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Research article

The Hausdorff dimension of the Julia sets concerning generated renormalization transformation

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Abstract: Considering a family of rational map $U_{mn\lambda}$ of the renormalization transformation of the generalized diamond hierarchical Potts model, we give the asymptotic formula of the Hausdorff dimension of the Julia sets of $U_{mn\lambda}$ as the parameter λ tends to infinity, here

$$U_{mn\lambda} = \left[\frac{(z+\lambda-1)^n + (\lambda-1)(z-1)^n}{(z+\lambda-1)^n - (z-1)^n} \right]^m,$$

where $m \geq 2,\ n \geq 2$ are two natural numbers, $\lambda \in \mathbb{C}$.

Keywords: renormlization transformation; Potts model; asymptotic formula; Hausdorff dimension; Julia set
Mathematics Subject Classification: 37F10, 37F45

1. Introduction

It is an important aspect of statistical mechanics to reveal the nature of phase transition by establishing statistical mechanical model. In fact, the statistical mechanical models on hierarchical lattices have exhibit a deep connection between their limiting sets of the zeros of the partition functions and the Julia sets of rational maps in complex dynamics [1–7]. In 1952, Yang and Lee [8, 9] proved the celebrated Unite circle theorem. This theorem deals with the analytic continuation of the free energy on the complex plane. Here the free energy means the logarithm of the partition function. They proved the famous circle theorem in an exact mathematical way for an Ising ferromagnet model in statistical mechanics, which asserts that the zeros of the partition function for some magnetic materials lie on the unit circle in the complex plane. An important problem stated in Lee-Yang's paper is to study the limit distribution of zeros of the function (Lee-Yang zeros). Here the complex singularities of free energy lie on this unit circle.

In 1965, Fisher [3, 10] initiated the investigation of zeros of the partition function in the complex temperature plane (Fisher zeros). However, compared with the Lee-Yang zeros, Fisher zeros do not lie on the unit circle any more.

After that, people investigated various properties of point distribution of zeros of partition function of ferromagnet model and antimagnetic model. In 1983, Derrida et al. [4] found fractal patterns in λ-state Potts model in diamand lattice. In fact, by Migdal-Kadanoff renormalization group theory, they proved that the limit distribution of this physical model are dense in the Julia set $J(U_{22\lambda})$ of a family of rational map $U_{22\lambda}$, here

$$U_{22\lambda}(z) = \left(\frac{z^2 + \lambda - 1}{2z + \lambda - 2}\right)^2, \tag{1.1}$$

with $\lambda \in \mathbb{N}$ is a positive integer. After this, it was shown that the model for non-integer λ may be describe properties of some physical system. Many examples of physical model show the limit of zeros of partition function are located in the Julia set of the family of rational function [1, 5–7, 11–14]. Bleher and Lyubich [15] investigated the analytic continuation of free energy or complex temperature plane for Ising model on diamond-like hierarchical lattices. For general models, an important problem in [15] is that how are the limit set of zeros of partition function and what is their global structure in complex space?

In recent years, many works have been devoted to the dynamics of a family of rational maps $U_{2n\lambda}$ of λ-state diamond-like hierarchical Potts models. Recently, for a λ-state Potts model on a generalized diamond hierarchical, Qiao [16] proved the limit set of the zeros of the partition function is indeed the Julia set $J(U_{mn\lambda})$ of a family of rational map $U_{mn\lambda}$. Here

$$U_{mn\lambda} = \left[\frac{(z + \lambda - 1)^n + (\lambda - 1)(z - 1)^n}{(z + \lambda - 1)^n - (z - 1)^n}\right]^m, \tag{1.2}$$

where $m \geq 2,\ n \geq 2$ are two natural numbers, $\lambda \in \mathbb{R}\setminus\{0\}$. The standard diamond lattice $U_{22\lambda}$ and the diamond-like lattice $U_{2n\lambda}$ are the special cases of $U_{mn\lambda}$.

It is well known that the research on the Hausdorff dimension of the Julia set is an important topic in complex dynamics and fractal theory. Many works had devoted to the asymptotic formula about the Hausdorff dimension of the Julia set.

The first heart-stirring formula on the Hausdorff dimension of Julia sets was due to Ruelle [22]. For polynomials $P_c = z^d + c$ with degree $d \geq 2$, he proved if c is small the Hausdorff dimension $\dim_H(J_c)$ of the Julia set J_c of P_c is given by

$$\dim_H(J_c) = 1 + \frac{|c|^2}{4\log d} + O\left(|c|^3\right). \tag{1.3}$$

Moreover, Widom et al. [17] improve Ruelle's result and obtain

$$\dim_H(J_c) = 1 + \frac{|c|^2}{4\log d} + \delta_{d,2}\frac{3\left(c^2\overline{c} + \overline{c}^2 c\right)}{16\log d} + O\left(|c|^4\right). \tag{1.4}$$

In 2012, Yang and Wang [18] use the iterated function system to show the Hausdorff dimension of the boundary of the immediate basin of infinity of the McMullen maps $f_p(z) = z^Q + p\backslash z^Q$. They proved that if $Q \geq 3$, then for sufficiently small p such that $J\left(f_p\right)$ is a Cantor circle, the Hausdorff dimension $\dim_H\left(\partial B_p\right)$ of ∂B_p is

$$\dim_H\left(\partial B_p\right) = 1 + \frac{|p|^2}{\log Q} + O\left(|p|^3\right). \tag{1.5}$$

For $U_{mn\lambda}$ defined as (1.2), $J(U_{mn\lambda}(z))$ is the Julia sets of $U_{mn\lambda}$ and $\dim_H(J_{mn\lambda})$ is the Hausdorff dimension of $J(U_{mn\lambda}(z))$. For $m = n = 2$, Osbaldestin [19] gives the following asymptotic formula for sufficiently large $|\lambda|$

$$\dim_H(J_{22\lambda}) = 1 + \frac{|\lambda|^{-\frac{2}{3}}}{4\log 2} + O\left(|\lambda|^{-1}\right). \tag{1.6}$$

Moreover, Yang and Zeng [20] show the following asymptotic formula for $m = n = d \geq 2$

$$\dim_H(J_{nn\lambda}) = 1 + \frac{|\lambda|^{-\frac{2}{d+1}}}{4\log d} + O\left(|\lambda|^{-\frac{3}{d+1}}\right). \tag{1.7}$$

Furthermore, Gao [21] obtains the following result for sufficiently large $|\lambda|$

$$\dim_H(J_{m2\lambda}) = \begin{cases} 1 + \frac{|\lambda|^{-\frac{2}{3}}}{4\log 2} + O\left(|\lambda|^{-1}\right), & for\ \ m = 2, \\ 1 + \frac{|\lambda|^{-\frac{2}{2m-1}}}{4\log(2m)} + O\left(|\lambda|^{-\frac{3}{2m-1}}\right), & for\ \ m \geq 3. \end{cases} \tag{1.8}$$

Because of the complexity of the parameters in $U_{mn\lambda}$, it is difficult to get the $\dim_H(J_{mn\lambda})$ of the Julia sets $J(U_{mn\lambda}(z))$. In this paper, we investigate the $\dim_H(J_{mn\lambda})$ and obtain the following results.

Theorem 1. *Suppose $|\lambda|$ is sufficiently large, then the Hausdorff dimension of $J(U_{mn\lambda})$ is given by the following asymptotic formula, i.e.*

$$\dim_H(J_{mn\lambda}) = \begin{cases} 1 + \frac{|\lambda|^{-\frac{2(m-1)}{mn-1}}}{4\log(mn)} + O\left(|\lambda|^{-\frac{p}{mn-1}}\right), & for\ \ m < n, \\ 1 + \frac{|\lambda|^{-\frac{2(n-1)}{mn-1}}}{4\log(mn)} + O\left(|\lambda|^{-\frac{q}{mn-1}}\right), & for\ \ m > n, \end{cases} \tag{1.9}$$

where p, q are two natural numbers related to m and n.

