AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 18194–18217.
DOI: 10.3934/math.20221001
Received: 17 May 2022
Revised: 12 July 2022
Accepted: 26 July 2022
Published: 11 August 2022

Research article

Sum of some product-type operators from mixed-norm spaces to weighted-type spaces on the unit ball

Cheng-shi Huang[1], **Zhi-jie Jiang**[1,2,*] **and Yan-fu Xue**[1]

[1] School of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, P. R. China

[2] South Sichuan Center for Applied Mathematics, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, P. R. China

* **Correspondence:** Email: matjzj@126.com.

Abstract: Let u_j be the holomorphic functions on the open unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $j = \overline{0, m}$, φ a holomorphic self-map of $\mathbb{B}$, and $\mathfrak{R}^j$ the jth iterated radial derivative operator. In this paper, the boundedness and compactness of the sum operator $\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{j=0}^m M_{u_j} C_\varphi \mathfrak{R}^j$ from the mixed-norm space $H(p, q, \phi)$, where $0 < p, q < +\infty$, and ϕ is normal, to the weighted-type space H^∞_μ are characterized. For the mixed-norm space $H(p, q, \phi)$, $1 \le p < +\infty$, $1 < q < +\infty$, the essential norm estimate of the operator is given, and the Hilbert-Schmidt norm of the operator on the weighted Bergman space A^2_α is also calculated.

Keywords: product-type operator; boundedness; compactness; mixed-norm space; weighted-type space; essential norm; Hilbert-Schmidt norm
Mathematics Subject Classification: 30H05, 47B33, 47B37, 47B38

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $B(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}$ the open ball in the complex vector space $\mathbb{C}^n$ centered at a with radius r, $\mathbb{B}^n = \mathbb{B} = B(0, 1)$, $\mathbb{S}$ the boundary of $\mathbb{B}$ and $n \in \mathbb{N}$. Let $z = (z_1, z_2, \dots, z_n)$ and $w = (w_1, w_2, \dots, w_n)$ be two points in $\mathbb{C}^n$, then $\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$ and $|z|^2 = \langle z, z\rangle$.

Let $H(\mathbb{B})$ be the set of all holomorphic functions on $\mathbb{B}$ and $S(\mathbb{B})$ the set of all holomorphic self-maps of $\mathbb{B}$. If $\varphi \in S(\mathbb{B})$, then by $C_\varphi f(z) = f(\varphi(z))$, $z \in \mathbb{B}$, is defined a operator, usually called the composition operator. If $u \in H(\mathbb{B})$, then by $M_u f(z) = u(z)f(z)$, $z \in \mathbb{B}$, is defined a operator, usually called the multiplication operator.

When $n = 1$, the open unit ball $\mathbb{B}$ is reduced to the open unit disk $\mathbb{D}$. Let $m \in \mathbb{N}_0$, the mth differentiation operator D^m on some subspaces of $H(\mathbb{D})$ is defined by

$$D^m f(z) = f^{(m)}(z),$$

where $f^{(0)} = f$. If $m = 1$, it is the classical differentiation operator, denoted by D. There have been a lot of studies on the products related to one of the differentiation operators. For example, products DC_φ and $C_\varphi D$ were first studied (see, for example, [5, 15–18, 22, 33–35]) containing the differentiation operator. What followed was the following six products of the operators were studied (see, for example, [25,40,41])

$$DM_uC_\varphi,\ \ DC_\varphi M_u,\ \ C_\varphi DM_u,\ \ C_\varphi M_u D,\ \ M_uC_\varphi D,\ \ M_u DC_\varphi. \tag{1.1}$$

Afterwards, operators in (1.1) were studied in terms of replacing D by D^m (see, for example, [9,10,44]), that is, the following six products of the operators were studied

$$D^m M_uC_\varphi,\ \ D^m C_\varphi M_u,\ \ C_\varphi D^m M_u,\ \ C_\varphi M_u D^m,\ \ M_uC_\varphi D^m,\ \ M_u D^m C_\varphi. \tag{1.2}$$

For some other products of the operators containing differentiation operators can be found in [11, 12, 29, 46] and the related references therein.

There are other ways to extend differentiation operators on domains in $\mathbb{C}^n$. For example, the radial derivative operator

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

is one of natural extensions of the differentiation operators. The products of the composition, multiplication, and radial derivative operators

$$M_uC_\varphi\Re,\ C_\varphi\Re M_u,\ C_\varphi M_u\Re,\ \ \Re M_uC_\varphi,\ M_u\Re C_\varphi,\ \Re C_\varphi M_u \tag{1.3}$$

were studied in [19–21, 47]. Operators $M_uC_\varphi\Re$ and $\Re M_uC_\varphi$ were also studied in [8]. Operators in (1.3) naturally generalize operators in (1.1) from the unit disk setting to the unit ball setting. Recently, they have been continuously studied in [45]. Some other products of the operators containing the radial derivative operators can be found, for example, in [31,49].

An advantage of the radial derivative operator is that it can be employed iteratively, that is, if $\Re^{m-1}f$ is defined for some $m \in \mathbb{N} \setminus \{1\}$, then $\Re^m f$ is naturally defined by $\Re^m f = \Re(\Re^{m-1}f)$. If $m = 0$, then we regard that $\Re^0 f = f$. By using the mth iterated radial derivative operator, we obtain the related product-type operators

$$M_uC_\varphi\Re^m,\ C_\varphi\Re^m M_u,\ C_\varphi M_u\Re^m,\ \ \Re^m M_uC_\varphi,\ M_u\Re^m C_\varphi,\ \Re^m C_\varphi M_u. \tag{1.4}$$

Operators in (1.4) are more complicated than those in (1.3). Clearly, the operator $M_uC_\varphi\Re^m$ can be regarded as the simplest one in (1.4), which was first studied and denoted by $\Re^m_{u,\varphi}$ in [32]. Recently, we have been reconsidered such operator in [38, 39]. The reason why we consider the operator $\Re^m_{u,\varphi}$ again is that we need to get more information on the related function spaces in order to characterize its properties.

Operators in (1.4) generalize operators in (1.2) from the unit disk case to the unit ball case. For the unit disk case, by using the famous Faà di Bruno's formula of $(f \circ \varphi)^{(m)}$ (see [13]), operators $D^m M_u C_\varphi$, $D^m C_\varphi M_u$ and $M_u D^m C_\varphi$ have been studied (see, for example, [9, 10]). But, we don't find any result on the operators $\mathfrak{R}^m M_u C_\varphi$, $\mathfrak{R}^m C_\varphi M_u$ and $M_u \mathfrak{R}^m C_\varphi$. By a direct calculation, it is easy to see that

$$C_\varphi M_u \mathfrak{R}^m = \mathfrak{R}^m_{u\circ\varphi,\varphi} \quad \text{and} \quad C_\varphi \mathfrak{R}^m M_u = \sum_{i=0}^{m} C_m^i \mathfrak{R}^i_{(\mathfrak{R}^{m-i}u)\circ\varphi,\varphi}. \tag{1.5}$$

Motivated by this interesting observation, we directly define the operator

$$\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{j=0}^{m} \mathfrak{R}^j_{u_j,\varphi} \tag{1.6}$$

on some subspaces of $H(\mathbb{B})$, where $u_j \in H(\mathbb{B})$, $j \in \{0, 1, \ldots, m\}$ and $\varphi \in S(\mathbb{B})$.

In this paper, the boundedness and compactness of the operator $\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{j=0}^{m} \mathfrak{R}^j_{u_j,\varphi}$ from the mixed-norm space to the weighted-type space on $\mathbb{B}$ are characterized. The essential norm estimate of the operator from the mixed-norm space to the weighted-type space on $\mathbb{B}$ is given, and the Hilbert-Schmidt norm of the operator on the weighted Hilbert-Bergman space is calculated. As corollaries, the corresponding results of the operator $C_\varphi \mathfrak{R}^m M_u$ from mixed-norm space to weighted-type space are also obtained. This study can be viewed as a continuation and extension of our previous work.

A positive continuous function ϕ on the interval $[0, 1)$ is called normal (see [28]), if there are $\lambda \in [0, 1)$, a and b $(0 < a < b)$ such that

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\lambda, 1), \quad \lim_{r\to 1} \frac{\phi(r)}{(1-r)^a} = 0;$$
$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\lambda, 1), \quad \lim_{r\to 1} \frac{\phi(r)}{(1-r)^b} = +\infty.$$

The functions $\{\phi, \psi\}$ will be called a normal pair if ϕ is normal and for b in above definition of normal function, there exists $\beta > b$, such that

$$\phi(r)\psi(r) = (1 - r^2)^\beta.$$

If ϕ is normal, then there exists ψ such that $\{\phi, \psi\}$ is normal pair (see [28]). Note that if $\{\phi, \psi\}$ is normal pair, then ψ is also normal.

For $0 < p, q < +\infty$ and a normal function ϕ, the mixed-norm space $H(p, q, \phi)(\mathbb{B}) := H(p, q, \phi)$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|^p_{H(p,q,\phi)} = \int_0^1 M_q^p(f, r) \frac{\phi^p(r)}{1-r} dr < +\infty,$$

where

$$M_q(f, r) = \left(\int_{\mathbb{S}} |f(r\zeta)|^q d\sigma(\zeta) \right)^{\frac{1}{q}},$$

and $d\sigma$ is the normalized surface measure on $\mathbb{S}$.

If $p = q$, $\phi(r) = (1 - r^2)^{(\alpha+1)/p}$ and $\alpha > -1$, then it is equivalent with the weighted Bergman space A^p_α (see [48]), which is defined by

$$A^p_\alpha = \left\{ f \in H(\mathbb{B}) : \|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}} |f(z)|^p \left(1 - |z|^2\right)^\alpha \mathrm{d}v(z) < +\infty \right\},$$

where dv denotes the normalized volume measure on $\mathbb{B}$. Some facts can be found about the mixed-norm spaces, for example, in [1, 2, 26, 27, 36] (see also the references therein).

