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Research article

Some properties for certain class of bi-univalent functions defined by q-Cătaş operator with bounded boundary rotation

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Abstract: Throughout the paper, we introduce a new subclass $\mathcal{H}^{n,q,\lambda,l}_{\alpha,\mu,\rho,m,\beta}\ f(z)$ by using the Bazilevič functions with the idea of bounded boundary rotation and q-analogue Cătaş operator. Also we find the estimate of the coefficients for functions in this class. Finally, in the concluding section, we have chosen to reiterate the well-demonstrated fact that any attempt to produce the rather straightforward (p, q)-variations of the results, which we have presented in this article, will be a rather trivial and inconsequential exercise, simply because the additional parameter p is obviously redundant.

Keywords: bi-univalent functions; q-analogue Cătaş operator, Bazilevič functions; bounded boundary rotation
Mathematics Subject Classification: 30C45

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k (z \in \mathbb{U} : \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}). \tag{1.1}$$

Let S be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ and let K, S_λ, S^* and C be the usual subclasses of S consisting of functions which are, respectively, close-to-convex, λ -spiral-like, starlike ($w.r.t.$ the origin) and convex in $\mathbb{U}$.

Definition 1. ([4], with $p = 1$.) Let $\mathcal{P}^{\mu}_{m}(\rho)$,$0 \le \rho < 1$, $m \ge 2$ and $|\mu| < \frac{\pi}{2}$, be the class of analytic functions $p(z) = 1 + \sum\limits_{k=1}^{\infty} c_k z^k$ and satisfy the conditions:

$$p(0) = 1$$

and

$$\int_0^{2\pi}\left|\frac{Re e^{i\mu}p(z)-\rho\cos\mu}{1-\rho}\right| \leq m\pi\cos\mu, \tag{1.2}$$

$$\text{for every } r<1(z=re^{i\theta}\in\mathbb{U}),\ 0\leq\rho<1,\ m\geq 2 \text{ and } |\mu|<\frac{\pi}{2}.$$

We note that:
(i) $\mathcal{P}_m^{\mu}(0)=\mathcal{P}_m^{\mu}$ ($m\geq 2$ and $|\mu|<\frac{\pi}{2}$), the class of functions introduced by Robertson [34].
(ii) $\mathcal{P}_m^{0}(\rho)=\mathcal{P}_m(\rho)$ $(0\leq\rho<1,\ m\geq 2)$, the class of functions introduced by Padmanabhan and Parvatham [31].
(iii) $\mathcal{P}_m^{0}(0)=\mathcal{P}_m(m\geq 2)$, the class of functions having their real parts bounded in the mean on $\mathbb{U}$, introduced by Robertson [34] and further studied by Pinchuk [32]. (iv) $\mathcal{P}_2^{0}(\rho)=\mathcal{P}(\rho)$ $(0\leq\rho<1)$, the class of functions with positive real part of order ρ, $0\leq\rho<1$. (v) $\mathcal{P}_2^{0}(0)=\mathcal{P}$, the class of functions having positive real part for $z\in\mathbb{U}$.

Definition 2. ([12, 36, 51]). The class of Bazilevič functions in the open unit disc $\mathbb{U}$ was introduced by Bazilevič [12], he defined Bazilevič functions by the following relation

$$f(z)=\left\{(\beta+i\tau)\int_0^z p(t)g(t)^{\beta}t^{i\tau-1}dt\right\}^{1/(\beta+i\tau)},$$

where $g(z)\in S^*$, $p(z)\in\mathcal{P}_m^{\mu}(\rho)$ and $\beta,\tau>0$. Singh [36] studied the class $B_1(\beta)$ of Bazilevič functions by putting $g(z)=z$ and $\tau=0$ in above equation. If $f(z)\in B_1(\beta)$ then $f(z)$ is a Bazilevič function of type β.