Corollary 1. *If $m = n \geq 2$, we can get that*

$$\dim_H(J(U_{mn\lambda})) = 1 + \frac{|\lambda|^{-\frac{2}{n+1}}}{4\log n} + O\left(|\lambda|^{-\frac{3}{n+1}}\right). \tag{1.10}$$

Remark. Yang and Wang [18] proved the same result of Corollary 1 by the factorization [16] of $U_{mn\lambda}$.

2. Perturbation theorems

Qiao [16] has dealt with topological properties of the Fatou components of $U_{mn\lambda}$. It is proved that all components of the Fatou set of $U_{mn\lambda}$ are Jordan domains with at most one exception which is a completely invariant domain. When $|\lambda|$ is large enough, it was shown that the Julia set $J(U_{mn\lambda})$ is actually a quasicircle. In this case the Fatou set $F(U_{mn\lambda})$ consists of two Jordan domains. Qiao [16] has given the following theorem.

Theorem 2.1. *For any given natural numbers $m \geq 2$ and $n \geq 2$, there exists a constant $\lambda_0 > 0$ such that $J(U_{mn\lambda})$ is a quasicircle when $|\lambda| > \lambda_0$. Furthermore, there exists an annulus $H_{mn\lambda} = \{z \,|r_{mn\lambda} < |z| < R_{mn\lambda}\}$ satisfying* $\mathrm{mod}\,(H_\lambda) \to 0$ *as $|\lambda| \to +\infty$ such that*

$$J(U_{mn\lambda}) \subset H_{mn\lambda}, \quad \dim_H(J(U_{mn\lambda})) \to 1 \quad as \quad |\lambda| \to +\infty.$$

If the parameter λ lies in the unbounded capture domain H_0, then the Julia set $J_{mn\lambda}$ is a quasicircle. In this case, $J_{mn\lambda}$ moves holomorphically in H_0 and its Hausdorff dimension depends real analytically on λ by a classic result of Ruelle [22]. The following Theorem 2.2 is a weak version of [22].

Theorem 2.2. *[20] Let $f_\lambda : \Lambda \times C$ be a holomorphic family of hyperbolic rational maps parameterized by Λ, where Λ is a complex manifold. Then the Hausdorff dimension of the Julia set of f_λ depends real analytically on $\lambda \in \Lambda$.*

Definition 2.1. *[20] Let V be a closed subset of $\mathbb{R}^n$. A map $S : V \to V$ is called a contraction on V if there exists a real number $c \in (0,1)$ such that $|S(x) - S(y)| \leq c\,|x - y|$ for all $x, y \in V$. A finite family of contractions $\{S_1, \ldots, S_m\}$ defined on $V \subset \mathbb{R}^n$, with $m \geq 2$, is called an iterated function system or IFS in short.*

To compute the Hausdorff dimension of $J_{mn\lambda}$ with $\lambda \in \Lambda$, we need the following results.

Theorem 2.3. *[23] Let $\{S_1, \ldots, S_m\}$ be an IFS on a closed set $\Omega \subset \mathbb{R}^n$ such that $|S_i(x) - S_i(y)| \leq c_i\,|x - y|$ with $0 < c_i < 1$. Then*
(1) There exists a unique non-empty compact set J such that $J = \cup_{i=1}^m S_i(J)$.
(2) The Hausdorff dimension $\dim_H(J)$ *of J satisfies* $\dim_H(J) \leq s$, *where $\sum_{i=1}^m c_i^s = 1$.*
(3) If we require further $|S_i(x) - S_i(y)| \geq b_i\,|x - y|$ for $0 < b_i < 1$, then $\dim_H(J) \geq s'$, *where $\sum_{i=1}^m b_i^{s'} = 1$.*

The non-empty compact set J appearing in Theorem 2.3(1) is called the *attractor* of the IFS $\{S_1, \ldots, S_m\}$.

3. Conjugation and solutions

In order to proof Theorem 1, we do some setting first.

Let $v = \lambda^{-\frac{1}{mn-1}}$, $\varphi_v = v^{m(n-1)}(z-1)$. Then $\lambda v^{mn} = v$. We define a new rational map with parameter v as

$$f_v(z) = \varphi_v \circ U_{mn\lambda} \circ \varphi_v^{-1}(z), \tag{3.1}$$

we have

$$f_v(z) = \frac{\left[z^n + v^{n-1}\left(\left(zv^{m-1}+1\right)^n - \left(zv^{m-1}\right)^n\right)\right]^m - \left[v^{n-1}\left(\left(zv^{m-1}+1\right)^n - \left(zv^{m-1}\right)^n\right)\right]^m}{\left[(zv^{m-1}+1)^n - (zv^{m-1})^n\right]^m}. \tag{3.2}$$

Then the family $\{U_{mn\lambda} : \lambda \in U_\infty^* = U_\infty \backslash \{\infty\}\}$ becomes $\left\{f_v : v \in V_0^* = V_0 \backslash \{0\}\right\}$ for sufficiently large λ, where U_∞ and V_0 is a neighborhood of ∞ and 0 respectively. Furthermore, we can assume that the map $v \to \lambda = v^{-mn+1}$ is a proper map with degree $mn - 1$ from V_0^* to U_∞^*.

Since for any $\varepsilon \in (0,1)$, there exists $\delta > 0$ such that when $|v| < \delta$, we have $f_v(\mathbb{D}_{1-\varepsilon}) \subset \mathbb{D}_{1-\varepsilon}$, $f_v\left(\overline{\mathbb{C}} \backslash \overline{\mathbb{D}}_{1-\varepsilon}\right) \subset \overline{\mathbb{C}} \backslash \mathbb{D}_{1-\varepsilon}$, where $\mathbb{D}_r = \{z : |z| < r\}$. Hence $\mathbb{D}_{1-\varepsilon} \subset F(f_v)$ and $\overline{\mathbb{D}}_{1-\varepsilon} \subset F(f_v)$, where $F(f_v)$ is the Fatou sets of f_v. So we conclude $\sigma_H(J(f_v), \mathbb{S}) \leq \varepsilon$, $\mathbb{S}$ is the unit circle. Where $\sigma_H(X, Y)$ is the Hausdorff distance of two compact sets X and Y is defined by $\sigma_H(X, Y) = \max\left\{\max_{x\in X} \sigma(x, Y), \max_{y\in Y} \sigma(X, y)\right\}$, $\sigma(\cdot, \cdot)$ denotes the spherical distance. This implies that the Julia sets $J(f_v)$ move continuously at $v = 0$ in the Hausdorff topology. So we get $\sigma_H(J(f_v), \mathbb{S}) \to 0$ as $v \to 0$.

It is obvious that the Julia set $J(f_v)$ moves continuously on V_0^* in the Hausdorff topology since f_v is hyperbolic for $v \in V_0^*$. Then the Julia set $J(f_v)$ moves continuously on V_0 in the Hausdorff topology by adding an new map $f_v(z) = z^{mn}$ to the family $\left\{f_v : v \in V_0^*\right\}$. By characterizations of stability [24], the Julia set $J(f_v)$ moves holomorphically on V_0. So there is a holomorphic motion $h : V_0 \times \mathbb{S} \to \overline{\mathbb{C}}$ parameterized by V_0 with base point 0 such that $h(0, \cdot)$ is identity and $h(V, \mathbb{S}) = J(f_v)$ for all $v \in V_0$. The above discussion implies that $J(f_v)$ is a quasicircle for sufficiently small v.