A positive continuous function μ on $\mathbb{B}$ is called weight. The weighted-type space $H^\infty_\mu(\mathbb{B}) := H^\infty_\mu$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < +\infty.$$

With the norm $\|\cdot\|_{H^\infty_\mu}$, H^∞_μ is a Banach space. In particular, if $\mu(z) = (1 - |z|^2)^\sigma$ ($\sigma > 0$), then the space H^∞_μ is called the classical weighted-type space H^∞_σ. If $\mu \equiv 1$, then the space H^∞_μ is reduced to the space H^∞ of bounded holomorphic functions on $\mathbb{B}$. Many operators acting from or to the weighted-type spaces have been studied (see, for example, [14, 16, 24, 46] and the related references therein).

Let X and Y be two Banach spaces. It is said that a linear operator $T : X \to Y$ is bounded if there exists a positive constant K such that $\|Tf\|_Y \le K\|f\|_X$ for all $f \in X$. The bounded operator $T : X \to Y$ is compact if it maps bounded sets into relatively compact sets. The norm of the operator $T : X \to Y$, usually denoted by $\|T\|_{X\to Y}$, is defined by

$$\|T\|_{X\to Y} = \sup_{\|f\|_X\le 1} \|Tf\|_Y.$$

In this paper, we use the notation $j = \overline{k, l}$ instead of writing $j = k, ..., l$, where $k, l \in \mathbb{N}_0$ and $k \le l$. Some positive constants are denoted by C, and they may differ from one occurrence to the other. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a positive constant C such that $a \le Cb$ (resp. $a \ge Cb$). When $a \lesssim b$ and $b \gtrsim a$, we write $a \asymp b$.

2. Preliminary results

Here, we give several lemmas which are used in the proofs of the main results. First, we have the following point-evaluation estimate for the functions in $H(p, q, \phi)$ (see [36]).

Lemma 2.1. *Let $0 < p, q < +\infty$ and ϕ normal. Then there is a positive constant C independent of $f \in H(p, q, \phi)$ and $z \in \mathbb{B}$ such that*

$$|f(z)| \le \frac{C}{\phi(|z|)(1 - |z|^2)^{\frac{n}{q}}} \|f\|_{H(p,q,\phi)}. \tag{2.1}$$

Now, we cite a point-evaluation estimate for the jth iterated radial derivatives of the functions in $H(p, q, \phi)$ (see [32]).

Lemma 2.2. *Let $j \in \mathbb{N}$, $0 < p, q < +\infty$ and ϕ normal. Then there is a positive constant C independent of $f \in H(p, q, \phi)$ and $z \in \mathbb{B}$ such that*

$$|\Re^j f(z)| \le \frac{C|z|}{\phi(|z|)(1-|z|^2)^{\frac{n}{q}+j}} \|f\|_{H(p,q,\phi)}. \tag{2.2}$$

Remark 2.1. From (2.1) and (2.2), it follows that if $\{f_k\}$ is bounded in $H(p, q, \phi)$, then $\{f_k\}$ and $\{\Re^j f_k\}$ are uniformly bounded on any compact subset of $\mathbb{B}$, and if $\{f_k\}$ converges to zero in $H(p, q, \phi)$, then $\{f_k\}$ and $\{\Re^j f_k\}$ converge to zero uniformly on any compact subset of $\mathbb{B}$ as $k \to \infty$.

To obtain a criterion for the compactness of a bounded linear operator $T : H(p, q, \phi) \to H_\mu^\infty$, we need to cite the following result, which can be found in [3] for the unit disk version. For the unit ball case, whose proof can be directly obtained by replacing the unit disk by the unit ball, and so the proof is omitted.

Lemma 2.3. *Let X be a reflexive Banach space of holomorphic functions on $\mathbb{B}$, and Y a Banach space. Then a bounded linear operator $T : X \to Y$ is compact if and only if for any bounded sequence $\{f_k\}$ in X such that $f_k \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $k \to \infty$, it follows that $\{T f_k\}$ converges to zero in the norm of Y as $k \to \infty$.*

To give the conditions such that $H(p, q, \phi)$ is reflexive, we recall some studies of the dual spaces of $H(p, q, \phi)$. Jevtić in [7] showed that the dual space of $H(p, q, \phi)$, where $1 \le p \le +\infty$ and $1 \le q < +\infty$, is topologically isomorphic to $H(p', q', \psi)$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Shi in [27] considered the dual spaces of $H(p, q, \phi)$ for two cases: (i) if $0 < p \le 1$ and $1 < q \le +\infty$, then the dual space of $H(p, q, \phi)$ is topologically isomorphic to $H(+\infty, q', \psi)$; (ii) if $0 < p \le 1$ and $0 < q \le 1$, then the dual space of $H(p, q, \phi)$ is topologically isomorphic to $H(+\infty, +\infty, \psi)$.

From above facts, we obtain the following remark.

Remark 2.2. Considering the assumption $0 < p, q < +\infty$, we see that if $1 \le p < +\infty$ and $1 < q < +\infty$, then $H(p, q, \phi)$ is reflexive.

In Lemma 2.3, Banach space X is assumed to be reflexive. The author in [42] gave the following general result.

Lemma 2.4. *Let X, Y be Banach spaces of holomorphic functions on $\mathbb{B}$. Suppose that*

(a) The point evaluation functionals on X are continuous.

(b) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(c) $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then the bounded operator $T : X \to Y$ is compact if and only if for every bounded sequence $\{f_k\}$ in X such that $f_k \to 0$ uniformly on compact sets as $k \to \infty$, it follows that $\{T f_k\}$ converges to zero in the norm of Y as $k \to \infty$.

We obtain the following result, which can be proved similar to Proposition 3.11 in [4]. However, here we prove this result by using Lemma 2.4.

Lemma 2.5. *Let $0 < p, q < +\infty$, ϕ normal, and T be one of the operators in (1.4) and (1.6). Then the bounded operator $T : H(p,q,\phi) \to H_\mu^\infty$ is compact if and only if for any bounded sequence $\{f_k\}$ in $H(p,q,\phi)$ such that $f_k \to 0$ uniformly on any compact subset of $\mathbb{B}$ as $k \to \infty$, it follows that $\{Tf_k\}$ converges to zero in the norm of H_μ^∞ as $k \to \infty$.*

Proof. Without lose of generality, we choose $T = M_u C_\varphi \mathfrak{R}^m$ to prove this result. Let $X = H(p,q,\phi)$ and $Y = H_\mu^\infty$. By Remark 2.1, it is easy to see that conditions (a) and (c) in Lemma 2.4 hold (here, we would like to mention that, for an abstract bounded linear operator $T : H(p,q,\phi) \to H_\mu^\infty$, condition (c) may not be valid.)

Let $\{f_k\}$ be a sequence in the closed unit ball of X. From (2.1), we see that $\{f_k\}$ is uniformly bounded on every compact subset of $\mathbb{B}$. Therefore, by Montel's theorem, there is a subsequence $\{f_{k_i}\}$ such that $f_{k_i} \to g$ uniformly on every compact subset of $\mathbb{B}$ for some $g \in H(\mathbb{B})$ as $i \to \infty$. By the definition of $M_q^p(f,r)$ and Fatou's theorem, we have

$$
\begin{aligned}
M_q^p(g,r) &= \left(\int_{\mathbb{S}} |g(r\zeta)|^q d\sigma(\zeta)\right)^{\frac{p}{q}} = \left(\int_{\mathbb{S}} \lim_{k\to\infty} |f_{k_i}(r\zeta)|^q d\sigma(\zeta)\right)^{\frac{p}{q}} \\
&\leq \liminf_{i\to\infty} \left(\int_{\mathbb{S}} |f_{k_i}(r\zeta)|^q d\sigma(\zeta)\right)^{\frac{p}{q}} = \liminf_{i\to\infty} M_q^p(f_{k_i}, r). \qquad (2.3)
\end{aligned}
$$

Hence, from (2.3) it follows that

$$
\int_0^1 M_q^p(g,r)\frac{\phi^p(r)}{1-r}dr \leq \liminf_{i\to\infty} \int_0^1 M_q^p(f_{k_i},r)\frac{\phi^p(r)}{1-r}dr = \liminf_{i\to\infty} \|f_{k_i}\|_{H(p,q,\phi)} = 1,
$$

which shows that $g \in H(p,q,\phi)$, that is, condition (b) in Lemma 2.4 holds. From Lemma 2.4, the desired result follows. □

The following result was proved in [6] (also see [36]). Hence, its proof is omitted.

Lemma 2.6. *Let $\tau > b$. Then for each $t \geq 0$ and fixed $w \in \mathbb{B}$, the following function is in $H(p,q,\phi)$*

$$
f_{w,t}(z) = \frac{(1-|w|^2)^{t+1+\tau}}{\phi(|w|)(1-\langle z,w\rangle)^{\frac{n}{q}+t+1+\tau}}. \qquad (2.4)
$$

Moreover,

$$
\sup_{w\in\mathbb{B}} \|f_{w,t}\|_{H(p,q,\phi)} \lesssim 1.
$$

We need the following result, which can be found in [30].

Lemma 2.7. *Let $s > 0$, $w \in \mathbb{B}$, and*

$$
g_{w,s}(z) = \frac{1}{(1-\langle z,w\rangle)^s}, \quad z \in \mathbb{B}.
$$

Then

$$
\mathfrak{R}^k g_{w,s}(z) = \sum_{t=1}^{k} a_t^{(k)} \Big(\prod_{j=0}^{t-1}(s+j)\Big)\frac{\langle z,w\rangle^t}{(1-\langle z,w\rangle)^{s+t}},
$$

where the sequences $\{a_t^{(k)}\}_{t=\overline{1,k}}$, $k \in \mathbb{N}$, *are defined by the relations*

$$a_k^{(k)} = a_1^{(k)} = 1 \tag{2.5}$$

for $k \in \mathbb{N}$ *and*

$$a_t^{(k)} = ta_t^{(k-1)} + a_{t-1}^{(k-1)} \tag{2.6}$$

for $2 \le t \le k-1, k \ge 3.$

We use the idea essentially obtained in [30] to construct some suitable linear combinations of the functions in Lemma 2.6, which will be used in the proofs of the main results.