By the Koebe one-quarter theorem [15], we know that the image of $\mathbb{U}$ under every univalent function $f\in\mathcal{A}$ contains the disc with the center in the origin and radius 1/4. Therefore, every univalent function f has an inverse f^{-1} satisfies

$$f^{-1}(f(z))=z\ (z\in\mathbb{U}) \quad\text{and}\quad f(f^{-1}(w))=w\ (|w|<r_0(f),\ r_0(f)\geq 1/4). \tag{1.3}$$

It is easy to see that the inverse function has the form

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\dots\ . \tag{1.4}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and its inverse map $g=f^{-1}$are univalent in $\mathbb{U}$. Let $\sum$ denote the class of bi-univalent functions in $\mathbb{U}$ in the form (1.1). For interesting examples about the class $\sum$ (see [8, 11, 16, 18, 25, 46]).

The pioneering work of Srivastava et al. [45] actually revived the study of bi-univalent functions in recent years. In a substantially large number of work subsequent to the work of Srivastava et al. [45], several distinct subclasses of the bi-univalent function class were presented and examined similarly by many authors. For example, the function classes $H_{\sum}(\tau,\mu,\lambda,\delta;\alpha)$ and $H_{\sum}(\tau,\mu,\lambda,\gamma;\beta)$ were defined and the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were obtained by Srivastava et al. [43]. The upper bounds for the second Hankel determinant for certain subclasses of analytic and bi-univalent functions were obtained by Caglar et al. [13]. Several new subclasses of the class of m-fold symmetric bi-univalent functions were introduced and the initial estimates of the Taylor-Maclaurin series as well as some Fekete-Szegö functional problems for each of their defined function classes were obtained by Tang et al. [50] and Srivastava et al. [42]. Several other well-known mathematicians gave their findings on this subject (e.g. [40, 41, 44]).

As we know that the fractional q-calculus and the fractional of q-derivative operators in Geometric Function Theory were investigated by sturdy of researchers (see [2,3,7,9,10,17,19–25,37–39,47–49]). The q-calculus is an important tool which is used to study various applications in mathematics, physics and chemistry and some basic sciences subjects. In the study of Geometric Function Theory, the versatile applications of the q-derivative operator D_q make it remarkably significant. Inspired by the above-mentioned works, in recent years, important researches have played a significant part in the development of Geometric Function Theory of complex analysis. Several convolutional and fractional calculus q-operators were defined by many researchers, which were surveyed in the above-cited work by Srivastava [37]. For a function $f(z) \in \mathcal{A}$ given by (1.1) and $0 < q < 1$. Jackson's q-derivative (or q-difference) D_q of a function defined on a subset of the complex space $\mathbb{C}$ is defined as follows:

$$\begin{aligned} D_q f(z) &= \frac{f(z) - f(qz)}{(1-q)z} \ \ (z \neq 0,\ 0 < q < 1), \\ &= 1 + \sum_{k=2}^{\infty} [k]_q \, a_k z^{k-1}, \end{aligned} \tag{1.5}$$

where $D_q f(0) = f'(0)$, $D_q^2 f(z) = D_q \left(D_q f(z)\right)$ and $[k]_q = \frac{1-q^k}{1-q}$, as $q \to 1-$, then $[k]_q \to k$, hence we have

$$\lim_{q \to 1-} D_q f(z) = f'(z) \quad (z \in \mathbb{U}). \tag{1.6}$$

For a function $f(z) \in \mathcal{A}$, Aouf and Madian [10] defined the q-analogue Cătaş operator as follows (see also [7, 9] with $p = 1$]):

$$I_q^n(\lambda, l) f(z) = z + \sum_{k=2}^{\infty} \Psi_q^n(k, \lambda, l) a_k z^k, \tag{1.7}$$

where

$$\begin{aligned} \Psi_q^n(k, \lambda, l) &= \left[\frac{[1+l]_q + \lambda \left([k+l]_q - [1+l]_q \right)}{[1+l]_q} \right]^n \\ (n &\in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\},\ l, \lambda \geq 0,\ 0 < q < 1). \end{aligned} \tag{1.8}$$

And introduced recurrent relation as follows:

$$\lambda q^l z D_q \left(I_q^n(\lambda, l) f(z) \right) = [1+l]_q \, I_q^{n+1}(\lambda, l) f(z) - [(1-\lambda) q^l + [l]_q] I_q^n(\lambda, l) f(z) \ (\lambda > 0).$$

We note that $I_q^n(\lambda, l) f(z)$ generalized many operators such as Cătaş operator, Multiplier operator and Sălăgean operator etc., for more details see [1, 5–7, 14, 28, 35].