Note that the Hausdorff dimension is invariant under a conformal isomorphism. This means that we only need to calculate the Hausdorff dimension of the Julia set $J(f_v)$ of f_v since $\dim_H(J(f_v)) = \dim_H(J(U_{mn\lambda}))$.

Note that the Julia set J_v of f_v is the unit circle if $v = 0$. For $z \in J_0 = \mathbb{S}$, we have $f_0(z) = z^{mn}$. There exists a holomorphic motion $\varphi_v : J_0 \to \overline{\mathbb{C}}$ of J_0 parametrized by $\mathbb{D}_\varepsilon := \{z : |z| < \varepsilon\}$ and with a base point 0 such that $\varphi_v(J_0) = J_\lambda$ and

$$f_v \circ \varphi_v(z) = \varphi_v \circ f_0(z) = \varphi_v(z^{mn}), \tag{3.3}$$

for all $z \in J_0$. Since every point on J_0 moves holomorphically, we can write φ_v in a power series of v. It is obviously to know that some coefficients are 0 of φ_v. In the following, we adopt the notation $d := mn$ for convenience.

We distinguish the following two cases.

(I) If $m < n$, we discuss in the following three subcase.

case $(I-1)$. If $m - 1 < \frac{1}{2}(n-1)$, we can get

$$f_v(z) = z^d - dz^{d+1}v^{m-1} + \frac{d(d+1)}{2} z^{d+2} v^{2(m-1)} + O\left(v^k\right), \tag{3.4}$$

it is easy to see that the nonzero higher order in (3.4) is $n-1$ for $\frac{1}{3}(n-1) < m-1 < \frac{1}{2}(n-1)$, and the nonzero higher order in (3.4) is $3(m-1)$ for $\frac{1}{3}(n-1) > m-1$. That implies $k = \min\{3(m-1), n-1\}$.

Since every point on J_0 moves holomorphically, we can write $\varphi_v(z)$ in a power series of v, i.e.

$$\varphi_v(z) = z\left(1 + u_1(z) v^{m-1} + u_2(z) v^{2(m-1)} + O\left(v^k\right)\right). \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), and comparing the terms to the second nonzero order about v, we obtain the following equations

$$u_1\left(z^d\right) - du_1(z) = -dz, \tag{3.6}$$

$$u_2\left(z^d\right) - du_2(z) = \frac{d(d-1)}{2} u_1^2(z) - d(d+1) z u_1(z) + \frac{d(d+1)}{2} z^2. \tag{3.7}$$

For each non-zero integer $q, l \in \mathbb{Z}$, the functional equation

$$u(z^q) - qu(z) = -qz^l \tag{3.8}$$

has the formal solution

$$u(z) = \sum_{k=0}^{+\infty} \frac{z^{lq^k}}{q^k}. \tag{3.9}$$

Note that the solution (3.9) is convergent if $|z| \leq 1$. This means that the solutions of (3.6) is

$$u_1(z) = \sum_{k=0}^{+\infty} \frac{z^{d^k}}{d^k}. \tag{3.10}$$

Therefore, Eq (3.7) can be reduced to

$$u_2\left(z^d\right) - du_2(z) = \frac{d(d-1)}{2}\left(\sum_{l=0}^{+\infty} \frac{z^{d^l}}{d^l}\right)^2 - d(d+1)\sum_{l=0}^{+\infty} \frac{z^{d^l+1}}{d^l} + \frac{d(d+1)}{2}z^2. \tag{3.11}$$

By (3.9) and (3.11), the solution of $u_2(z)$ is

$$u_2(z) = \sum_{k=0}^{+\infty}\left((d+1)\sum_{l=0}^{+\infty} \frac{z^{d^{l+k}+d^k}}{d^{l+k}} - \frac{d-1}{2d^k}\left(\sum_{l=0}^{+\infty} \frac{z^{d^{l+k}}}{d^l}\right)^2 - \frac{d+1}{2d^k}z^{2d^k}\right). \tag{3.12}$$

case **$(I-2)$.** If $m-1 = \frac{1}{2}(n-1)$, we can get

$$f_v(z) = z^d - dz^{d+1}v^{m-1} + \left(\frac{d(d+1)}{2}z^{d+2} + mz^{n(m-1)}\right)v^{2(m-1)} + O\left(v^{3(m-1)}\right). \tag{3.13}$$

Since every point on J_0 moves holomorphically, we can write $\varphi_v(z)$ in a power series about v, i.e.

$$\varphi_v(z) = z\left(1 + u_1(z)v^{m-1} + u_2(z)v^{2(m-1)} + O\left(v^{3(m-1)}\right)\right). \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.3), we obtain the following equations

$$u_1\left(z^d\right) - du_1(z) = -dz, \tag{3.15}$$

$$u_2\left(z^d\right) - du_2(z) = \frac{d(d-1)}{2}u_1^2(z) - d(d+1)zu_1(z) + \frac{d(d+1)}{2}z^2 + mz^{-n}. \tag{3.16}$$

By (3.8) and (3.9), we get the solutions of (3.15) is

$$u_1(z) = \sum_{k=0}^{+\infty} \frac{z^{d^k}}{d^k}. \tag{3.17}$$

Therefore, Eq (3.16) can be reduced to

$$u_2\left(z^d\right) - du_2(z) = \frac{d(d-1)}{2}\left(\sum_{l=0}^{+\infty} \frac{z^{d^l}}{d^l}\right)^2 - d(d+1)\sum_{l=0}^{+\infty} \frac{z^{d^l+1}}{d^l} + \frac{d(d+1)}{2}z^2 + mz^{-n}. \tag{3.18}$$

By (3.9) and (3.18), the solution of $u_2(z)$ is

$$u_2(z) = \sum_{k=0}^{+\infty}\left((d+1)\sum_{l=0}^{+\infty} \frac{z^{d^{l+k}+d^k}}{d^{l+k}} - \frac{d-1}{2d^k}\left(\sum_{l=0}^{+\infty} \frac{z^{d^{l+k}}}{d^l}\right)^2 - \frac{d+1}{2d^k}z^{2d^k} - \frac{1}{n}\frac{z^{-nd^k}}{d^k}\right). \tag{3.19}$$

case $(I-3)$**.** If $\frac{1}{2}(n-1) < m-1$, we can get

$$f_v(z) = z^d - dz^{d+1}v^{m-1} + mz^{n(m-1)}v^{n-1} + \frac{d(d+1)}{2}z^{d+2}v^{2(m-1)} + O\left(v^{m+n-2}\right). \tag{3.20}$$

Since every point on J_0 moves holomorphically, we can write $\varphi_v(z)$ in a power series about v, i.e.

$$\varphi_v(z) = z\left(1 + u_1(z)v^{m-1} + u_2(z)v^{n-1} + u_3(z)v^{2(m-1)} + O\left(v^{m+n-2}\right)\right). \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.3), we obtain the following equations

$$u_1\left(z^d\right) - du_1(z) = -dz, \tag{3.22}$$

$$u_2\left(z^d\right) - du_2(z) = mz^{-n}, \tag{3.23}$$

$$u_3\left(z^d\right) - du_3(z) = \frac{d(d-1)}{2}u_1^2(z) - d(d+1)zu_1(z) + \frac{d(d+1)}{2}z^2. \tag{3.24}$$

