Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 855–868.
DOI: 10.3934/math.2022051
Received: 20 May 2021
Accepted: 13 October 2021
Published: 18 October 2021

Research article

Imploring interior GE-filters in GE-algebras

Sun Shin Ahn[1], **Ravikumar Bandaru**[2,*] **and Young Bae Jun**[3]

[1] Department of Mathematics Education, Dongguk University, Seoul 04620, Korea
[2] Department of Mathematics, GITAM (Deemed to be University), Hyderabad Campus, Telangana-502329, India
[3] Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

* **Correspondence:** Email: ravimaths83@gmail.com.

Abstract: The concept of an imploring interior GE-filter is introduced, and their properties are investigated. The relationship between an interior GE-filter and an imploring interior GE-filter are discussed. Example to show that any interior GE-filter is not an imploring interior GE-filter is provided. Conditions for an interior GE-filter to be an imploring interior GE-filter are given. Examples to show that an imploring interior GE-filter is independent to a belligerent interior GE-filter are provided. Conditions for an imploring interior GE-filter to be a belligerent interior GE-filter are given. The relationship between imploring interior GE-filter and prominent interior GE-filter are discussed. Example to show that any imploring interior GE-filter is not a prominent interior GE-filter is provided. Conditions for an imploring interior GE-filter to be a prominent interior GE-filter are given. Also, conditions under which an interior GE-filter larger than a given interior GE-filter can become an imploring interior GE-filter are considered.

Keywords: (pre-transitive) GE-algebra; GE-filter; interior GE-filter; imploring interior GE-filter; prominent interior GE-filter
Mathematics Subject Classification: 03G25, 06F35

1. Introduction

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [5] proved that Hilbert algebras form a variety which is locally finite. Bandaru et al. introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1–3, 8, 9]). The notion of interior operator is introduced by Vorster [13] in an arbitrary category, and it is used in [4] to study the notions of connectedness and disconnectedness in topology. Interior algebras are a certain type of algebraic structure that encodes the idea of the topological interior of a set, and are a generalization of topological spaces defined by means of topological interior operators. Rachůnek and Svoboda [7] studied interior operators on bounded residuated lattices, and Svrcek [12] studied multiplicative interior operators on GMV-algebras. Lee et al. [6] applied the interior operator theory to GE-algebras, and they introduced the concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras, and investigated their relations and properties. Later, Song et al. [10, 11] introduced the notions of an interior GE-filter, a weak interior GE-filter, a belligerent interior GE-filter, prominent interior GE-filter and investigate their relations and properties. They provided relations between a belligerent interior GE-filter and an interior GE-filter and conditions for an interior GE-filter to be a belligerent interior GE-filter is considered. Also, they provided relations between a prominent interior GE-filter and an interior GE-filter and conditions for an interior GE-filter to be a prominent interior GE-filter is considered. Given a subset and an element, they established an interior GE-filter, and they considered conditions for a subset to be a belligerent interior GE-filter. They studied the extensibility of the belligerent interior GE-filter and established relationships between weak interior GE-filter and belligerent interior GE-filter of type 1–type 3. Also, they introduced the concept of a prominent interior GE-filter of type 1 and type 2, and investigate their properties. They studied the relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1.

In this manuscript, we introduce the concept of an imploring interior GE-filter, and investigate their properties. We discuss the relationship between an interior GE-filter and an imploring interior GE-filter. We provide conditions for an interior GE-filter to be an imploring interior GE-filter and give examples to show that an imploring interior GE-filter is independent to a belligerent interior GE-filter. We provide conditions for an imploring interior GE-filter to be a belligerent interior GE-filter. We discuss the relationship between imploring interior GE-filter and prominent interior GE-filter and give example to show that any imploring interior GE-filter is not a prominent interior GE-filter. We provide conditions for an imploring interior GE-filter to be a prominent interior GE-filter. Also, we consider conditions under which an interior GE-filter larger than a given interior GE-filter to become an imploring interior GE-filter.

2. Preliminaries

2.1. Default background for GE-algebras

Definition 2.1 ([1]). By a *GE-algebra* we mean a non-empty set X with a constant 1 and a binary operation $*$ satisfying the following axioms:

(GE1) $\tilde{x} * \tilde{x} = 1$,

(GE2) $1 * \tilde{x} = \tilde{x}$,

(GE3) $\tilde{x} * (\tilde{y} * \tilde{z}) = \tilde{x} * (\tilde{y} * (\tilde{x} * \tilde{z}))$,

for all $\tilde{x}, \tilde{y}, \tilde{z} \in X$.

