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Research article

Solving a nonlinear integral equation via orthogonal metric space

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Abstract: We propose the concept of orthogonally triangular α-admissible mapping and demonstrate some fixed point theorems for self-mappings in orthogonal complete metric spaces. Some of the well-known outcomes in the literature are generalized and expanded by our results. An instance to help our outcome is presented. We also explore applications of our key results.

Keywords: orthogonal set; orthogonal complete metric space; orthogonal continuous; orthogonal preserving; orthogonally triangular α-admissible; fixed point
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

One of the most important results of mathematical analysis is the famous fixed point result, called the Banach contraction theory. In several branches of mathematics, it is the most commonly used fixed point result and it is generalized in many different directions. The substitution of the metric space by other generalized metric spaces is one natural way of reinforcing the Banach contraction principle. Wardowski [15], who generalized the Banach contraction principle in metric spaces, defined the fixed point result in the setting of complete metric spaces. In other branches of mathematics, on the other hand, the notion of an orthogonal set has many applications and has several kinds of orthogonality. Eshaghi Gordji, Ramezani, De la Sen and Cho [3] have imported the current concept of orthogonality in metric spaces and demonstrated some fixed point results equipped with the new orthogonality for contraction mappings in metric spaces. Furthermore, they used these results to claim the presence and uniqueness of the solution of the first-ordinary differential equation, while the Banach contraction mapping cannot be applied to this problem. In generalized orthogonal metric space, Eshaghi Gordji and Habibi [4] investigated the theory of fixed points. The new definition of orthogonal F-contraction mappings was introduced by Sawangsup, Sintunavarat and Cho [12], and some fixed point theorems on orthogonal-complete metric space were proved by them. Many authors have investigated orthogonal contractive mappings and significant results have been obtained in [2, 5–11, 13, 14, 16, 17].

In this paper, we prove fixed point theorems in orthogonal metric spaces.

In 2014, Alsulami, Gülyaz, Karapinar and Erhan [1] introduced the concepts of α-admissible contraction mappings and proved some fixed point theorems.

On the other hand, the definition of an orthogonal set (or O-set), some examples and some premises of orthogonal sets were introduced by Eshaghi Gordji, Ramezani, De la Sen and Cho [3], as follows:

Definition 1.1. [3] Let $\mathfrak{W} \neq \phi$ and $\perp \subseteq \mathfrak{W} \times \mathfrak{W}$ be a binary relation. If $\perp$ satisfies the consecutive condition:

$$\exists \mathcal{X}_0 \in \mathfrak{W} : (\forall \mathcal{X} \in \mathfrak{W}, \mathcal{X} \perp \mathcal{X}_0) \quad \text{or} \quad (\forall \mathcal{X} \in \mathfrak{W}, \mathcal{X}_0 \perp \mathcal{X}),$$

then it is said to be an orthogonal set (briefly, O-set). We indicate this O-set by $(\mathfrak{W}, \perp)$.

Example 1.2. [3] Let $\mathfrak{W} = [0, \infty)$ and define $\mathcal{X} \perp \mathcal{Y}$ if $\mathcal{X}\mathcal{Y} \in \{\mathcal{X}, \mathcal{Y}\}$. Set $\mathcal{X}_0 = 0$ or $\mathcal{X}_0 = 1$. Then $(\mathfrak{W}, \perp)$ is an O-set.

Definition 1.3. [3] A triplet $(\mathfrak{W}, \perp, \varphi)$ is said to be an O-metric space if $(\mathfrak{W}, \perp)$ is an O-set and $(\mathfrak{W}, \varphi)$ is a metric space.

Definition 1.4. [3] Let $(\mathfrak{W}, \perp, \varphi)$ be an O-metric space. Then $\mathfrak{W}$ is said to be O-complete if every Cauchy O-sequence is convergent.

Definition 1.5. [3] Let $(\mathfrak{W}, \perp)$ be an O-set. A mapping $\mathfrak{G} : \mathfrak{W} \to \mathfrak{W}$ is said to be $\perp$-preserving if $\mathfrak{G}\mathcal{X} \perp \mathfrak{G}\mathcal{Y}$ for $\mathcal{X} \perp \mathcal{Y}$.

Definition 1.6. [8] Let $(\mathfrak{W}, \perp)$ be an O-set and φ be a metric on $\mathfrak{W}$, $\mathfrak{G} : \mathfrak{W} \to \mathfrak{W}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be two mappings. We say that $\mathfrak{G}$ is orthogonally α-admissible if $\mathcal{X} \perp \mathcal{Y}$ and $\alpha(\mathcal{X}, \mathcal{Y}) \geq 1$ imply that $\alpha(\mathfrak{G}(\mathcal{X}), \mathfrak{G}(\mathcal{Y})) \geq 1$.

Definition 1.7. Let $(\mathfrak{W}, \perp)$ be an O-set and φ be a metric on $\mathfrak{W}$, $\mathfrak{G} : \mathfrak{W} \to \mathfrak{W}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to (-\infty, \infty)$. We say that $\mathfrak{G}$ is an orthogonally triangular α-admissible mapping if

(i) $\mathcal{X} \perp \mathcal{Y}$ and $\alpha(\mathcal{X}, \mathcal{Y}) \geq 1$ imply that $\alpha(\mathfrak{G}(\mathcal{X}), \mathfrak{G}(\mathcal{Y})) \geq 1$;

(ii) $\mathcal{X} \perp \mathcal{Z}$, $\alpha(\mathcal{X}, \mathcal{Z}) \geq 1$ and $\mathcal{Z} \perp \mathcal{Y}$, $\alpha(\mathcal{Z}, \mathcal{Y}) \geq 1$ imply that $\mathcal{X} \perp \mathcal{Y}$, $\alpha(\mathfrak{G}(\mathcal{X}), \mathfrak{G}(\mathcal{Y})) \geq 1$.

We modify the concept of triangular α-admissible to orthogonal sets in this article. To illustrate our results, we also give some examples and application.

2. Main results

Inspired by the triangular α-admissible contraction mappings defined by Alsulami, Gülyaz, Karapinar and Erhan [1], we implement a new orthogonally triangular α-admissible contraction mapping and demonstrate some fixed point theorems in an orthogonal complete metric space for this contraction mapping.