By (3.8) and (3.9), we get that the solutions of (3.22) and (3.23) are

$$u_1(z) = \sum_{k=0}^{+\infty}\frac{z^{d^k}}{d^k}, \tag{3.25}$$

$$u_2(z) = -\frac{1}{n}\sum_{k=0}^{+\infty}\frac{z^{-nd^k}}{d^k}. \tag{3.26}$$

Therefore, Eq (3.24) can be reduced to

$$u_3\left(z^d\right) - du_3(z) = \frac{d(d-1)}{2}\left(\sum_{l=0}^{+\infty}\frac{z^{d^l}}{d^l}\right)^2 - d(d+1)\sum_{l=0}^{+\infty}\frac{z^{d^l+1}}{d^l} + \frac{d(d+1)}{2}z^2. \tag{3.27}$$

By (3.9) and (3.27), the solution of $u_3(z)$ is

$$u_3(z) = \sum_{k=0}^{+\infty}\left((d+1)\sum_{l=0}^{+\infty}\frac{z^{d^{l+k}+d^k}}{d^{l+k}} - \frac{d-1}{2d^k}\left(\sum_{l=0}^{+\infty}\frac{z^{d^{l+k}}}{d^l}\right)^2 - \frac{d+1}{2d^k}z^{2d^k}\right). \tag{3.28}$$

(II) If $m > n$, we discuss in the following three subcase.

case $(II-1)$**.** If $m-1 < 2(n-1)$, we can get

$$f_v(z) = z^d + mz^{n(m-1)}v^{n-1} - dz^{d+1}v^{m-1} + \frac{m(m-1)}{2}z^{n(m-2)}v^{2(n-1)} + O\left(v^{m+n-2}\right). \tag{3.29}$$

Since every point on J_0 moves holomorphically, we have

$$\varphi_v(z) = z\left(1 + u_1(z)v^{n-1} + u_2(z)v^{m-1} + u_3(z)v^{2(n-1)} + O\left(v^{m+n-2}\right)\right). \tag{3.30}$$

Substituting (3.29) and (3.30) into (3.3), we obtain the following equations

$$u_1\left(z^d\right) - du_1(z) = mz^{-n}, \tag{3.31}$$

$$u_2\left(z^d\right) - du_2(z) = -dz, \tag{3.32}$$

$$u_3\left(z^d\right) - du_3(z) = \frac{d(d-1)}{2}u_1^2(z) + d(m-1)z^{-n}u_1(z) + \frac{m(m-1)}{2}z^{-2n}. \tag{3.33}$$

By (3.8) and (3.9), we get that the solutions of (3.31) and (3.32) are

$$u_1(z) = -\frac{1}{n}\sum_{k=0}^{+\infty}\frac{z^{-nd^k}}{d^k}, \tag{3.34}$$

$$u_2(z) = \sum_{k=0}^{+\infty}\frac{z^{d^k}}{d^k}. \tag{3.35}$$

Therefore, Eq (3.33) can be reduced to

$$\begin{aligned} u_3\left(z^d\right) - du_3(z) = {} & \frac{d(d-1)}{2}\left(\frac{1}{n}\sum_{l=0}^{+\infty}\frac{z^{-nd^l}}{d^l}\right)^2 \\ & - d(m-1)z^{-n}\left(\frac{1}{n}\sum_{l=0}^{+\infty}\frac{z^{-nd^l}}{d^l}\right) + \frac{m(m-1)}{2}z^{-2n}. \end{aligned} \tag{3.36}$$

By (3.9) and (3.36), the solution of $u_3(z)$ is

$$u_3(z) = \sum_{k=0}^{+\infty}\left(\frac{m-1}{n}\sum_{l=0}^{+\infty}\frac{z^{-n\left(d^l+1\right)d^k}}{d^{l+k}} - \frac{d-1}{2n^2d^k}\left(\sum_{l=0}^{+\infty}\frac{z^{-nd^{l+k}}}{d^l}\right)^2 - \frac{m-1}{2nd^k}z^{-2nd^k}\right). \tag{3.37}$$

case $(II-2)$**.** If $m-1 = 2(n-1)$, we can get

$$f_v(z) = z^d + mz^{n(m-1)}v^{n-1} + \left(\frac{m(m-1)}{2}z^{n(m-2)} - dz^{d+1}\right)v^{2(n-1)} + O\left(v^{3(n-1)}\right). \tag{3.38}$$

Since every point on J_0 moves holomorphically, we have

$$\varphi_v(z) = z\left(1 + u_1(z)v^{n-1} + u_2(z)v^{2(n-1)} + O\left(v^{3(n-1)}\right)\right). \tag{3.39}$$

Substituting (3.38) and (3.39) into (3.3), we obtain the following equations

$$u_1\left(z^d\right) - du_1(z) = mz^{-n}, \tag{3.40}$$

$$\begin{aligned} u_2\left(z^d\right) - du_2(z) = {} & \frac{d(d-1)}{2}u_1^2(z) + d(m-1)z^{-n}u_1(z) \\ & + \frac{m(m-1)}{2}z^{-2n} - dz. \end{aligned} \tag{3.41}$$

By (3.8) and (3.9), we get that the solutions of (3.40) is

$$u_1(z) = -\frac{1}{n}\sum_{k=0}^{+\infty}\frac{z^{-nd^k}}{d^k}, \tag{3.42}$$

Therefore, Eq (3.41) can be reduced to

$$
\begin{aligned}
u_2\left(z^d\right) - du_2(z) = {} & \frac{d(d-1)}{2}\left(\frac{1}{n}\sum_{l=0}^{+\infty}\frac{z^{-nd^l}}{d^l}\right)^2 \\
& - d(m-1)z^{-n}\left(\frac{1}{n}\sum_{l=0}^{+\infty}\frac{z^{-nd^l}}{d^l}\right) + \frac{m(m-1)}{2}z^{-2n} - dz.
\end{aligned} \tag{3.43}
$$

By (3.9) and (3.43), the solution of $u_2(z)$ is

$$
u_2(z) = \sum_{k=0}^{+\infty}\left(\frac{m-1}{n}\sum_{l=0}^{+\infty}\frac{z^{-n\left(d^l+1\right)d^k}}{d^{l+k}} - \frac{d-1}{2n^2d^k}\left(\sum_{l=0}^{+\infty}\frac{z^{-nd^{l+k}}}{d^l}\right)^2 - \frac{m-1}{2nd^k}z^{-2nd^k} + \frac{z^{d^k}}{d^k}\right). \tag{3.44}
$$

case $(II-3)$**.** If $m-1 > 2(n-1)$, we can get

$$
f_v(z) = z^d + mz^{n(m-1)}v^{n-1} + \frac{m(m-1)}{2}z^{n(m-2)}v^{2(n-1)} + O\left(v^k\right), \tag{3.45}
$$

it is easy to know that the nonzero higher order in (3.45) is $m-1$ for $2(n-1) < m-1 < 3(n-1)$, and the nonzero higher order in (3.45) is $3(n-1)$ for $m-1 > 3(n-1)$. That implies $k = \min\{3(n-1), m-1\}$.