In a GE-algebra X, a binary relation "$\leq$" is defined by

$$(\forall \tilde{x}, \tilde{y} \in X)\,(\tilde{x} \leq \tilde{y} \;\Leftrightarrow\; \tilde{x} * \tilde{y} = 1)\,. \tag{2.1}$$

Definition 2.2 ([1, 2]). A GE-algebra X is said to be

- *transitive* if it satisfies:

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\, (\tilde{x} * \tilde{y} \leq (\tilde{z} * \tilde{x}) * (\tilde{z} * \tilde{y}))\,; \tag{2.2}$$

- *belligerent* if it satisfies:

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\, (\tilde{x} * (\tilde{y} * \tilde{z}) = (\tilde{x} * \tilde{y}) * (\tilde{x} * \tilde{z}))\,. \tag{2.3}$$

Proposition 2.3 ([1]). *Every GE-algebra X satisfies the following items.*

$$(\forall \tilde{x} \in X)\, (\tilde{x} * 1 = 1)\,. \tag{2.4}$$
$$(\forall \tilde{x}, \tilde{y} \in X)\, (\tilde{x} * (\tilde{x} * \tilde{y}) = \tilde{x} * \tilde{y})\,. \tag{2.5}$$
$$(\forall \tilde{x}, \tilde{y} \in X)\, (\tilde{x} \leq \tilde{y} * \tilde{x})\,. \tag{2.6}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\, (\tilde{x} * (\tilde{y} * \tilde{z}) \leq \tilde{y} * (\tilde{x} * \tilde{z}))\,. \tag{2.7}$$
$$(\forall \tilde{x} \in X)\, (1 \leq \tilde{x} \;\Rightarrow\; \tilde{x} = 1)\,. \tag{2.8}$$
$$(\forall \tilde{x}, \tilde{y} \in X)\, (\tilde{x} \leq (\tilde{y} * \tilde{x}) * \tilde{x})\,. \tag{2.9}$$
$$(\forall \tilde{x}, \tilde{y} \in X)\, (\tilde{x} \leq (\tilde{x} * \tilde{y}) * \tilde{y})\,. \tag{2.10}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\, (\tilde{x} \leq \tilde{y} * \tilde{z} \;\Leftrightarrow\; \tilde{y} \leq \tilde{x} * \tilde{z})\,. \tag{2.11}$$

If X is transitive, then

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\, (\tilde{x} \leq \tilde{y} \;\Rightarrow\; \tilde{z} * \tilde{x} \leq \tilde{z} * \tilde{y},\; \tilde{y} * \tilde{z} \leq \tilde{x} * \tilde{z})\,. \tag{2.12}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\, (\tilde{x} * \tilde{y} \leq (\tilde{y} * \tilde{z}) * (\tilde{x} * \tilde{z}))\,. \tag{2.13}$$
$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)\, (\tilde{x} \leq \tilde{y},\; \tilde{y} \leq \tilde{z} \;\Rightarrow\; \tilde{x} \leq \tilde{z})\,. \tag{2.14}$$

Lemma 2.4 ([1]). *A GE-algebra X is transitive if and only if X satisfies the condition* (2.13).

Definition 2.5 ([1]). A subset F of a GE-algebra X is called a *GE-filter* of X if it satisfies:

$$1 \in F, \tag{2.15}$$
$$(\forall \tilde{x}, \tilde{y} \in X)(\tilde{x} * \tilde{y} \in F,\; \tilde{x} \in F \;\Rightarrow\; \tilde{y} \in F). \tag{2.16}$$

Lemma 2.6 ([1]). *In a GE-algebra X, every GE-filter F of X satisfies:*

$$(\forall \tilde{x}, \tilde{y} \in X)\, (\tilde{x} \leq \tilde{y},\; \tilde{x} \in F \;\Rightarrow\; \tilde{y} \in F)\,. \tag{2.17}$$

Definition 2.7 ([2,8,9]). A subset F of a GE-algebra X containing the constant 1 is called:

- A *belligerent GE-filter* of X if it satisfies

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)(\tilde{x} * (\tilde{y} * \tilde{z}) \in F,\; \tilde{x} * \tilde{y} \in F \;\Rightarrow\; \tilde{x} * \tilde{z} \in F). \tag{2.18}$$

- A *prominent GE-filter* of X if it satisfies

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)(\tilde{x} * (\tilde{y} * \tilde{z}) \in F,\; \tilde{x} \in F \;\Rightarrow\; ((\tilde{z} * \tilde{y}) * \tilde{y}) * \tilde{z} \in F). \tag{2.19}$$

- An *imploring GE-filter* of X if it satisfies

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)(\tilde{x} * ((\tilde{y} * \tilde{z}) * \tilde{y}) \in F,\; \tilde{x} \in F \;\Rightarrow\; \tilde{y} \in F). \tag{2.20}$$