Lemma 2.8. *Let* $m \in \mathbb{N}$ *and* $w \in \mathbb{B}$, $f_{w,t}$ *be the set of functions in (2.4), and the sequences* $\{a_t^{(k)}\}_{t=\overline{1,k}}$, $k = \overline{1,m}$, *are defined by the relations in (2.5) and (2.6). Then, for each* $l \in \{1, 2, ..., m\}$, *there are constants* $c_j = c_j^{(l)}$, $j = \overline{0,m}$ *such that the function*

$$h_w^{(l)}(z) = \sum_{k=0}^{m} c_k^{(l)} f_{w,k}(z)$$

satisfies

$$\Re^j h_w^{(l)}(w) = 0, \quad 0 \le j < l, \tag{2.7}$$

and

$$\Re^i h_w^{(l)}(w) = a_l^{(i)} \frac{|w|^{2l}}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}+l}}, \quad l \le i \le m. \tag{2.8}$$

Moreover,

$$\sup_{w\in\mathbb{B}} \|h_w^{(l)}\|_{H(p,q,\phi)} < +\infty. \tag{2.9}$$

Proof. For the sake of simplicity, we write $d_k = \frac{n}{q} + k + 1 + \tau$. By some calculations and using

Lemma 2.7, we have

$$
\begin{aligned}
h_w^{(l)}(w) =& \frac{c_0 + c_1 + \cdots + c_m}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}}} \\
\Re h_w^{(l)}(w) =& \frac{(d_0 c_0 + d_1 c_1 + \cdots d_m c_m)\,|w|^2}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}+1}} \\
\Re^2 h_w^{(l)}(w) =& \frac{(d_0 c_0 + d_1 c_1 + \cdots + d_m c_m)\,|w|^2}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}+1}} \\
&+ \frac{(d_0 d_1 c_0 + d_1 d_2 c_1 + \cdots + d_m d_{m+1} c_m)\,|w|^4}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}+2}} \\
&\vdots \\
\Re^m h_w^{(l)}(w) =& a_1^{(m)} \frac{(d_0 c_0 + d_1 c_1 + \cdots + d_m c_m)\,|w|^2}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}+1}} \\
&+ a_2^{(m)} \frac{(d_0 d_1 c_0 + d_1 d_2 c_1 + \cdots + d_m d_{m+1} c_m)\,|w|^4}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}+2}} + \cdots \\
&+ a_l^{(m)} \frac{(d_0 \cdots d_{l-1} c_0 + d_1 \cdots d_l c_1 + \cdots + d_m \cdots d_{m+l-1} c_m)\,|w|^{2l}}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}+l}} + \cdots \\
&+ a_m^{(m)} \frac{(d_0 \cdots d_{m-1} c_0 + d_1 \cdots d_m c_1 + \cdots + d_m \cdots d_{2m-1} c_m)\,|w|^{2m}}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}+m}}.
\end{aligned}
\tag{2.10}
$$

From (2.10), we obtain that the system consists of (2.7) and (2.8) is equivalent to the following $m + 1$ linear equations

$$
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
d_0 & d_1 & \cdots & d_m \\
\vdots & \vdots & \ddots & \vdots \\
\prod_{k=0}^{l-1} d_k & \prod_{k=0}^{l-1} d_{k+1} & \cdots & \prod_{k=0}^{l-1} d_{m+k} \\
\vdots & \vdots & \ddots & \vdots \\
\prod_{k=0}^{m-1} d_k & \prod_{k=0}^{m-1} d_{k+1} & \cdots & \prod_{k=0}^{m-1} d_{m+k}
\end{pmatrix}
\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_l \\ \vdots \\ c_m \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}.
\tag{2.11}
$$

Since $d_k > 0$, $k = \overline{0, m}$, by Lemma 5 in [35], the determinant of system (2.11) is $D_{m+1}(d_0) = \prod_{j=1}^{m} j!$, which is different from zero. Hence, there are unique constants $c_j = c_j^{(l)}$, $j = \overline{0, m}$ such that the system (2.11) holds. Moreover, we have that the relations in (2.7) and (2.8) hold when such obtained constants c_j, $j = \overline{0, m}$, are used in (2.11). Finally, by Lemma 2.6, it is clear that (2.9) holds. The proof is finished. □

Lemma 2.9. *Let $m \in \mathbb{N}$ and $w \in \mathbb{B}$, $f_{w,t}$ be the set of functions in (2.4), and the sequences $\{a_t^{(k)}\}_{t=\overline{1,k}}$, $k = \overline{1, m}$, are defined by the relations in (2.5) and (2.6). Then, there are constants $c_j = c_j^{(0)}$, $j = \overline{0, m}$,* *such that the function*

$$h_w^{(0)}(z) = \sum_{k=0}^{m} c_k^{(0)} f_{w,k}(z)$$

satisfies

$$\mathfrak{R}^i h_w^{(0)}(w) = 0, \quad i = \overline{1,m}, \tag{2.12}$$

and

$$h_w^{(0)}(w) = \frac{1}{\phi(|w|)(1-|w|^2)^{\frac{n}{q}}}. \tag{2.13}$$

Moreover,

$$\sup_{w\in\mathbb{B}} \|h_w^{(0)}\|_{H(p,q,\phi)} < +\infty. \tag{2.14}$$

Proof. From the proof of Lemma 2.8, we know that the determinant of system consists of (2.12) and (2.13) is not zero. Therefore, there are unique constants $c_j^{(0)}$, $j = \overline{0,m}$, in the system of (2.12) and (2.13). For the same reason, we also have that the relations in (2.12) and (2.13) hold, and moreover, (2.14) holds. □

3. Boundedness and compactness of the operator $\mathfrak{S}_{\vec{u},\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$

First, we need to characterize the boundedness of the operator $\mathfrak{R}_{u,\varphi}^0 : H(p,q,\phi) \to H_\mu^\infty$. Although it is a folklore, we still give a proof for the completeness and benefit of the reader.

Theorem 3.1. *Let $0 < p, q < +\infty$, ϕ normal, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and μ a weight function on $\mathbb{B}$. Then the operator $\mathfrak{R}_{u,\varphi}^0 : H(p,q,\phi) \to H_\mu^\infty$ is bounded if and only if*

$$I_0 := \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}} < +\infty. \tag{3.1}$$

Moreover, if the operator $\mathfrak{R}_{u,\varphi}^0 : H(p,q,\phi) \to H_\mu^\infty$ is bounded, then the following asymptotic relationship holds

$$\left\|\mathfrak{R}_{u,\varphi}^0\right\|_{H(p,q,\phi)\to H_\mu^\infty} \asymp I_0. \tag{3.2}$$

Proof. Assume (3.1) holds. By Lemma 2.1, for every $f \in H(p,q,\phi)$ and $z \in \mathbb{B}$, we have

$$\mu(z)\left|\mathfrak{R}_{u,\varphi}^0 f(z)\right| = \mu(z)|u(z)f(\varphi(z))| \le C\frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}}\|f\|_{H(p,q,\phi)}. \tag{3.3}$$

By taking the supremum in inequality (3.3) over the unit ball in the space $H(p,q,\phi)$, and using (3.1), we obtain that the operator $\mathfrak{R}_{u,\varphi}^0 : H(p,q,\phi) \to H_\mu^\infty$ is bounded. Moreover, we have

$$\left\|\mathfrak{R}_{u,\varphi}^0\right\|_{H(p,q,\phi)\to H_\mu^\infty} \le CI_0. \tag{3.4}$$

Assume that the operator $\mathfrak{R}^0_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded. Let $g_w(z) = f_{\varphi(w),1}(z)$ and $w \in \mathbb{B}$. Then we have

$$L := \sup_{w\in\mathbb{B}} \|g_w\|_{H(p,q,\phi)} < +\infty,$$

and

$$\left\|\mathfrak{R}^0_{u,\varphi} g_w\right\|_{H^\infty_\mu} = \sup_{z\in\mathbb{B}} \mu(z)\left|\mathfrak{R}^0_{u,\varphi} g_w(z)\right| \geq \mu(w)\,|u(w)g_w(\varphi(w))| = \frac{\mu(w)|u(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}}}. \tag{3.5}$$

From (3.5) and the boundedness of the operator $\mathfrak{R}^0_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$, we have

$$L\left\|\mathfrak{R}^0_{u,\varphi}\right\|_{H(p,q,\phi)\to H^\infty_\mu} \geq \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}}. \tag{3.6}$$

From (3.6), condition (3.1) follows, and moreover,

$$I_0 \leq L\left\|\mathfrak{R}^0_{u,\varphi}\right\|_{H(p,q,\phi)\to H^\infty_\mu}. \tag{3.7}$$

Also (3.4) and (3.7) imply asymptotic relationship (3.2), finishing the proof. □

Remark 3.1. When $k \in \mathbb{N}$, from [32] it follows that the operator $\mathfrak{R}^k_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded if and only if

$$I_k := \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+k}} < +\infty.$$

Now, we consider the boundedness of the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p,q,\phi) \to H^\infty_\mu$.

Theorem 3.2. *Let $m \in \mathbb{N}$, $0 < p,q < +\infty$, ϕ normal, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$ and μ a weight function on $\mathbb{B}$. Then the operators $\mathfrak{R}^j_{u_j,\varphi} : H(p,q,\phi) \to H^\infty_\mu$, $j = \overline{0,m}$, are bounded if and only if the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and*

$$\sup_{z\in\mathbb{B}} \mu(z)|u_j(z)||\varphi(z)| < +\infty, \quad j = \overline{1,m}. \tag{3.8}$$

Proof. Assume that the operators $\mathfrak{R}^j_{u_j,\varphi} : H(p,q,\phi) \to H^\infty_\mu$, $j = \overline{0,m}$, are bounded, then the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is obviously bounded. By using the functions $f_i(z) = z_i \in H(p,q,\phi)$, $i = \overline{1,n}$, we have that $\mathfrak{R}^j_{u_j,\varphi} f_i \in H(p,q,\phi)$, $i = \overline{1,n}$. From this and since $\mathfrak{R} f_i = f_i$, $i = \overline{1,n}$, we have

$$\|\mathfrak{R}^j_{u_j,\varphi} f_i\|_{H^\infty_\mu} = \sup_{z\in\mathbb{B}} \mu(z)|u_j(z)||\varphi_i(z)| \leq \|\mathfrak{R}^j_{u_j,\varphi}\|_{H(p,q,\phi)\to H^\infty_\mu} \|z_i\|_{H(p,q,\phi)},$$

which shows that

$$\sup_{z\in\mathbb{B}} \mu(z)|u_j(z)||\varphi(z)| \leq \|\mathfrak{R}^j_{u_j,\varphi}\|_{H(p,q,\phi)\to H^\infty_\mu} \sum_{i=1}^n \|z_i\|_{H(p,q,\phi)} < +\infty. \tag{3.9}$$

From (3.9), we have that (3.8) holds.