Since every point on J_0 moves holomorphically, we have

$$
\varphi_v(z) = z\left(1 + u_1(z)v^{n-1} + u_2(z)v^{2(n-1)} + O\left(v^k\right)\right). \tag{3.46}
$$

Substituting (3.45) and (3.46) into (3.3), we obtain the following equations

$$
u_1\left(z^d\right) - du_1(z) = mz^{-n}, \tag{3.47}
$$

$$
u_2\left(z^d\right) - du_2(z) = \frac{d(d-1)}{2}u_1^2(z) + d(m-1)z^{-n}u_1(z) + \frac{m(m-1)}{2}z^{-2n}. \tag{3.48}
$$

By (3.8) and (3.9), we get that the solutions of (3.47) is

$$
u_1(z) = -\frac{1}{n}\sum_{k=0}^{+\infty}\frac{z^{-nd^k}}{d^k}. \tag{3.49}
$$

Therefore, Eq (3.48) can be reduced to

$$
\begin{aligned}
u_2\left(z^d\right) - du_2(z) = {} & \frac{d(d-1)}{2}\left(\frac{1}{n}\sum_{l=0}^{+\infty}\frac{z^{-nd^l}}{d^l}\right)^2 \\
& - d(m-1)z^{-n}\frac{1}{n}\sum_{l=0}^{+\infty}\frac{z^{-nd^l}}{d^l} + \frac{m(m-1)}{2}z^{-2n}.
\end{aligned} \tag{3.50}
$$

By (3.9) and (3.50), the solution of $u_2(z)$ is

$$
u_2(z) = \sum_{k=0}^{+\infty}\left(\frac{m-1}{n}\sum_{l=0}^{+\infty}\frac{z^{-n\left(d^l+1\right)d^k}}{d^{l+k}} - \frac{d-1}{2n^2d^k}\left(\sum_{l=0}^{+\infty}\frac{z^{-nd^{l+k}}}{d^l}\right)^2 - \frac{m-1}{2nd^k}z^{-2nd^k}\right). \tag{3.51}
$$

4. Proof of the Theorem 1

In fact, the proof of the asymptotic formula (1.9) is based on the calculation of an explicit iterated function system. We only give the proof of case $(II-3)$ in Section 3, the proofs of the other cases are the same as this case. Without loss of generality, we suppose that $k = 3(n-1)$ in (3.45).

For each $p \geq 1$, the collection of the fixed points of f_v^p on the Julia set J_v forms the finite set

$$Fix\left(f_v^p\right) = \left\{\varphi_v\left(e^{2\pi i t_j}\right) : t_j = \frac{j}{d^p - 1}, 1 \leq j \leq d^p - 1\right\}. \tag{4.1}$$

By (3.3) and the chain rule, we have $\left(f_v^p\right)'\left(\varphi_v\left(e^{2\pi i t_j}\right)\right) = \prod_{k=0}^{p-1}\left(f_v\right)'\left(\varphi_v\left(e^{2\pi i d^k t_j}\right)\right)$.

Firstly, we need proof the following proposition.

Proposition 4.1. *For every $D > 0$ and all sufficiently large n, the following holds*

$$\frac{1}{d^p - 1}\sum_{j=1}^{d^p-1}\prod_{k=0}^{p-1}\left|\left(f_v\right)'\left(\varphi_v\left(e^{2\pi i d^k t_j}\right)\right)\right|^{-D} = d^{-pD}\left(1 + \frac{D^2 p}{4}\left|v^{n-1}\right|^2 + O\left(v^{3(n-1)}\right)\right). \tag{4.2}$$

Proof. By (3.45), we have

$$\begin{aligned} f_v{}'(z) &= dz^{d-1} + d(m-1)z^{n(m-1)-1}v^{n-1} \\ &\quad + \frac{d(m-1)(m-2)}{2}z^{n(m-2)-1}v^{2(n-1)} + O\left(v^{3(n-1)}\right). \end{aligned} \tag{4.3}$$

Substituting (3.46) into (4.3), we have

$$\begin{aligned} f_v{}'\left(\varphi_v(z)\right) &= dz^{d-1} + dz^{d-1}\left((d-1)u_1(z) + (m-1)z^{-n}\right) \\ &\quad + dz^{d-1}\left((d-1)u_2(z) + \frac{(d-1)(d-2)}{2}u_1^2(z)\right. \\ &\quad + (m-1)(n(m-1)-1)z^{-n}u_1(z) \\ &\quad \left.+ \frac{(m-1)(m-2)}{2}z^{-2n}\right)v^{2(n-1)} + O\left(v^{3(n-1)}\right). \end{aligned} \tag{4.4}$$

Define $\sigma := \sigma(t) = e^{2\pi i t}$. Then $\sigma\overline{\sigma} = 1$. For $0 \leq m \leq n-1$, by (4.4), we have

$$\begin{aligned} \left|f_v{}'\left(\varphi_v\left(\sigma^{d^k}\right)\right)\right|^2 &= f_v{}'\left(\varphi\left(\sigma^{d^k}\right)\right)\overline{f_v{}'\left(\varphi\left(\sigma^{d^k}\right)\right)} \\ &= d^2 + A_k v^{n-1} + \overline{A_k}\,\overline{v^{n-1}} + A_k\overline{A_k}\left|v^{n-1}\right|^2 \big/ d^2 \\ &\quad + B_k v^{2(n-1)} + \overline{B_k}\,\overline{v^{2(n-1)}} + O\left(v^{3(n-1)}\right), \end{aligned} \tag{4.5}$$

where

$$A_k = d^2(d-1)u_1\left(\sigma^{d^k}\right) + (m-1)\left(\sigma^{d^k}\right)^{-n} \tag{4.6}$$

and

$$\begin{aligned} B_k &= d^2(d-1)u_2\left(\sigma^{d^k}\right) + \frac{d^2(d-1)(d-2)}{2}u_2^2\left(\sigma^{d^k}\right) \\ &\quad + d^2(m-1)(n(m-1)-1)\left(\sigma^{d^k}\right)^{-n}u_1\left(\sigma^{d^k}\right) \\ &\quad + \frac{d^2(m-1)(m-2)}{2}\left(\sigma^{d^k}\right)^{-2n}. \end{aligned} \tag{4.7}$$

For every $D > 0$, by (4.5), we have

$$\begin{aligned}
\prod_{k=0}^{p-1}\left|f_v'\left(\varphi_v\left(\sigma^{d^k}\right)\right)\right|^{-D} &= \prod_{k=0}^{p-1}\left(\left|f_v'\left(\varphi_v\left(\sigma^{d^k}\right)\right)\right|^2\right)^{-\frac{D}{2}} \\
&= d^{-pD}\prod_{k=0}^{p-1}\left(1+\frac{A_k v^{n-1}+\overline{A_k}\,\overline{v^{n-1}}+B_k v^{2(n-1)}+\overline{B_k}\,\overline{v^{2(n-1)}}}{d^2}\right. \\
&\left.+\frac{A_k\overline{A_k}\left|v^{n-1}\right|^2}{d^4}+\mathrm{O}\left(v^{2n-1}\right)\right)^{-\frac{D}{2}} \\
&= d^{-pD}-\frac{D}{2}d^{-pD-2}\sum_{k=0}^{p-1}\left(A_k v^{n-1}+\overline{A_k}\,\overline{v^{n-1}}+B_k v^{2(n-1)}+\overline{B_k}\,\overline{v^{2(n-1)}}\right) \qquad (4.8) \\
&-\frac{D}{2}d^{-pD-4}\left(\sum_{0\le k_1<k_2\le p-1}\left(A_{k_1}A_{k_2}v^{2(n-1)}+\overline{A_{k_1}}\,\overline{A_{k_2}}v^{2(n-1)}\right)\right. \\
&\left.+\sum_{0\le k_1,k_2\le p-1}A_{k_1}\overline{A_{k_2}}\left|v^{n-1}\right|^2\right) \\
&+\frac{D(D+2)}{8}d^{-pD-4}\left(\sum_{k=0}^{p-1}\left(A_k v^{n-1}+\overline{A_k}\,\overline{v^{n-1}}\right)\right)^2+O\left(v^{3(n-1)}\right).
\end{aligned}$$