Definition 3. Let $f \in \sum$, $\alpha \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$, $l, \lambda\ \beta \geq 0$, $0 \leq \rho < 1$, $m \geq 2$, $|\mu| < \frac{\pi}{2}, n \in \mathbb{N}_0$ and $0 < q < 1$, then $I_q^n(\lambda, l) f(z) \in \sum$ is said to be in $\mathcal{H}_{\alpha,\mu,\rho,m,\beta}^{n,q,\lambda,l} f(z)$, if it satisfies the following conditions:

$$\left\{ (1-\alpha) \left(\frac{I_q^n(\lambda, l) f(z)}{z} \right)^{\beta} + \alpha \frac{z D_q (I_q^n(\lambda, l) f(z))}{I_q^n(\lambda, l) f(z)} \left(\frac{I_q^n(\lambda, l) f(z)}{z} \right)^{\beta} \right\} \in \mathcal{P}_m^{\mu}(\rho) \ (z \in \mathbb{U}), \tag{1.9}$$

and

$$\left\{(1-\alpha)\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{wD_q(I_q^n(\lambda,l)f^{-1}(w))}{I_q^n(\lambda,l)f^{-1}(w)}\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta}\right\}\in\mathcal{P}_m^{\mu}(\rho)\ (w\in\mathbb{U}). \tag{1.10}$$

By choosing different values for $\alpha,\mu,\rho,m,\beta,n,q,\lambda$ and l, in the a above definition we have:

(1) $\lim_{q\to 1^-}\mathcal{H}_{1,0,\rho,m,\beta}^{0,q,\lambda,l}f(z)=K_{\sum}(\rho,m,\beta)f(z)$ $(f\in\sum,\ \beta\geq 0,\ 0\leq\rho<1,\ m\geq 2)$ (see [29], with $\gamma=1$ and $\delta=0$).

(2) $\lim_{q\to 1^-}\mathcal{H}_{\alpha,0,\rho,2,\beta}^{0,q,\lambda,l}f(z)=\mathcal{F}_{\rho,\beta}f(z)$ $(f\in\sum,\ \beta\geq 0,\ 0\leq\rho<1)$ (see [33]).

(3) $\mathcal{H}_{1,0,\rho,2,1}^{0,q,\lambda,l}f(z)=\mathcal{F}_{\rho,q}(f)(z)(\,f\in\sum,\ 0\leq\rho<1\,)$ (see [46]).

(4) $\lim_{q\to 1^-}\mathcal{H}_{\alpha,0,\rho,2,\beta}^{0,q,\lambda,l}f(z)=\mathcal{MP}_{\sum}^{\beta,\alpha}(0,\rho)f(z)$ $(f\in\sum,\ \beta,\alpha\geq 0,\ 0\leq\rho<1\,)$ (see [30], with $\beta=0$]).

(5) $\lim_{q\to 1^-}\mathcal{H}_{\alpha,\mu,\rho,2,\beta}^{0,q,\lambda,l}f(z)=\mathcal{L}_{\sum}(\beta,\rho)f(z)$ $(\,f\in\sum,\ \beta\geq 0,\ 0\leq\rho<1\,)$ (see [27]).

(6) $\lim_{q\to 1^-}\mathcal{H}_{\alpha,\mu,\rho,m,\beta}^{0,q,\lambda,l}f(z)=\mathcal{K}(\alpha,\mu,\rho,m,\beta)f(z)$ $(\,f\in\sum,\ \alpha\in\mathbb{C}^*,\ \beta\geq 0,\ 0\leq\rho<1,\ m\geq 2,\ |\mu|<\frac{\pi}{2},\ 0\leq\rho<1)$ (see [11], with $h=\frac{z}{1-z}$ and [8], with $h=\frac{z}{1-z}$, $b=1$ and $\delta=0$).

