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Research article

Some new Hardy-Hilbert-type inequalities with multiparameters

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Abstract: The purpose of this paper is to build some new Hardy-Hilbert-type inequalities with multiparameters and their equivalent forms and variants, which generalize some existing results. Similarly, the corresponding Hardy-Hilbert-type integral inequalities are also given.

Keywords: Hardy–Hilbert-type inequality; Hölder's inequality; Beta function
Mathematics Subject Classification: 26D15

1. Introduction

Let $a_n, b_n \geq 0, p > 1, 1/p + 1/q = 1$. If $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\Big[\sum_{n=1}^{\infty} a_n^p\Big]^{1/p}\Big[\sum_{n=1}^{\infty} b_n^q\Big]^{1/q}, \tag{1.1}$$

where the constant $\pi/\sin(\pi/p)$ is the best possible. The inequality (1.1) can be called as the well known Hardy-Hilbert's inequality [1]. An equivalent form of inequality (1.1) is presented as follows.

$$\sum_{n=1}^{\infty}\Big[\sum_{m=1}^{\infty}\frac{a_m}{m+n}\Big]^p < \Big[\frac{\pi}{\sin(\pi/p)}\Big]^p \sum_{n=1}^{\infty} a_n^p, \tag{1.2}$$

where the constant $[\pi/\sin(\pi/p)]^p$ is also the best possible. In connection with applications in analysis, their generalizations and variants have received considerable interest recent years [2–12]. By introducing some parameters, Yang [13] obtained a generalization of Hardy-Hilbert's integral inequality with a best constant factor that involves the beta function. In the paper [14], Das and sahoo considered a generalization of multiple Hardy-Hilbert's inequality with the best constant factor. Sroysang [15] established a generalization on the kinds of Hardy-Hilbert's integral inequality with the weight homogeneous function.

By introducing a parameter, the following extension of (1.1) was obtained by Yang [3]. Let $a_n, b_n \geq 0, p > 1, 1/p + 1/q = 1$. If $0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda\sin(\pi/p)}\Big[\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p\Big]^{1/p}\Big[\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q\Big]^{1/q}, \tag{1.3}$$

where the constat $\pi/(\lambda\sin(\pi/p))$ is the best possible.

Nextly, Sun [8] gave an extension of (1.3) as follows. Let $a_n, b_n \geq 0$, $p > 1$, $1/p + 1/q = 1$, $0 < \lambda < \min(p, q)$, $c > 0$. If $0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\lambda + n^\lambda)^{\frac{c}{\lambda}}} < c_{\lambda,p}\Big[\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p\Big]^{1/p}\Big[\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q\Big]^{1/q}, \tag{1.4}$$

where the constant $c_{\lambda,p} = (1/c)B(\lambda/(cp), \lambda/(cq))$ is the best possible, $B(\bullet, *)$ denotes the beta function.

Finally, by introducing two parameters, Xu [9] presented another extension of (1.3) as follows. Let $a_n, b_n \geq 0, p > 1$, $1/p + 1/q = 1$, $0 < \lambda_1 < p$, $0 < \lambda_2 < q$. If $0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda_1)} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda_2)} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\lambda_1} + n^{\lambda_2}} < \frac{\pi}{{\lambda_1}^{1/q}{\lambda_2}^{1/p}\sin(\pi/p)}\Big[\sum_{n=1}^{\infty} n^{(p-1)(1-\lambda_1)} a_n^p\Big]^{1/p}\Big[\sum_{n=1}^{\infty} n^{(q-1)(1-\lambda_2)} b_n^q\Big]^{1/q}, \tag{1.5}$$

where the constant $\pi/(\lambda_1^{1/q}\lambda_2^{1/p}\sin(\pi/p))$ is the best possible.

Recently, some Hilbert-type integral inequalities with quasi-homogeneous integral kernels and multiple functions were established [16]. By means of the technique of real analysis and the weight functions, Xin [17] obtained an equivalent statements of a Hilbert-type integral inequality with the nonhomogeneous kernel in the whole plane and the best constant factor related to the beta function. By using the weight function and the technique of real analysis, Liu [18, 19] established some multi-parameter Hilbert-type integral inequalities with the hybrid kernel and the best constant factors, respectively. By using weight functions and introducing parameters, Chen and Yang [20] presented a reverse Hardy-Hilbert-type integral inequality involving one derivative function and the beta function. Based on the theory of operators, Hong et al. [21] obtained a necessary and sufficient condition and the best constant factor for the Hilbert-type multiple integral inequality with kernel. Liao et al. [22] investigated a new half-discrete Hilbert-type inequality involving the variable upper limit integral and partial sums and proved the equivalent conditions of the best possible constant factor related to several parameters.

In the paper, motivated by the mentioned references above, we will obtain a Hardy-Hilbert-type inequality with multiparameters, which can be see as a new generalization of (1.3)–(1.5). And its equivalent form and variant are given. Furthermore, their integral forms are also presented.

2. New Hardy-Hilbert-type inequalities with multiparameters for double series

For convenience, we assume always that $B(\bullet, *)$ represents the beta function throughout the paper.

Theorem 2.1. *Let $a_n, b_n \geq 0$, and $p > 1$, $1/p + 1/q = 1$, $0 < \lambda_1 < p$, $0 < \lambda_2 < q$, $0 < c \leq \lambda$. If $0 < \sum_{n=1}^{\infty} n^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q < \infty$, then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} < c_{\lambda,p}\Big[\sum_{n=1}^{\infty} n^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_n^p\Big]^{1/p}\Big[\sum_{n=1}^{\infty} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q\Big]^{1/q}, \tag{2.1}$$

where the constant $c_{\lambda,p} = (1/\lambda_1^{1/q}\lambda_2^{1/p})B(c/(\lambda p), c/(\lambda q))$ is the best possible.

Remark 2.1. The above inequality presented in (2.1) concretely produces some known Hardy-Hilbert type inequalities based on different settings of the parameters λ_1, λ_2 and λ.

(A1) If $\lambda_1 = \lambda_2 = \lambda = c$, then inequality (2.1) reduces to the inequality (1.3) given in Yang [3].
(A2) If $\lambda_1 = \lambda_2 = \lambda$, then inequality (2.1) develops into the inequality (1.4) presented in Sun [8].
(A3) If $\lambda = c$, then inequality (2.1) converts into the inequality (1.5) given in Xu [9].

The idea of proof of Theorem 2.1 comes similarly from [3] and [9]. To prove the Theorem 2.1, we need some lemmas in the same way, which are new generalizations of some lemmas given in [8, 9].

Lemma 2.1. *Let $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2 > 0$, $0 < c \leq \lambda$. Define the weight function $\omega_{\lambda_1,\lambda_2}(y,p) = \int_0^{\infty}(1/(x^{\lambda_1} + y^{\lambda_2})^{c/\lambda})(y^{\lambda_2}/x^{\lambda_1})^{c/(\lambda p)} x^{c\lambda_1/\lambda - 1}dx$, $y \in (0,\infty)$ and weight coefficient $\tilde{\omega}_{\lambda_1,\lambda_2}(n,p) = \sum_{m=1}^{\infty}(1/(m^{\lambda_1} + n^{\lambda_2})^{c/\lambda})(n^{\lambda_2}/m^{\lambda_1})^{c/(\lambda p)} m^{c\lambda_1/\lambda - 1}$.*
Then (a) *$\omega_{\lambda_1,\lambda_2}(y,p) = (1/\lambda_1)B(c/(\lambda p), c/(\lambda q))$, and* (b) *If $0 < \lambda_1 < p$, and $n \in N$, then $\tilde{\omega}_{\lambda_1,\lambda_2}(n,p) < \omega_{\lambda_1,\lambda_2}(n,p)$ holds.*

Proof. (a) Set $t = x^{\lambda_1}/y^{\lambda_2}$. Then

$$\omega_{\lambda_1,\lambda_2}(y,p) = \int_0^{\infty}\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}\Big(\frac{y^{\lambda_2}}{x^{\lambda_1}}\Big)^{c/(\lambda p)} x^{\frac{c\lambda_1}{\lambda}-1}dx = \frac{1}{\lambda_1}\int_0^{\infty}\frac{1}{(1+t)^{\frac{c}{\lambda}}} t^{\frac{c}{\lambda q}-1}dt = \frac{1}{\lambda_1}B\Big(\frac{c}{\lambda p}, \frac{c}{\lambda q}\Big).$$

(b) Since $1 - (c\lambda_1/\lambda) + (c\lambda_1/\lambda p) > 0$, foxed λ_1, λ_2,and $n \in N$, the function $f(x) = (1/(x^{\lambda_1} + n^{\lambda_2})^{c/\lambda})(n^{\lambda_2}/x^{\lambda_1})^{c/(\lambda p)} x^{c\lambda_1/\lambda - 1}$ is strict deceasing in $(0,\infty)$. Then

$$f(m) < \int_{m-1}^{m} f(x)dx \Rightarrow \sum_{m=1}^{\infty} f(m) < \int_0^{\infty} f(x)dx.$$

So we have $\tilde{\omega}_{\lambda_1,\lambda_2}(n,p) < \omega_{\lambda_1,\lambda_2}(n,p)$. This completes the proof of Lemma 2.1. □

Similar to Lemma 2.1, we can introduce the following lemma.