Definition 2.1. A function $\psi : [0,\infty) \to [0,\infty)$ is called an orthogonal altering distance function if the following properties are satisfied:

(1) ψ is orthogonally continuous and nondecreasing;

(2) $\psi(\mathfrak{t}) = 0$ if and only if $\mathfrak{t} = 0$.

First we define the following two classes of contractions which are investigated throughout the paper.

Definition 2.2. Let $(\mathfrak{W}, \perp, \varphi)$ be an O-metric space, ψ be an orthogonal altering distance function, and $\phi\colon [0,+\infty) \to [0,+\infty)$ be an orthogonally continuous function satisfying $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ for all $\mathfrak{t} > 0$.

(i) A mapping $\mathfrak{G} : \mathfrak{W} \to \mathfrak{W}$ is said to be an orthogonal ψ-ϕ contraction of type (A) if it satisfies, for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$ with $\mathcal{X} \perp \mathcal{Y}$,

$$\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Y}) > 0 \Rightarrow \alpha(\mathcal{X}, \mathcal{Y})\psi(\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Y})) \quad \leq \phi(\mathfrak{M}(\mathcal{X}, \mathcal{Y})), \tag{2.1}$$

where

$$\mathfrak{M}(\mathcal{X}, \mathcal{Y}) = \max\{\varphi(\mathcal{X}, \mathcal{Y}), \varphi(\mathcal{X}, \mathfrak{G}\mathcal{X}), \varphi(\mathcal{Y}, \mathfrak{G}\mathcal{Y}), \frac{1}{2}[\varphi(\mathcal{X}, \mathfrak{G}\mathcal{Y}) + \varphi(\mathcal{Y}, \mathfrak{G}\mathcal{X})]\}.$$

(ii) A mapping $\mathfrak{G} : \mathfrak{W} \to \mathfrak{W}$ is said to be an orthogonal ψ-ϕ contraction of type (B) if it satisfies, for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$ with $\mathcal{X} \perp \mathcal{Y}$,

$$\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Y}) > 0 \Rightarrow \alpha(\mathcal{X}, \mathcal{Y})\psi(\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Y})) \leq \phi(\mathfrak{N}(\mathcal{X}, \mathcal{Y})),$$

where

$$\mathfrak{N}(\mathcal{X}, \mathcal{Y}) = \max\{\varphi(\mathcal{X}, \mathcal{Y}), \frac{1}{2}[\varphi(\mathcal{X}, \mathfrak{G}\mathcal{X}) + \varphi(\mathcal{Y}, \mathfrak{G}\mathcal{Y})], \frac{1}{2}[\varphi(\mathcal{X}, \mathfrak{G}\mathcal{Y}) + \varphi(\mathcal{Y}, \mathfrak{G}\mathcal{X})]\}.$$

Remark 2.3. Note that $\mathfrak{N}(\mathcal{X}, \mathcal{Y}) \leq \mathfrak{M}(\mathcal{X}, \mathcal{Y})$ for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$.

The following theorem gives conditions for the existence of a fixed point for mappings in orthogonal ψ-ϕ contraction of type (A).

Theorem 2.4. *Let $(\mathfrak{W}, \perp, \varphi)$ be an O-complete metric space and $\mathfrak{G}$ be a self mapping on $\mathfrak{W}$ satisfying the following conditions:*

(i) $\mathfrak{G}$ is $\perp$-preserving;

(ii) $\mathfrak{G}$ is an orthogonal ψ-ϕ contraction of type (A);

(iii) $\mathfrak{G}$ is orthogonally triangular α-admissible;

(iv) there exists $\mathcal{X}_0 \in \mathfrak{W}$ such that $\mathcal{X}_0 \perp \mathfrak{G}\mathcal{X}_0$ and $\alpha(\mathcal{X}_0, \mathfrak{G}\mathcal{X}_0) \geq 1$;

(v) $\mathfrak{G}$ is orthogonally continuous.

Then $\mathfrak{G}$ has a fixed point in $\mathfrak{W}$.

Proof. By the condition (iv), there exists $\mathcal{X}_0 \in \mathfrak{W}$ such that $\mathcal{X}_0 \perp \mathfrak{G}\mathcal{X}_0$ and $\alpha(\mathcal{X}_0, \mathfrak{G}\mathcal{X}_0) \geq 1$. Let

$$\mathcal{X}_1 := \mathfrak{G}\mathcal{X}_0, \mathcal{X}_2 := \mathfrak{G}\mathcal{X}_1 = \mathfrak{G}^2\mathcal{X}_0, \cdots, , \mathcal{X}_{n+1} := \mathfrak{G}\mathcal{X}_n = \mathfrak{G}^{\mathfrak{n}+1}\mathcal{X}_0$$

for all $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. Since $\mathfrak{G}$ is $\perp$-preserving, $\{\mathcal{X}_n\}$ is an O-sequence in $\mathfrak{W}$. Since $\mathfrak{G}$ is an α-admissible mapping, we have $\alpha(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1}) \geq 1$ for all $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. If $\mathcal{X}_\mathfrak{n} = \mathcal{X}_{\mathfrak{n}+1}$ for some $\mathfrak{n} \in \mathbb{N} \cup \{0\}$, then $\mathcal{X}_\mathfrak{n}$ is a fixed point of $\mathfrak{G}$.

Assume that $\mathcal{X}_\mathfrak{n} \neq \mathcal{X}_{\mathfrak{n}+1}$ for all $\mathfrak{n} \in \mathbb{N} \cup \{0\}$. Then $\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1}) > 0$ for all $\mathfrak{n} \in \mathbb{N} \cup \{0\}$.