Lemma 2.8 ([8]). *Let F be a GE-filter of a GE-algebra X. Then F is a prominent GE-filter of X if and only if it satisfies:*

$$(\forall \tilde{x}, \tilde{y} \in X)\, (\tilde{x} * \tilde{y} \in F \;\Rightarrow\; ((\tilde{y} * \tilde{x}) * \tilde{x}) * \tilde{y} \in F)\,. \tag{2.21}$$

2.2. Default background for interior GE-algebras

Definition 2.9 ([6]). By an *interior GE-algebra* we mean a pair (X, ξ) in which X is a GE-algebra and $\xi : X \to X$ is a mapping such that

$$(\forall \tilde{x} \in X)(\tilde{x} \leq \xi(\tilde{x})), \tag{2.22}$$

$$(\forall \tilde{x} \in X)((\xi \circ \xi)(\tilde{x}) = \xi(\tilde{x})), \tag{2.23}$$

$$(\forall \tilde{x}, \tilde{y} \in X)(\tilde{x} \leq \tilde{y} \;\Rightarrow\; \xi(\tilde{x}) \leq \xi(\tilde{y})). \tag{2.24}$$

Definition 2.10 ([6]). An interior GE-algebra (X, ξ) is said to be

- *transitive* if it satisfies:

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)(\xi(\tilde{x} * \tilde{y}) \leq \xi((\tilde{z} * \tilde{x}) * (\tilde{z} * \tilde{y}))); \tag{2.25}$$

- *belligerent* if it satisfies:

$$(\forall \tilde{x}, \tilde{y}, \tilde{z} \in X)(\xi(\tilde{x} * (\tilde{y} * \tilde{z})) = \xi((\tilde{x} * \tilde{y}) * (\tilde{x} * \tilde{z}))). \tag{2.26}$$

Definition 2.11 ([11]). Let (X, ξ) be an interior GE-algebra. A GE-filter F of X is said to be interior if it satisfies:

$$(\forall \tilde{x} \in X)(\xi(\tilde{x}) \in F \;\Rightarrow\; \tilde{x} \in F). \tag{2.27}$$

Definition 2.12 ([11]). Let (X, ξ) be an interior GE-algebra. Then a subset F of X is called a *belligerent interior GE-filter* in (X, ξ) if F is a belligerent GE-filter of X which satisfies the condition (2.27).

Definition 2.13 ([10]). Let (X, ξ) be an interior GE-algebra. Then a subset F of X is called a *prominent interior GE-filter* in (X, ξ) if F is a prominent GE-filter of X which satisfies the condition (2.27).

3. Imploring interior GE-filters

Definition 3.1. An interior GE-algebra (X, ξ) is said to be *pre-transitive* (resp., *pre-belligerent*, if X itself is transitive (resp., belligerent).

It is clear that every pre-transitive (resp., pre-belligerent) interior GE-algebra is a transitive (resp., belligerent) interior GE-algebra, but the converse is not true (see [6]).

In what follows, let (X, ξ) denote an interior GE-algebra unless otherwise specified.

Definition 3.2. A subset F of X in (X, ξ) is called an *imploring interior GE-filter* in (X, ξ) if F contains the constant "1" and satisfies (2.20) and (2.27).

Example 3.3. Consider a set $X = \{1, a, b, c, d\}$ with the binary operation $*$ given as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	d	d
c	1	a	a	1	1
d	1	a	a	1	1

If we define a mapping ξ on X as follows:

$$\xi : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } \tilde{x} = 1, \\ a & \text{if } \tilde{x} \in \{a, b\}, \\ c & \text{if } \tilde{x} \in \{c, d\}, \end{cases}$$

then (X, ξ) is an interior GE-algebra and $F := \{1, a, b\}$ is an imploring interior GE-filter in in (X, ξ).

We first discuss the relationship between interior GE-filter and imploring interior GE-filter.

Theorem 3.4. *Every imploring interior GE-filter is an interior GE-filter.*

Proof. Let F be an imploring interior GE-filter in (X, ξ). Then $1 \in F$ and (2.27) is clearly true. Let $x, y \in X$ be such that $x * y \in F$ and $x \in F$. The combination of (GE1) and (GE2) leads to

$$x * ((y * y) * y) = x * y \in F.$$

It follows from (2.20) that $y \in F$. Hence F is an interior GE-filter in (X, ξ). □

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.5. Let $X = \{1, a, b, c, d\}$ and consider a binary operation $*$ and a mapping ξ on X given as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	d
b	1	1	1	c	d
c	1	a	a	1	d
d	1	b	b	1	1

and

$$\xi : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } \tilde{x} = 1, \\ a & \text{if } \tilde{x} \in \{a, b\}, \\ c & \text{if } \tilde{x} \in \{c, d\}. \end{cases}$$

Then (X, ξ) is an interior GE-algebra and the set $F := \{1, a, b\}$ is an interior GE-filter in (X, ξ). But it is not imploring interior GE-filter in (X, ξ) since

$$1 * ((c * d) * c) = 1 * (d * c) = 1 * 1 = 1 \in F$$

and $1 \in F$ but $c \notin F$.