Let $k, k_1, k_2 \in \mathbb{N}$. If $p \ge 1$, we can get the following results.
(1) Since $(d, d^p-1) = 1$, we have $\left(d^k, d^q-1\right) = 1$, $k \ge 0$. That is

$$d^k \not\equiv 0 \ \ mod \ \ d^p-1. \qquad (4.9)$$

(2) Since d^p-1 is relative prime to $d^{k'}$ $(k' \ge 0)$ by (1), it shows that $d^k+1 \not\equiv 0$ mod d^p-1 $(k \ge 0)$. Let $k = lp+t$ $(l \ge 0, 0 \le t \le p-1)$. We have $d^k+1 = d^{lp+t}-d^t+d^t+1 \equiv d^t-1 \ne 0$ mod d^p-1 since $0 < \left|d^t+1\right| < \left|d^t-1\right|$ That shows

$$d^{k_1}+d^{k_2} \not\equiv 0 \ \ mod \ \ d^p-1. \qquad (4.10)$$

(3) Since d^q-1 is relative prime to $d^{k'}$, $k' \ge 0$, we can find k such that $d^k-1 \equiv 0$ mod d^p-1 for fix $p \ge 1$. Let $k = lp+t$, where $l \ge 0$ and $0 \le t \le p-1$. We have $d^k-1 = d^{lp+t}-d^t+d^t-1 \equiv d^t-1$ mod d^p-1. This means that $d^k-1 \equiv 0$ mod d^p-1 if and only if $t = 0$ since $\left|d^t-1\right| < |d^p-1|$. We get

$$d^{k_1}-d^{k_2} \ \ mod \ \ d^p-1 \ if \ and \ only \ if \ d^{k_1}-d^{k_2} = lp \ for \ some \ l \in \mathbb{N}. \qquad (4.11)$$

It is convenient to introduce the average notation

$$\langle G(t)\rangle_p := \frac{j}{d^p-1}\sum_{j=1}^{d^p-1}G\left(t_j\right), \qquad (4.12)$$

where G is a continuous function defined on the interval $[0, 1)$ and $t_j = \frac{j}{d^p-1}$ is defined in (4.1).

For each $p \geq 1$ and any $l \in \mathbb{N}$, it is straightforward to verify that the average in (4.12) has following useful property

$$\left\langle \sigma^{l} \right\rangle_{p} = \left\langle e^{2\pi i l t} \right\rangle_{p} = \begin{cases} 1, & \text{if } l \equiv 0 \mod d^{n}-1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.13}$$

By (3.49) and (3.50), the average property (4.13) and (4.9), (4.10), it can be get the following results.

Suppose $0 \leq k, k_1, k_2 \leq p-1$, then

$$\begin{aligned} &\left\langle \sigma^{d^{k}} \right\rangle_{p} = 0, \ \left\langle \sigma^{\left(d^{k_1}+d^{k_2}\right)} \right\rangle_{p} = 0, \\ &\left\langle u_1\left(\sigma^{d^{k}}\right) \right\rangle_{p} = 0, \ \left\langle \sigma^{d^{k_1}} u_1\left(\sigma^{d^{k_2}}\right) \right\rangle_{p} = 0, \\ &\left\langle u_1\left(\sigma^{d^{k_1}}\right) u_1\left(\sigma^{d^{k_2}}\right) \right\rangle_{p} = 0 \ \textit{and} \ \left\langle u_2\left(\sigma^{d^{k}}\right) \right\rangle_{p} = 0. \end{aligned} \tag{4.14}$$

As an immediate result of (4.14), if $0 \leq k, k_1, k_2 \leq p-1$, we have

$$\langle A_k \rangle_p = \left\langle \overline{A_k} \right\rangle_p = 0, \ \langle B_k \rangle_p = \left\langle \overline{B_k} \right\rangle_p = 0 \ \textit{and} \ \langle A_{k_1} A_{k_2} \rangle_p = \left\langle \overline{A_{k_1}} \ \overline{A_{k_2}} \right\rangle_p = 0. \tag{4.15}$$

By (4.8) and (4.15), we have

$$\left\langle \prod_{k=0}^{p-1} \left| f_v{}'\left(\varphi_v\left(\sigma^{d^{k}}\right)\right) \right|^{-D} \right\rangle_p = d^{-nD} \left(1 + \frac{D^2}{4d^4} \sum_{0 \leq k_1, k_2 \leq p-1} \left\langle A_{k_1} \overline{A_{k_2}} \right\rangle_p \left| v^{n-1} \right|^2 \right) + O\left(v^{3(n-1)}\right). \tag{4.16}$$

By (4.6) and (4.7), we have

$$\begin{aligned} \left\langle A_{k_1} \overline{A_{k_2}} \right\rangle_p = &\ d^4(d-1)^2 \left\langle u_1\left(\sigma^{d^{k_1}}\right) \overline{u_1\left(\sigma^{d^{k_2}}\right)} \right\rangle_p + d^4(m-1)^2 \left\langle \sigma^{-n\left(d^{k_1}-d^{k_2}\right)} \right\rangle_p \\ &- d^4(d-1)(m-1) \left\langle u_1\left(\sigma^{d^{k_1}}\right) \sigma^{n d^{k_2}} + \overline{u_1\left(\sigma^{d^{k_2}}\right)} \sigma^{-n d^{k_1}} \right\rangle_p. \end{aligned} \tag{4.17}$$

Since $0 \leq k_1, k_2 \leq p-1$, it follows that $k_1 - k_2 = lp$ for $l \in \mathbb{N}$ if and only if $k_1 = k_2$. By (4.11), we have

$$\left\langle \sigma^{-n\left(d^{k_1}-d^{k_2}\right)} \right\rangle_p = \begin{cases} 1, & \text{if } k_1 = k_2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.18}$$

That means

$$\sum_{0 \leq k_1, k_2 \leq p-1} \left\langle \sigma^{-n\left(d^{k_1}-d^{k_2}\right)} \right\rangle_p = p. \tag{4.19}$$

Similarly, by (4.11), we have

$$\begin{aligned} \left\langle u_1\left(\sigma^{d^{k_1}}\right) \sigma^{n d^{k_2}} \right\rangle_p &= -\frac{1}{n} \sum_{l=0}^{+\infty} \frac{\left\langle \sigma^{-n\left(d^{k_1+l}-d^{k_2}\right)} \right\rangle_p}{d^l} \\ &= \begin{cases} \frac{1}{n} \frac{d^{k_1-k_2}}{d^p-1}, & \text{if } k_1 \geq k_2, \\ \frac{1}{n} \frac{d^{p-(k_2-k_1)}}{d^p-1}, & \text{if } k_1 < k_2. \end{cases} \end{aligned} \tag{4.20}$$

That means

$$\begin{aligned}\sum_{0\le k_1,k_2\le p-1}\left\langle u_1\left(\sigma^{d^{k_1}}\right)\sigma^{nd^{k_2}}\right\rangle_p &= -\frac{1}{n}\sum_{0\le k_2\le k_1\le p-1}\frac{d^{k_1-k_2}}{d^p-1}-\frac{1}{n}\sum_{0\le k_1<k_2\le p-1}\frac{d^{p-(k_2-k_1)}}{d^p-1}\\ &= -\frac{1}{n}\frac{p}{d^p-1}\left(d+\cdots+d^p\right)\\ &= -\frac{pm}{d-1}.\end{aligned} \tag{4.21}$$

Moreover, by (4.11), we have

$$\left\langle u_1\left(\sigma^{d^{k_1}}\right)\overline{u_1\left(\sigma^{nd^{k_2}}\right)}\right\rangle_p = \sum_{l_1=1}^{+\infty}\sum_{l_2=1}^{+\infty}\frac{\left\langle\sigma^{-n\left(d^{k_1+l_1}-d^{k_2+l_2}\right)}\right\rangle_p}{d^{l_1+l_2}}. \tag{4.22}$$