Letting $\mathcal{X} = \mathcal{X}_\mathfrak{n}$ and $\mathcal{Y} = \mathcal{X}_{\mathfrak{n}-1}$ in (2.1), we obtain

$$\begin{aligned} \psi(\varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n})) &\leq \alpha(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1})\psi(\varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n})) \\ &= \alpha(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1})\psi(\varphi(\mathfrak{G}\mathcal{X}_\mathfrak{n}, \mathfrak{G}\mathcal{X}_{\mathfrak{n}-1})) \\ &\leq \phi(\mathfrak{M}(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1})), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathfrak{M}(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1}) &= \max\{\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1}), \varphi(\mathcal{X}_\mathfrak{n}, \mathfrak{G}\mathcal{X}_\mathfrak{n}), \varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathfrak{G}\mathcal{X}_{\mathfrak{n}-1}), \frac{1}{2}[\varphi(\mathcal{X}_\mathfrak{n}, \mathfrak{G}\mathcal{X}_{\mathfrak{n}-1}) + \varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathfrak{G}\mathcal{X}_\mathfrak{n})]\} \\ &= \max\{\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1}), \varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1}), \varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathcal{X}_\mathfrak{n}), \frac{1}{2}[\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_\mathfrak{n}) + \varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathcal{X}_{\mathfrak{n}+1})]\} \\ &= \max\left\{\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1}), \varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1}), \frac{\varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathcal{X}_{\mathfrak{n}+1})}{2}\right\}. \end{aligned}$$

Note that $\frac{\varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathcal{X}_{\mathfrak{n}+1})}{2} \leq \frac{1}{2}[\varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathcal{X}_\mathfrak{n}) + \varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1})]$, which is smaller than both $\varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathcal{X}_\mathfrak{n})$ and $\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1})$. Then $\mathfrak{M}(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1})$ can be either $\varphi(\mathcal{X}_{\mathfrak{n}-1}, \mathcal{X}_\mathfrak{n})$ or $\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1})$. If $\mathfrak{M}(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1}) = \varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1})$ for some $\mathfrak{n}$, then the expression (2.2) implies that

$$0 < \psi(\varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n})) \leq \phi(\varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n})),$$

which contradicts the condition $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ for $\mathfrak{t} > 0$. Hence $\mathfrak{M}(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1}) = \varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1})$ for all $\mathfrak{n} \geq 1$ and we have

$$0 < \psi(\varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n})) \leq \phi(\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1})) < \psi(\varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1})),$$

which implies

$$\varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n}) < \varphi(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}-1})$$

since ψ is nondecreasing. Thus we conclude that the nonnegative sequence $\varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n})$ is decreasing. Therefore, there exists $\mathfrak{r} \geq 0$ such that $\lim_{\mathfrak{n}\to\infty} \varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n}) = \mathfrak{r}$. Letting $\mathfrak{n} \to \infty$ in (2.2), we get

$$\psi(\mathfrak{r}) \leq \phi(\mathfrak{r}).$$

By the hypothesis of the theorem, since $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ for all $\mathfrak{t} > 0$, this inequality is possible only if $\mathfrak{r} = 0$ and hence

$$\lim_{\mathfrak{n}\to\infty} \varphi(\mathcal{X}_{\mathfrak{n}+1}, \mathcal{X}_\mathfrak{n}) = \mathfrak{r} = 0. \tag{2.3}$$

Next, we will prove that $\{\mathcal{X}_\mathfrak{n}\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{\mathcal{X}_\mathfrak{n}\}$ is not Cauchy. Then, for some $\epsilon > 0$, there exist subsequences $\{\mathcal{X}_{\mathfrak{m}_\mathfrak{k}}\}$ and $\{\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}\}$ of $\{\mathcal{X}_\mathfrak{n}\}$ such that

$$\mathfrak{n}_\mathfrak{k} > \mathfrak{m}_\mathfrak{k} > \mathfrak{k}, \quad \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) \geq \epsilon \tag{2.4}$$

for all $\mathfrak{k} \geq 1$, where, corresponding to each $\mathfrak{m}_\mathfrak{k}$, we can choose $\mathfrak{n}_\mathfrak{k}$ as the smallest integer with $\mathfrak{n}_\mathfrak{k} > \mathfrak{m}_\mathfrak{k}$ for which (2.4) holds. Thus

$$\varphi(\mathcal{X}_{\mathfrak{n}_{\mathfrak{k}-1}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) < \epsilon. \tag{2.5}$$

Employing the triangle inequality and using (2.4) and (2.5), we obtain

$$\epsilon \leq \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) \leq \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}) + \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) < \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}) + \epsilon.$$

Taking the limit as $\mathfrak{k} \to \infty$ and using (2.3), we get

$$\lim_{\mathfrak{k}\to\infty} \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) = \epsilon. \tag{2.6}$$

From the triangular inequality, we also have

$$\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) \leq \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}) + \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) + \varphi(\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}),$$
$$\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) \leq \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}}) + \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) + \varphi(\mathcal{X}_{\mathfrak{m}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}).$$

Taking the limit as $\mathfrak{k} \to \infty$ in the above two inequalities and using (2.3) and (2.6), we get

$$\lim_{\mathfrak{k}\to\infty} \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) = \epsilon. \tag{2.7}$$

In a similar way, we obtain that

$$\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) \leq \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}) + \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}),$$
$$\varphi(\mathcal{X}_{\mathfrak{n}_{\mathfrak{k}-1}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) \leq \varphi(\mathcal{X}_{\mathfrak{n}_{\mathfrak{k}-1}}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}}) + \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}).$$

Taking the limit as $\mathfrak{k} \to \infty$ in the above two inequalities and using (2.3) and (2.6), we get

$$\lim_{\mathfrak{k}\to\infty} \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) = \epsilon. \tag{2.8}$$

In a similar way, we obtain that

$$\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}) \leq \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) + \varphi(\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}}),$$
$$\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) \leq \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}) + \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}).$$

Letting $\mathfrak{k} \to \infty$ and taking into account (2.3) and (2.6), we obtain

$$\lim_{\mathfrak{k}\to\infty} \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) = \epsilon. \tag{2.9}$$

By the definition of $\mathfrak{M}(\mathcal{X}, \mathcal{Y})$ and using the limits found above, we get

$$\lim_{\mathfrak{k}\to\infty} \mathfrak{M}(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}, \mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) = \epsilon. \tag{2.10}$$

Indeed, since

$$\begin{aligned}\mathfrak{M}(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) &= \max\{\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}),\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathfrak{G}\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}),\varphi(\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1},\mathfrak{G}\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}),\\ &\quad \frac{1}{2}\left[\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathfrak{G}\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1})+\varphi(\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1},\mathfrak{G}\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1})\right]\}\\ &= \max\{\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}),\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}),\varphi(\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}}),\\ &\quad \frac{1}{2}\left[\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}})+\varphi(\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{n}_\mathfrak{k}})\right]\},\end{aligned} \tag{2.11}$$

by passing to the limit as $\mathfrak{k}\to\infty$ in (2.11) and using (2.3), (2.6), (2.7), (2.8) and (2.9), we obtain

$$\lim_{\mathfrak{k}\to\infty}\mathfrak{M}(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}) = \max\left\{\epsilon,0,0,\frac{1}{2}(\epsilon+\epsilon)\right\}=\epsilon.$$