Lemma 2.2. *Let $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2 > 0$, $0 < c \leq \lambda$. Define the weight function $\bar{\omega}_{\lambda_1,\lambda_2}(x,p) = \int_0^{\infty}(1/(x^{\lambda_1} + y^{\lambda_2})^{c/\lambda})(x^{\lambda_1}/y^{\lambda_2})^{c/(\lambda p)} y^{c\lambda_2/\lambda - 1}dy$, $x \in (0,\infty)$ and weight coefficient $\hat{\omega}_{\lambda_1,\lambda_2}(m,p) = \sum_{n=1}^{\infty}(1/(m^{\lambda_1} + n^{\lambda_2})^{c/\lambda})(m^{\lambda_1}/n^{\lambda_2})^{c/(\lambda p)} n^{c\lambda_2/\lambda - 1}$.*
Then (a) *$\bar{\omega}_{\lambda_1,\lambda_2}(x,p) = (1/\lambda_2)B(c/(\lambda p), c/(\lambda q))$, and* (b) *If $0 < \lambda_2 < q$, and $m \in N$, then $\hat{\omega}_{\lambda_1,\lambda_2}(m,p) < \bar{\omega}_{\lambda_1,\lambda_2}(m,p)$ holds.*

Lemma 2.3. *Let $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2 > 0$, $0 < c \leq \lambda$, $\varepsilon \in (0, q/(2p))$. Then*

$$\int_1^{\infty}\frac{1}{x^{1+\frac{c\lambda_1\varepsilon}{\lambda}}}\int_0^{\frac{1}{x^{\lambda_1}}}\frac{1}{(1+u)^{\frac{c}{\lambda}}} u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx = O(1), \quad \varepsilon \to 0^+. \tag{2.2}$$

Proof. It follows from $\varepsilon \in (0, q/(2p))$ that $1/p - \varepsilon/q > 1/(2p)$. Then we have

$$
\begin{aligned}
0 &< \int_1^\infty \frac{1}{x^{1+\frac{c\lambda_1\varepsilon}{\lambda}}} \int_0^{\frac{1}{x^{\lambda_1}}} \frac{1}{(1+u)^{\frac{c}{\lambda}}} u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1} du dx < \int_1^\infty \frac{1}{x} \int_0^{\frac{1}{x^{\lambda_1}}} \frac{1}{(1+u)^{\frac{c}{\lambda}}} u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1} du dx \\
&< \int_1^\infty \frac{1}{x} \int_0^{\frac{1}{x^{\lambda_1}}} \frac{1}{(1+u)^{\frac{c}{\lambda}}} u^{\frac{c}{2p\lambda}-1} du dx < \frac{2p\lambda}{c} \int_1^\infty x^{-1-\frac{c\lambda_1}{2p\lambda}} dx = \frac{4p^2\lambda^2}{\lambda_1 c^2},
\end{aligned}
$$

which implies (2.2). This finishes the proof of Lemma 2.3. □

Nextly, we will give the proof of Theorem 2.1.

Proof. By using the Hölder's inequality and Lemmas 2.1 and 2.2, we can capture

$$
\begin{aligned}
\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} &= \sum_{n=1}^\infty \sum_{m=1}^\infty \left[\frac{a_m}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda p}}} \left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)^{\frac{c}{\lambda pq}} \frac{n^{\frac{\frac{c\lambda_2}{\lambda}-1}{p}}}{m^{\frac{\frac{c\lambda_1}{\lambda}-1}{q}}} \right] \left[\frac{b_n}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda q}}} \left(\frac{n^{\lambda_2}}{m^{\lambda_1}}\right)^{\frac{c}{\lambda pq}} \frac{m^{\frac{\frac{c\lambda_1}{\lambda}-1}{q}}}{n^{\frac{\frac{c\lambda_2}{\lambda}-1}{p}}} \right] \\
&\leq \left[\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} \left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)^{\frac{c}{\lambda q}} \frac{n^{\frac{c\lambda_2}{\lambda}-1}}{m^{\frac{p(\frac{c\lambda_1}{\lambda}-1)}{q}}} \right]^{1/p} \left[\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{b_n}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} \left(\frac{n^{\lambda_2}}{m^{\lambda_1}}\right)^{\frac{c}{\lambda p}} \frac{m^{\frac{c\lambda_1}{\lambda}-1}}{n^{\frac{q(\frac{c\lambda_2}{\lambda}-1)}{p}}} \right]^{1/q} \\
&= \left[\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} \left(\frac{m^{\lambda_1}}{n^{\lambda_2}}\right)^{\frac{c}{\lambda q}} n^{\frac{c\lambda_2}{\lambda}-1} m^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_m^p \right]^{1/p} \\
&\quad \times \left[\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} \left(\frac{n^{\lambda_2}}{m^{\lambda_1}}\right)^{\frac{c}{\lambda p}} m^{\frac{c\lambda_1}{\lambda}-1} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q \right]^{1/q} \\
&= \left[\sum_{m=1}^\infty \hat{\omega}_{\lambda_1,\lambda_2}(m,q) m^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_m^p \right]^{1/p} \left[\sum_{n=1}^\infty \tilde{\omega}_{\lambda_1,\lambda_2}(n,p) n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q \right]^{1/q} \\
&< \left[\sum_{m=1}^\infty \bar{\omega}_{\lambda_1,\lambda_2}(m,q) m^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_m^p \right]^{1/p} \left[\sum_{n=1}^\infty \omega_{\lambda_1,\lambda_2}(n,p) n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q \right]^{1/q} \\
&= \left[\frac{1}{\lambda_2} B\left(\frac{c}{\lambda p}, \frac{c}{\lambda q}\right) \right]^{1/p} \left[\sum_{m=1}^\infty m^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_m^p \right]^{1/p} \left[\frac{1}{\lambda_1} B\left(\frac{c}{\lambda p}, \frac{c}{\lambda q}\right) \right]^{1/q} \left[\sum_{n=1}^\infty n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q \right]^{1/q} \\
&= c_{\lambda,p} \left[\sum_{m=1}^\infty m^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_m^p \right]^{1/p} \left[\sum_{n=1}^\infty n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q \right]^{1/q},
\end{aligned}
\tag{2.3}
$$

which implies (2.1). Now we prove that the constant $c_{\lambda,p}$ is the best possible. Suppose that the constant $c_{\lambda,p}$ in inequality (2.1) is not the best possible, then there exists positive $k < c_{\lambda,p}$ such that k replaces with $c_{\lambda,p}$ in inequality (2.1) holds all the same. Especially, for $\varepsilon \in (0, q/2p)$, let $\hat{a}_n = (1/n)^{\frac{1+(c\lambda_1\varepsilon/\lambda)+(p-1)(1-c\lambda_1/\lambda)}{p}}$ and $\hat{b}_n = (1/n)^{\frac{1+(c\lambda_2\varepsilon/\lambda)+(q-1)(1-c\lambda_2/\lambda)}{q}}$ for $n \in N$, then we obtain

$$
\sum_{n=1}^\infty n^{(p-1)(1-\frac{c\lambda_1}{\lambda})} \hat{a}_n^p = \sum_{n=1}^\infty \frac{1}{n^{1+\frac{\lambda_1\varepsilon}{\lambda}}} \quad \text{and} \quad \sum_{n=1}^\infty n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} \hat{b}_n^q = \sum_{n=1}^\infty \frac{1}{n^{1+\frac{\lambda_2\varepsilon}{\lambda}}}. \tag{2.4}
$$

On the other hand, we have

$$
\frac{1}{\sigma} = \int_1^\infty \frac{1}{x^{1+\sigma}} dx < \sum_{n=1}^\infty \frac{1}{n^{1+\sigma}} = 1 + \sum_{n=2}^\infty \frac{1}{n^{1+\sigma}} < 1 + \int_1^\infty \frac{1}{x^{1+\sigma}} dx = 1 + \frac{1}{\sigma} (\sigma > 0). \tag{2.5}
$$