Similar to the reduction process of (4.21), we have

$$\sum_{0\le k_1,k_2\le p-1}\left\langle u_1\left(\sigma^{d^{k_1}}\right)\overline{u_1\left(\sigma^{d^{k_2}}\right)}\right\rangle_p = \frac{pm^2}{(d-1)^2}. \tag{4.23}$$

By substituting (4.19), (4.21) and (4.23) into (4.17), we have

$$\sum_{0\le k_1,k_2\le p-1}\left\langle A_{k_1}\overline{A_{k_2}}\right\rangle_p = pd^4. \tag{4.24}$$

For every $D>0$ and sufficiently large p, the following holds

$$\left\langle\prod_{k=0}^{p-1}\left|f_v'\left(\varphi_v\left(\sigma^{d^k}\right)\right)\right|^{-D}\right\rangle_p = |d|^{-pD}\left(1+\frac{D^2p\left|v^{n-1}\right|^2}{4}+O\left(v^{3(n-1)}\right)\right). \tag{4.25}$$

That is (4.2) holds. The proof of Proposition 4.1 is complete. □

Note that J_v is a quasicircle and 0 and ∞ are two attracting fixed points of f_v, and f_v^p has d^p+1 fixed points in $\widehat{\mathbb{C}}$, then we get f_v^p has d^p-1 fixed points in J_v. Set $Fix(f)$ be the collection of all the repelling fixed points of f_v with period p. We get the following proposition.

Proposition 4.2. *Let $D_v := \dim_H(J_v)$ be the Hausdorff dimension of J_v, we claim that D_v satisfies the following equation*

$$\sum_{z\in Fix\left(f_v^p\right)}\left|\left(f_v^p\right)'(z)\right|^{-D_v} = O(1). \tag{4.26}$$

Proof. Since f_v is hyperbolic and the Julia set J_v of f_v is a quasicircle, there exist a pair of closed annular neighborhoods W_1, W_2 of J_v and a quasiconformal mapping $\phi : W_1 \to A_\varepsilon$, $\varepsilon>0$ is small enough and $A_\varepsilon := \{z : 1-\varepsilon\le|z|\le 1+\varepsilon\}$, such that ϕ conjugates $f_v : W_1\to W_2$ to $z\mapsto z^d$ or $z\mapsto z^{-d}$. Without loss of generality, we only consider the first case.

The Julia set of a hyperbolic rational map can be seen as the limit of a sequence of IFS. These IFS are defined in terms of the inverse branches of the iterations of the rational map. Since J_v separates 0 and ∞, we define a curve $\gamma := \phi^{-1}\left(\left[(1-\varepsilon)^d,(1+\varepsilon)^d\right]\right)\subset W_2$. In order to define IFS, we lift J_v and f_v under the exponential map. Fix a component of $\exp^{-1}(W_2\backslash\gamma)$ and denote it by U. Then U is topologically a strip and $\exp : U \to W_2\backslash\gamma$ is conformal in the interior of U. For each $p \geq 1$, the map $f_v^p : W_1 \to W_2$ has d^p inverse branches, say $T_1, \cdots, T_{d^p}$, each maps $W_2\backslash\gamma$ onto a half open quadrilateral such that their images are arranged in anticlockwise order one by one. Define $S_i := \log \circ T_i \circ \exp$, $1 \leq i \leq d^p$, be the map in U. Then each S_i is conformal in the interior of U and can be conformally extended to an open neighborhood of $\bar{U}$. Since f_v is strictly expanding on W_1, $\{S_1, \cdots, S_{d^p}\}$ is an IFS defined on $\bar{U}$.

The attractor $J_v{}'$ of $\{S_1, \cdots, S_{d^p}\}$ is a closed set satisfying $J_v = \exp(J_v{}')$. Moreover, $J_v\backslash\{z_1\}$ is the conformal image of $J_v{}'$ with two ends removed, where $z_1 \in J_v \cap \gamma$ is a fixed point of f_v. This means that the Hausdorff dimensions of $J_v{}'$ and J_v satisfy $\dim_H(J_v{}') = \dim_H(J_v)$. Let $F_p|_U := \cup_{i=1}^{d^p} S_i^{-1}\big|_{S_i(U)}$ be the lift of f_v^p under exp. Then for $1 < i < d^p$ each $S_i(U)$ contains only one fixed point $\zeta_i \in J_v{}'$ of F_p in its interior and for $i = 1$ or $i = d^p$ the fixed point on its boundary. Since S_i can be conformally extended to an open neighborhood of $\bar{U}$, by Koebes distortion theorem, there exist two constants C_1, C_2 $(0 < C_1 \leq 1 \leq C_2)$ both independent of p, such that

$$\frac{C_1}{\left|F_p{}'(\zeta_i)\right|} \leq \frac{|S_i(x) - S_i(y)|}{|x-y|} \leq \frac{C_2}{\left|F_p{}'(\zeta_i)\right|}, \tag{4.27}$$

where $1 \leq i \leq d^p$, $x, y \in \bar{U}$. By *Theorem* 2.3 we get $s_1 \leq D_v \leq s_2$, where $\sum_{i=0}^{d^p} C_j^{s_j}\left|F_p{}'(\zeta_i)\right|^{-s_j} = 1$, $j = 1, 2$. Then

$$\frac{1}{C_2^{D_v}} \leq \frac{1}{C_2^{s_2}} \leq \sum_{i=1}^{d^p} \frac{1}{\left|F_p{}'(\zeta_i)\right|^{s_2}} \leq \sum_{i=1}^{d^p} \frac{1}{\left|F_p{}'(\zeta_i)\right|^{D_v}} \leq \sum_{i=1}^{d^p} \frac{1}{\left|F_p{}'(\zeta_i)\right|^{s_1}} \leq \frac{1}{C_1^{s_1}} \leq \frac{1}{C_1^{D_v}}. \tag{4.28}$$

Since F_p is conformally conjugate to f_v^p in the interior of each $S_i(U)$, we have $F_n{}'(\zeta_i) = (f^p)'(\exp(\zeta_i))$ for $1 \leq i \leq d^p$. Therefore, by (4.28), we have

$$\sum_{z \in Fix(f_v^p)} \frac{1}{\left|(f_v^p)'(z)\right|^{D_v}} = \sum_{i=1}^{d^p} \frac{1}{\left|(f_v^p)'(\exp(\zeta_i))\right|^{D_v}} = \sum_{i=1}^{d^p} \frac{1}{\left|F_p{}'(\zeta_i)\right|^{D_v}} - \left|F_p{}'(\zeta_{d^p})\right|^{-D_v} = O(1). \tag{4.29}$$

The proof of Proposition 4.2 is complete. □

The Proof of Theorem 1. By Proposition 4.1 and Proposition 4.2, we have

$$|d^p - 1|\,|d|^{-pD_v}\left(1 + \frac{D^2 p\left|v^{n-1}\right|^2}{4} + \mathrm{O}\left(v^{3(n-1)}\right)\right) = O(1). \tag{4.30}$$

Fix some large p when v is small enough. Then (4.30) is equivalent to

$$\exp\left(p\left(D_v^2\left|v^{n-1}\right|^2 - (D_v - 1)\log|d|\right) + O\left(v^{3(n-1)}\right)\right) = O(1). \tag{4.31}$$

By Theorem 2.1 and Theorem 2.2, D_v depends real analytically on v in a small neighborhood of the origin and $D_0 = 1$. This means that in a small neighborhood of 0, D_v can be written as

$$D_v = 1 + a_{10}v^{n-1} + a_{01}\overline{v^{n-1}} + a_{20}v^{2(n-1)} + a_{02}\overline{v^{2(n-1)}} + a_{11}\left|v^{n-1}\right|^2 + O\left(v^{3(n-1)}\right). \tag{4.32}$$

Substituting (4.32) into (4.31) and comparing the corresponding coefficients, we have

$$a_{10} = a_{01} = a_{20} = a_{02} = 0 \quad \text{and} \quad a_{11} = \frac{1}{4\log|d|}. \tag{4.33}$$

That means

$$D_v = 1 + \frac{\left|v^{n-1}\right|^2}{4\log|d|} + O\left(v^{2n-1}\right). \tag{4.34}$$

So the Hausdorff dimension D_λ of $J\left(U_{mn\lambda}\right)$ is

$$D_\lambda = 1 + \frac{|\lambda|^{-\frac{2(n-1)}{mn-1}}}{4\log mn} + O\left(|\lambda|^{-\frac{3(n-1)}{mn-1}}\right). \tag{4.35}$$

This ends the proof of the case $(II-3)$ in Theorem 1.