Assume that the operator $\mathfrak{S}_{\vec{u},\varphi}^{m}: H(p,q,\phi) \to H_{\mu}^{\infty}$ is bounded and (3.8) holds. Then there exists a positive constant C such that

$$\|\mathfrak{S}_{\vec{u},\varphi}^{m} f\|_{H_{\mu}^{\infty}} \leq C\|f\|_{H(p,q,\phi)} \tag{3.10}$$

for each $f \in H(p,q,\phi)$. By Theorem 3.1 and Remark 3.1, we need to prove

$$M_j = \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_j(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+j}} < +\infty, \quad j = \overline{1,m}, \tag{3.11}$$

and

$$M_0 = \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_0(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}} < +\infty. \tag{3.12}$$

By Lemma 2.8, if $\varphi(w) \neq 0$, then there is a function $h_{\varphi(w)}^{(m)} \in H(p,q,\phi)$ such that

$$\mathfrak{R}^i h_{\varphi(w)}^{(m)}(\varphi(w)) = 0, \quad 0 \leq i < m, \tag{3.13}$$

and

$$\mathfrak{R}^m h_{\varphi(w)}^{(m)}(\varphi(w)) = a_m^{(m)} \frac{|\varphi(w)|^{2m}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}+m}}. \tag{3.14}$$

Let $L_m = \sup_{w\in\mathbb{B}} \|h_{\varphi(w)}^{(m)}\|_{H(p,q,\phi)}$. Then $L_m < +\infty$. From this, (3.10), (3.13) and (3.14), we have that

$$\begin{aligned} L_m\|\mathfrak{S}_{\vec{u},\varphi}^{m}\|_{H(p,q,\phi)\to H_{\mu}^{\infty}} &\geq \|\mathfrak{S}_{\vec{u},\varphi}^{m} h_{\varphi(w)}^{(m)}\|_{H_{\mu}^{\infty}} = \sup_{z\in\mathbb{B}} \mu(z)\Big|\sum_{j=0}^{m} \mathfrak{R}_{u_j,\varphi}^{j} h_{\varphi(w)}^{(m)}(z)\Big| \\ &\geq \mu(w)\Big|\sum_{j=0}^{m} u_j(w)\mathfrak{R}^j h_{\varphi(w)}^{(m)}(\varphi(w))\Big| \\ &\geq \mu(w)\Big|u_m(w)\mathfrak{R}^m h_{\varphi(w)}^{(m)}(\varphi(w))\Big| \\ &= \frac{\mu(w)|u_m(w)||\varphi(w)|^{2m}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}+m}}. \end{aligned} \tag{3.15}$$

From (3.15), we have

$$\begin{aligned} L_m\|\mathfrak{S}_{\vec{u},\varphi}^{m}\|_{H(p,q,\phi)\to H_{\mu}^{\infty}} &\geq \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u_m(z)||\varphi(z)|^{2m}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+m}} \\ &\geq \frac{1}{2^{2m-1}} \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u_m(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+m}}. \end{aligned} \tag{3.16}$$

From (3.16) and (3.8) with $j = m$, since ϕ is normal, we have

$$\sup_{|\varphi(z)|\leq 1/2} \frac{\mu(z)|u_m(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+m}} \leq (4/3)^{\frac{n}{q}+m} \max_{|z|\leq 1/2} \phi(|z|) \sup_{|\varphi(z)|\leq 1/2} \mu(z)|u_m(z)||\varphi(z)|$$

$$\leq (4/3)^{\frac{n}{q}+m} \max_{|z|\leq 1/2} \phi(|z|) \sup_{z\in\mathbb{B}} \mu(z)|u_m(z)||\varphi(z)|$$
$$< +\infty. \tag{3.17}$$

From (3.16) and (3.17), it follows that $M_m < +\infty$.

Assume that (3.11) holds for $j = \overline{s+1, m}$, for some $s \in \{1, 2, \ldots, m-1\}$. By using the function $h^{(s)}_{\varphi(w)}(z)$ in Lemma 2.8, we have $L_s := \sup_{w\in\mathbb{B}} \|h^{(s)}_{\varphi(w)}\|_{H(p,q,\phi)} < +\infty$ and

$$\begin{aligned} L_s\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{H(p,q,\phi)\to H^\infty_\mu} &\geq \sup_{z\in\mathbb{B}} \mu(z)\Big|\sum_{j=s}^m u_j(z)\mathfrak{R}^j h^{(s)}_{\varphi(w)}(\varphi(z))\Big| \\ &\geq \mu(w)\Big|\sum_{j=s}^m a^{(j)}_s u_j(w)\frac{|\varphi(w)|^{2s}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}+s}}\Big| \\ &\geq a^{(s)}_s \frac{\mu(w)|u_s(w)||\varphi(w)|^{2s}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}+s}} \\ &\quad - \sum_{j=s+1}^m a^{(j)}_s \frac{\mu(w)|u_j(w)||\varphi(w)|^{2s}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}+s}}, \end{aligned}$$

from which along with (2.5) we obtain

$$\frac{\mu(w)|u_s(w)||\varphi(w)|^{2s}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}+s}} \leq L_s\big\|\mathfrak{S}^m_{\vec{u},\varphi}\big\|_{H(p,q,\phi)\to H^\infty_\mu} + \sum_{j=s+1}^m a^{(j)}_s \frac{\mu(w)|u_j(w)||\varphi(w)|^{2s}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}+s}}. \tag{3.18}$$

From (3.18) and since $s \geq 1$, it follows that

$$\begin{aligned} \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u_s(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+s}} &\leq 2^{2s-1} \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u_s(z)||\varphi(z)|^{2s}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+s}} \\ &\leq 2^{2s-1} L_s\big\|\mathfrak{S}^m_{\vec{u},\varphi}\big\|_{H(p,q,\phi)\to H^\infty_\mu} \\ &\quad + 2^{2s-1}\sum_{j=s+1}^m a^{(j)}_s \sup_{|\varphi(z)|>1/2} \frac{\mu(z)|u_j(z)||\varphi(z)|^{2s}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+s}} \\ &\leq 2^{2s-1} L_s\big\|\mathfrak{S}^m_{\vec{u},\varphi}\big\|_{H(p,q,\phi)\to H^\infty_\mu} + 2^{2s-1}\sum_{j=s+1}^m a^{(j)}_s M_j. \end{aligned} \tag{3.19}$$

From (3.8) with $j = s$ and (3.19), we have

$$\begin{aligned} \sup_{|\varphi(z)|\leq 1/2} \frac{\mu(z)|u_s(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+s}} &\leq (4/3)^{\frac{n}{q}+s} \max_{|z|\leq 1/2} \phi(|z|) \sup_{|\varphi(z)|\leq 1/2} \mu(z)|u_s(z)||\varphi(z)| \\ &\leq (4/3)^{\frac{n}{q}+s} \max_{|z|\leq 1/2} \phi(|z|) \sup_{z\in\mathbb{B}} \mu(z)|u_s(z)||\varphi(z)| \\ &< +\infty. \end{aligned} \tag{3.20}$$

So, (3.19) and (3.20) imply that $M_s < +\infty$. Hence, from induction it follows that for each $j \in \{1, \ldots, m\}$, (3.11) holds.

By Lemma 2.9, we have that for each $w \in \mathbb{B}$ there exists a function $h^{(0)}_{\varphi(w)} \in H(p,q,\phi)$ such that

$$h^{(0)}_{\varphi(w)}(\varphi(w)) = \frac{1}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}}}, \quad \Re^j h^{(0)}_{\varphi(w)}(\varphi(w)) = 0, \quad j = \overline{1,m}, \tag{3.21}$$

and

$$L_0 := \sup_{w\in\mathbb{B}} \|h^{(0)}_{\varphi(w)}\|_{H(p,q,\phi)} < +\infty.$$

From this, (3.10) and (3.21), we have

$$\begin{aligned} L_0\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{H(p,q,\phi)\to H^\infty_\mu} &\geq \|\mathfrak{S}^m_{\vec{u},\varphi} h^{(0)}_{\varphi(w)}\|_{H^\infty_\mu} = \sup_{z\in\mathbb{B}} \mu(z)\Big|\sum_{j=0}^m \Re^j_{u_j,\varphi} h^{(0)}_{\varphi(w)}(z)\Big| \\ &\geq \mu(w)\Big|\sum_{j=0}^m u_j(w)\Re^j h^{(0)}_{\varphi(w)}(\varphi(w))\Big| \\ &\geq \mu(w)|u_0(w)h^{(0)}_{\varphi(w)}(\varphi(w))| \\ &= \frac{\mu(w)|u_0(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{q}}}. \end{aligned} \tag{3.22}$$

Hence,

$$L_0\left\|\mathfrak{S}^m_{\vec{u},\varphi}\right\|_{H(p,q,\phi)\to H^\infty_\mu} \geq \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_0(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}}. \tag{3.23}$$

From (3.23), we see that (3.12) holds. This completes the proof. □

From Theorem 3.2 and (1.5), we obtain the following result.

Corollary 3.1. *Let $m \in \mathbb{N}$, $0 < p,q < +\infty$, ϕ normal, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and μ a weight function on $\mathbb{B}$. Then the operators $M_{(\Re^{m-j}u)\circ\varphi}C_\varphi\Re^j : H(p,q,\phi) \to H^\infty_\mu$, $j = \overline{0,m}$, are bounded if and only if the operator $C_\varphi\Re^m M_u : H(p,q,\phi) \to H^\infty_\mu$ is bounded and*

$$\sup_{z\in\mathbb{B}} \mu(z)|\Re^{m-j}u(\varphi(z))||\varphi(z)| < +\infty, \quad j = \overline{1,m}. \tag{3.24}$$

Remark 3.2. When $k \in \mathbb{N}$, the author in [32] proved that the operator $\Re^k_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is compact if and only if $\Re^k_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+k}} = 0.$$

Here, we need to characterize the compactness of the operator $\Re^0_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$.