Since there exists $\mathcal{X}_0\in\mathfrak{W}$ such that $\mathcal{X}_0\perp\mathfrak{G}\mathcal{X}_0$ and $\alpha(\mathcal{X}_0,\mathfrak{G}\mathcal{X}_0)\geq 1$, by using the condition (iii), we obtain that $\mathcal{X}_1\perp\mathcal{X}_2,\ \alpha(\mathcal{X}_1,\mathcal{X}_2)=\alpha(\mathfrak{G}\mathcal{X}_0,\mathfrak{G}^2\mathcal{X}_0)\geq 1$. By continuing this process, we get

$$\mathcal{X}_\mathfrak{n}\perp\mathcal{X}_{\mathfrak{n}+1},\ \alpha(\mathcal{X}_\mathfrak{n},\mathcal{X}_{\mathfrak{n}+1})\geq 1$$

for all $\mathfrak{n}\in\mathbb{N}\cup\{0\}$. Suppose that $\mathfrak{m}<\mathfrak{n}$. Since

$$\begin{cases}\mathcal{X}_\mathfrak{m}\perp\mathcal{X}_{\mathfrak{m}+1},\ \alpha(\mathcal{X}_\mathfrak{m},\mathcal{X}_{\mathfrak{m}+1})\geq 1\\ \mathcal{X}_{\mathfrak{m}+1}\perp\mathcal{X}_{\mathfrak{m}+2},\ \alpha(\mathcal{X}_{\mathfrak{m}+1},\mathcal{X}_{\mathfrak{m}+2})\geq 1,\end{cases}$$

by the definition of orthogonally triangular α-admissible mapping $\mathfrak{G}$, we have

$$\mathcal{X}_\mathfrak{m}\perp\mathcal{X}_{\mathfrak{m}+2},\ \alpha(\mathcal{X}_\mathfrak{m},\mathcal{X}_{\mathfrak{m}+2})\geq 1.$$

Again, since

$$\begin{cases}\mathcal{X}_\mathfrak{m}\perp\mathcal{X}_{\mathfrak{m}+2},\ \alpha(\mathcal{X}_\mathfrak{m},\mathcal{X}_{\mathfrak{m}+2})\geq 1\\ \mathcal{X}_{\mathfrak{m}+2}\perp\mathcal{X}_{\mathfrak{m}+3},\ \alpha(\mathcal{X}_{\mathfrak{m}+2},\mathcal{X}_{\mathfrak{m}+3})\geq 1,\end{cases}$$

by the definition of orthogonally triangular α-admissible mapping $\mathfrak{G}$, we have

$$\mathcal{X}_\mathfrak{m}\perp\mathcal{X}_{\mathfrak{m}+3},\ \alpha(\mathcal{X}_\mathfrak{m},\mathcal{X}_{\mathfrak{m}+3})\geq 1.$$

By continuing this process, we get $\mathcal{X}_\mathfrak{m}\perp\mathcal{X}_\mathfrak{n},\ \alpha(\mathcal{X}_\mathfrak{m},\mathcal{X}_\mathfrak{n})\geq 1$ and so

$$\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}\perp\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1},\alpha(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1})\geq 1.$$

Therefore, we can apply the condition (2.1) to $\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1}$ and $\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1}$ to obtain

$$0<\psi(\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}}))\leq\alpha(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1})\psi(\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}}))\leq\phi(\mathfrak{M}(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}-1},\mathcal{X}_{\mathfrak{m}_\mathfrak{k}-1})).$$

Letting $\mathfrak{k}\to\infty$ and taking into account (2.6) and (2.10), we have

$$0<\psi(\epsilon)\leq\phi(\epsilon).$$

However, since $\psi(\mathfrak{t})>\phi(\mathfrak{t})$ for $\mathfrak{t}>0$, we deduce that $\epsilon=0$, which contradicts the assumption that $\{\mathcal{X}_\mathfrak{n}\}$ is not a Cauchy sequence. Thus $\{\mathcal{X}_\mathfrak{n}\}$ is Cauchy. Due to the fact that $(\mathfrak{W},\perp,\varphi)$ is an O-complete metric space, there exists $\mathfrak{u}\in\mathfrak{W}$ such that $\lim_{\mathfrak{n}\to\infty}\mathcal{X}_\mathfrak{n}=\mathfrak{u}$. Finally, orthogonally continuity of $\mathfrak{G}$ gives

$$\mathfrak{u}=\lim_{\mathfrak{n}\to\infty}\mathcal{X}_\mathfrak{n}=\lim_{\mathfrak{n}\to\infty}\mathfrak{G}\mathcal{X}_{\mathfrak{n}-1}=\mathfrak{G}\mathfrak{u}.$$

Hence $\mathfrak{u}$ is a fixed point of $\mathfrak{G}$. □

One of the advantages of orthogonally α-admissible mappings is that the the orthogonally continuity is no longer required for the existence of a fixed point provided that the space under consideration has the following property.

(C) If $\{\mathcal{X}_\mathfrak{n}\}$ is an O-sequence in $\mathfrak{W}$ such that

$$\mathcal{X}_\mathfrak{n} \to \mathcal{X}, \quad \mathcal{X}_\mathfrak{n} \perp \mathcal{X}_{\mathfrak{n}+1}, \quad \alpha(\mathcal{X}_\mathfrak{n}, \mathcal{X}_{\mathfrak{n}+1}) \geq 1 \quad \forall\, \mathfrak{n} \in \mathbb{N},$$

then there exists an O-subsequence $\{\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}\}$ of $\{\mathcal{X}_\mathfrak{n}\}$ for which

$$\mathcal{X}_{\mathfrak{n}_\mathfrak{k}} \perp \mathcal{X}, \quad \alpha(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}) \geq 1 \quad \forall\, \mathfrak{k} \in \mathbb{N}.$$

Theorem 2.5. *Let $(\mathfrak{W}, \perp, \varphi)$ be an O-complete metric space and $\mathfrak{G}$ be a self mapping on $\mathfrak{W}$ satisfying the following conditions*

(i) *$\mathfrak{G}$ is $\perp$-preserving;*

(ii) *$\mathfrak{G}$ is an orthogonal ψ-ϕ contraction of type* (A)*;*

(iii) *$\mathfrak{G}$ satisfies the condition* (C)*;*

(iv) *$\mathfrak{G}$ is orthogonally triangular α-admissible;*

(v) *there exists $\mathcal{X}_0 \in \mathfrak{W}$ such that $\mathcal{X}_0 \perp \mathfrak{G}\mathcal{X}_0$ and $\alpha(\mathcal{X}_0, \mathfrak{G}\mathcal{X}_0) \geq 1$.*

Then $\mathfrak{G}$ has a fixed point in $\mathfrak{W}$.