We find and present the conditions under which interior GE-filter becomes imploring interior GE-filter.

Theorem 3.6. *Given an interior GE-filter F in (X, ξ), the following arguments are equivalent.*

(i) *F is an imploring interior GE-filter in (X, ξ).*

(ii) *F satisfies:*

$$(\forall x, y \in X)(\xi((x * y) * x) \in F \;\Rightarrow\; x \in F). \tag{3.1}$$

Proof. Assume that F is an imploring interior GE-filter in (X, ξ) and let $x, y \in X$ be such that $\xi((x * y) * x) \in F$. Then

$$1 * ((x * y) * x) = (x * y) * x \in F$$

by (GE2) and (2.27). It follows from (2.20) that $x \in F$.

Conversely, let F be an interior GE-filter in (X, ξ) which satisfies the condition (3.1). It is clear that F contains the constant 1 and satisfies the condition (2.27). Let $x, y, z \in X$ be such that $x*((y*z)*y) \in F$ and $x \in F$. Then $(y*z)*y \in F$ by (2.16), and so $(y*z)*y \leq \xi((y*z)*y)$ by (2.22). Hence $\xi((y*z)*y) \in F$ since F is a GE-filter of X. Thus $y \in F$ by (2.27). Therefore F is an imploring interior GE-filter in (X, ξ). □

Given a subset F of X in (X, ξ), consider the following argument.

$$(\forall x, y \in X)(\xi((x * y) * y) \in F \;\Rightarrow\; (y * x) * x \in F). \tag{3.2}$$

The following example shows that (imploring) interior GE-filter F in (X, ξ) does not satisfy the argument (3.2).

Example 3.7. (1) If we consider the interior GE-algebra (X, ξ) in Example 3.5, then the set $F := \{1, a, b\}$ is an interior GE-filter in (X, ξ). But it does not satisfy the argument (3.2) since $\xi((c * d) * d) = \xi(d * d) = \xi(1) = 1 \in F$ but $(d * c) * c = 1 * c = c \notin F$.

(2) Let $X = \{1, a, b, c, d, e\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	d	d
b	1	1	1	c	d	d
c	1	a	a	1	e	e
d	1	a	a	1	1	1
e	1	a	a	c	1	1

If we define a mapping ξ on X as follows:

$$\xi : X \to X,\; x \mapsto \begin{cases} 1 & \text{if } \tilde{x} = 1, \\ a & \text{if } \tilde{x} \in \{a, b\}, \\ c & \text{if } \tilde{x} = c, \\ d & \text{if } \tilde{x} \in \{d, e\}, \end{cases}$$

then (X, ξ) is an interior GE-algebra and the set $F := \{1, a, b\}$ is an imploring interior GE-filter in (X, ξ). But it does not satisfy the argument (3.2) since $\xi((c * d) * d) = \xi(e * d) = \xi(1) = 1 \in F$ but $(d * c) * c = 1 * c = c \notin F$.

We explore conditions under which imploring interior GE-filter can satisfy the argument (3.2).

Proposition 3.8. *If* (X, ξ) *is a pre-transitive interior GE-algebra, then every imploring interior GE-filter F satisfies the argument* (3.2).

Proof. Let F be an imploring interior GE-filter in a pre-transitive interior GE-algebra (X, ξ). Then F is an interior GE-filter in (X, ξ) by Theorem 3.4. Let $x, y \in X$ be such that $\xi((x * y) * y) \in F$. Then $(x * y) * y \in F$ by (2.27). Since $x \leq (y * x) * x$ by (2.9), it follows from (2.12) that $((y * x) * x) * y \leq x * y$. Hence

$$\begin{aligned}(x * y) * y &\leq (y * x) * ((x * y) * x)\\ &\leq (x * y) * ((y * x) * x)\\ &\leq (((y * x) * x) * y) * ((y * x) * x)\end{aligned}$$

by (2.7), (2.12) and (2.13). Using (GE2) and Lemma 2.6 derive

$$1 * ((((y * x) * x) * y) * ((y * x) * x)) = (((y * x) * x) * y) * ((y * x) * x) \in F,$$

which implies from (2.20) that $(y * x) * x \in F$. □

Corollary 3.9. *If* (X, ξ) *is a pre-belligerent interior GE-algebra, then every imploring interior GE-filter F satisfies the argument* (3.2).