Theorem 3.3. *Let $0 < p,q < +\infty$, ϕ normal, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and μ a weight function on $\mathbb{B}$. Then the operator $\Re^0_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is compact if and only if $\Re^0_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}} = 0. \tag{3.25}$$

Proof. Suppose that the operator $\mathfrak{R}^0_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is compact. Then, it is bounded. If $\|\varphi\|_\infty < 1$, then (3.25) holds. Let $\|\varphi\|_\infty = 1$ and $\{z_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$, and $h_i(z) = f_{\varphi(z_i),1}(z)$, where $f_{w,t}$ is defined in Lemma 2.6. Then $\sup_{i\in\mathbb{N}} \|h_i\|_{H(p,q,\phi)} < +\infty$. Since $\lim_{i\to\infty}(1 - |\varphi(z_i)|^2)^{t+1+\tau} = 0$, we see that $h_i \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $i \to \infty$. Hence, by Lemma 2.5 we have

$$\lim_{i\to\infty} \left\|\mathfrak{R}^0_{u,\varphi} h_i\right\|_{H^\infty_\mu} = 0. \tag{3.26}$$

On the other hand, by (3.5), we see that sufficiently large i

$$\frac{\mu(z_i)|u(z_i)|}{\phi(|\varphi(z_i)|)(1 - |\varphi(z_i)|^2)^{\frac{n}{q}}} \le C\|\mathfrak{R}^0_{u,\varphi} h_i\|_{H^\infty_\mu}. \tag{3.27}$$

Letting $i \to \infty$ in (3.27) and using (3.26), equality (3.25) follows.

Assume that the operator $\mathfrak{R}^0_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and (3.25) holds. By using the function $f(z) \equiv 1$, we obtain that

$$\widehat{M} := \sup_{z\in\mathbb{B}} \mu(z)|u(z)| \le \|\mathfrak{R}^0_{u,\varphi}\|_{H(p,q,\phi)\to H^\infty_\mu} \|1\|_{H(p,q,\phi)} < +\infty. \tag{3.28}$$

From (3.25), we have that for each $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that

$$\frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{n}{q}}} < \varepsilon \tag{3.29}$$

on the set $\{z \in \mathbb{B} : \delta < |\varphi(z)| < 1\}$. Suppose that $\{f_i\}$ is a sequence such that $\sup_{i\in\mathbb{N}} \|f_i\|_{H(p,q,\phi)} \le M$, and $f_i \to 0$ uniformly on compacts of $\mathbb{B}$ as $i \to \infty$. Then, by Lemma 2.1 and using (3.28) and (3.29), we have

$$\begin{aligned}
\left\|\mathfrak{R}^0_{u,\varphi} f_i\right\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{B}} \mu(z) \left|u(z)\mathfrak{R}^0 f_i(\varphi(z))\right| \\
&\le \sup_{z\in K} \mu(z) \left|u(z) f_i(\varphi(z))\right| + \sup_{z\in\mathbb{B}\setminus K} \mu(z) \left|u(z) f_i(\varphi(z))\right| \\
&\le \sup_{z\in K} \mu(z)|u(z)|\, |f_i(\varphi(z))| \\
&\quad + C\,\|f_i\|_{H(p,q,\phi)} \sup_{z\in\mathbb{B}\setminus K} \frac{\mu(z)|u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{\frac{n}{q}}} \\
&\le \widehat{M} \sup_{|z|\le\delta} |f_i(z)| + CM\varepsilon,
\end{aligned} \tag{3.30}$$

where $K = \{z \in \mathbb{B} : |\varphi(z)| \le \delta\}$. Since $\{z : |z| \le \delta\}$ is a compact subset of $\mathbb{B}$ and $f_i \to 0$ uniformly on compacts of $\mathbb{B}$ as $i \to \infty$, we have

$$\lim_{i\to\infty} \left\|\mathfrak{R}^0_{u,\varphi} f_i\right\|_{H^\infty_\mu} = 0. \tag{3.31}$$

From (3.31) and Lemma 2.5, it follows that the operator $\mathfrak{R}^0_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is compact. □

Next, we characterize the compactness of the operator $\mathfrak{S}^m_{\vec{u},\varphi}: H(p,q,\phi)\to H^\infty_\mu$.

Theorem 3.4. *Let $m\in\mathbb{N}$, $0<p,q<+\infty$, ϕ normal, $u_j\in H(\mathbb{B})$, $j=\overline{0,m}$, $\varphi\in S(\mathbb{B})$ and μ a weight function on $\mathbb{B}$. Then the operator $\mathfrak{S}^m_{\vec{u},\varphi}: H(p,q,\phi)\to H^\infty_\mu$ is compact and (3.8) holds if and only if the operators $\mathfrak{R}^j_{u_j,\varphi}: H(p,q,\phi)\to H^\infty_\mu$ are compact for $j=\overline{0,m}$.*

Proof. Assume that every operator $\mathfrak{R}^j_{u_j,\varphi}: H(p,q,\phi)\to H^\infty_\mu$ is compact, then we have that the operator $\mathfrak{S}^m_{\vec{u},\varphi}: H(p,q,\phi)\to H^\infty_\mu$ is clearly compact. Furthermore, from Theorem 3.2 it follows that (3.8) holds.

Now, assume that the operator $\mathfrak{S}^m_{\vec{u},\varphi}: H(p,q,\phi)\to H^\infty_\mu$ is compact and (3.8) holds. Then the operator $\mathfrak{S}^m_{\vec{u},\varphi}: H(p,q,\phi)\to H^\infty_\mu$ is bounded. In order to prove the operator $\mathfrak{R}^j_{u_j,\varphi}: H(p,q,\phi)\to H^\infty_\mu$ is compact, from Remark 3.2 and Theorem 3.3, we only need to prove

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|u_j(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+j}}=0,\quad j=\overline{1,m}, \tag{3.32}$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|u_0(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}}=0. \tag{3.33}$$

If $\|\varphi\|_\infty<1$, then (3.32) and (3.33) are obviously true. Hence, assume that $\|\varphi\|_\infty=1$. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$, and $h^{(s)}_k(z)=h^{(s)}_{\varphi(z_k)}(z)$, where $h^{(s)}_w$ is defined in Lemma 2.8 for a fixed $s\in\{1,2,\dots,m\}$. Then, we have $\sup_{k\in\mathbb{N}}\|h^{(s)}_k\|_{H(p,q,\phi)}<+\infty$. Since $\lim_{k\to\infty}(1-|\varphi(z_k)|^2)^{t+1+\tau}=0$, we see that $h^{(s)}_k\to 0$ uniformly on any compact subset of $\mathbb{B}$ as $k\to\infty$. Hence, by Lemma 2.5 we have

$$\lim_{k\to\infty}\|\mathfrak{S}^m_{\vec{u},\varphi}h^{(s)}_k\|_{H^\infty_\mu}=0. \tag{3.34}$$

On the other hand, from (3.15) we have that for sufficiently large k

$$\frac{\mu(z_k)|u_m(z_k)||\varphi(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{\frac{n}{q}+m}}\le C\|\mathfrak{S}^m_{\vec{u},\varphi}h^{(m)}_k\|_{H^\infty_\mu}. \tag{3.35}$$

Letting $k\to\infty$ in (3.35) and using (3.34) with $s=m$, then we obtain that (3.32) holds for $j=m$.

Now, suppose that (3.32) holds for $j=\overline{s+1,m}$, for some $s\in\mathbb{N}$. Then, from (3.18), we easily have that

$$\frac{\mu(z_k)|u_s(z_k)||\varphi(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{\frac{n}{q}+s}}\lesssim\left\|\mathfrak{S}^m_{\vec{u},\varphi}h^{(s)}_k\right\|_{H(p,q,\phi)\to H^\infty_\mu}+\sum_{j=s+1}^{m}\frac{\mu(z_k)|u_j(z_k)||\varphi(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{\frac{n}{q}+j}} \tag{3.36}$$

for sufficiently large k. Letting $k\to\infty$ in (3.36), using (3.34) and the induction hypothesis, we have that (3.32) holds for $j=s$, from which by induction it follows that (3.32) holds for each $s\in\{1,2,\dots,m\}$.

Let $h^{(0)}_k(z)=h^{(0)}_{\varphi(z_k)}(z)$, where $h^{(0)}_w$ is defined in Lemma 2.9. Then we have that $\sup_{k\in\mathbb{N}}\|h^{(0)}_k\|_{H(p,q,\phi)}<+\infty$ and $h^{(0)}_k\to 0$ uniformly on any compact subset of $\mathbb{B}$ as $k\to\infty$. Hence, by Lemma 2.5 we have

$$\lim_{k\to\infty}\|\mathfrak{S}^m_{\vec{u},\varphi}h^{(0)}_k\|_{H^\infty_\mu}=0. \tag{3.37}$$

From (3.22), for sufficiently large k it follows that

$$\frac{\mu(z_k)|u_0(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{\frac{n}{q}}} \le C\|\mathfrak{S}^m_{\vec{u},\varphi} h^{(0)}_k\|_{H^\infty_\mu}. \tag{3.38}$$

Letting $k\to\infty$ in (3.38) and using (3.37), (3.33) follows. The proof is finished. □

We also have the following result.

Corollary 3.2. *Let $m\in\mathbb{N}$, $0<p,q<+\infty$, ϕ normal, $u\in H(\mathbb{B})$, $\varphi\in S(\mathbb{B})$ and μ a weight function on $\mathbb{B}$. Then the operator $C_\varphi\mathfrak{R}^m M_u : H(p,q,\phi)\to H^\infty_\mu$ is compact and (3.24) holds if and only if the operators $M_{(\mathfrak{R}^{m-j}u)\circ\varphi}C_\varphi\mathfrak{R}^j : H(p,q,\phi)\to H^\infty_\mu$, $j=\overline{0,m}$, are compact.*

4. Essential norm estimate of the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p,q,\phi)\to H^\infty_\mu$

In this section, we estimate the essential norm of $\mathfrak{S}^m_{\vec{u},\varphi} : H(p,q,\phi)\to H^\infty_\mu$. Let us recall the definition of the essential norm of the bounded linear operators. Assume that X and Y are Banach spaces and $T : X\to Y$ is a bounded linear operator, then the essential norm of the operator $T : X\to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K\in\mathcal{K}\},$$

where $\mathcal{K}$ denotes the set of all compact linear operators from X to Y. It is easy to see that $\|T\|_{e,X\to Y}=0$ if and only if the bounded operator $T : X\to Y$ is compact.