Proof. Following the proof of Theorem 2.4, it is clear that the sequence $\{\mathcal{X}_\mathfrak{n}\}$ defined by $\mathcal{X}_\mathfrak{n} = \mathfrak{G}\mathcal{X}_{\mathfrak{n}-1}$ for $\mathfrak{n} \in \mathbb{N}$, converges to a limit $\mathfrak{u} \in \mathfrak{W}$. The only thing which remains to show that is $\mathfrak{G}\mathfrak{u} = \mathfrak{u}$. Since $\lim_{\mathfrak{n}\to\infty} \mathcal{X}_\mathfrak{n} = \mathfrak{u}$, the condition ($C$) implies $\mathcal{X}_{\mathfrak{n}_\mathfrak{k}} \perp \mathfrak{u}$, $\alpha(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathfrak{u}) \geq 1$ for all $\mathfrak{k} \in \mathbb{N}$. Consequently, the condition (ii) with $\mathcal{X} = \mathcal{X}_{\mathfrak{n}_\mathfrak{k}}$ and $\mathcal{Y} = \mathfrak{u}$ becomes

$$\psi(\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}+1}, \mathfrak{G}\mathfrak{u})) \leq \alpha(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathfrak{u})\psi(\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}+1}, \mathfrak{G}\mathfrak{u})) = \alpha(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathfrak{u})\psi(\varphi(\mathfrak{G}\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathfrak{G}\mathfrak{u})) \leq \phi(\mathfrak{M}(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathfrak{u})),$$

where

$$\mathfrak{M}(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathfrak{u}) = \max\{\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathfrak{u}), \varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}+1}), \varphi(\mathfrak{u}, \mathfrak{G}\mathfrak{u}), \frac{1}{2}\left[\varphi(\mathcal{X}_{\mathfrak{n}_\mathfrak{k}}, \mathfrak{G}\mathfrak{u}) + \varphi(\mathfrak{u}, \mathcal{X}_{\mathfrak{n}_\mathfrak{k}+1})\right]\}.$$

Passing to the limit as $\mathfrak{k} \to \infty$ and taking into account the continuity of ψ and ϕ, we get

$$\psi(\varphi(\mathfrak{u}, \mathfrak{G}\mathfrak{u})) \leq \phi(\varphi(\mathfrak{u}, \mathfrak{G}\mathfrak{u})).$$

From the condition $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ for $\mathfrak{t} > 0$, we conclude that $\varphi(\mathfrak{u}, \mathfrak{G}\mathfrak{u}) = 0$ and hence $\mathfrak{G}\mathfrak{u} = \mathfrak{u}$, which completes the proof. □

Similar results can be stated for a mapping $\mathfrak{G} : \mathcal{X} \to \mathcal{X}$ in the orthogonal ψ-ϕ contraction of type (B). More precisely, the conditions for existence of a fixed point of a mapping in orthogonal ψ-ϕ contraction of type (B) are given in the next two theorems.

Theorem 2.6. *Let $(\mathfrak{W}, \perp, \varphi)$ be an O-complete metric space and $\mathfrak{G}$ be a self mapping on $\mathfrak{W}$ satisfying the following conditions:*

(i) $\mathfrak{G}$ *is* $\perp$*-preserving;*

(ii) $\mathfrak{G}$ *is an orthogonal* ψ*-*ϕ *contraction of type* (B)*;*

(iii) $\mathfrak{G}$ *is orthogonally triangular* α*-admissible;*

(iv) there exists $\mathcal{X}_0 \in \mathfrak{W}$ *such that* $\mathcal{X}_0 \perp \mathfrak{G}\mathcal{X}_0$ *and* $\alpha(\mathcal{X}_0, \mathfrak{G}\mathcal{X}_0) \geq 1$*;*

(v) $\mathfrak{G}$ *is orthogonally continuous.*

Then $\mathfrak{G}$ *has a fixed point in* $\mathfrak{W}$*.*

Theorem 2.7. *Let* $(\mathfrak{W}, \perp, \varphi)$ *be an O-complete metric space and* $\mathfrak{G}$ *be a self mapping on* $\mathfrak{W}$ *satisfying the following conditions:*

(i) $\mathfrak{G}$ *is* $\perp$*-preserving;*

(ii) $\mathfrak{G}$ *is an orthogonal* ψ*-*ϕ *contraction of type* (B)*;*

(iii) $\mathfrak{G}$ *satisfies the condition* (C)*;*

(iv) $\mathfrak{G}$ *is orthogonally triangular* α*-admissible;*

(v) there exists $\mathcal{X}_0 \in \mathfrak{W}$ *such that* $\mathcal{X}_0 \perp \mathfrak{G}\mathcal{X}_0$ *and* $\alpha(\mathcal{X}_0, \mathfrak{G}\mathcal{X}_0) \geq 1$.

Then $\mathfrak{G}$ *has a fixed point in* $\mathfrak{W}$*.*

Note that the proofs of Theorems 2.6 and 2.7 can be easily done by mimicking the proofs of Theorems 2.4 and 2.5, respectively.

Next, we discuss the conditions for the uniqueness of the fixed point. A sufficient condition for the uniqueness of the fixed point in Theorems 2.6 and 2.7 can be stated as follows:

(D) For $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$, there exists $\mathcal{Z} \in \mathfrak{W}$ such that

$$\mathcal{X} \perp \mathcal{Z} \,\&\, \mathcal{Y} \perp \mathcal{Z}, \;\; \alpha(\mathcal{X}, \mathcal{Z}) \geq 1, \;\; \alpha(\mathcal{Y}, \mathcal{Z}) \geq 1.$$

Note, however, that this condition is not sufficient for the uniqueness of fixed point.