As well as, we obtain new subclasses as follows:

(a)

$$\begin{aligned}&H_{\alpha,\mu,0,m,\beta}^{n,q,\lambda,l}f(z)\\=&P_{\alpha,\mu,m,\beta}^{n,q,\lambda,l}f(z)\\=&\left\{f\in\sum:(1-\alpha)\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}+\alpha\frac{zD_q(I_q^n(\lambda,l)f(z))}{I_q^n(\lambda,l)f(z)}\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}\in\mathcal{P}_m^{\mu}\right.\\&\left.\text{and }(1-\alpha)\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{wD_q(I_q^n(\lambda,l)f^{-1}(w))}{I_q^n(\lambda,l)f^{-1}(w)}\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta}\in\mathcal{P}_m^{\mu}\right\}.\end{aligned}$$

(b)

$$\begin{aligned}&\mathcal{H}_{\alpha,0,\rho,2,\beta}^{n,q,\lambda,l}f(z)\\&=\mathcal{D}_{\alpha,\rho,\beta}^{n,q,\lambda,l}f(z)\\&=\left\{f\in\sum:\left[(1-\alpha)\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}+\alpha\frac{zD_q(I_q^n(\lambda,l)f(z))}{I_q^n(\lambda,l)f(z)}\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}\right]>\rho\right.\\&\left.\text{and}\left[(1-\alpha)\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{wD_q(I_q^n(\lambda,l)f^{-1}(w))}{I_q^n(\lambda,l)f^{-1}(w)}\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta}\right]>\rho\right\}.\end{aligned}$$

(c)

$$\begin{aligned}&\mathcal{H}_{1,\mu,\rho,m,\beta}^{n,q,\lambda,l}f(z)\\&=\mathcal{U}_{\mu,\rho,m,\beta}^{n,q,\lambda,l}f(z)\\&=\left\{f\in\sum:\frac{zD_q(I_q^n(\lambda,l)f(z))}{I_q^n(\lambda,l)f(z)}\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}\in\mathcal{P}_m^{\mu}(\rho)\right.\\&\left.\text{and }\frac{wD_q(I_q^n(\lambda,l)f^{-1}(w))}{I_q^n(\lambda,l)f^{-1}(w)}\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta}\in\mathcal{P}_m^{\mu}(\rho)\right\}.\end{aligned}$$

In order to obtain our main results, we have to recall here the following lemma.

Lemma 1. ([4] with p=1.) If $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n \in \mathcal{P}_m^{\mu}(\rho)$, then

$$|c_n| \leq (1-\rho)\, m\, \cos\mu. \tag{1.11}$$

The result is sharp. Equality is attained for the odd coefficients and even coefficients, respectively, for the functions

$$p_1(z) = 1 + (1-\rho)\cos\mu\, e^{-i\mu}\left[\left(\frac{m+2}{4}\right)\left(\frac{1-z}{1+z}\right) - \left(\frac{m-2}{4}\right)\left(\frac{1+z}{1-z}\right) - 1\right],$$

$$p_2(z) = 1 + (1-\rho)\cos\mu\, e^{-i\mu}\left[\left(\frac{m+2}{4}\right)\left(\frac{1-z^2}{1+z^2}\right) - \left(\frac{m-2}{4}\right)\left(\frac{1+z^2}{1-z^2}\right) - 1\right].$$

We note that for $\mu = \rho = 0$ in Lemma 1, we obtain the result obtained by Goswami et al. [16, Lemma 1] for the class $\mathcal{P}_m$.

The object of this paper is to introduce a new subclass of the class $\sum$ by using the definition of Bazilevič functions, bi-univalent functions with bounded boundary rotation and q-analogue Cătaş operator. As well as I calculate the coefficient estimates for functions in the subclass $\mathcal{H}_{\alpha,\mu,\rho,m,\beta}^{n,q,\lambda,l} f(z)$. Also I get coefficients bounds for the subclasses of our main class.