It follows from (2.4) and (2.5) that we observe

$$\Big[\sum_{n=1}^{\infty} n^{(p-1)(1-\frac{c\lambda_1}{\lambda})}\hat{a}_n^p\Big]^{1/p}\Big[\sum_{n=1}^{\infty} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})}\hat{b}_n^q\Big]^{1/q} = \Big[\frac{\lambda}{c\lambda_1\varepsilon}+O(1)\Big]^{1/p}\Big[\frac{\lambda}{c\lambda_2\varepsilon}+O(1)\Big]^{1/q}$$
$$=\Big[\frac{\lambda}{c\lambda_1\varepsilon}(1+o(1))\Big]^{1/p}\Big[\frac{\lambda}{c\lambda_2\varepsilon}(1+o(1))\Big]^{1/p} = \frac{\lambda}{c\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}(1+o(1)), \tag{2.6}$$

as ε tends to 0^+. According to the assumption, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\hat{a}_m\hat{b}_n}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}}\Big(\frac{1}{m}\Big)^{\frac{1+(c\lambda_1\varepsilon/\lambda)+(p-1)(1-c\lambda_1/\lambda)}{p}}\Big(\frac{1}{n}\Big)^{\frac{1+(c\lambda_2\varepsilon/\lambda)+(q-1)(1-c\lambda_2/\lambda)}{q}}$$
$$> \int_1^{\infty}\int_1^{\infty}\frac{1}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}\Big(\frac{1}{x}\Big)^{\frac{1+(c\lambda_1\varepsilon/\lambda)+(p-1)(1-c\lambda_1/\lambda)}{p}}\Big(\frac{1}{y}\Big)^{\frac{1+(c\lambda_2\varepsilon/\lambda)+(q-1)(1-c\lambda_2/\lambda)}{q}}dxdy$$
$$=\frac{1}{\lambda_2}\int_1^{\infty}\frac{1}{x^{1+\frac{c\lambda_1\varepsilon}{\lambda}}}\int_{\frac{1}{x^{\lambda_1}}}^{\infty}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx$$
$$=\frac{1}{\lambda_2}\int_1^{\infty}\frac{1}{x^{1+\frac{c\lambda_1\varepsilon}{\lambda}}}\int_0^{\infty}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx - \frac{1}{\lambda_2}\int_1^{\infty}\frac{1}{x^{1+\frac{c\lambda_1\varepsilon}{\lambda}}}\int_0^{\frac{1}{x^{\lambda_1}}}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx,$$

where in the third equality we set that $u = y^{\lambda_2}/x^{\lambda_1}$. At the same time, when ε tends to 0^+, we have

$$\lim_{\varepsilon\to 0^+} B\Big(\frac{c}{\lambda}\Big(\frac{1}{q}+\frac{\varepsilon}{q}\Big),\frac{c}{\lambda}\Big(\frac{1}{p}-\frac{\varepsilon}{q}\Big)\Big) =B\Big(\frac{c}{\lambda q},\frac{c}{\lambda p}\Big) = B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big),$$
$$\int_0^{\infty}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}du =B\Big(\frac{c}{\lambda}\Big(\frac{1}{q}+\frac{\varepsilon}{q}\Big),\frac{c}{\lambda}\Big(\frac{1}{p}-\frac{\varepsilon}{q}\Big)\Big) = B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big)+o(1),$$

which implies that if ε tends to 0^+, the following equations holds true

$$\frac{1}{\lambda_2}\int_1^{\infty}\frac{1}{x^{1+\frac{c\lambda_1\varepsilon}{\lambda}}}\int_0^{\infty}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx = \frac{\lambda}{c\lambda_1\lambda_2\varepsilon}\Big[B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big)+o(1)\Big].$$

It follows from Lemma 2.3 that when ε tends to 0^+, we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\hat{a}_m\hat{b}_n}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} >\frac{\lambda}{c\lambda_1\lambda_2\varepsilon}\Big[B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big)+o(1)\Big]-O(1)$$
$$=\frac{\lambda}{c\lambda_1\lambda_2\varepsilon}B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big)(1+o(1)) = \frac{\lambda}{c\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}c_{\lambda,p}[1+o(1)]. \tag{2.7}$$

Due to the equations (2.6) and (2.7), we can obtain

$$\frac{\lambda}{c\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}c_{\lambda,p}(1+o(1)) < \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\hat{a}_m\hat{b}_n}{(m^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}} < \frac{\lambda}{c\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}k(1+o(1)),$$

as ε tends to 0^+. We have $c_{\lambda,p} < k$, in contradiction with supposition. So the constant $c_{\lambda,p}$ in inequality (2.1) is the best possible. This finishes the proof of Theorem 2.1. □

Theorem 2.2. *Let* $a_m \geq 0$,*and* $p > 1$, $1/p + 1/q = 1$, $0 < \lambda_1 < p$, $0 < \lambda_2 < q$, $0 < c \leq \lambda$. $c_{\lambda,p} = (1/\lambda_1^{1/q}\lambda_2^{1/p})B(c/(\lambda p), c/(\lambda q))$. *If* $0 < \sum_{m=1}^{\infty} m^{(p-1)(1-\lambda_1)} a_m^p < \infty$, *then*

$$0 < \sum_{n=1}^{\infty} n^{\frac{c\lambda_2}{\lambda}-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^{\frac{c}{\lambda}}} \right]^p < c_{\lambda,p}^p \sum_{m=1}^{\infty} m^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_m^p, \tag{2.8}$$

where the constant $c_{\lambda,p}^p$ *is the best possible. Moreover, inequality* (2.8) *is equivalent to inequality* (2.1).

Proof. There exists $k_0 \in N$, such that $\sum_{m=1}^{k}(a_m/(m^{\lambda_1} + n^{\lambda_2})^{c/\lambda}) > 0$ for $k > k_0$. Setting $b_n(k) = n^{c\lambda_2/\lambda-1}[\sum_{m=1}^{k}(a_m/(m^{\lambda_1} + n^{\lambda_2})^{c/\lambda})]^{p-1}$ for $k > k_0$. Thanks to (2.1), then we get

$$\begin{aligned} 0 < \sum_{n=1}^{k} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q(k) &= \sum_{n=1}^{k} n^{\frac{c\lambda_2}{\lambda}-1} \Big[\sum_{m=1}^{k} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^{\frac{c}{\lambda}}} \Big]^p = \sum_{n=1}^{k}\sum_{m=1}^{k} \frac{a_m b_n(k)}{(m^{\lambda_1} + n^{\lambda_2})^{\frac{c}{\lambda}}} \\ &< c_{\lambda,p} \Big[\sum_{n=1}^{k} n^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_n^p \Big]^{1/p} \Big[\sum_{n=1}^{k} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q(k) \Big]^{1/q}, \end{aligned}$$

which imply the following inequality

$$\sum_{n=1}^{k} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q(k) < c_{\lambda,p}^p \sum_{n=1}^{k} n^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_n^p.$$

Letting k tends to ∞, we have $0 < \sum\limits_{n=1}^{\infty} n^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_n^p < \infty$. Then we obtain

$$0 < \sum_{n=1}^{\infty} n^{\frac{c\lambda_2}{\lambda}-1} \Big[\sum_{m=1}^{\infty} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^{\frac{c}{\lambda}}} \Big]^p < c_{\lambda,p}^p \sum_{m=1}^{\infty} m^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_m^p.$$

Therefore, inequality (2.8) holds from inequality (2.1).

On the other hand, if inequality (2.8) holds, by applying the Hölder's inequality, we have

$$\begin{aligned} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m^{\lambda_1} + n^{\lambda_2})^{\frac{c}{\lambda}}} &= \sum_{n=1}^{\infty} n^{\frac{\frac{c\lambda_2}{\lambda}-1}{p}} \sum_{m=1}^{\infty} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^{\frac{c}{\lambda}}} \Big[n^{\frac{1-\frac{c\lambda_2}{\lambda}}{p}} b_n \Big] \\ &< \Big[\sum_{n=1}^{\infty} n^{\frac{c\lambda_2}{\lambda}-1} \Big[\sum_{m=1}^{\infty} \frac{a_m}{(m^{\lambda_1} + n^{\lambda_2})^{\frac{c}{\lambda}}} \Big]^p \Big]^{1/p} \Big[\sum_{n=1}^{\infty} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q \Big]^{1/q} \\ &< c_{\lambda,p} \Big[\sum_{n=1}^{\infty} n^{(p-1)(1-\frac{c\lambda_1}{\lambda})} a_n^p \Big]^{1/p} \Big[\sum_{n=1}^{\infty} n^{(q-1)(1-\frac{c\lambda_2}{\lambda})} b_n^q \Big]^{1/q}, \end{aligned}$$

which means that inequality (2.1) holds. Since the constant $c_{\lambda,p}$ in inequality (2.1) is the best possible, and inequality (2.1) is the equivalent to inequality (2.8), the constant $c_{\lambda,p}$ in inequality (2.8) is also the best possible. This completes the proof of Theorem 2.2. □

Now we present a variant of Theorem 2.1 as follows.