Theorem 2.8. *If the condition* (D) *is added to the hypothesis of Theorem 2.6 (respectively Theorem 2.7), then the fixed point of* $\mathfrak{W}$ *is unique.*

Proof. Since $\mathfrak{W}$ satisfies the hypothesis of Theorem 2.6 (respectively, Theorem 2.7), the fixed point of $\mathfrak{W}$ exists. Suppose that we have two different fixed points, say, $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$. From the condition (D), there exists $\mathcal{Z} \in \mathfrak{W}$ such that

$$\mathcal{X} \perp \mathcal{Z} \,\&\, \mathcal{Y} \perp \mathcal{Z}, \;\; \alpha(\mathcal{X}, \mathcal{Z}) \geq 1, \;\; \alpha(\mathcal{Y}, \mathcal{Z}) \geq 1. \tag{2.12}$$

Since $\mathfrak{G}$ is $\perp$-preserving and orthogonally triangular α-admissible, we have from (2.12)

$$\mathcal{X} \perp \mathfrak{G}^{\mathfrak{n}}\mathcal{Z}, \; \alpha(\mathcal{X}, \mathfrak{G}^{\mathfrak{n}}\mathcal{Z}) \geq 1 \text{ and } \mathcal{Y} \perp \mathfrak{G}^{\mathfrak{n}}\mathcal{Z}, \; \alpha(\mathcal{Y}, \mathfrak{G}^{\mathfrak{n}}\mathcal{Z}) \geq 1, \quad \forall \mathfrak{n} \in \mathbb{N}.$$

Thus, for the sequence $\{\mathcal{Z}_{\mathfrak{n}}\} \in \mathfrak{W}$ defined as $\mathcal{Z}_{\mathfrak{n}} = \mathfrak{G}^{\mathfrak{n}}\mathcal{Z}$, we have

$$0 < \psi(\varphi(\mathcal{X}, \mathcal{Z}_{\mathfrak{n}+1})) \leq \alpha(\mathcal{X}, \mathcal{Z}_{\mathfrak{n}})\psi(\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Z}_{\mathfrak{n}})) \leq \phi(\mathfrak{R}(\mathcal{X}, \mathcal{Z}_{\mathfrak{n}})), \tag{2.13}$$

where

$$\begin{aligned}\mathfrak{R}(\mathcal{X},\mathcal{Z}_{\mathfrak{n}}) &= \max\{\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}}),\frac{1}{2}\left[\varphi(\mathcal{X},\mathfrak{G}\mathcal{X})+\varphi(\mathcal{Z}_{\mathfrak{n}},\mathfrak{G}\mathcal{Z}_{\mathfrak{n}})\right],\frac{1}{2}\left[\varphi(\mathcal{X},\mathfrak{G}\mathcal{Z}_{\mathfrak{n}})+\varphi(\mathcal{Z}_{\mathfrak{n}},\mathfrak{G}\mathcal{X})\right]\}\\ &= \max\{\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}}),\frac{1}{2}\left[\varphi(\mathcal{Z}_{\mathfrak{n}},\mathcal{Z}_{\mathfrak{n}+1})\right],\frac{1}{2}\left[\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1})+\varphi(\mathcal{Z}_{\mathfrak{n}},\mathcal{X})\right]\}.\end{aligned}$$

Observe that $\frac{\varphi(\mathcal{Z}_{\mathfrak{n}},\mathcal{Z}_{\mathfrak{n}+1})}{2} \leq \frac{1}{2}\left[\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1})+\varphi(\mathcal{Z}_{\mathfrak{n}},\mathcal{X})\right]$. Thus we deduce that $\mathfrak{R}(\mathcal{X},\mathcal{Y}) = \max\{\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}}),\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1})\}$. Without loss of generality, we may assume that $\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}}) > 0$ for all $\mathfrak{n}\in\mathbb{N}$. If $\mathfrak{R}(\mathcal{X},\mathcal{Y}) = \varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1})$, then the inequality (2.13) becomes

$$0 < \psi(\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1})) \leq \alpha(\mathcal{X},\mathcal{Z}_{\mathfrak{n}})\psi(\varphi(\mathfrak{G}\mathcal{X},\mathfrak{G}\mathcal{Z}_{\mathfrak{n}})) \leq \phi(\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1})) < \psi(\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1})).$$

This is a contradiction. So we have $\mathfrak{R}(\mathcal{X},\mathcal{Y}) = \varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}})$ for all $\mathfrak{n}\in\mathbb{N}$, which implies

$$0 < \psi(\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1})) \leq \alpha(\mathcal{X},\mathcal{Z}_{\mathfrak{n}})\psi(\varphi(\mathfrak{G}\mathcal{X},\mathfrak{G}\mathcal{Z}_{\mathfrak{n}})) \leq \phi(\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}})) < \psi(\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}})), \tag{2.14}$$

due to the fact that $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ for $\mathfrak{t} > 0$. On the other hand, since ψ is nondecreasing, $\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}+1}) \leq \varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}})$ for all $\mathfrak{n}\in\mathbb{N}$. Thus the O-sequence $\{\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}})\}$ is a positive nonincreasing sequence and hence the sequence converges to a limit, say, $\mathfrak{L} \geq 0$. Taking the limit as $\mathfrak{n}\to\infty$ in (2.14) and regarding the orthogonally continuity of ψ and ϕ, we deduce

$$0 \leq \psi(\mathfrak{L}) \leq \phi(\mathfrak{L}),$$

which is possible only if $\mathfrak{L} = 0$. Hence we conclude that

$$\lim_{\mathfrak{n}\to\infty}\varphi(\mathcal{X},\mathcal{Z}_{\mathfrak{n}}) = 0. \tag{2.15}$$

In a similar way, we obtain

$$\lim_{\mathfrak{n}\to\infty}\varphi(\mathcal{Y},\mathcal{Z}_{\mathfrak{n}}) = 0. \tag{2.16}$$

From (2.15) and (2.16), it follows that $\mathcal{X} = \mathcal{Y}$, which completes the proof of the uniqueness. □

Theorem 2.9. *Let* $(\mathfrak{W},\perp,\varphi)$ *be an O-complete metric space and* $\mathfrak{G}$ *be a self mapping on* $\mathfrak{W}$ *satisfying the following conditions:*