2. Coefficient estimates for functions in the subclass $\mathcal{H}_{\alpha,\mu,\rho,m,\beta}^{n,q,\lambda,l} f(z)$

Theorem 1. Let $f \in \sum$, $\alpha \in \mathbb{C}^*\backslash\{-1, \frac{-1}{2}\}$, $l, \lambda, \beta \geq 0$, $0 \leq \rho < 1$, $m \geq 2$, $|\mu| < \frac{\pi}{2}, n \in \mathbb{N}_0$ and $0 < q < 1$, then $I_q^n(\lambda, l) f(z) \in \mathcal{H}_{\alpha,\mu,\rho,m,\beta}^{n,q,\lambda,l} f(z)$ if satisfies

$$|a_2| \leq \min\left\{ \begin{array}{c} \sqrt{\frac{2m(1-\rho)\cos\mu}{\left|2\alpha(([2]_q-1)\beta+q^2)+\beta(\beta+1)\right|\left|\Psi_q^n(2,\lambda,l)\right|^2}}; \\ \frac{m(1-\rho)\cos\mu}{\left|\alpha([2]_q-1)+\beta\right|\left|\Psi_q^n(2,\lambda,l)\right|} \end{array} \right\} \tag{2.1}$$

and

$$|a_3| \leq \frac{m(1-\rho)\cos\mu}{\left|\alpha([3]_q-1)+\beta\right|\left|\Psi_q^n(3,\lambda,l)\right|} + \frac{[m(1-\rho)\cos\mu]^2}{\left|\alpha+\beta([2]_q-1)\right|^2\left|\Psi_q^n(3,\lambda,l)\right|}. \tag{2.2}$$

Proof. Let $I_q^n(\lambda, l) f(z) \in \mathcal{H}_{\alpha,\mu,\rho,m,\beta}^{n,q,\lambda,l} f(z)$, then from Definition 1, we have

$$(1-\alpha)\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta} + \alpha\frac{zD_q(I_q^n(\lambda,l)f(z))}{I_q^n(\lambda,l)f(z)}\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta} = p(z),\ p \in \mathcal{P}_m^{\mu}(\rho) \tag{2.3}$$

and

$$(1-\alpha)\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta} + \alpha\frac{wD_q(I_q^n(\lambda,l)f^{-1}(w))}{I_q^n(\lambda,l)f^{-1}(w)}\left(\frac{I_q^n(\lambda,l)f^{-1}(w)}{w}\right)^{\beta} = q(w),\ q \in \mathcal{P}_m^{\mu}(\rho), \tag{2.4}$$

where p and q have Taylor expansions as follows:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ..., z \in \mathbb{U}, \tag{2.5}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + ..., w \in \mathbb{U}. \tag{2.6}$$

By comparing coefficients in (2.3) with (2.5) and coefficients in (2.4) with (2.6), we obtain

$$p_1 = \left[\alpha([2]_q - 1) + \beta\right] a_2 \Psi_q^n(2, \lambda, l), \tag{2.7}$$

$$p_2 = \left[\alpha([3]_q - 1) + \beta\right] a_3 \Psi_q^n(3, \lambda, l) + \frac{\left[\beta + 2\alpha([2]_q - 1)\right](\beta - 1)}{2} a_2^2 \left(\Psi_q^n(2, \lambda, l)\right)^2, \tag{2.8}$$

$$q_1 = -\left[\alpha([2]_q - 1) + \beta\right] a_2 \Psi_q^n(2, \lambda, l), \tag{2.9}$$

and

$$\begin{aligned} q_2 =& \left[\beta + \alpha([3]_q - 1)\right]\left[2a_2^2 \left(\Psi_q^n(2, \lambda, l)\right)^2 - a_3 \Psi_q^n(3, \lambda, l)\right] \\ &+ \frac{\left[\beta + 2\alpha([2]_q - 1)\right](\beta - 1)}{2} a_2^2 \left(\Psi_q^n(2, \lambda, l)\right)^2. \end{aligned} \tag{2.10}$$

Since $p, q \in \mathcal{P}_m^\mu(\rho)$ and by applying Lemma 1, we have

$$|p_n| \leq m(1 - \rho) \cos \mu \ (n \geq 1) \tag{2.11}$$

and

$$|q_n| \leq m(1 - \rho) \cos \mu \ (n \geq 1). \tag{2.12}$$

From (2.8), (2.10) and using inequalities (2.11) and (2.12), we obtain

$$\begin{aligned} |a_2|^2 &\leq \frac{1}{\left|2\alpha([2]_q - 1)\beta + q^2 + \beta(\beta + 1)\right|} \frac{|p_2| + |q_2|}{\left|\Psi_q^n(2, \lambda, l)\right|^2} \\ &\leq \frac{2m(1 - \rho) \cos \mu}{\left|2\alpha([2]_q - 1)\beta + q^2 + \beta(\beta + 1)\right| \left|\Psi_q^n(2, \lambda, l)\right|^2}. \end{aligned} \tag{2.13}$$