Theorem 2.3. *Let $a_n, b_n \geq 0$, and $p > 1$, $1/p + 1/q = 1$, $0 < \lambda_1 < p$, $0 < \lambda_2 < q$, $0 < c \leq \lambda$. If $0 < \sum_{n=2}^{\infty}(\ln n^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}n^{p-1}a_n^p < \infty$, and $0 < \sum_{n=2}^{\infty}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q < \infty$, then*

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}} < c_{\lambda,p}\left[\sum_{n=2}^{\infty}(\ln n^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}n^{p-1}a_n^p\right]^{1/p}\left[\sum_{n=2}^{\infty}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q\right]^{1/q}, \tag{2.9}$$

where the constant $c_{\lambda,p} = (1/\lambda_1^{1/q}\lambda_2^{1/p})B(c/(\lambda p), c/(\lambda q))$ is the best possible.

Remark 2.2. The above inequality presented in (2.9) concretely produces some known Hardy-Hilbert type inequalities based on different settings of the parameters λ_1, λ_2 and λ.

(B1) If $\lambda_1 = \lambda_2 = \lambda = c = 1$, Theorem 2.3 reduces to Theorem 2.1 given by Yang [4].
(B2) If $\lambda = c$, then Theorem 2.3 is converted into Theorem 4 obtained by Xu [9].
(B3) If $\lambda_1 = \lambda_2 = \lambda = c = 1$ and a_m, b_n are replaced by $a_m/m, b_n/n$, respectively, then the inequality (2.9) is developed into the Muholland's inequality presented in [10].

When a_m, b_n are replaced by $a_m/m, b_n/n$ in inequality (2.9), respectively, it is easy to obtain that inequality (2.9) is equivalent to the following new variant of Muholland's inequality.

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{mn(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}} < c_{\lambda,p}\left[\sum_{n=2}^{\infty}(\ln n^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}n^{-1}a_n^p\right]^{1/p}\left[\sum_{n=2}^{\infty}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{-1}b_n^q\right]^{1/q},$$

where $0 < \sum_{n=2}^{\infty}(\ln n^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}n^{-1}a_n^p < \infty$, $0 < \sum_{n=2}^{\infty}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{-1}b_n^q < \infty$, and the constant $c_{\lambda,p} = (1/\lambda_1^{1/q}\lambda_2^{1/p})B(c/(\lambda p), c/(\lambda q))$ is also the best possible.

The method of proof of Theorem 2.3 comes from [4, 9]. To prove Theorem 2.3, we need the following two lemmas, which are new generalizations of Lemma 3 and Lemma 4 in [9].

Lemma 2.4. *Let $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2 > 0, 0 < c \leq \lambda$. Define the weight function $\omega'_{\lambda_1,\lambda_2}(y,p) = \int_1^{\infty}(1/(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}})(\ln y^{\lambda_2}/\ln x^{\lambda_1})^{\frac{c}{\lambda p}}(\ln x^{\lambda_1})^{\frac{c}{\lambda}-1}x^{-1}dx$, $y \in (0,\infty)$, and weight coefficient $\tilde{\omega}'_{\lambda_1,\lambda_2}(n,p) = \sum_{m=2}^{\infty}(1/(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}})(\ln n^{\lambda_2}/\ln m^{\lambda_1})^{\frac{c}{\lambda p}}(\ln m^{\lambda_1})^{\frac{c}{\lambda}-1}m^{-1}$. Then* (a) *$\omega'_{\lambda_1,\lambda_2}(y,p) = (1/\lambda_1)B(c/(\lambda p), c/(\lambda q))$, and* (b) *If $n \in N$ and $n \geq 2$, then $\tilde{\omega}'_{\lambda_1,\lambda_2}(n,p) < \omega'_{\lambda_1,\lambda_2}(n,p)$ holds.*

Proof. (a) Set $t = \ln x^{\lambda_1}/\ln y^{\lambda_2}$. Then

$$\omega'_{\lambda_1,\lambda_2}(y,p) = \int_1^{\infty}\frac{1}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{\ln y^{\lambda_2}}{\ln x^{\lambda_1}}\right)^{\frac{c}{\lambda p}}(\ln x^{\lambda_1})^{\frac{c}{\lambda}-1}x^{-1}dx = \frac{1}{\lambda_1}\int_0^{\infty}\frac{1}{(1+t)^{\frac{c}{\lambda}}}t^{\frac{c}{\lambda q}-1}dt = \frac{1}{\lambda_1}B\left(\frac{c}{\lambda p}, \frac{c}{\lambda q}\right).$$

(b) For fixed λ_1, λ_2,and $2 \leq n \in N$, the function $f(x) = (1/(\ln x^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}})(\ln n^{\lambda_2}/\ln x^{\lambda_1})^{\frac{c}{\lambda s}}(\ln x^{\lambda_1})^{\frac{c}{\lambda}-1}x^{-1}$ is strict deceasing in $(1,\infty)$. Then $f(m) < \int_{m-1}^{m}f(x)dx$. Furthermore, we can obtain $\sum_{m=1}^{\infty}f(m) < \int_0^{\infty}f(x)dx$. So we have $\tilde{\omega}'_{\lambda_1,\lambda_2}(n,s) < \omega'_{\lambda_1,\lambda_2}(n,s)$. □

Similar to Lemma 2.4, we can get the following lemma.

Lemma 2.5. *Let $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2 > 0, 0 < c \leq \lambda$. Define the weight function $\bar{\omega}'_{\lambda_1,\lambda_2}(x,p) = \int_1^{\infty}(1/(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}})(\ln x^{\lambda_1}/\ln y^{\lambda_2})^{\frac{c}{\lambda p}}(\ln y^{\lambda_2})^{\frac{c}{\lambda}-1}y^{-1}dy$, $x \in (0,\infty)$, and weight coefficient $\hat{\omega}'_{\lambda_1,\lambda_2}(m,p) = \sum_{n=2}^{\infty}(1/(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}})(\ln m^{\lambda_1}/\ln n^{\lambda_2})^{\frac{c}{\lambda p}}(\ln n^{\lambda_2})^{\frac{c}{\lambda}-1}n^{-1}$. Then* (a) *$\bar{\omega}'_{\lambda_1,\lambda_2}(x,p) = (1/\lambda_2)B(c/(\lambda p), c/(\lambda q))$, and* (b) *If $m \in N$, and $m \geq 2$ then $\hat{\omega}'_{\lambda_1,\lambda_2}(m,p) < \bar{\omega}'_{\lambda_1,\lambda_2}(m,p)$ holds.*

Nextly, we will give the proof of Theorem 2.1.