Theorem 4.1. *Let $m\in\mathbb{N}$, $1\le p<+\infty$, $1<q<+\infty$, $u_j\in H(\mathbb{B})$, $j=\overline{0,m}$, $\varphi\in S(\mathbb{B})$, ϕ normal, μ a weight function on $\mathbb{B}$ and (3.8) hold. If the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p,q,\phi)\to H^\infty_\mu$ is bounded, then*

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{e,H(p,q,\phi)\to H^\infty_\mu} \asymp \limsup_{|\varphi(z)|\to 1}\Big(\frac{\mu(z)|u_0(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}} + \sum_{j=1}^m \frac{\mu(z)|u_j(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+j}}\Big).$$

Proof. Assume that $\{z_i\}_{i\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ such that $|\varphi(z_i)|\to 1$ as $i\to\infty$. Take the function $h^{(s)}_i = h^{(s)}_{\varphi(z_i)}$, where $h^{(s)}_w$ is defined in Lemma 2.8 for a fixed $s\in\{1,2,\dots,m\}$. Then we have $\sup_{i\in\mathbb{N}}\|h^{(s)}_i\|_{H(p,q,\phi)}<+\infty$ and $h^{(s)}_i\to 0$ uniformly on compacts of $\mathbb{B}$ as $i\to\infty$. Hence, by Lemma 2.3 and Remark 2.2 we have $\|Kh^{(s)}_i\|_{H(p,q,\phi)}\to 0$ as $i\to\infty$. Thus, from (3.15) it follows that

$$\begin{aligned}
\|\mathfrak{S}^m_{\vec{u},\varphi}-K\|_{H(p,q,\phi)\to H^\infty_\mu} &= \sup_{\|h\|_{H(p,q,\phi)}=1}\|(\mathfrak{S}^m_{\vec{u},\varphi}-K)h\|_{H^\infty_\mu}\\
&\ge \limsup_{i\to\infty}\frac{\|(\mathfrak{S}^m_{\vec{u},\varphi}-K)h^{(m)}_i\|_{H^\infty_\mu}}{\|h^{(m)}_i\|_{H(p,q,\phi)}}\\
&\ge \limsup_{i\to\infty}\frac{\|\mathfrak{S}^m_{\vec{u},\varphi}h^{(m)}_i\|_{H^\infty_\mu}-\|Kh^{(m)}_i\|_{H^\infty_\mu}}{\|h^{(m)}_i\|_{H(p,q,\phi)}}\\
&\ge \limsup_{i\to\infty}\frac{\mu(z_i)|u_m(z_i)||\varphi(z_i)|}{\phi(|\varphi(z_i)|)(1-|\varphi(z_i)|^2)^{\frac{n}{q}+m}}.
\end{aligned}$$

From induction and combining these inequalities, it follows that

$$\|\mathfrak{S}^m_{\vec{u},\varphi} - K\|_{H(p,q,\phi)\to H^\infty_\mu} \gtrsim \limsup_{i\to\infty} \sum_{j=1}^m \frac{\mu(z_i)|u_j(z_i)||\varphi(z_i)|}{\phi(|\varphi(z_i)|)(1-|\varphi(z_i)|^2)^{\frac{n}{q}+j}}. \tag{4.1}$$

Take the function $h_i^{(0)} = h^{(0)}_{\varphi(z_i)}$, where $h_w^{(0)}$ is defined in Lemma 2.9. Then we have that $\sup_{i\in\mathbb{N}} \left\|h_i^{(0)}\right\|_{H(p,q,\phi)} < +\infty$, and $h_i^{(0)} \to 0$ uniformly on compacts of $\mathbb{B}$ as $i\to\infty$. Then from (3.22), we have

$$\|\mathfrak{S}^m_{\vec{u},\varphi} - K\|_{H(p,q,\phi)\to H^\infty_\mu} \gtrsim \limsup_{i\to\infty} \frac{\mu(z_i)|u_0(z_i)|}{\phi(|\varphi(z_i)|)(1-|\varphi(z_i)|^2)^{\frac{n}{q}}}. \tag{4.2}$$

By taking the infimum in (4.1) and (4.2) over the set of all compact operators $K : H(p,q,\phi) \to H^\infty_\mu$, we have

$$\|\mathfrak{S}^m_{\vec{u},\varphi}\|_{e,H(p,q,\phi)\to H^\infty_\mu} \gtrsim \limsup_{|\varphi(z_i)|\to 1} \left(\frac{\mu(z_i)|u_0(z_i)|}{\phi(|\varphi(z_i)|)(1-|\varphi(z_i)|^2)^{\frac{n}{q}}} + \sum_{j=1}^m \frac{\mu(z_i)|u_j(z_i)||\varphi(z_i)|}{\phi(|\varphi(z_i)|)(1-|\varphi(z_i)|^2)^{\frac{n}{q}+j}} \right).$$

Now, assume that $\{r_i\}_{i\in\mathbb{N}}$ is a positive sequence which increasingly converges to 1. For each r_i, we define the operator by

$$\mathfrak{S}^m_{\vec{u},r_i\varphi} = \sum_{j=0}^m \mathfrak{R}^j_{u_j,r_i\varphi}.$$

Since the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded, by Theorem 3.2 one can obtain that the operator $\mathfrak{S}^m_{\vec{u},r_i\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded. Since $|r_i\varphi(z)| \le r_i < 1$, by Lemma 2.5 the operator $\mathfrak{S}^m_{\vec{u},r_i\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is also compact. Hence, from Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\left\|\mathfrak{S}^m_{\vec{u},\varphi} - \mathfrak{S}^m_{\vec{u},r_i\varphi}\right\|_{H(p,q,\phi)\to H^\infty_\mu} &= \sup_{\|f\|_{H(p,q,\phi)}=1} \left\|(\mathfrak{S}^m_{\vec{u},\varphi} - \mathfrak{S}^m_{\vec{u},r_i\varphi})f\right\|_{H^\infty_\mu} \\
&= \sup_{\|f\|_{H(p,q,\phi)}=1} \sup_{z\in\mathbb{B}} \mu(z)\Big|\sum_{j=0}^m \mathfrak{R}^j_{u_j,\varphi}f - \sum_{j=0}^m \mathfrak{R}^j_{u_j,r_i\varphi}f\Big| \\
&\le \sup_{\|f\|_{H(p,q,\phi)}=1} \sup_{z\in\mathbb{B}} \sum_{j=0}^m \mu(z)\big|u_j(z)\big|\big|\mathfrak{R}^j f(\varphi(z)) - \mathfrak{R}^j f(r_i\varphi(z))\big| \\
&\le \sup_{\|f\|_{H(p,q,\phi)}=1} \sup_{|\varphi(z)|\le\delta} \sum_{j=0}^m \mu(z)\big|u_j(z)\big|\big|\mathfrak{R}^j f(\varphi(z)) - \mathfrak{R}^j f(r_i\varphi(z))\big| \\
&\quad + \sup_{\|f\|_{H(p,q,\phi)}=1} \sup_{|\varphi(z)|>\delta} \sum_{j=0}^m \mu(z)\big|u_j(z)\big|\big|\mathfrak{R}^j f(\varphi(z)) - \mathfrak{R}^j f(r_i\varphi(z))\big| \\
&\le \sup_{\|f\|_{H(p,q,\phi)}=1} \sup_{|\varphi(z)|\le\delta} \sum_{j=0}^m \mu(z)\big|u_j(z)\big|\big|\mathfrak{R}^j f(\varphi(z)) - \mathfrak{R}^j f(r_i\varphi(z))\big| \\
&\quad + \sup_{|\varphi(z)|>\delta} \left(\frac{\mu(z)|u_0(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}} + \sum_{j=1}^m \frac{\mu(z)|u_j(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+j}} \right)
\end{aligned}$$

$$+\sup_{|\varphi(z)|>\delta}\Big(\frac{\mu(z)|u_0(z)|}{\phi(|r_i\varphi(z)|)(1-|r_i\varphi(z)|^2)^{\frac{n}{q}}}+\sum_{j=1}^{m}\frac{\mu(z)|u_j(z)||r_i\varphi(z)|}{\phi(|r_i\varphi(z)|)(1-|r_i\varphi(z)|^2)^{\frac{n}{q}+j}}\Big). \tag{4.3}$$

For each $f\in H(\mathbb{B})$ and $|\varphi(z)|\le\delta$, we have

$$\begin{aligned}
&\Big|f(\varphi(z))-f(r_i\varphi(z))\Big| = \Big|f(\varphi_1(z),\varphi_2(z),\dots,\varphi_n(z))-f(r_i\varphi_1(z),r_i\varphi_2(z),\dots,r_i\varphi_n(z))\Big|\\
&\le\sum_{k=1}^{n}|f(r_i\varphi_1(z),\dots,r_i\varphi_{k-1}(z),\varphi_k(z),\varphi_{k+1}(z)\dots,\varphi_n(z))\\
&\quad - f(r_i\varphi_1(z),\dots,r_i\varphi_{k-1}(z),r_i\varphi_k(z),\varphi_{k+1}(z)\dots,\varphi_n(z))|\\
&=\sum_{k=1}^{n}\Big|(1-r_i)\varphi_k(z)\int_0^1\frac{\partial f}{\partial z_k}(r_i\varphi_1(z),\dots,r_i\varphi_{k-1}(z),\theta_k(t),\varphi_{k+1}(z)\dots,\varphi_n(z))\,dt\Big|\\
&\le(1-r_i)\sum_{k=1}^{n}\int_0^1\Big|\frac{\partial f}{\partial z_k}(r_i\varphi_1(z),\dots,r_i\varphi_{k-1}(z),\theta_k(t),\varphi_{k+1}(z)\dots,\varphi_n(z))\Big|dt\\
&\le(1-r_i)\sum_{k=1}^{n}\sup_{|w|\le\delta}\Big|\frac{\partial f}{\partial z_k}(w)\Big|\\
&\le C(1-r_i),
\end{aligned} \tag{4.4}$$

where $\theta_k(t)=(1-t)r_i\varphi_k(z)+t\varphi_k(z)$. Let $g=\nabla\mathfrak{R}^{j-1}f$, $j=\overline{1,m}$, from (4.4) and $|\varphi(z)|\le\delta$, we have that

$$|\mathfrak{R}^jf(\varphi(z))-\mathfrak{R}^jf(r_i\varphi(z))|\le|g(\varphi(z))-g(r_i\varphi(z))||\varphi(z)|\le(1-r_i)|\varphi(z)|\sum_{k=1}^{n}\sup_{|w|\le\delta}\Big|\frac{\partial g}{\partial z_k}(w)\Big|\le C(1-r_i)|\varphi(z)|. \tag{4.5}$$

By letting $i\to\infty$ in (4.3), from (3.8), (3.28), (4.4) and (4.5), we obtain

$$\Big\|\mathfrak{S}^m_{\vec{u},\varphi}-\mathfrak{S}^m_{\vec{u},r_i\varphi}\Big\|_{H(p,q,\phi)\to H^\infty_\mu}\lesssim\sup_{|\varphi(z)|>\delta}\Big(\frac{\mu(z)|u_0(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}}+\sum_{j=1}^{m}\frac{\mu(z)|u_j(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+j}}\Big)$$

as $i\to\infty$. Since

$$\Big\|\mathfrak{S}^m_{\vec{u},\varphi}\Big\|_{e,H(p,q,\phi)\to H^\infty_\mu}\le\Big\|\mathfrak{S}^m_{\vec{u},\varphi}-\mathfrak{S}^m_{\vec{u},r_i\varphi}\Big\|_{H(p,q,\phi)\to H^\infty_\mu},$$

we finish the proof. □

From Theorem 4.1 and (1.5), we obtain the following result.