Also, from (2.7) and (2.11), we obtain

$$|a_2| \leq \frac{m(1 - \rho) \cos \mu}{\left|\alpha([2]_q - 1) + \beta\right| \left|\Psi_q^n(2, \lambda, l)\right|}. \tag{2.14}$$

By subtracting (2.10) from (2.8), we have

$$\begin{aligned} p_2 - q_2 =& 2\left[\alpha([3]_q - 1) + \beta\right] a_3 \Psi_q^n(3, \lambda, l) \\ &- 2\left[\alpha([3]_q - 1) + \beta\right] a_2^2 \left(\Psi_q^n(2, \lambda, l)\right)^2. \end{aligned} \tag{2.15}$$

Also, we have

$$p_1^2 + q_1^2 = 2\left[\alpha + \beta([2]_q - 1)\right]^2 a_2^2 \left(\Psi_q^n(2, \lambda, l)\right)^2. \tag{2.16}$$

After using (2.15), (2.16), (2.11) and (2.12) and some easily calculations, we obtain

$$|a_3| \leq \frac{m(1 - \rho) \cos \mu}{\left|\alpha([3]_q - 1) + \beta\right| \left|\Psi_q^n(3, \lambda, l)\right|} + \frac{\left[m(1 - \rho) \cos \mu\right]^2}{\left|\alpha + \beta([2]_q - 1)\right|^2 \left|\Psi_q^n(3, \lambda, l)\right|}, \tag{2.17}$$

which completes the proof of Theorem 1. The result is sharp in view of the fact that assertion (1.11) of Lemma 1 is sharp. By following the observation of Thomas [52], the equality is attained for the odd coefficients and even coefficients, respectively, for the functions

$$\begin{aligned}
&(1-\alpha)\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}+\alpha\frac{zD_q(I_q^n(\lambda,l)f(z))}{I_q^n(\lambda,l)f(z)}\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}\\
=\;& 1+(1-\rho)\cos\mu\, e^{-i\mu}\left[\left(\frac{m+2}{4}\right)\left(\frac{1-z}{1+z}\right)-\left(\frac{m-2}{4}\right)\left(\frac{1+z}{1-z}\right)-1\right],
\end{aligned}$$

$$\begin{aligned}
&(1-\alpha)\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}+\alpha\frac{zD_q(I_q^n(\lambda,l)f(z))}{I_q^n(\lambda,l)f(z)}\left(\frac{I_q^n(\lambda,l)f(z)}{z}\right)^{\beta}\\
=\;& 1+(1-\rho)\cos\mu\, e^{-i\mu}\left[\left(\frac{m+2}{4}\right)\left(\frac{1-z^2}{1+z^2}\right)-\left(\frac{m-2}{4}\right)\left(\frac{1+z^2}{1-z^2}\right)-1\right].
\end{aligned}$$

By using Ma and Minda Lemma (see [26]), we obtain the following corollary.

Corollary 1. *Let* $I_q^n(\lambda,l)f(z)\in H_{\alpha,\mu,\rho,m,\beta}^{n,q,\lambda,l}f(z),\ \alpha\in\mathbb{C}^*\backslash\{-1,\frac{-1}{2}\},\ l,\lambda,\beta\geq 0, n\in\mathbb{N}_0$ *and* $0<q<1$, *then*
(i) For any real number ζ, we have