Similarly, the other cases can be proved by the similar method as used in the case $(II-3)$. Hence the Hausdorff dimension of $\dim_H\left(J_{mn\lambda}\right)$ is given by the following asymptotic formula.

(I) If $m < n$

$$\dim_H\left(J_{mn\lambda}\right) = \begin{cases} 1 + \frac{|\lambda|^{-\frac{2(m-1)}{mn-1}}}{4\log(mn)} + O\left(|\lambda|^{-\frac{3(m-1)}{mn-1}}\right), \ for\ m-1 < \frac{1}{3}(n-1)\ \ or\ m-1 = \frac{1}{2}(n-1), \\ 1 + \frac{|\lambda|^{-\frac{2(m-1)}{mn-1}}}{4\log(mn)} + O\left(|\lambda|^{-\frac{n-1}{mn-1}}\right), \ for\ \frac{1}{3}(n-1) < m-1 < \frac{1}{2}(n-1), \\ 1 + \frac{|\lambda|^{-\frac{2(m-1)}{mn-1}}}{4\log(mn)} + O\left(|\lambda|^{-\frac{m+n-2}{mn-1}}\right), \ for\ m-1 > \frac{1}{2}(n-1). \end{cases} \tag{4.36}$$

(II) If $m > n$

$$\dim_H\left(J_{mn\lambda}\right) = \begin{cases} 1 + \frac{|\lambda|^{-\frac{2(n-1)}{mn-1}}}{4\log(mn)} + O\left(|\lambda|^{-\frac{3(n-1)}{mn-1}}\right), \ for\ m-1 > 3(n-1)\ \ or\ m-1 = 2(n-1), \\ 1 + \frac{|\lambda|^{-\frac{2(n-1)}{mn-1}}}{4\log(mn)} + O\left(|\lambda|^{-\frac{m-1}{mn-1}}\right), \ for\ 2(n-1) < m-1 < 3(n-1), \\ 1 + \frac{|\lambda|^{-\frac{2(n-1)}{mn-1}}}{4\log(mn)} + O\left(|\lambda|^{-\frac{m+n-2}{mn-1}}\right), \ for\ m-1 < 2(n-1). \end{cases} \tag{4.37}$$

The proof of the Theorem 1 is complete. □

Proof of Corollary 1. If $m = n$, then $d = n^2$, we can get

$$\begin{aligned} f_v(z) = z^d + \left(nz^{n(n-1)} - dz^{d+1}\right)v^{n-1} + \Bigg(&\frac{d(d+1)}{2}z^{d+2} \\ &+ nz^{n(n-1)} + (1-n)dz^{n^2-n+1}\Bigg)v^{2(n-1)} + O\left(v^{3(n-1)}\right). \end{aligned} \tag{4.38}$$

Since every point on J_0 moves holomorphically, we have

$$\varphi_v(z) = z\left(1 + u_1(z)v^{n-1} + u_2(z)v^{2(n-1)} + O\left(v^{3(n-1)}\right)\right). \tag{4.39}$$

Substituting (4.38) and (4.39) into (3.3), we obtain the following equations

$$u_1\left(z^d\right) - du_1(z) = nz^{-n} - dz, \tag{4.40}$$

$$\begin{aligned} u_2\left(z^d\right) - du_2(z) = &\frac{d(d-1)}{2}u_1^2(z) + \left(d(n-1)z^{-n} - d(d+1)z\right)u_1(z) \\ &+ \frac{n(n-1)}{2}z^{-2n} + \frac{d(d+1)}{2}z^2 + (1-n)dz^{1-n}. \end{aligned} \tag{4.41}$$

By (3.8) and (3.9), we get that the solutions of (4.40) is

$$u_1(z) = \sum_{k=0}^{+\infty}\left(\frac{z^{d^k}}{d^k} - \frac{z^{-nd^k}}{nd^k}\right). \tag{4.42}$$

Therefore, Eq (4.41) can be reduced to

$$\begin{aligned} u_2\left(z^d\right) - du_2(z) = {}& \frac{d(d-1)}{2}\left(\sum_{l=0}^{+\infty}\left(\frac{z^{d^l}}{d^l} - \frac{z^{-nd^l}}{nd^l}\right)\right)^2 + d(n-1)\sum_{l=0}^{+\infty}\left(\frac{z^{d^l-n}}{d^l} - \frac{z^{-n\left(d^l+1\right)}}{nd^l}\right) \\ & - d(d+1)\sum_{l=0}^{+\infty}\left(\frac{z^{d^l+1}}{d^l} - \frac{z^{-nd^l+1}}{nd^l}\right) + \frac{n(n-1)}{2}z^{-2n} \\ & + \frac{d(d+1)}{2}z^2 + (1-n)dz^{1-n}. \end{aligned} \tag{4.43}$$

By (3.9) and (4.43), the solution of $u_2(z)$ is

$$\begin{aligned} u_2(z) = \sum_{k=0}^{+\infty}\Bigg(& (d+1)\sum_{l=0}^{+\infty}\left(\frac{z^{d^{l+k}+d^k}}{d^{l+k}} - \frac{z^{-nz^{d^{l+k}+d^k}}}{nd^{l+k}}\right) - \frac{d-1}{2}\left(\sum_{l=0}^{+\infty}\left(\frac{z^{d^{l+k}}}{d^{l+k}} - \frac{z^{-nd^{l+k}}}{nd^{l+k}}\right)\right)^2 \\ & - (n-1)\sum_{l=0}^{+\infty}\left(\frac{z^{\left(d^l-n\right)d^k}}{d^{l+k}} - \frac{z^{-n\left(d^l+1\right)d^k}}{nd^{l+k}}\right) - \frac{m-1}{2n}\frac{z^{-2nd^k}}{d^k} \\ & - \frac{d+1}{2}\frac{z^{2d^k}}{d^k} - (1-n)\frac{z^{(1-n)d^k}}{d^k}\Bigg). \end{aligned} \tag{4.44}$$

It can be proofed by the method as Theorem 1 that if $|\lambda|$ is sufficiently large, the Hausdorff dimension D_λ of $J(U_{mn\lambda})$ is

$$D_\lambda = 1 + \frac{2|\lambda|^{-\frac{2(n-1)}{n^2-1}}}{4\log n^2} + O\left(|\lambda|^{-\frac{3(n-1)}{n^2-1}}\right). \tag{4.45}$$

i.e.

$$D_\lambda = 1 + \frac{|\lambda|^{-\frac{2}{n+1}}}{4\log n} + O\left(|\lambda|^{-\frac{3}{n+1}}\right). \tag{4.46}$$

□

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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