Corollary 4.1. *Let $m\in\mathbb{N}$, $1\le p<+\infty$, $1<q<+\infty$, ϕ normal, $u\in H(\mathbb{B})$, $\varphi\in S(\mathbb{B})$, μ a weight function on $\mathbb{B}$ and (3.24) hold. If the operator $C_\varphi\mathfrak{R}^mM_u: H(p,q,\phi)\to H^\infty_\mu$ is bounded, then*

$$\|C_\varphi\mathfrak{R}^mM_u\|_{e,H(p,q,\phi)\to H^\infty_\mu}\asymp\limsup_{|\varphi(z)|\to1}\Big(\frac{\mu(z)|\mathfrak{R}^mu(\varphi(z))|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}}}+\sum_{j=1}^{m}\frac{\mu(z)|\mathfrak{R}^{m-j}u(\varphi(z))||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{q}+j}}\Big).$$

5. Hilbert-Schmidt norm of the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^2_\alpha \to A^2_\alpha$

In this section, we calculate Hilbert-Schmidt norm of the operator $\mathfrak{S}^m_{\vec{u},\varphi} : A^2_\alpha \to A^2_\alpha$. For some related results, one can see [37, 43]. If $\mathcal{H}$ is a separable Hilbert space, then the Hilbert-Schmidt norm $\|T\|_{\mathrm{HS},\mathcal{H}\to\mathcal{H}}$ of an operator $T : \mathcal{H} \to \mathcal{H}$ is defined by

$$\|T\|_{\mathrm{HS},\mathcal{H}\to\mathcal{H}} = \Big(\sum_{n=1}^{\infty} \|Te_n\|^2\Big)^{\frac{1}{2}}, \tag{5.1}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis on $\mathcal{H}$. The right-hand side in (5.1) does not depend on the choice of basis. Hence, we have that $\|T\|_{\mathrm{HS},\mathcal{H}\to\mathcal{H}} \ge \|T\|_{\mathcal{H}\to\mathcal{H}}$ the operator norm.

Theorem 5.1. *Let $m \in \mathbb{N}$ and $\alpha > -1$. Then Hilbert-Schmidt norm of the operator $\mathfrak{S}^m_{\vec{u},\varphi}$ on A^2_α is*

$$\big\|\mathfrak{S}^m_{\vec{u},\varphi}\big\|_{\mathrm{HS},A^2_\alpha\to A^2_\alpha} = \left(\sum_{j=0}^{m} \int_{\mathbb{B}} |u_j(z)|^2 \Big(\mathfrak{R}^j \frac{1}{(1-\sum_{i=1}^n w_i)^{n+\alpha+1}}\Big)\Big|_{w_i=|\varphi_i(z)|^2} dv_\alpha(z)\right)^{\frac{1}{2}}.$$

Proof. From Proposition 1.4.9 in [23] (or Lemma 1.11 in [48]), for each multi-index $\beta = (\beta_1, \dots, \beta_n)$, we have

$$\int_{\mathbb{B}} \big|z^\beta\big|^2 dv_\alpha(z) = \frac{\beta!\,\Gamma(n+\alpha+1)}{\Gamma(n+|\beta|+\alpha+1)},$$

where $\beta! = \beta_1! \cdots \beta_n!$, and

$$\int_{\mathbb{B}} z^\beta z^\gamma dv_\alpha(z) = 0, \quad \beta \ne \gamma.$$

From this, we have that the vectors

$$e_\beta(z) = \sqrt{\frac{\Gamma(n+|\beta|+\alpha+1)}{\beta!\,\Gamma(n+\alpha+1)}}\, z^\beta$$

form an orthonormal basis in A^2_α. By using the definition of the Hilbert-Schmidt norm and the monotone convergence theorem, we have

$$\begin{aligned}
\big\|\mathfrak{S}^m_{\vec{u},\varphi}\big\|^2_{\mathrm{HS},A^2_\alpha\to A^2_\alpha} &= \sum_\beta \big\|\mathfrak{S}^m_{\vec{u},\varphi} e_\beta\big\|^2_{A^2_\alpha} = \sum_\beta \frac{\Gamma(n+|\beta|+\alpha+1)}{\beta!\,\Gamma(n+\alpha+1)} \big\|\mathfrak{S}^m_{\vec{u},\varphi} z^\beta\big\|^2_{A^2_\alpha} \\
&= \sum_\beta \frac{\Gamma(n+|\beta|+\alpha+1)}{\beta!\,\Gamma(n+\alpha+1)} \sum_{j=0}^m |\beta|^j \int_{\mathbb{B}} |u_j(z)|^2 \prod_{i=1}^n |\varphi_i(z)|^{2\beta_i} dv_\alpha(z) \\
&= \sum_{j=0}^m \int_{\mathbb{B}} |u_j(z)|^2 \sum_\beta |\beta|^j \frac{\Gamma(n+|\beta|+\alpha+1)}{\beta!\,\Gamma(n+\alpha+1)} \prod_{i=1}^n |\varphi_i(z)|^{2\beta_i} dv_\alpha(z).
\end{aligned} \tag{5.2}$$

For each $f \in H(\mathbb{B})$, by using the Taylor expansion

$$f(z) = \sum_{|\beta|\ge 0} a_\beta z^\beta,$$

and taking the jth radial derivatives, we have

$$\mathfrak{R}^j f(z) = \sum_{|\beta|\geq 0} |\beta|^j a_\beta z^\beta, \tag{5.3}$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$.

On the other hand, from Proposition 1.14 in [48] (also see [32]), we have

$$\Big(1 - \sum_{i=1}^n w_i\Big)^{-(n+\alpha+1)} = \sum_l \frac{\Gamma(n+\alpha+|l|+1)}{l!\,\Gamma(n+\alpha+1)} \prod_{i=1}^n w_i^{l_i}. \tag{5.4}$$

Hence, from (5.3) and (5.4), we have

$$\mathfrak{R}^j \frac{1}{(1-\sum_{i=1}^n w_i)^{n+\alpha+1}} = \sum_l |l|^j \frac{\Gamma(n+\alpha+|l|+1)}{l!\,\Gamma(n+\alpha+1)} \prod_{i=1}^n w_i^{l_i}. \tag{5.5}$$

From (5.2) and (5.5), the desired result follows. □

From Theorem 5.1 and (1.5), we obtain the following result.

Corollary 5.1. *Let $m \in \mathbb{N}$ and $\alpha > -1$. Then Hilbert-Schmidt norm of the operator $C_\varphi \mathfrak{R}^m M_u$ on A^2_α is*

$$\left\|C_\varphi \mathfrak{R}^m M_u\right\|_{\mathrm{HS}, A^2_\alpha \to A^2_\alpha} = \left(\sum_{j=0}^m \int_{\mathbb{B}} |\mathfrak{R}^{m-j} u(\varphi(z))|^2 \Big(\mathfrak{R}^j \frac{1}{(1-\sum_{i=1}^n w_i)^{n+\alpha+1}} \Big)\Big|_{w_i = |\varphi_i(z)|^2} dv_\alpha(z) \right)^{\frac{1}{2}}.$$

6. Conclusions

In this paper, we define the operator $\mathfrak{S}^m_{\vec{u},\varphi} = \sum_{j=0}^m M_{u_j} C_\varphi \mathfrak{R}^j$ on some subspaces of $H(\mathbb{B})$, where $u_j \in H(\mathbb{B})$, $j \in \{0, 1, \ldots, m\}$ and $\varphi \in S(\mathbb{B})$. We completely characterized the boundedness and compactness of the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p, q, \phi) \to H^\infty_\mu$ in terms of the behaviours of the symbols u_j and φ. In order to study the essential norm estimate of the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p, q, \phi) \to H^\infty_\mu$, we considered the conditions for the reflexivity of $H(p, q, \phi)$. By using a criterion of the compactness for a bounded linear operator $T : X \to Y$, where X is a reflexive Banach space of holomorphic functions on $\mathbb{B}$ and Y is any Banach space, we obtained the essential norm estimate of the operator $\mathfrak{S}^m_{\vec{u},\varphi} : H(p, q, \phi) \to H^\infty_\mu$. Moreover, we also calculated the Hilbert-Schmidt norm of the operator on the weighted Bergman space A^2_α. As an application, the corresponding results of the operator $C_\varphi \mathfrak{R}^m M_u : H(p, q, \phi) \to H^\infty_\mu$ are obtained. This paper can be viewed as a continuation and extension of our previous work. We hope that this study can attract people's more attention for such operators and mixed-norm spaces.

Acknowledgments

This work was supported by the Innovation Fund of Postgraduate, Sichuan University of Science and Engineering (Y2021098) and the Key Project of Zigong City (2020YGJC24).

Conflict of interest

The authors declare that they have no competing interests.