$$\begin{aligned}
&\left|a_3\Psi_q^n(3,\lambda,l)-\zeta a_2^2\left(\Psi_q^n(2,\lambda,l)\right)^2\right|\\
\leq&\begin{cases}
\frac{-4\zeta\left[\alpha([3]_q-1)+\beta\right]-2\{[2\alpha([2]_q-1)+\beta](\beta-1)\}}{\left[\alpha([3]_q-1)+\beta\right][\alpha([2]_q-1)+\beta]^2}+\frac{2}{\left[\alpha([3]_q-1)+\beta\right]}, if\\
\zeta\leq\frac{[2\alpha([2]_q-1)+\beta](\beta-1)}{2\left[\alpha([3]_q-1)+\beta\right]}\\
\frac{2}{\left[\alpha([3]_q-1)+\beta\right]},\ if\\
\frac{[2\alpha([2]_q-1)+\beta](\beta-1)}{2\left[\alpha([3]_q-1)+\beta\right]}\leq\zeta\leq\frac{2\{\alpha([2]_q-1)+\beta\}^2-[2\alpha([2]_q-1)+\beta](\beta-1)}{2\left[\alpha([3]_q-1)+\beta\right]}\\
\frac{4\zeta\left[\alpha([3]_q-1)+\beta\right]+2\{[2\alpha([2]_q-1)+\beta](\beta-1)\}}{\left[\alpha([3]_q-1)+\beta\right][\alpha([2]_q-1)+\beta]^2}-\frac{2}{\left[\alpha([3]_q-1)+\beta\right]},\ if\\
\zeta\geq\frac{2\{\alpha([2]_q-1)+\beta\}^2-[2\alpha([2]_q-1)+\beta](\beta-1)}{2\left[\alpha([3]_q-1)+\beta\right]}.
\end{cases}
\end{aligned}$$

(ii) For any complex number ζ, we have

$$\begin{aligned}
&\left|a_3\Psi_q^n(3,\lambda,l)-\zeta a_2^2\left(\Psi_q^n(2,\lambda,l)\right)^2\right|\\
\leq&\frac{2}{\left[\alpha([3]_q-1)+\beta\right]}\times\max\left\{1;\ \left|\frac{2\zeta\left[\alpha([3]_q-1)+\beta\right]+\{[2\alpha([2]_q-1)+\beta](\beta-1)\}}{[\alpha([2]_q-1)+\beta]^2}-1\right|\right\}.
\end{aligned}$$

Putting $\rho=0$, in Theorem 1, we obtain the following corollary.

Corollary 2. Let $f\in\sum,\ \alpha\in\mathbb{C}^*\backslash\{-1,\frac{-1}{2}\},\ l,\lambda,\beta\geq 0,\ m\geq 2,\ |\mu|<\frac{\pi}{2}, n\in\mathbb{N}_0$ and $0<q<1$, then $I_q^n(\lambda,l)f(z)\in\mathcal{P}_{\alpha,\mu,m,\beta}^{n,q,\lambda,l}f(z)$ if satisfies

$$|a_2|\leq\min\left\{\begin{array}{c}\sqrt{\frac{2m\cos\mu}{\left|2\alpha(([2]_q-1)\beta+q^2)+\beta(\beta+1)\right|\left|\Psi_q^n(2,\lambda,l)\right|^2}};\\ \frac{m\cos\mu}{\left|\alpha([2]_q-1)+\beta\right|\left|\Psi_q^n(2,\lambda,l)\right|}\end{array}\right\}$$

and

$$|a_3| \leq \frac{m\cos\mu}{\left|\alpha([3]_q-1)+\beta\right|\left|\Psi_q^n(3,\lambda,l)\right|}+\frac{[m\cos\mu]^2}{\left|\alpha+\beta([2]_q-1)\right|^2\left|\Psi_q^n(3,\lambda,l)\right|}.$$

Putting $\mu = 0$ and $m = 2$ in Theorem 1, we obtain the following corollary.