Proof. By using the Hölder's inequality and Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(\ln m^{\lambda_1} n^{\lambda_2})^{\frac{c}{\lambda}}} =& \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\left[\frac{a_m}{(\ln m^{\lambda_1} n^{\lambda_2})^{\frac{c}{\lambda p}}}\left(\frac{\ln m^{\lambda_1}}{\ln n^{\lambda_2}}\right)^{\frac{c}{\lambda pq}}(\ln n^{\lambda_2})^{\frac{1}{p}(\frac{c}{\lambda}-1)}(\ln m^{\lambda_1})^{\frac{1}{q}(1-\frac{c}{\lambda})}\frac{m^{\frac{1}{q}}}{n^{\frac{1}{p}}}\right] \\
&\times\left[\frac{b_n}{(\ln m^{\lambda_1} n^{\lambda_2})^{\frac{c}{\lambda q}}}\left(\frac{\ln n^{\lambda_2}}{\ln m^{\lambda_1}}\right)^{\frac{c}{\lambda pq}}(\ln n^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})}(\ln m^{\lambda_1})^{\frac{1}{q}(\frac{c}{\lambda}-1)}\frac{n^{\frac{1}{p}}}{m^{\frac{1}{q}}}\right] \\
&\leq\left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m^p}{(\ln m^{\lambda_1} n^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{\ln m^{\lambda_1}}{\ln n^{\lambda_2}}\right)^{\frac{c}{\lambda q}}(\ln n^{\lambda_2})^{\frac{c}{\lambda}-1}(\ln m^{\lambda_1})^{\frac{p}{q}(1-\frac{c}{\lambda})}\frac{m^{p-1}}{n}\right]^{1/p} \\
&\times\left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{b_n^q}{(\ln m^{\lambda_1} n^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{\ln n^{\lambda_2}}{\ln m^{\lambda_1}}\right)^{\frac{c}{\lambda p}}(\ln n^{\lambda_2})^{\frac{q}{p}(1-\frac{c}{\lambda})}(\ln m^{\lambda_1})^{\frac{c}{\lambda}-1}\frac{n^{q-1}}{m}\right]^{1/q} \\
&=\left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{(\ln m^{\lambda_1} n^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{\ln m^{\lambda_1}}{\ln n^{\lambda_2}}\right)^{\frac{c}{\lambda q}}(\ln n^{\lambda_2})^{\frac{c}{\lambda}-1}n^{-1}(\ln m^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}m^{p-1}a_m^p\right]^{1/p} \\
&\times\left[\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{(\ln m^{\lambda_1} n^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{\ln n^{\lambda_2}}{\ln m^{\lambda_1}}\right)^{\frac{c}{\lambda p}}(\ln m^{\lambda_1})^{\frac{c}{\lambda}-1}m^{-1}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q\right]^{1/q} \\
&=\left[\sum_{m=2}^{\infty}\hat{\omega}'_{\lambda_1,\lambda_2}(m,q)(\ln m^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}m^{p-1}a_m^p\right]^{1/p}\left[\sum_{n=2}^{\infty}\tilde{\omega}'_{\lambda_1,\lambda_2}(n,p)(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q\right]^{1/q} \\
&<\left[\sum_{m=2}^{\infty}\bar{\omega}'_{\lambda_1,\lambda_2}(m,q)(\ln m^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}m^{p-1}a_m^p\right]^{1/p}\left[\sum_{n=2}^{\infty}\omega'_{\lambda_1,\lambda_2}(n,p)(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q\right]^{1/q} \\
&=c_{\lambda,p}\left[\sum_{m=2}^{\infty}(\ln m^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}m^{p-1}a_m^p\right]^{1/p}\left[\sum_{n=2}^{\infty}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q\right]^{1/q},
\end{aligned}$$

which implies (2.9). Now we prove that the constant $c_{\lambda,p}$ is the best possible. Suppose that the constant $c_{\lambda,p}$ in inequality (2.9) is not the best possible, then there exists positive $k < c_{\lambda,p}$, such that k replaces with $c_{\lambda,p}$, inequality (2.9) holds all the same. Especially, for $\varepsilon \in (0, q/(2p))$, set $\hat{a}_n = 1/(n(\ln n^{\lambda_1})^{(1-c/\lambda)/p+(1+c\varepsilon/\lambda)/p)})$ and $\hat{b}_n = 1/(n(\ln n^{\lambda_2})^{(1-c/\lambda)/p+(1+c\varepsilon/\lambda)/q})$ for $n \in N$, then we have

$$\sum_{n=2}^{\infty}(\ln n^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}n^{p-1}\hat{a}_n^p = \sum_{n=2}^{\infty}\frac{1}{n(\ln n^{\lambda_1})^{1+\frac{c\varepsilon}{\lambda}}} \text{ and } \sum_{n=2}^{\infty}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}\hat{b}_n^q = \sum_{n=2}^{\infty}\frac{1}{n(\ln n^{\lambda_2})^{1+\frac{c\varepsilon}{\lambda}}}.$$

As σ tends to 0^+, we have

$$\begin{aligned}
\frac{1}{\lambda^{1+\sigma}\sigma} = \int_e^{\infty}\frac{1}{x(\ln x^{\lambda})^{1+\sigma}}dx < \sum_{n=2}^{\infty}\frac{1}{n(\ln n^{\lambda})^{1+\sigma}} =& \frac{1}{2(\ln 2^{\lambda})^{1+\sigma}} + \frac{1}{3(\ln 3^{\lambda})^{1+\sigma}} + \sum_{n=4}^{\infty}\frac{1}{n(\ln n^{\lambda})^{1+\sigma}} \\
&< O(1) + \int_e^{\infty}\frac{1}{x(\ln x^{\lambda})^{1+\sigma}}dx = O(1) + \frac{1}{\lambda^{1+\sigma}\sigma},
\end{aligned}$$

which implies that as ε tends to 0^+, we obtain

$$\Big[\sum_{n=2}^{\infty}(\ln n^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}n^{p-1}\hat{a}_n^p\Big]^{1/p}\Big[\sum_{n=2}^{\infty}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}\hat{b}_n^q\Big]^{1/q}$$
$$=\Big[\frac{\lambda}{c\lambda_1{}^{1+c\varepsilon/\lambda}\varepsilon}+O(1)\Big]^{1/p}\Big[\frac{\lambda}{c\lambda_2{}^{1+c\varepsilon/\lambda}\varepsilon}+O(1)\Big]^{1/q}=\frac{\lambda}{c\lambda_1^{(1+c\varepsilon/\lambda)/p}\lambda_2^{(1+c\varepsilon/\lambda)/q}\varepsilon}(1+o(1)). \quad (2.10)$$

From the assumptions, we have

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\hat{a}_m\hat{b}_n}{\ln(m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}&=\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{\ln(m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}\frac{1}{m(\ln m^{\lambda_1})^{\frac{1}{q}(1-\frac{c}{\lambda})+\frac{1+\frac{c\varepsilon}{\lambda}}{p}}}\frac{1}{n(\ln n^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})+\frac{1+\frac{c\varepsilon}{\lambda}}{q}}}\\
&>\int_e^{\infty}\int_e^{\infty}\frac{1}{\ln(x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}\frac{1}{x(\ln x^{\lambda_1})^{\frac{1}{q}(1-\frac{c}{\lambda})+\frac{1+\frac{c\varepsilon}{\lambda}}{p}}}\frac{1}{y(\ln y^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})+\frac{1+\frac{c\varepsilon}{\lambda}}{q}}}dxdy\\
&=\frac{1}{\lambda_2}\int_e^{\infty}\frac{1}{x(\ln x^{\lambda_1})^{1+\frac{c\varepsilon}{\lambda}}}\int_{\frac{\lambda_2}{\ln x^{\lambda_1}}}^{\infty}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx\\
&=\frac{1}{\lambda_2}\int_e^{\infty}\frac{1}{x(\ln x^{\lambda_1})^{1+\frac{c\varepsilon}{\lambda}}}\int_0^{\infty}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx\\
&\quad-\frac{1}{\lambda_2}\int_e^{\infty}\frac{1}{x(\ln x^{\lambda_1})^{1+\frac{c\varepsilon}{\lambda}}}\int_0^{\frac{\lambda_2}{\ln x^{\lambda_1}}}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx,\end{aligned}$$

where in the third equality we set that $u=\ln y^{\lambda_2}/\ln x^{\lambda_1}$. Since

$$\begin{aligned}\frac{1}{\lambda_2}\int_e^{\infty}\frac{1}{x(\ln x^{\lambda_1})^{1+\frac{c\varepsilon}{\lambda}}}\int_0^{\infty}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx&=\frac{1}{\lambda_2}\Big\{\frac{\lambda}{c\lambda_1{}^{1+\frac{c\varepsilon}{\lambda}}\varepsilon}\Big[B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big)+o(1)\Big]\Big\}\\
&=\frac{\lambda}{c\lambda_1{}^{1+\frac{c\varepsilon}{\lambda}}\lambda_2\varepsilon}\Big[B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big)+o(1)\Big],\quad \varepsilon\to0^+,\end{aligned}$$

which implies

$$\begin{aligned}0<\int_e^{\infty}\frac{1}{x(\ln x^{\lambda_1})^{1+\frac{c\varepsilon}{\lambda}}}\int_0^{\frac{\lambda_2}{\ln x^{\lambda_1}}}\frac{1}{(1+u)^{\frac{c}{\lambda}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx&<\int_e^{\infty}\frac{1}{x(\ln x^{\lambda_1})^{1+\frac{c\varepsilon}{\lambda}}}\int_0^{\frac{\lambda_2}{\ln x^{\lambda_1}}}u^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})-1}dudx\\
&=\frac{qp^2\lambda^2\lambda_2^{\frac{c}{\lambda}(\frac{1}{p}-\frac{\varepsilon}{q})}}{c^2(1+\varepsilon)(q-p\varepsilon)\lambda_1{}^{\frac{c}{\lambda p}(1+\varepsilon)}}=O(1),\quad \varepsilon\to0^+.\end{aligned}$$