(i) $\mathfrak{G}$ *is* $\perp$*-preserving;*

(ii) there exists $\alpha : \mathfrak{W}\times\mathfrak{W}\to[0,\infty)$ *such that, for all* $\mathcal{X},\mathcal{Y}\in\mathfrak{W}$ *with* $\mathcal{X}\perp\mathcal{Y}$,

$$\varphi(\mathfrak{G}\mathcal{X},\mathfrak{G}\mathcal{Y}) > 0 \Rightarrow \alpha(\mathcal{X},\mathcal{Y})\psi(\varphi(\mathfrak{G}\mathcal{X},\mathfrak{G}\mathcal{Y})) \leq \phi(\varphi(\mathcal{X},\mathcal{Y})),$$

where ψ *is an orthogonal altering distance function and* $\phi\colon [0,+\infty)\to[0,+\infty)$ *is an orthogonal continuous function satisfying* $\psi(\mathfrak{t}) > \phi(\mathfrak{t})$ *for all* $\mathfrak{t} > 0$*;*

(iii) $\mathfrak{G}$ *satisfies the condition* (C)*;*

(iv) $\mathfrak{G}$ *is orthogonally triangular* α*-admissible;*

(v) *there exists* $\mathcal{X}_0 \in \mathfrak{W}$ *such that* $\mathcal{X}_0 \perp \mathfrak{G}\mathcal{X}_0$ *and* $\alpha(\mathcal{X}_0, \mathfrak{G}\mathcal{X}_0) \geq 1$.

Then $\mathfrak{G}$ *has a fixed point in* $\mathfrak{W}$. *If, in addition,* $\mathfrak{G}$ *satisfies the condition* (D)*, then the fixed point is unique.*

The proof of Theorem 2.9 can be done by following the lines of proofs of Theorems 2.4, 2.5, and 2.8. Hence it is omitted.

Corollary 2.10. *Let* $(\mathfrak{W}, \perp, \varphi)$ *be an O-complete metric space and* $\mathfrak{G}$ *be a self mapping on* $\mathfrak{W}$ *satis fying the following conditions:*

(i) $\mathfrak{G}$ *is* $\perp$*-preserving;*

(ii) *there exists* $\alpha : \mathfrak{W} \times \mathfrak{W} \to [0, \infty)$ *such that, for all* $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$ *with* $\mathcal{X} \perp \mathcal{Y}$*,*

$$\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Y}) > 0 \Rightarrow \alpha(\mathcal{X}, \mathcal{Y})\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Y}) \leq k\mathfrak{M}(\mathcal{X}, \mathcal{Y}),$$

$$\text{where } 0 < k < 1 \text{ and } \mathfrak{M}(\mathcal{X}, \mathcal{Y}) = \max\{\varphi(\mathcal{X}, \mathcal{Y}), \varphi(\mathcal{X}, \mathfrak{G}\mathcal{X}), \varphi(\mathcal{Y}, \mathfrak{G}\mathcal{Y}), \frac{1}{2}[\varphi(\mathcal{X}, \mathfrak{G}\mathcal{Y}) + \varphi(\mathcal{Y}, \mathfrak{G}\mathcal{X})]\};$$

(iii) $\mathfrak{G}$ *is orthogonally triangular* α*-admissible;*

(iv) *there exists* $\mathcal{X}_0 \in \mathfrak{W}$ *such that* $\mathcal{X}_0 \perp \mathfrak{G}\mathcal{X}_0$ *and* $\alpha(\mathcal{X}_0, \mathfrak{G}\mathcal{X}_0) \geq 1$*;*

(v) $\mathfrak{G}$ *is orthogonally continuous.*

Then $\mathfrak{G}$ *has a fixed point in* $\mathfrak{W}$.

Proof. The proof is obvious by choosing $\psi(t) = t$ and $\phi(t) = kt$ in Theorem 2.4. □

Example 2.11. Let $\mathfrak{W} = [0, \infty)$ with usual metric $\varphi(\mathcal{X}, \mathcal{Y}) = |\mathcal{X} - \mathcal{Y}|$. Suppose $\mathcal{X} \perp \mathcal{Y}$ if $\mathcal{X}, \mathcal{Y} \geq 0$. It is easy to see that $(\mathfrak{W}, \perp, \varphi)$ is an O-complete metric space. Define $\mathfrak{G} : \mathfrak{W} \to \mathfrak{W}$ and $\alpha : \mathfrak{W} \times \mathfrak{W} \to [0, \infty)$ by $\mathfrak{G}(\mathcal{X}) = \frac{\mathcal{X}}{\sqrt{2}\sqrt{1+\mathcal{X}}}$ for all $\mathcal{X} \in \mathfrak{W}$ and $\alpha(\mathcal{X}, \mathcal{Y}) = 1$ for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$.
Take the orthogonal altering functions $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$ with $\psi(t) > \phi(t)$ for all $t > 0$. Then $\mathfrak{G}$ is an orthogonally triangular α-admissible. Clearly, $\mathfrak{G}$ is $\perp$-preserving and orthogonally continuous. For all $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$ with $\mathfrak{G}\mathcal{X} \neq \mathfrak{G}\mathcal{Y}$, we obtain

$$\begin{aligned}\alpha(\mathcal{X}, \mathcal{Y})\psi(\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Y})) = \psi(|\mathfrak{G}\mathcal{X} - \mathfrak{G}\mathcal{Y}|) &= \left|\frac{\mathcal{X}}{\sqrt{2}\sqrt{1+\mathcal{X}}} - \frac{\mathcal{Y}}{\sqrt{2}\sqrt{1+\mathcal{Y}}}\right| \leq \frac{1}{2}\left|\mathcal{X} - \mathcal{Y}\right| = \phi(\left|\mathcal{X} - \mathcal{Y}\right|) \\ &= \phi(\varphi(\mathcal{X}, \mathcal{Y})).\end{aligned}$$

Hence all the conditions of Theorem 2.9 are satisfied and so $\mathfrak{G}$ has a unique fixed point $\mathcal{X} = 0$.

