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Research article

Nonisotropic symplectic graphs over finite commutative rings

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Abstract: In this paper, we study two types of nonisotropic symplectic graphs over finite commutative rings defined by nonisotropic free submodules of rank 2 and McCoy rank of matrices. We prove that the graphs are quasi-strongly regular or Deza graphs and we find their parameters. The diameter and vertex transitivity are also analyzed. Moreover, we study subconstituents of these nonisotropic symplectic graphs.

Keywords: local ring; McCoy rank, nonisotropic subspace; symplectic space
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1. Introduction

Throughout the paper, all rings are assumed to be with identity $1 \neq 0$. Let V be a symplectic space over a finite commutative ring R of rank 2ν. Define

$$K = \begin{pmatrix} 0 & I_\nu \\ -I_\nu & 0 \end{pmatrix},$$

where I_ν is the $\nu \times \nu$ identity matrix. A free submodule X of V is called *totally isotropic* if $XKX^T = 0$ and is called *nonisotropic* if $\det(XKX^T)$ is a unit in R.

The relation between the geometry of classical groups over finite fields have been widely explored. The earliest record mention for the collinearity graphs of finite classical polar spaces was by Hubaut [1] and also in [2]. Graphs arising from symplectic spaces over finite fields, namely sympletic graphs were studied in [3–6]. A special case of these graphs is related to simple Lie algebras [7]. These symplectic graphs have 1-dimensional subspaces as its vertices and two vertices $\mathbb{F}_q\vec{x}$ and $\mathbb{F}_q\vec{y}$ are adjacent if and only if $\vec{x}K\vec{y} \neq 0$. One can see that all one dimensional subspaces are totally isotropic. The graphs were classified and their parameters were found. Later, Meemark and Prinyasart studied symplectic graphs over the ring of integers modulo n in [8] in the same manner of [3]. After that, these graphs were generalized to general finite local ring and finite commutative ring cases in [9] and [10], respectively. Many related works of graphs on symplectic spaces were announced (see [11], [12], [13]). All of these graphs are concerned with totally isotropic subspaces.

Recently, Su et al. studied a symplectic graph constructed from nonisotropic subspaces of a symplectic space over a finite field [14]. Since all 1-dimensional subspaces are totally isotropic, a nonisotropic subspace must be of dimension at least 2. The authors defined the vertex set of this graph to be the set of all 2-dimensional nonisotropic subspaces and two vertices X and Y are adjacent if and only if $\dim(X \cap Y) = 1$. We observe that for any two distinct 2-dimensional nonisotropic subspaces X and Y of V, $\dim(X \cap Y)$ is either 0 or 1. Thus, a graph whose vertex set is the same as above while the adjacency condition is replaced by $\dim(X \cap Y) = 0$ can also be considered. It is natural to generalize these graphs to cases over finite commutative rings. In this paper, we define two types of nonisotropic symplectic graphs over finite commutative rings using the McCoy rank of matrices for the adjacency condition. Then we find their graph parameters and we study some of their properties.

One concept of generalization from rank of matrices over fields to the rank of matrices over commutative rings were introduced by McCoy [15] as follows. Let R be a commutative ring with identity. For an ideal I of R, the annihilator of I is the ideal $\operatorname{Ann}_R I = \{r \in R \mid ra = 0 \text{ for all } a \in I\}$. Let A be an $m \times n$ matrix over R. Define $I_0 = R$ and $I_s(A)$ the ideal of R generated by the $s \times s$ minors of A for $1 \le s \le \min\{m, n\}$. We have

$$\{0\} = \operatorname{Ann}_R I_0(A) \subseteq \operatorname{Ann}_R I_1(A) \subseteq \cdots \subseteq \operatorname{Ann}_R I_{\min\{m,n\}}(A).$$

The *rank of* A, denoted by $\operatorname{rank} A$, is the largest integer s such that

$$\operatorname{Ann}_R I_s(A) = \{0\}.$$

Note that if R is a field, then $\operatorname{rank} A$ is the usual rank.

Let R be a finite commutative ring with identity. For each $s = 1, 2$, the *nonisotropic symplectic graph of type* s *of* V, denoted by $\Gamma_s(R)$, is the graph whose vertices are nonisotropic free submodules of V of rank 2 and

$$X \text{ is adjacent to } Y \text{ if and only if } \operatorname{rank}\begin{pmatrix} X \\ Y \end{pmatrix} = 2 + s,$$

for any two distinct vertices X and Y. Note that for $R = \mathbb{F}_q$ and two subspaces X and Y of V, $\operatorname{rank}\begin{pmatrix} X \\ Y \end{pmatrix} = 2 + s$ if and only if $\dim(X \cap Y) = 2 - s$. Thus, the graph $\Gamma_1(\mathbb{F}_q)$ is the graph studied in [14].

The rest of the paper is organized as follows. In Section 2, we study the graphs $\Gamma_s, s = 1, 2$ over finite fields, finite local rings and finite commutative rings by computing their parameters. We also consider their diameter and vertex transitivity. Finally, we work on subconstituents of these graphs.

