^{20} - 4^{20} + 1 \equiv 9 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{21} - 3^{21} - 4^{21} + 1 \equiv 14 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{30} - 3^{30} - 4^{30} + 1 \equiv 4 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{31} - 3^{31} - 4^{31} + 1 \equiv 9 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{40} - 3^{40} - 4^{40} + 1 \equiv 17 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{41} - 3^{41} - 4^{41} + 1 \equiv 4 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{50} - 3^{50} - 4^{50} + 1 \equiv 18 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{51} - 3^{51} - 4^{51} + 1 \equiv 17 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{60} - 3^{60} - 4^{60} + 1 \equiv 1 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{61} - 3^{61} - 4^{61} + 1 \equiv 18 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{70} - 3^{70} - 4^{70} + 1 \equiv 17 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{71} - 3^{71} - 4^{71} + 1 \equiv 1 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{80} - 3^{80} - 4^{80} + 1 \equiv 11 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{81} - 3^{81} - 4^{81} + 1 \equiv 17 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{90} - 3^{90} - 4^{90} + 1 \equiv 12 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{91} - 3^{91} - 4^{91} + 1 \equiv 11 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv 6^{101} - 3^{101} - 4^{101} + 1 \equiv 12 \not\equiv 0 \pmod{23}.\end{aligned}$$

Since

$$l \equiv 2, 3, 4, 5, 7, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21 \pmod{23},$$

one can derive that

$$\begin{aligned}l^b - 1 &\equiv l^{10} - 1 \equiv 11, 7, 5, 8, 12, 17, 15, 2 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{20} - 1 \equiv 5, 17, 12, 11, 7, 1, 2, 8 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{21} - 1 \equiv 11, 7, 5, 13, 9, 17, 6, 16, 15, 4, 19, 12, 10, 8, 14 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{30} - 1 \equiv 2, 5, 8, 15, 11, 12, 1, 3 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{31} - 1 \equiv 5, 17, 12, 10, 14, 19, 2, 20, 13, 7, 11, 9, 4, 16 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{40} - 1 \equiv 12, 1, 7, 5, 17, 3, 8, 11 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{41} - 1 \equiv 2, 5, 8, 6, 10, 12, 20, 19, 1, 9, 18, 11, 21, 15, 13, 16 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{50} - 1 \equiv 17, 15, 1, 7, 3, 2, 5, 12 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{51} - 1 \equiv 12, 1, 7, 16, 4, 3, 13, 18, 8, 10, 17, 5, 14, 20, 9 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{60} - 1 \equiv 8, 12, 11, 2, 5, 7, 3, 2, 15 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{61} - 1 \equiv 17, 15, 1, 14, 18, 2, 16, 9, 19, 3, 20, 7, 6, 4 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{70} - 1 \equiv 15, 11, 2, 3, 8, 5, 17, 1 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{71} - 1 \equiv 8, 12, 11, 19, 16, 7, 18, 20, 3, 14, 6, 5, 9, 2, 10, 13 \not\equiv 0 \pmod{23}\\
&\text{or} \equiv l^{80} - 1 \equiv 7, 3, 17, 12, 1, 15, 11, 5 \not\equiv 0 \pmod{23}\end{aligned}$$

$$\text{or} \equiv l^{81} - 1 \equiv 15, 11, 2, 18, 13, 5, 4, 17, 16, 20, 8, 3, 19, 10, 6 \not\equiv 0 \pmod{23}$$
$$\text{or} \equiv l^{90} - 1 \equiv 3, 8, 15, 1, 2, 11, 7, 17 \not\equiv 0 \pmod{23}$$
$$\text{or} \equiv l^{91} - 1 \equiv 7, 3, 17, 9, 20, 15, 10, 21, 11, 6, 16, 1, 19, 12, 4, 18, 14 \not\equiv 0 \pmod{23}$$
$$\text{or} \equiv l^{101} - 1 \equiv 3, 8, 15, 20, 19, 11, 14, 13, 7, 10, 4, 2, 1, 6, 18 \not\equiv 0 \pmod{23}.$$

So, $\frac{\det[S_3^b]}{\det[S_3]} \notin \mathbf{Z}$ holds as one expects in this case.

This finishes the proof of Theorem 1.3. □

4. Conclusions

Let a, b and n be positive integers. Parts (i) and (ii) of Theorem 1.3 in this paper tell us that if $a|b$ and $n \leq 3$, then for any gcd-closed set S with $|S| = n$, one has $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$. Furthermore, if $a|b$ and $n \geq 4$, then for any $(n-3)$-fold gcd-closed set S with $|S| = n$, one has $(S^a)|(S^b)$, $(S^a)|[S^b]$ and $[S^a]|[S^b]$.

On the other hand, let $n \geq 4$, $b \geq 2$ be integers with $36 \nmid b$ (resp. $b \not\equiv 0, 35 \pmod{36}$, or $b \not\equiv 0, 11, 100 \pmod{110}$). By part (iii) of Theorem 1.3 in this paper, we know that there exist some $(n-4)$-fold gcd-closed sets S with $\max_{x \in S}\{|G_S(x)|\} = 2$ such that in the ring $M_{|S|}(\mathbf{Z})$, one has $(S) \nmid (S^b)$ (resp. $(S) \nmid [S^b]$, or $[S] \nmid [S^b]$). However, when $36|b$ (resp. $b \equiv 0, 35 \pmod{36}$, or $b \equiv 0, 11, 100 \pmod{110}$), does there exist an $(n-4)$-fold gcd-closed set S with $\max_{x \in S}\{|G_S(x)|\} = 2$ such that in the ring $M_{|S|}(\mathbf{Z})$, we have $(S) \nmid (S^b)$ (resp. $(S) \nmid [S^b]$, or $[S] \nmid [S^b]$)? This question remains open.

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Conflict of interest

We declare that we have no conflict of interest.

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