From the above inequalities, we can obtain

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\hat{a}_m\hat{b}_n}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}&>\frac{\lambda}{c\lambda_1{}^{1+c\varepsilon/\lambda}\lambda_2\varepsilon}\Big[B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big)+o(1)\Big]-O(1)\\
&=\frac{\lambda}{c\lambda_1{}^{1+c\varepsilon/\lambda}\lambda_2\varepsilon}B\Big(\frac{c}{\lambda p},\frac{c}{\lambda q}\Big)(1+o(1))=\frac{\lambda}{c\lambda_1^{1/p+c\varepsilon/\lambda}\lambda_2^{1/q}\varepsilon}c_{\lambda,p}(1+o(1)),\quad \varepsilon\to0^+. \quad (2.11)\end{aligned}$$

According to (2.10) and (2.11), then we give

$$\frac{\lambda}{c\lambda_1^{1/p+c\varepsilon/\lambda}\lambda_2^{1/q}\varepsilon}c_{\lambda,p}(1+o(1))<\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\hat{a}_m\hat{b}_n}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}<\frac{\lambda}{c\lambda_1^{(1+c\varepsilon/\lambda)/p}\lambda_2^{(1+c\varepsilon/\lambda)/q}\varepsilon}k(1+o(1)),\quad \varepsilon\to0^+,$$

which impies $c_{\lambda,p} < k$, in contradiction with supposition. So the constant $c_{\lambda,p}$ in inequality (2.9) is the best possible. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let $a_n, b_n \geq 0$, and $p > 1$, $1/p + 1/q = 1$, $0 < \lambda_1 < p$, $0 < \lambda_2 < q$, $0 < c \leq \lambda$, $c_{\lambda,p} = (1/\lambda_1^{1/q}\lambda_2^{1/p})B(c/(\lambda p), c/(\lambda q))$. If $0 < \sum_{n=2}^{\infty}(\ln n^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}n^{p-1}a_n^p < \infty$, then*

$$0 < \sum_{n=2}^{\infty}\frac{1}{n(\ln n^{\lambda_2})^{1-\frac{c}{\lambda}}}\left[\sum_{m=2}^{\infty}\frac{a_m}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}\right]^p < c_{\lambda,p}^p\sum_{m=2}^{\infty}(\ln m^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}m^{p-1}a_m^p, \tag{2.12}$$

where the constant $c_{\lambda,p}^p$ is the best possible. And inequality (2.12) *is equivalent to inequality* (2.9).

Proof. There exists $k_0 \in N$ such that $\sum_{m=2}^{k}a_m/(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}} > 0$ for $k > k_0$. Setting $b_n(k) = 1/(n(\ln n^{\lambda_2})^{1-\frac{c}{\lambda}})[\sum_{m=2}^{k}a_m/(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}]^{p-1}$ for $k > k_0$, then

$$\begin{aligned}0 &< \sum_{n=2}^{k}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q(k) = \sum_{n=2}^{k}\frac{1}{n(\ln n^{\lambda_2})^{1-\frac{c}{\lambda}}}\Big[\sum_{m=2}^{k}\frac{a_m}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}\Big]^p\\ &= \sum_{n=2}^{k}\sum_{m=2}^{k}\frac{a_m b_n(k)}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}} < c_{\lambda,p}\Big[\sum_{m=2}^{k}(\ln m^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}m^{p-1}a_m^p\Big]^{1/p}\Big[\sum_{n=2}^{k}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q(k)\Big]^{1/q},\end{aligned}$$

which implies that the following inequality holds

$$\sum_{n=2}^{k}(\ln n^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}n^{q-1}b_n^q(k) < c_{\lambda,p}^p\sum_{m=2}^{k}(\ln m^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}m^{p-1}a_m^p.$$

As k tends to ∞, we have

$$0 < \sum_{n=2}^{\infty}\frac{1}{n(\ln n^{\lambda_2})^{1-\frac{c}{\lambda}}}\Big[\sum_{m=2}^{\infty}\frac{a_m}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}\Big]^p < c_{\lambda,p}^p\sum_{m=2}^{\infty}(\ln m^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}m^{p-1}a_m^p,$$

which implies inequality (2.12). If inequality (2.12) holds, by applying the Hölder's inequality, then

$$\begin{aligned}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}} &= \sum_{n=2}^{\infty}\Big[\frac{1}{n^{\frac{1}{p}}(\ln n^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})}}\sum_{m=2}^{\infty}\frac{a_m}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}\Big]\Big[n^{\frac{1}{p}}(\ln n^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})}b_n\Big]\\ &< \Big[\sum_{n=2}^{\infty}\frac{1}{n(\ln n^{\lambda_2})^{1-\frac{c}{\lambda}}}\Big[\sum_{m=2}^{\infty}\frac{a_m}{(\ln m^{\lambda_1}n^{\lambda_2})^{\frac{c}{\lambda}}}\Big]^p\Big]^{1/p}\Big[\sum_{n=2}^{\infty}(\ln n^{\lambda_2})^{(q-1)(1-\lambda)}n^{q-1}b_n^q\Big]^{1/q}.\end{aligned}$$

Thanks to the inequality (2.12), this means the inequality (2.9) holds. Since the inequalities (2.9) and (2.12) are equivalent, then $c_{\lambda,p}^p$ is also the best possible. This completes the proof of Theorem 2.4. □

3. New corresponding Hardy-Hilbert-type integral inequalities

In this section, we will give new double integral forms of double series inequalities.

Theorem 3.1. *Let $f(x), g(x) \geq 0$, $p > 1$, $1/p + 1/q = 1$, $\lambda_1, \lambda_2, \lambda, c > 0$. $c_{\lambda,p} = (1/(\lambda_1^{1/q}\lambda_2^{1/p}))B(c/(\lambda p), c/(\lambda q))$. If $0 < \int_0^{\infty}x^{(p-1)(1-\frac{c\lambda_1}{\lambda})}f^p(x)dx < \infty, 0 < \int_0^{\infty}y^{(q-1)(1-\frac{c\lambda_2}{\lambda})}g^q(y)dy < \infty$, then*

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy < c_{\lambda,p}\Big[\int_0^{\infty}x^{(p-1)(1-\frac{c\lambda_1}{\lambda})}f^p(x)dx\Big]^{1/p}\Big[\int_0^{\infty}y^{(q-1)(1-\frac{c\lambda_2}{\lambda})}g^q(y)dy\Big]^{1/q}, \tag{3.1}$$

$$\int_0^\infty y^{\frac{c\lambda_2}{\lambda}-1}\left[\int_0^\infty \frac{f(x)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dx\right]^p dy < c_{\lambda,p}^p \int_0^\infty x^{(p-1)(1-\frac{c\lambda_1}{\lambda})} f^p(x)dx, \tag{3.2}$$

where the constant factors in the inequalities (3.1) *and* (3.2) *are also the best possible. Moreover, the inequalities* (3.1) *and* (3.2) *are equivalent.*