Question. *If* (X, ξ) *is a pre-transitive interior GE-algebra, then*

1. is any interior GE-filter an imploring interior GE-filter?
2. does any interior GE-filter F satisfy the argument (3.2)*?*

The following example provides a negative answer to the above Question.

Example 3.10. Let $X = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	d
b	1	1	1	c	d
c	1	a	a	1	d
d	1	a	b	1	1

If we define a mapping ξ on X as follows:

$$\xi : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1,\\ a & \text{if } x \in \{a, b\},\\ c & \text{if } x \in \{c, d\},\end{cases}$$

then (X, ξ) is a pre-transitive interior GE-algebra. It is routine to verify that the set $F := \{1, a, b\}$ is an interior GE-filter in (X, ξ). But it is not an imploring interior GE-filter in (X, ξ) since $1 * ((c * d) * c) = 1 * (d * c) = 1 * 1 = 1 \in F$ and $1 \in F$ but $c \notin F$. Also, it does not satisfy (3.2) since $\xi((c * d) * d) = \xi(d * d) = \xi(1) = 1 \in F$ but $(d * c) * c = 1 * c = c \notin F$.

We consider conditions for an interior GE-filter to be an imploring interior GE-filter.

Theorem 3.11. *Let F be an interior GE-filter in a pre-transitive interior GE-algebra (X, ξ). If F satisfies the argument* (3.2)*, then it is an imploring interior GE-filter in* (X, ξ)*.*

Proof. Assume that F is an interior GE-filter in a pre-transitive interior GE-algebra (X, ξ) which satisfies the argument (3.2). Let $x, y \in X$ be such that $\xi((x * y) * x) \in F$. Since the combination of (2.5), (2.10) and (2.12) induces

$$(x * y) * x \leq (x * y) * ((x * y) * y) = (x * y) * y,$$

we get $\xi((x * y) * y) \in F$ by (2.24) and Lemma 2.6. Hence $(y * x) * x \in F$ by (3.2). Combining (2.6) and (2.12), we get $(x * y) * x \leq y * x$, and so $\xi((x * y) * x) \leq \xi(y * x)$ by (2.24). Since $\xi((x * y) * x) \in F$, we obtain $\xi(y * x) \in F$ by Lemma 2.6. If follows from (2.27) that $y * x \in F$. Since $(y * x) * x \in F$, we have $x \in F$ by (2.16). Therefore F is an imploring interior GE-filter in (X, ξ) by Theorem 3.6. □

Corollary 3.12. *Let F be an interior GE-filter in a pre-belligerent interior GE-algebra (X, ξ). If F satisfies the the argument* (3.2)*, then it is an imploring interior GE-filter in* (X, ξ)*.*

In the following example, we can confirm that an imploring interior GE-filter is independent to a belligerent interior GE-filter.

Example 3.13. (1) Let $X = \{1, a, b, c, d, e\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	d	d
b	1	1	1	c	d	d
c	1	a	a	1	d	d
d	1	a	a	c	1	1
e	1	a	a	1	1	1

If we define a mapping ξ on X as follows:

$$\xi : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \\ d & \text{if } x = d, \\ e & \text{if } x = e, \end{cases}$$

then (X, ξ) is an interior GE-algebra which is not pre-transitive since

$$(e * c) * ((a * e) * (a * c)) = 1 * (d * c) = 1 * c = c \neq 1.$$

We can observe that the set $F := \{1, a, b\}$ is an imploring interior GE-filter in (X, ξ). But F can not be a belligerent interior GE-filter in (X, ξ) because $d * (e * c) = d * 1 = 1 \in F$ and $d * e = 1 \in F$ but $d * c = c \notin F$.

(2) Let $X = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	1	b	1	1

If we define a mapping ξ on X as follows:

$$\xi : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \\ d & \text{if } x = d, \end{cases}$$

then (X, ξ) is an interior GE-algebra which is not pre-transitive since

$$(a * b) * ((d * a) * (d * b)) = 1 * (1 * b) = 1 * b = b \neq 1.$$

Let $F = \{1, a, b\}$. Then we can observe that F is a belligerent interior GE-filter in (X, ξ). But F is not imploring interior GE-filter in (X, ξ) since $1 * ((c * d) * c) = 1 * (d * c) = 1 * 1 = 1 \in F$ and $1 \in F$ but $c \notin F$.

We explore the conditions under which imploring interior GE-filter becomes belligerent interior GE-filter.