3. Application

Let $\mathfrak{W} = C[\leftthreetimes_1, \leftthreetimes_2]$ be a set of all real continuous functions on $[\leftthreetimes_1, \leftthreetimes_2]$ equipped with metric $\varphi(\mathcal{X}, \mathcal{Y}) = |\mathcal{X} - \mathcal{Y}|$ for all $\mathcal{X}, \mathcal{Y} \in C[\leftthreetimes_1, \leftthreetimes_2]$. Then $(\mathfrak{W}, \varphi)$ is a complete metric space. Define the orthogonality relation $\perp$ on $\mathfrak{W}$ by

$$\mathcal{X} \perp \mathcal{Y} \iff \mathcal{X}(\sqcup)\mathcal{Y}(\sqcup) \geq \mathcal{X}(\sqcup) \quad or \quad \mathcal{X}(\sqcup)\mathcal{Y}(\sqcup) \geq \mathcal{Y}(\sqcup), \forall \sqcup \in [\leftthreetimes_1, \leftthreetimes_2].$$

Now, we consider the nonlinear Fredholm integral equation

$$\mathcal{X}(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1}\int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}(\sqcup, \mathfrak{s}, \mathcal{X}(\mathfrak{s}))ds,$$

where $\sqcup, \mathfrak{s} \in [\leftthreetimes_1, \leftthreetimes_2]$. Assume that $\mathfrak{K} : [\leftthreetimes_1, \leftthreetimes_2] \times [\leftthreetimes_1, \leftthreetimes_2] \times \mathcal{X} \to \mathbb{R}$ and $\mathfrak{v} : [\leftthreetimes_1, \leftthreetimes_2] \to \mathbb{R}$ are continuous, where $\mathfrak{v}(\sqcup)$ is a given function in $\mathfrak{W}$.

Theorem 3.1. *Suppose that* $(\mathfrak{W}, d)$ *is an orthogonal metric space equipped with metric* $\varphi(\mathcal{X}, \mathcal{Y}) = |\mathcal{X} - \mathcal{Y}|$ *for all* $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$ *and* $\mathfrak{G} : \mathfrak{W} \to \mathfrak{W}$ *is an orthogonal continuous operator on* $\mathfrak{W}$ *defined by*

$$\mathfrak{G}\mathcal{X}(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1}\int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}(\sqcup, \mathfrak{s}, \mathcal{X}(\mathfrak{s}))ds. \tag{3.1}$$

If there exists $\wr > 0$ *such that for all* $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$ *with* $\mathcal{X} \neq \mathcal{Y}$ *and* $\mathfrak{s}, \sqcup \in [\leftthreetimes_1, \leftthreetimes_2]$*, the following inequality*

$$|\mathfrak{K}(\sqcup, \mathfrak{s}, \mathfrak{G}\mathcal{X}(\mathfrak{s})) - \mathfrak{K}(\sqcup, \mathfrak{s}, \mathfrak{G}\mathcal{Y}(\mathfrak{s}))| \leq \frac{|\mathcal{X} - \mathcal{Y}|}{2}$$

holds, then the integral operator defined by (3.1) *has a unique solution*

Proof. We define $\alpha : \mathfrak{W} \times \mathfrak{W} \to [0, \infty)$ such that $\alpha(\mathcal{X}, \mathcal{Y}) = 1$ for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$. Then $\mathfrak{G}$ is orthogonally triangular α-admissible. Now, we show that $\mathfrak{G}$ is $\perp$-preserving. For each $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$ with $\mathcal{X} \perp \mathcal{Y}$ and $\sqcup \in [a, b]$, we have

$$\mathfrak{G}\mathcal{X}(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\leftthreetimes_2 - \leftthreetimes_1}\int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}(\sqcup, \mathfrak{s}, \mathcal{X}(\mathfrak{s}))ds \geq 1.$$

Accordingly, $[(\mathfrak{G}\mathcal{X})(\sqcup)][(\mathfrak{G}\mathcal{Y})(\sqcup)] \geq (\mathfrak{G}\mathcal{Y})(\sqcup)$ and so $(\mathfrak{G}\mathcal{Y})(\sqcup) \perp (\mathfrak{G}\mathcal{Y})(\sqcup)$. Thus $\mathfrak{G}$ is $\perp$-preserving. Take the orthogonal altering functions $\psi(\mathrm{t}) = \mathrm{t}$ and $\phi(\mathrm{t}) = \frac{\mathrm{t}}{2}$ with $\psi(\mathrm{t}) > \phi(\mathrm{t})$ for all $\mathrm{t} > 0$.

Let $\mathcal{X}, \mathcal{Y} \in \mathfrak{W}$ with $\mathcal{X} \perp \mathcal{Y}$. Suppose that $\mathfrak{G}(\mathcal{X}) \neq \mathfrak{G}(\mathcal{Y})$. Using (3.1), we get

$$\begin{aligned}
\alpha(\mathcal{X}, \mathcal{Y})\psi(\varphi(\mathfrak{G}\mathcal{X}, \mathfrak{G}\mathcal{Y})) &= \psi(|\mathfrak{G}\mathcal{X} - \mathfrak{G}\mathcal{Y}|) = |\mathfrak{G}\mathcal{X} - \mathfrak{G}\mathcal{Y}| \\
&= \frac{1}{|\leftthreetimes_2 - \leftthreetimes_1|}\left|\int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}(\sqcup, \mathfrak{s}, \mathfrak{G}\mathcal{X}(\mathfrak{s}))ds - \int_{\leftthreetimes_1}^{\leftthreetimes_2} \mathfrak{K}(\sqcup, \mathfrak{s}, \mathfrak{G}\mathcal{Y}(\mathfrak{s}))ds\right| \\
&\leq \frac{1}{|\leftthreetimes_2 - \leftthreetimes_1|}\int_{\leftthreetimes_1}^{\leftthreetimes_2} |\mathfrak{K}(\sqcup, \mathfrak{s}, \mathfrak{G}\mathcal{X}(\mathfrak{s})) - \mathfrak{K}(\sqcup, \mathfrak{s}, \mathfrak{G}\mathcal{Y}(\mathfrak{s}))|ds \\
&\leq \frac{1}{|\leftthreetimes_2 - \leftthreetimes_1|}\int_{\leftthreetimes_1}^{\leftthreetimes_2} \frac{|\mathcal{X} - \mathcal{Y}|}{2}ds \\
&= \frac{|\mathcal{X} - \mathcal{Y}|}{2} \\
&= \phi(\varphi(\mathcal{X}, \mathcal{Y})).
\end{aligned}$$

Hence all the conditions of Theorem 2.9 are satisfied and so the integral operator $\mathfrak{G}$ defined by (3.1) has a unique solution. □

4. Conclusions

The idea of new orthogonal ψ-ϕ contraction of type (A) and new orthogonal ψ-ϕ contraction of type (B) in O-complete metric spaces was introduced in this article and some fixed point theorems were demonstrated. An illustrative example was provided to show the validity of the hypothesis and the degree of usefulness of our findings.

Conflict of interest

The authors declare that they have no competing interest.

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