Proof. By mean of Hölder's inequality and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy \\
=&\int_0^\infty \int_0^\infty \left[\frac{f(x)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda p}}}\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)^{\frac{c}{\lambda pq}} \frac{y^{\frac{\frac{c\lambda_2}{\lambda}-1}{p}}}{x^{\frac{\frac{c\lambda_1}{\lambda}-1}{q}}}\right]\left[\frac{g(y)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda q}}}\left(\frac{y^{\lambda_2}}{x^{\lambda_1}}\right)^{\frac{c}{\lambda pq}} \frac{x^{\frac{\frac{c\lambda_1}{\lambda}-1}{q}}}{y^{\frac{\frac{c\lambda_2}{\lambda}-1}{p}}}\right]dxdy \\
\le&\left[\int_0^\infty \int_0^\infty \frac{f^p(x)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)^{\frac{c}{\lambda q}} \frac{y^{\frac{c\lambda_2}{\lambda}-1}}{x^{p(\frac{c\lambda_1}{\lambda}-1)/q}}dxdy\right]^{1/p}\left[\int_0^\infty \int_0^\infty \frac{g^q(y)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{y^{\lambda_2}}{x^{\lambda_1}}\right)^{\frac{c}{\lambda p}} \frac{x^{\frac{c\lambda_1}{\lambda}-1}}{y^{q(\frac{c\lambda_2}{\lambda}-1)/p}}dxdy\right]^{1/q} \\
=&\left[\int_0^\infty dx \int_0^\infty \frac{1}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{x^{\lambda_1}}{y^{\lambda_2}}\right)^{\frac{c}{\lambda q}} y^{\frac{c\lambda_2}{\lambda}-1} x^{(p-1)(1-\frac{c\lambda_1}{\lambda})} f^p(x)dy\right]^{1/p} \\
&\times\left[\int_0^\infty dy \int_0^\infty \frac{1}{(x^{\lambda_1}+n^{\lambda_2})^{\frac{c}{\lambda}}}\left(\frac{y^{\lambda_2}}{x^{\lambda_1}}\right)^{\frac{c}{\lambda p}} x^{\frac{c\lambda_1}{\lambda}-1} y^{(q-1)(1-\frac{c\lambda_2}{\lambda})} g^q(y)dx\right]^{1/q} \\
=&\left[\int_0^\infty \bar{\omega}_{\lambda_1,\lambda_2}(x,q) x^{(p-1)(1-\frac{c\lambda_1}{\lambda})} f^p(x)dx\right]^{1/p}\left[\int_0^\infty \omega_{\lambda_1,\lambda_2}(y,p) y^{(q-1)(1-\frac{c\lambda_2}{\lambda})} g^q(y)dy\right]^{1/q} \\
=&\left[\frac{1}{\lambda_2}B\left(\frac{c}{\lambda p},\frac{c}{\lambda q}\right)\right]^{1/p}\left[\int_0^\infty x^{(p-1)(1-\frac{c\lambda_1}{\lambda})} f^p(x)dx\right]^{1/p}\left[\frac{1}{\lambda_1}B\left(\frac{c}{\lambda p},\frac{c}{\lambda q}\right)\right]^{\frac{1}{q}}\left[\int_0^\infty y^{(q-1)(1-\frac{c\lambda_2}{\lambda})} g^q(y)dy\right]^{1/q} \\
=&c_{\lambda,p}\left[\int_0^\infty x^{(p-1)(1-\frac{c\lambda_1}{\lambda})} f^p(x)dx\right]^{1/p}\left[\int_0^\infty y^{(q-1)(1-\frac{c\lambda_2}{\lambda})} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

According to the hypotheses, it is easy to see that the equality in the above the second inequality is not possible. Now we prove that the constant $c_{\lambda,p}$ is the best possible. If the constant $c_{\lambda,p}$ is not the best possible, then there exists $k < c_{\lambda,p}$ such that k replaces with $c_{\lambda,p}$, inequality (3.1) holds all the same. Especially, for $\varepsilon \in (0, q/(2p))$, we set

$$\tilde{f}(x) = \begin{cases} \frac{1}{x^{\frac{1+\frac{c\lambda_1\varepsilon}{\lambda}+(p-1)(1-\frac{c\lambda_1}{\lambda})}{p}}}, & x \in [0,1), \\ 0, & x \in [1,\infty), \end{cases} \quad \text{and} \quad \tilde{g}(x) = \begin{cases} \frac{1}{x^{\frac{1+\frac{c\lambda_2\varepsilon}{\lambda}+(q-1)(1-\frac{c\lambda_2}{\lambda})}{q}}}, & x \in [0,1), \\ 0, & x \in [1,\infty). \end{cases}$$

Based on the course of the proof of Theorem 2.1, we have

$$\frac{\lambda}{c\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}c_{\lambda,p}(1+o(1)) < \int_0^\infty \int_0^\infty \frac{\tilde{f}(x)\tilde{g}(y)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy < \frac{\lambda}{c\lambda_1^{1/p}\lambda_2^{1/q}\varepsilon}k(1+o(1)),$$

as ε tends to 0^+. We have $c_{\lambda,p} < k$, in contradiction with supposition. So the constant $c_{\lambda,p}$ in the inequality (3.1) is the best possible.

There exists $t_0 > 0$ such that $\int_0^T f(x)/(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}dx > 0$ for $T > t_0$, setting $g(y,T) = y^{\frac{c\lambda_2}{\lambda}-1}[\int_0^T f(x)/(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}dx]^{p-1}$, $y \in (0,\infty)$. Thanks to the inequality (3.1), thus we get

$$\int_0^T y^{(q-1)(1-\frac{c\lambda_2}{\lambda})} g^q(y,T)dy = \int_0^T y^{\frac{c\lambda_2}{\lambda}-1}\left[\int_0^T \frac{f(x)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dx\right]^p dy = \int_0^T \int_0^T \frac{f(x)g(y,T)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy$$

$$<c_{\lambda,p}\Big[\int_0^T x^{(p-1)(1-\frac{c\lambda_1}{\lambda})}f^p(x)dx\Big]^{1/p}\Big[\int_0^T y^{(q-1)(1-\frac{c\lambda_2}{\lambda})}g^q(y,T)dy\Big]^{1/q},$$

which implies that as T tends to ∞, then we have

$$\int_0^\infty y^{\frac{c\lambda_2}{\lambda}-1}\Big[\int_0^\infty \frac{f(x)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dx\Big]^p dy < c_{\lambda,p}^p\int_0^\infty x^{(p-1)(1-\frac{c\lambda_1}{\lambda})}f^p(x)dx,$$

which means the inequality (3.2) holds. If inequality (3.2) holds, by using the Hölder's inequality, then

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy &= \int_0^\infty\Big[y^{\frac{\frac{c\lambda_2}{\lambda}-1}{p}}\int_0^\infty \frac{f(x)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dx\Big]\Big[y^{\frac{1-\frac{c\lambda_2}{\lambda}}{p}}g(y)\Big]dy\\ &<\Big[\int_0^\infty y^{\frac{c\lambda_2}{\lambda}-1}\Big(\int_0^\infty \frac{f(x)}{(x^{\lambda_1}+y^{\lambda_2})^{\frac{c}{\lambda}}}dx\Big)^p dy\Big]^{1/p}\Big[\int_0^\infty x^{(p-1)(1-\frac{c\lambda_1}{\lambda})}f^p(x)dx\Big]^{1/q}.\end{aligned}$$

which implies that the inequality (3.1) holds. Therefore, inequalities (3.1) and (3.2) are equivalent. It is easy to see that the constant $c_{\lambda,p}^p$ in the inequality (3.2) is also the best possible. □

Now we present a new variant of Theorem 3.1 as follows.

Theorem 3.2. *Let $f(x), g(x)\geq 0$, $p>1$, $1/p+1/q=1$, $\lambda_1,\lambda_2,\lambda,c>0$. $c_{\lambda,p}=(1/(\lambda_1^{1/q}\lambda_2^{1/p}))B(c/(\lambda p), c/(\lambda q))$. If $0<\int_1^\infty(\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}x^{p-1}f^p(x)dx<\infty$, and $0<\int_1^\infty(\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}y^{q-1}g^q(y)dy<\infty$, then*

$$\begin{aligned}\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy &< c_{\lambda,p}\Big[\int_1^\infty(\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}x^{p-1}f^p(x)dx\Big]^{1/p}\\ &\times\Big[\int_1^\infty(\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}y^{q-1}g^q(y)dy\Big]^{1/q},\end{aligned}\tag{3.3}$$

$$\int_1^\infty \frac{1}{y(\ln y^{\lambda_2})^{1-\frac{c}{\lambda}}}\Big[\int_1^\infty \frac{f(x)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}dx\Big]^p dy < c_{\lambda,p}^p\int_1^\infty(\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}x^{p-1}f^p(x)dx,\tag{3.4}$$

where the constant factors in the inequalities (3.3) *and* (3.4) *are also the best possible. Furthermore, inequalities* (3.3) *and* (3.4) *are equivalent.*