Corollary 3. Let $f \in \sum$, $\alpha \in \mathbb{C}^*\backslash\{-1, \frac{-1}{2}\}$, $l, \lambda, \beta \geq 0$, $0 \leq \rho < 1, n \in \mathbb{N}_0$ and $0 < q < 1$, then $I_q^n(\lambda, l)f(z) \in \mathcal{D}_{\alpha,\rho,\beta}^{n,q,\lambda,l} f(z)$ if satisfies

$$|a_2| \leq \min\left\{ \begin{array}{c} \sqrt{\frac{4(1-\rho)}{\left|2\alpha(([2]_q-1)\beta+q^2)+\beta(\beta+1)\right|\left|\Psi_q^n(2,\lambda,l)\right|^2}}; \\ \frac{2(1-\rho)}{\left|\alpha([2]_q-1)+\beta\right|\left|\Psi_q^n(2,\lambda,l)\right|} \end{array} \right\}$$

and

$$|a_3| \leq \frac{2(1-\rho)}{\left|\alpha([3]_q-1)+\beta\right|\left|\Psi_q^n(3,\lambda,l)\right|}+\frac{4(1-\rho)^2}{\left|\alpha+\beta([2]_q-1)\right|^2\left|\Psi_q^n(3,\lambda,l)\right|}.$$

Putting $\alpha = 1$ in Theorem 1, we obtain the following corollary.

Corollary 4. Let $f \in \sum$, $l, \lambda, \beta \geq 0$, $0 \leq \rho < 1$, $m \geq 2$, $|\mu| < \frac{\pi}{2}, n \in \mathbb{N}_0$ and $0 < q < 1$, then $I_q^n(\lambda, l)f(z) \in \mathcal{U}_{\mu,\rho,m,\beta}^{n,q,\lambda,l} f(z)$ if satisfies

$$|a_2| \leq \min\left\{ \begin{array}{c} \sqrt{\frac{2m(1-\rho)\cos\mu}{\left|2(([2]_q-1)\beta+q^2)+\beta(\beta+1)\right|\left|\Psi_q^n(2,\lambda,l)\right|^2}}; \\ \frac{m(1-\rho)\cos\mu}{\left|[2]_q-1+\beta\right|\left|\Psi_q^n(2,\lambda,l)\right|} \end{array} \right\}$$

and

$$|a_3| \leq \frac{m(1-\rho)\cos\mu}{\left|[3]_q-1+\beta\right|\left|\Psi_q^n(3,\lambda,l)\right|}+\frac{[m(1-\rho)\cos\mu]^2}{\left|1+\beta([2]_q-1)\right|^2\left|\Psi_q^n(3,\lambda,l)\right|}.$$

Remarks. (i) Putting, $\beta = \alpha = 1, m = 2$ and $n = \mu = 0$ in Theorem 1, we obtain the result obtained by Srivastava et al. [46, Theorem 2].
(ii) Letting $q \to 1-$ and $n = 0$ in Theorem 1, we obtain the result obtained by Aouf and Madian [11, with $h = \frac{z}{1-z}$, Theorem 1] and [8 , with $h = \frac{z}{1-z}, b = 1$ and $\delta = 0$, Theorem 1]).

3. Conclusions

Throughout the paper, I used the definition of Bazilevič function, bi-univalent functions with bounded boundary rotation and the definition of q-analogue Cătaş operator to introduce the new subclass $\mathcal{H}_{\alpha,\mu,\rho,m,\beta}^{n,q,\lambda,l} f(z)$. I estimated the coefficients bounds for the functions belong to the subclass $\mathcal{H}_{\alpha,\mu,\rho,m,\beta}^{n,q,\lambda,l} f(z)$. In addition, through this paper we presented coefficients bounds for the functions belong to the subclasses of our main class.

Srivastava [37, p. 340] discussed the connection between the classical q-analysis, which we used in this article, and its so-called trivial and inconsequential (p, q)-variation involving an obviously superfluous parameter p. Specifically, the results in this article for the q-analogues $(0 < q < 1)$, can easily translated into the corresponding (p, q)-variants $(0 < q < p \leq 1)$ by following the observation by Srivastava [37, p. 340] who applied some obvious parametric and argument variations, the additional parameter p being redundant. As clearly and significantly pointed out by Srivastava et al. [46, 48], some group of authors have made use of the so-called trivial and inconsequential (p, q)-variation by introducing a seemingly redundant parameter p in the already known results dealing with the classical q-analysis. For further details, see the survey-cum-expository review article by Srivastava [37, p. 340].

Availability of data and material

During the current study the data sets are derived arithmetically.

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Conflict of interest

The authors don't have competing for any interests.

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