Lemma 3.14. *Let (X, ξ) be a pre-transitive interior GE-algebra. Then every interior GE-filter is a belligerent interior GE-filter.*

Proof. Let F be an interior GE-filter in (X, ξ). Clearly the argument (2.27) is valid and F contains the constant 1. Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x * y \in F$. By (2.7), (2.12) and (2.5), we have

$$x * (y * z) \leq y * (x * z) \leq (x * y) * (x * (x * z)) = (x * y) * (x * z).$$

Since F is a GE-filter of X and $x * (y * z) \in F$, we get $(x * y) * (x * z) \in F$. Hence $x * z \in F$ by Lemma 2.6. Therefore F is a belligerent interior GE-filter in (X, ξ). □

Corollary 3.15. *In a pre-transitive interior GE-algebra, every imploring interior GE-filter is a belligerent interior GE-filter.*

Corollary 3.16. *In a pre-belligerent interior GE-algebra, every imploring interior GE-filter is a belligerent interior GE-filter.*

The following example shows that the converse of Corollary 3.15 and Corollary 3.16 is not true in general.

Example 3.17. (1) Consider the pre-transitive interior GE-algebra (X, ξ) which is described in Example 3.10. As

$$d * (c * b) = d * a = a \neq b = 1 * b = (d * c) * (d * b),$$

it is not pre-belligerent. Then we can observe that $F := \{1, a, b\}$ is a belligerent interior GE-filter in (X, ξ). But F is not imploring interior GE-filter in (X, ξ) since $1 * ((c * d) * c) = 1 * (d * c) = 1 * 1 = 1 \in F$ and $1 \in F$ but $c \notin F$.

(2) Let $X = \{1, a, b, c, d\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	d
b	1	1	1	c	d
c	1	a	a	1	d
d	1	a	a	1	1

If we define a mapping ξ on X as follows:

$$\xi : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x \in \{c, d\}, \end{cases}$$

then (X, ξ) is a pre-belligerent interior GE-algebra. Let $F = \{1, a, b\}$. Then we can observe that F is a belligerent interior GE-filter in (X, ξ). But F is not imploring interior GE-filter in (X, ξ) since $1 * ((c * d) * c) = 1 * (d * c) = 1 * 1 = 1 \in F$ and $1 \in F$ but $c \notin F$.

We establish a relationship between imploring interior GE-filter and prominent interior GE-filter.

Theorem 3.18. *In a GE-algebra, every prominent interior GE-filter is an imploring interior GE-filter.*

Proof. Let F be a prominent interior GE-filter in (X, ξ). Then it is an interior GE-filter in (X, ξ) (see Theorem 3.4 in [10]), and so $1 \in F$ and F satisfies (2.27). Let $x, y, z \in X$ be such that $x * ((y * z) * y) \in F$ and $x \in F$. Since F is a GE-filter of X, we have $(y * z) * y \in F$. Since F is a prominent GE-filter of X, it follows from (GE1), (GE2), (2.5) and Lemma 2.8 that

$$y = 1 * y = ((y * z) * (y * z)) * y = (((y * (y * z)) * (y * z)) * y \in F.$$

Therefore F is an imploring interior GE-filter in (X, ξ). □

The converse of Theorem 3.18 is not true as seen in the following example.

Example 3.19. Consider the interior GE-algebra (X, ξ) in Example 3.7(2). It is not pre-transitive because

$$(d * c) * ((e * d) * (e * c)) = 1 * (1 * c) = 1 * c = c \neq 1.$$

Let $F := \{1, a, b\}$. Then we can observe that F is an imploring interior GE-filter in (X, ξ). But F is not a prominent interior GE-filter in (X, ξ) since $1 * (d * c) = 1 * 1 = 1 \in F$ and $1 \in F$ but $((c * d) * d) * c = (e * d) * c = 1 * c = c \notin F$.

The combination of Theorem 3.18 and Corollary 3.15 induces the next corollary.

Corollary 3.20. *In a pre-transitive GE-algebra, every prominent interior GE-filter is a belligerent interior GE-filter.*

Consider the pre-transitive interior GE-algebra (X, ξ) which is described in Example 3.10. As

$$d * (c * b) = d * a = a \neq b = 1 * b = (d * c) * (d * b),$$

it is not pre-belligerent. Then we can observe that $F := \{1, a, b\}$ is a belligerent interior GE-filter in (X, ξ). But F is not a prominent interior GE-filter in (X, ξ) since $1 * (d * c) = 1 * 1 = 1 \in F$ and $1 \in F$ but $((c * d) * d) * c = (d * d) * c = 1 * c = c \notin F$. Hence we know that the converse of Corollary 3.20 is not true in general.

We can strengthen the conditions of interior GE-algebra so that imploring interior GE-filter becomes prominent interior GE-filter.