Proof. By applying the Hölder's inequality and Lemmas 2.4 and 2.5, we have

$$\begin{aligned}\int_1^\infty\int_1^\infty \frac{f(x)g(y)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy &= \int_1^\infty\int_1^\infty\left[\frac{f(x)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda p}}}\left(\frac{\ln x^{\lambda_1}}{\ln y^{\lambda_2}}\right)^{\frac{c}{\lambda pq}}(\ln y^{\lambda_2})^{\frac{1}{p}(\frac{c}{\lambda}-1)}(\ln x^{\lambda_1})^{\frac{1}{q}(1-\frac{c}{\lambda})}\frac{x^{\frac{1}{q}}}{y^{\frac{1}{p}}}\right]\\ &\times\left[\frac{g(y)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda q}}}\left(\frac{\ln y^{\lambda_2}}{\ln x^{\lambda_1}}\right)^{\frac{c}{\lambda pq}}(\ln y^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})}(\ln x^{\lambda_1})^{\frac{1}{q}(\frac{c}{\lambda}-1)}\frac{y^{\frac{1}{p}}}{x^{\frac{1}{q}}}\right]dxdy\\ &\leq\Big[\int_1^\infty\int_1^\infty \frac{f^p(x)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}\Big(\frac{\ln x^{\lambda_1}}{\ln y^{\lambda_2}}\Big)^{\frac{c}{\lambda q}}(\ln y^{\lambda_2})^{\frac{c}{\lambda}-1}(\ln x^{\lambda_1})^{\frac{p}{q}(1-\frac{c}{\lambda})}\frac{x^{p-1}}{y}dxdy\Big]^{1/p}\\ &\times\Big[\int_1^\infty\int_1^\infty \frac{g^q(y)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}\Big(\frac{\ln y^{\lambda_2}}{\ln x^{\lambda_1}}\Big)^{\frac{c}{\lambda p}}(\ln y^{\lambda_2})^{\frac{q}{p}(1-\frac{c}{\lambda})}(\ln x^{\lambda_1})^{\frac{c}{\lambda}-1}\frac{y^{q-1}}{x}dxdy\Big]^{1/q}\end{aligned}$$

$$=\Big[\int_1^\infty dx\int_1^\infty \frac{1}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}\Big(\frac{\ln x^{\lambda_1}}{\ln y^{\lambda_2}}\Big)^{\frac{c}{\lambda q}}(\ln y^{\lambda_2})^{\frac{c}{\lambda}-1}y^{-1}(\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}x^{p-1}f^p(x)dy\Big]^{1/p}$$
$$\times\Big[\int_1^\infty dy\int_1^\infty \frac{1}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}\Big(\frac{\ln y^{\lambda_2}}{\ln x^{\lambda_1}})^{\frac{c}{\lambda p}}(\ln x^{\lambda_1}\Big)^{\frac{c}{\lambda}-1}x^{-1}(\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}y^{q-1}g^q(y)dx\Big]^{1/q}$$
$$=\Big[\int_1^\infty \bar{\omega}'_{\lambda_1,\lambda_2}(x,q)(\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}x^{p-1}f^p(x)dx\Big]^{1/p}\Big[\int_1^\infty \omega'_{\lambda_1,\lambda_2}(y,p)(\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}y^{q-1}g^q(y)dy\Big]^{1/q}$$
$$=c_{\lambda,p}\Big[\int_1^\infty(\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}x^{p-1}f^p(x)dx\Big]^{1/p}\Big[\int_1^\infty(\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}y^{q-1}g^q(y)dy\Big]^{1/q},$$

which implies the inequality (3.3). According to the hypotheses, it is easy to see that the equality in the above the second inequality is not possible. Now we prove that the constant $c_{\lambda,p}$ is the best possible. Suppose that the constant $c_{\lambda,p}$ in inequality (3.3) is not the best possible, then exists positive $k < c_{\lambda,p}$ such that when k replaces with $c_{\lambda,p}$, the inequality (3.3) holds all the same. Especially, for $\varepsilon \in (0, q/(2p))$, we set

$$\hat{f}(x)=\begin{cases}\frac{1}{x(\ln x^{\lambda_1})^{\frac{1}{q}(1-\frac{c}{\lambda})+\frac{1+\frac{c\varepsilon}{\lambda}}{p}}}, & x\in[0,1),\\ 0, & x\in[1,\infty),\end{cases}\quad \text{and}\quad \hat{g}(x)=\begin{cases}\frac{1}{x(\ln x^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})+\frac{1+\frac{c\varepsilon}{\lambda}}{q}}}, & x\in[0,1),\\ 0, & x\in[1,\infty).\end{cases}$$

Due to the course of the proof of Theorem 2.3, we have

$$\frac{\lambda}{c\lambda_1^{1/p+c\varepsilon/\lambda}\lambda_2^{1/q}\varepsilon}c_{\lambda,p}(1+o(1))<\int_1^\infty\int_1^\infty\frac{\hat{f}(x)\hat{g}(y)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy<\frac{\lambda}{c\lambda_1^{(1+c\varepsilon/\lambda)/p}\lambda_2^{(1+c\varepsilon/\lambda)/q}\varepsilon}k(1+o(1)),$$

as ε tends to 0^+. We have $c_{\lambda,p} < k$, in contradiction with supposition. So the constant $c_{\lambda,p}$ in inequality (3.3) is the best possible.

There exists $t_0 > 0$ such that $\int_1^T f(x)/(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}dx > 0$ for $T > t_0$. Setting $g(y,T) = (1/(y(\ln y^{\lambda_2})^{1-c/\lambda}))[\int_1^T f(x)/(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}dx]^{p-1}$ for $T > t_0$, then

$$\int_1^T(\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}y^{q-1}g^q(y,T)dy=\int_1^T\int_1^T\frac{f(x)g(y,T)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy$$
$$<c_{\lambda,p}\Big[\int_1^T(\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}x^{p-1}f^p(x)dx\Big]^{1/p}\Big[\int_1^T(\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})}y^{q-1}g^q(y,T)dy\Big]^{1/q},$$

which implies that as T tends to ∞, then we have

$$\int_1^\infty\frac{1}{y(\ln y^{\lambda_2})^{1-\frac{c}{\lambda}}}\Big[\int_1^\infty\frac{f(x)}{(\ln x^{\lambda_1}x^{\lambda_2})^{\frac{c}{\lambda}}}dx\Big]^p dy<c_{\lambda,p}^p\int_1^\infty(\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})}x^{p-1}f^p(x)dx,$$

which implies the inequality (3.4). If inequality (3.4) holds, by Hölder's inequality, then

$$\int_1^\infty\int_1^\infty\frac{f(x)g(y)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}dxdy=\int_1^\infty\Big[\frac{1}{y^{\frac{1}{p}}(\ln y^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})}}\int_1^\infty\frac{f(x)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}dx\Big]\Big[y^{\frac{1}{p}}(\ln y^{\lambda_2})^{\frac{1}{p}(1-\frac{c}{\lambda})}g(y)\Big]dy$$
$$<\Big[\int_1^\infty\frac{1}{y(\ln y^{\lambda_2})^{1-\frac{c}{\lambda}}}\Big(\int_1^\infty\frac{f(x)}{(\ln x^{\lambda_1}y^{\lambda_2})^{\frac{c}{\lambda}}}dx\Big)^p dy\Big]^{1/p}\Big[\int_1^\infty(\ln y^{\lambda_2})^{(q-1)(1-\lambda)}y^{q-1}g^q(y)dy\Big]^{1/q},$$

which means the inequality (3.3) holds. Since the inequalities (3.3) and (3.4) are equivalent, then $c_{\lambda,p}^p$ is also the best possible. The proof of Theorem 3.2 is completed. □

Remark 3.1. It is easy to see that inequality (3.3) is equivalent to the following new Mullholand's inequality associated integral.

$$\int_1^\infty \int_1^\infty \frac{f(x)g(y)}{xy(\ln x^{\lambda_1} y^{\lambda_2})^{\frac{c}{\lambda}}} dxdy < c_{\lambda,p} \Big[\int_1^\infty (\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})} x^{-1} f^p(x) dx \Big]^{1/p} \times \Big[\int_1^\infty (\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})} y^{-1} g^q(y) dy \Big]^{1/q},$$

where $0 < \int_1^\infty (\ln x^{\lambda_1})^{(p-1)(1-\frac{c}{\lambda})} x^{-1} f^p(x) dx < \infty$, $0 < \int_1^\infty (\ln y^{\lambda_2})^{(q-1)(1-\frac{c}{\lambda})} y^{-1} g^q(y) dy < \infty$, and the constant factor $c_{\lambda,p}$ is also the best possible.

Remark 3.2. In the paper [16], by using the quasi-homogeneous integral kernels, Cao et al. obtained some Hilbert-type integral inequalities involving multiple functions with the best constant factors. In this paper, we not only give some Hilbert-type integral inequalities with multiparameters, but also obtain some Hilbert-type inequalities with multiparameters for double series. Therefore, our results are different from Cao et al. [16].

4. Conclusions

In this paper, we have established a new Hardy-Hilbert-type inequality with multiparameters. Furthermore, its equivalent forms and variants, which generalize some existing results, have been also presented. Finally, the corresponding Hardy-Hilbert-type integral inequalities haven been obtained.

Conflict of interest

The authors declare that there is no conflict of interests.

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