References

1. K. L. Avetisyan, Fractional integro-differentiation in harmonic mixed norm spaces on a half-space, *Comment. Math. Univ. Ca.*, **42** (2001), 691–709.
2. K. L. Avetisyan, Continuous inclusions and Bergman type operators in n-harmonic mixed norm spaces on the polydisc, *J. Math. Anal. Appl.*, **291** (2004), 727–740. https://doi.org/10.1016/j.jmaa.2003.11.039
3. F. Colonna, M. Tjani, Operator norms and essential norms of weighted composition operators between Banach spaces of analytic functions, *J. Math. Anal. Appl.*, **434** (2016), 93–124. https://doi.org/10.1016/j.jmaa.2015.08.073
4. C. C. Cowen, B. D. Maccluer, *Composition operators on spaces of analytic functions*, Boca Raton: CRC Press, 1995.
5. R. A. Hibschweiler, N. Portnoy, Composition followed by differentiation between Bergman and Hardy spaces, *Rocky Mountain J. Math.*, **35** (2005), 843–855. https://doi.org/10.1216/rmjm/1181069709
6. Z. J. Hu, Extended Cesàro operators on mixed-norm spaces, *Proc. Amer. Math. Soc.*, **131** (2003), 2171–2179. https://doi.org/10.1090/S0002-9939-02-06777-1
7. M. Jević, Bounded projections and duality in mixed-norm spaces of analytic functions, *Complex Var. Theory Appl: Int. J.*, **8** (1987), 293–301. https://doi.org/10.1080/17476938708814239
8. Z. J. Jiang, X. F. Wang, Products of radial derivative and weighted composition operators from weighted Bergman-Orlicz spaces to weighted-type spaces, *Oper. Matrices*, **12** (2018), 301–319. https://doi.org/10.7153/oam-2018-12-20
9. Z. J. Jiang, Product-type operators from Zygmund spaces to Bloch-Orlicz spaces, *Complex Var. Elliptic*, **62** (2017), 1645–1664. https://doi.org/10.1080/17476933.2016.1278436
10. Z. J. Jiang, Product-type operators from Logarithmic Bergman-type spaces to Zygmund-Orlicz spaces, *Mediterr. J. Math.*, **13** (2016), 4639–4659. https://doi.org/10.1007/s00009-016-0767-8
11. Z. J. Jiang, Generalized product-type operators from weighted Bergman-Orlicz spaces to Bloch-Orlicz spaces, *Appl. Math. Comput.*, **268** (2015), 966–977. https://doi.org/10.1016/j.amc.2015.06.100
12. Z. J. Jiang, On a class of operators from weighted Bergman spaces to some spaces of analytic functions, *Taiwan. J. Math.*, **15** (2011), 2095–2121. https://doi.org/10.11650/twjm/1500406425
13. W. Johnson, The curious history of Faà di Bruno's formula, *Am. Math. Mon.*, **109** (2002), 217–234. https://doi.org/10.1080/00029890.2002.11919857
14. S. X. Li, S. Stević, Weighted differentiation composition operators from the logarithmic Bloch space to the weighted-type space, *An. Stiint. Univ. Ovidius Constanta, Ser. Mat.*, **24** (2016), 223–240. https://doi.org/10.1515/auom-2016-0056

15. S. X. Li, S. Stević, Products of composition and differentiation operators from Zygmund spaces to Bloch spaces and Bers spaces, *Appl. Math. Comput.*, **217** (2010), 3144–3154. https://doi.org/10.1016/j.amc.2010.08.047

16. S. X. Li, S. Stević, Composition followed by differentiation between H^∞ and α-Bloch spaces, *Houston J. Math.*, **35** (2009), 327–340.

17. S. X. Li, S. Stević, Composition followed by differentiation from mixed norm spaces to α-Bloch spaces, *Sb. Math.*, **199** (2008), 1847–1857. https://doi.org/10.1070/SM2008v199n12ABEH003983

18. S. X. Li, S. Stević, Composition followed by differentiation between Bloch type spaces, *J. Comput. Anal. Appl.*, **9** (2007), 195–206.

19. Y. M. Liu, X. M. Liu, Y. Y. Yu, On an extension of Stević-Sharma operator from the mixed-norm space to weighted-type spaces, *Complex Var. Elliptic*, **62** (2017), 670–694. https://doi.org/10.1080/17476933.2016.1238465

20. Y. M. Liu, Y. Y. Yu, On an extension of Stević-Sharma operator from the general spaces to weighted-type spaces on the unit ball, *Complex Anal. Oper. Theory*, **11** (2017), 261–288. https://doi.org/10.1007/s11785-016-0535-6

21. Y. M. Liu, Y. Y. Yu, Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball, *J. Math. Anal. Appl.*, **423** (2015), 76–93. https://doi.org/10.1016/j.jmaa.2014.09.069

22. S. Ohno, Products of composition and differentiation on Bloch spaces, *B. Korean Math. Soc.*, **46** (2009), 1135–1140. https://doi.org/10.4134/BKMS.2009.46.6.1135

23. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Berlin, Heidelberg: Springer, 2008. https://doi.org/10.1007/978-3-540-68276-9

24. B. Sehba, S. Stević, On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces, *Appl. Math. Comput.*, **233** (2014), 565–581. https://doi.org/10.1016/j.amc.2014.01.002

25. A. K. Sharma, Products of composition multiplication and differentiation between Bergman and Bloch type spaces, *Turk. J. Math.*, **35** (2011), 275–291. https://doi.org/10.3906/mat-0806-24

26. J. H. Shi, G. B. Ren, Boundedness of the Cesàro operator on mixed norm spaces, *Proc. Amer. Math. Soc.*, **126** (1998), 3553–3560. https://doi.org/10.1090/S0002-9939-98-04514-6

27. J. H. Shi, Duality and multipliers for mixed norm spaces in the ball (I), *Complex Var. Theory Appl: Int. J.*, **25** (1994), 119–130. https://doi.org/10.1080/17476939408814736

28. A. L. Shields, D. L. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, *T. Am. Math. Soc.*, **162** (1971), 287–302. https://doi.org/10.2307/1995754

29. S. Stević, Essential norm of some extensions of the generalized composition operators between kth weighted-type spaces, *J. Inequal. Appl.*, **2017** (2017), 220. https://doi.org/10.1186/s13660-017-1493-x

30. S. Stević, Weighted radial operator from the mixed-norm space to the nth weighted-type space on the unit ball, *Appl. Math. Comput.*, **218** (2012), 9241–9247. https://doi.org/10.1016/j.amc.2012.03.001

31. S. Stević, On some integral-type operators between a general space and Bloch-type spaces, *Appl. Math. Comput.*, **218** (2011), 2600–2618. https://doi.org/10.1016/j.amc.2011.07.077
32. S. Stevć, Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball, *Abstr. Appl. Anal.*, **2010** (2010), 801264. https://doi.org/10.1155/2010/801264
33. S. Stević, Composition followed by differentiation from H^∞ and the Bloch space to nth weighted-type spaces on the unit disk, *Appl. Math. Comput.*, **216** (2010), 3450–3458. https://doi.org/10.1016/j.amc.2010.03.117
34. S. Stević, Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H^∞_μ, *Appl. Math. Comput.*, **207** (2009), 225–229. https://doi.org/10.1016/j.amc.2008.10.032
35. S. Stević, Products of composition and differentiation operators on the weighted Bergman space, *Bull. Belg. Math. Soc. Simon Stevin*, **16** (2009), 623–635. https://doi.org/10.36045/bbms/1257776238
36. S. Stević, Weighted composition operators between mixed norm spaces and H^∞_α spaces in the unit ball, *J. Inequal. Appl.*, **2007** (2007), 28629. https://doi.org/10.1155/2007/28629
37. S. Stević, Continuity with respect to symbols of composition operators on the weighted Bergman space, *Taiwan. J. Math.*, **11** (2007), 1177–1188. https://doi.org/10.11650/twjm/1500404811
38. S. Stević, Z. J. Jiang, Weighted iterated radial composition operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball, *Math. Methods Appl. Sci.*, **45** (2021), 3083–3097. https://doi.org/10.1002/mma.7978
39. S. Stević, Z. J. Jiang, Weighted iterated radial composition operators from weighted Bergman-Orlicz spaces to weighted-type spaces on the unit ball, *Math. Methods Appl. Sci.*, **44** (2021), 8684–8696. https://doi.org/10.1002/mma.7298
40. S. Stević, A. K. Sharma, A. Bhat, Essential norm of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.*, **218** (2011), 2386–2397. https://doi.org/10.1016/j.amc.2011.06.055
41. S. Stević, A. K. Sharma, A. Bhat, Products of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.*, **217** (2011), 8115–8125. https://doi.org/10.1016/j.amc.2011.03.014
42. M. Tjani, Compact composition operators on some Möbius invariant Banach space, PhD Thesis, Michigan State University, 1996.
43. S. I. Ueki, Hilbert-Schmidt weighted composition operator on the Fock space, *Int. J. Math. Anal.*, **1** (2007), 769–774.
44. S. M. Wang, M. F. Wang, X. Guo, Products of composition, multiplication and iterated differentiation operators between Banach spaces of holomorphic functions, *Taiwan. J. Math.*, **24** (2020), 355–376. https://doi.org/10.11650/tjm/190405
45. S. Wang, M. F. Wang, X. Guo, Products of composition, multiplication and radial derivative operators between Banach spaces of holomorphic functions on the unit ball, *Complex Var. Elliptic*, **65** (2020), 2026–2055. https://doi.org/10.1080/17476933.2019.1687455

46. W. F. Yang, W. R. Yan, Generalized weighted composition operators from area Nevanlinna spaces to weighted-type spaces, *B. Korean Math. Soc.*, **48** (2011), 1195–1205. https://doi.org/10.4134/BKMS.2011.48.6.1195

47. J. Zhou, Y. M. Liu, Products of radial derivative and multiplication operators from $F(p,q,s)$ to weighted-type spaces on the unit ball, *Taiwan. J. Math.*, **17** (2013), 161–178. https://doi.org/10.11650/tjm.17.2013.2127

48. K. H. Zhu, *Spaces of holomorphic functions in the unit ball*, New York: Springer, 2005. https://doi.org/10.1007/0-387-27539-8

49. X. L. Zhu, On an integral-type operator from Privalov spaces to Bloch-type spaces, *Ann. Pol. Math.*, **101** (2011), 139–147. https://doi.org/10.4064/ap101-2-4