Theorem 3.21. *If (X, ξ) is a pre-transitive interior GE-algebra, then every imploring interior GE-filter is a prominent interior GE-filter.*

Proof. Let F be an imploring interior GE-filter in a pre-transitive interior GE-algebra (X, ξ). Then F satisfies (2.27) and it is an interior GE-filter in (X, ξ) (see Theorem 3.4), and so F is a GE-filter of X. Let $x, y \in X$ be such that $x * y \in F$. Note that $y \leq ((y * x) * x) * y$ by (2.6), and thus $(((y * x) * x) * y) * x \leq y * x$ by (2.12). It follows from (2.2), (2.7) and (2.12) that

$$\begin{aligned} x * y &\leq ((y * x) * x) * ((y * x) * y) \\ &\leq (y * x) * (((y * x) * x) * y) \\ &\leq ((((y * x) * x) * y) * x) * (((y * x) * x) * y). \end{aligned}$$

Hence $((((y * x) * x) * y) * x) * (((y * x) * x) * y) \in F$ by Lemma 2.6, and so

$$\begin{aligned} & 1 * (((((y * x) * x) * y) * x) * (((y * x) * x) * y)) \\ &= ((((y * x) * x) * y) * x) * (((y * x) * x) * y) \in F \end{aligned}$$

by (GE2). Since $1 \in F$, we have $((y * x) * x) * y \in F$ by (2.20). This shows that F is a prominent GE-filter of X by Lemma 2.8, and therefore F is a prominent interior GE-filter in (X, ξ). □

Corollary 3.22. *If (X, ξ) is a pre-belligerent interior GE-algebra, then every imploring interior GE-filter is a prominent interior GE-filter.*

The following example shows that prominent interior GE-filter and belligerent interior GE-filter are independent of each other.

Example 3.23. (1) Let $X = \{1, a, b, c, d, e\}$ and define binary operation $*$ as follows:

$*$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	c	d	e
b	1	1	1	c	d	e
c	1	a	a	1	d	1
d	1	a	a	c	1	1
e	1	a	a	1	1	1

If we define a mapping ξ on X as follows:

$$\xi : X \to X,\ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ a & \text{if } x \in \{a, b\}, \\ c & \text{if } x = c, \\ d & \text{if } x = d, \\ e & \text{if } x = e, \end{cases}$$

then (X, ξ) is an interior GE-algebra. We can observe that the set $F := \{1, a, b\}$ is a prominent interior GE-filter in (X, ξ). But it is not a belligerent interior GE-filter in (X, ξ) because of $d * (e * c) = d * 1 = 1 \in F$ and $d * e = 1 \in F$ but $d * c = c \notin F$.

(2) In Example 3.13(2), we can observe that $F := \{1, a, b\}$ is a belligerent interior GE-filter in (X, ξ). But it is not a prominent interior GE-filter in (X, ξ) since $1 * (d * c) = 1 * 1 = 1 \in F$ and $1 \in F$ but $((c * d) * d) * c = (d * d) * c = 1 * c = c \notin F$.

We build the extension property of imploring interior GE-filter.

Lemma 3.24. *In a pre-transitive interior GE-algebra (X, ξ), every interior GE-filter F satisfies:*

$$(\forall x, y, z \in X)(\xi(x * (y * z)) \in F \ \Rightarrow\ (x * y) * (x * z) \in F). \tag{3.3}$$

Proof. Let $x, y, z \in X$ be such that $\xi(x * (y * z)) \in F$. Since

$$\begin{aligned} x * (y * z) &\leq x * ((x * y) * (x * z)) \\ &\leq x * (x * ((x * y) * z)) \\ &= x * ((x * y) * z) \\ &\leq (x * y) * (x * z), \end{aligned}$$

we get $\xi(x * (y * z)) \leq \xi((x * y) * (x * z))$ by (2.24). It follows from Lemma 2.6 and (2.27) that $(x * y) * (x * z) \in F$. □

Theorem 3.25. *Let F and G be interior GE-filters in a pre-transitive interior GE-algebra (X, ξ). If F is contained in G and F is an imploring interior GE-filter in (X, ξ), then G is also an imploring interior GE-filter in (X, ξ).*

Proof. Assume that $F \subseteq G$ and F is an imploring interior GE-filter in (X, ξ). Let $x, y \in X$ be such that $\xi((x * y) * y) \in G$. Then $(x * y) * y \in G$ by (2.27). Since $\xi(((x * y) * y) * ((x * y) * y)) = \xi(1) = 1 \in F$, It follows from Lemma 3.24 that $(((x * y) * y) * (x * y)) * (((x * y) * y) * y) \in F$. Using (2.7) and (2.12), we have

$$\begin{aligned} &(((x * y) * y) * (x * y)) * (((x * y) * y) * y) \\ &\leq (x * (((x * y) * y) * y)) * (((x * y) * y) * y). \end{aligned}$$

Hence $(x * (((x * y) * y) * y)) * (((x * y) * y) * y) \in F$ by Lemma 2.6, and so

$$\xi((x * (((x * y) * y) * y)) * (((x * y) * y) * y)) \in F$$

by (2.22) and Lemma 2.6. Since F is an imploring interior GE-filter in (X, ξ), we have

$$((((x * y) * y) * y) * x) * x \in F \subseteq G$$

by Proposition 3.8. Note that

$$\begin{aligned}(x * y) * y &\leq (((x * y) * y) * y) * y \\ &\leq (y * x) * ((((x * y) * y) * y) * x) \\ &\leq (((((x * y) * y) * y) * x) * x) * ((y * x) * x).\end{aligned}$$

It follows from Lemma 2.6 that

$$(((((x * y) * y) * y) * x) * x) * ((y * x) * x) \in G.$$

Thus $(y*x)*x \in G$, and consequently G is an imploring interior GE-filter in (X, ξ) by Theorem 3.11. □

Corollary 3.26. *Let F and G be interior GE-filters in a pre-belligerent interior GE-algebra (X, ξ). If F is contained in G and F is an imploring interior GE-filter in (X, ξ), then G is also an imploring interior GE-filter in (X, ξ).*

4. Conclusions

We have introduced the concept of an imploring interior GE-filter and investigated their properties. We have discussed the relationship between an interior GE-filter and an imploring interior GE-filter. We have given an example to show that any interior GE-filter is not an imploring interior GE-filter. We have given conditions for an interior GE-filter to be an imploring interior GE-filter. We have provided examples to show that an imploring interior GE-filter is independent to a belligerent interior GE-filter. Conditions for an imploring interior GE-filter to be a belligerent interior GE-filter are given. We have discussed the relationship between imploring interior GE-filter and prominent interior GE-filter. We have provided an example to show that any imploring interior GE-filter is not a prominent interior GE-filter. Conditions for an imploring interior GE-filter to be a prominent interior GE-filter are given. Also, we have considered the conditions under which an interior GE-filter larger than a given interior GE-filter can become an imploring interior GE-filter. In future, we will study the prime and maximal imploring interior GE-filters and their topological properties. Moreover, based on the ideas and results obtained in this paper, we will study the interior operator theory in related algebraic systems such as MV-algebra, BL-algebra, EQ-algebra, etc. It will also be used for pseudo algebra systems and further to conduct research related to the very true operator theory.

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable suggestions.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. R. K. Bandaru, A. B. Saeid, Y. B. Jun, On GE-algebras, *Bull. Sect. Logic*, **50** (2021), 81–96. doi: 10.18778/0138-0680.2020.20.
2. R. K. Bandaru, A. B. Saeid, Y. B. Jun, Belligerent GE-filter in GE-algebras, *J. Indones. Math. Soc.*, in press.
3. R. K. Bandaru, M. A. Öztürk, Y. B. Jun, Bordered GE-algebras, *J. Alg. Sys.*, in press.
4. G. Castellini, J. Ramos, Interior operators and topological connectedness, *Quaest. Math.*, **33** (2010), 290–304. doi: 10.2989/16073606.2010.507322.
5. A. Diego, Sur algébres de Hilbert, *Collect. Logique Math. Ser. A*, **21** (1967), 177–198.
6. J. G. Lee, R. K. Bandaru, K. Hur, Y. B. Jun, Interior GE-algebras, *J. Math.*, **2021** (2021), 1–10. doi: 10.1155/2021/6646091.
7. J. Rachůnek, Z. Svoboda, Interior and closure operators on bounded residuated lattices, *Cent. Eur. J. Math.*, **12** (2014), 534–544. doi: 10.2478/s11533-013-0349-y.
8. A. Rezaei, R. K. Bandaru, A. B. Saeid, Y.B. Jun, Prominent GE-filters and GE-morphisms in GE-algebras, *Afr. Mat.*, **32** (2021), 1121–1136. doi: 10.1007/s13370-021-00886-6.
9. S. Z. Song, R. K. Bandaru, Y. B. Jun, Imploring GE-filters of GE-algebras, *J. Math.*, **2021** (2021), 1–7. doi: 10.1155/2021/6651531.
10. S. Z. Song, R. K. Bandaru, Y. B. Jun, Prominent interior GE-filters of GE-algebras, *AIMS Math.*, **6** (2021), 13432–13447. doi: 10.3934/math.2021778.
11. S. Z. Song, R. K. Bandaru, D. A. Romano, Y. B. Jun, Interior GE-filters of GE-algebras, *Discuss. Math. Gen. Algebra Appl.*, in press.
12. F. Svrcek, Operators on GMV-algebras, *Math. Bohem.*, **129** (2004), 337–347. doi: 10.21136/MB.2004.134044.
13. S. J. R. Vorster, Interior operators in general categories, *Quaest. Math.*, **23** (2000), 405–416. doi: 10.2989/16073600009485987.