2. Nonisotropic symplectic graphs

We first recall definitions of a strongly regular graph, a quasi-strongly regular graph, a Deza graph and vertex transitivity.

A *strongly regular graph* with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that for every pair of adjacent vertices there are λ vertices adjacent to both, and for every pair of nonadjacent vertices there are μ vertices adjacent to both. A k-regular graph G on n vertices such that for every pair of adjacent vertices there are λ vertices adjacent to both is called a *quasi-strongly regular graph* with parameters $(n, k, \lambda; c_1, \ldots, c_d)$ if every pair of nonadjacent vetices of G has $c_1, \ldots, c_d$ common adjecent vertices for some $d \geq 2$. A *t-Deza graph* with parameter $(n, k, \{d_1, \ldots, d_t\})$ is a k-regular graph of n vertices in which every pair of distinct vertices has $d_1, \ldots, d_t$ common adjacent vertices.

A graph G is *vertex transitive* if its automorphism group acts transitively on the vertex set. For convenience, we denote $\mathcal{V}(G)$ the vertex set of G.

2.1. Over finite fields

Su et. al. studied the type of graph $\Gamma_1(\mathbb{F}_q)$ and found its parameters as follows.

Theorem 2.1. [14] *The graph $\Gamma_1(\mathbb{F}_q)$ is a quasi-strongly regular graph with parameters*

$$\begin{aligned} n &= \frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1}, \\ k &= q^{2\nu-1}+q^{2\nu-2}-q-1, \\ \lambda &= q^{2\nu-2}+q^2-q-2, \\ c_1 &= q^2+q, \\ c_2 &= q^2, \\ c_3 &= 0. \end{aligned}$$

Moreover, the graph $\Gamma_1(\mathbb{F}_q)$ is a vertex transitive graph with diameter 3.

Indeed, the numbers c_1, c_2 and c_3 of common neighbors of two nonadjacent vertices occur in three cases according to the proof of Theorem 2.7 in [14] shown in the following theorem.

Theorem 2.2. [14] *If X and Y are two nonadjacent vertices of $\Gamma_1(\mathbb{F}_q)$, then the number of common neighbors of X and Y is*

$$\begin{cases} c_1 = q^2+q & \textit{if} \quad \dim(X^\perp \cap Y) = 0, \\ c_2 = q^2 & \textit{if} \quad \dim(X^\perp \cap Y) = 1, \\ c_3 = 0 & \textit{if} \quad \dim(X^\perp \cap Y) = 2. \end{cases}$$

We remark that for two distinct subspaces X and Y of V of dimension 2,

$$\begin{aligned} X \text{ is adjacent to } Y \text{ in } \Gamma_2(\mathbb{F}_q) &\Longleftrightarrow \dim(X \cap Y) = 0 \\ &\Longleftrightarrow \dim(X \cap Y) \neq 1 \\ &\Longleftrightarrow X \text{ is not adjacent to } Y \text{ in } \Gamma_1(\mathbb{F}_q). \end{aligned} \tag{2.1}$$

Thus, the graph $\Gamma_2(\mathbb{F}_q)$ is the complement graph of $\Gamma_1(\mathbb{F}_q)$ and they both have the same vertex set. We find the parameters of $\Gamma_2(\mathbb{F}_q)$ in the following theorem.

Theorem 2.3. *The graph* $\Gamma_2(\mathbb{F}_q)$ *is a* 4*-Deza graph with parameters* $(n, k, \{d_1, d_2, d_3, d_4\})$ *where*

$$\begin{aligned}
n &= \frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1},\\
k &= \frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - q^{2\nu-1} - q^{2\nu-2} + q,\\
d_1 &= \frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 2q^{2\nu-1} - q^{2\nu-2} + q^2 + q,\\
d_2 &= \frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 2q^{2\nu-1} - 2q^{2\nu-2} + q^2 + 3q,\\
d_3 &= \frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 2q^{2\nu-1} - 2q^{2\nu-2} + q^2 + 2q,\\
d_4 &= \frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 2q^{2\nu-1} - 2q^{2\nu-2} + 2q.
\end{aligned}$$

Proof. Let X be any vertex in $\Gamma_2(\mathbb{F}_q)$. From the valency of $\Gamma_1(\mathbb{F}_q)$, we have $|\{Y \in \mathcal{V}(\Gamma_2(\mathbb{F}_q)) \mid \dim(X \cap Y) = 1\}| = q^{2\nu-1} + q^{2\nu-2} - q - 1$. This implies that the number of vertices Y such that $\dim(X \cap Y) = 0$ equals

$$\frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 1 - (q^{2\nu-1} + q^{2\nu-2} - q - 1).$$

We obtain the parameter k.

Next, let X and Y be any two nonadjacent vertices in $\Gamma_2(\mathbb{F}_q)$. Then X and Y are adjacent in $\Gamma_1(\mathbb{F}_q)$. Let $A = \{Z \in \mathcal{V}(\Gamma_2(\mathbb{F}_q)) \mid \dim(Z \cap X) = 1\}$ and $B = \{Z \in \mathcal{V}(\Gamma_2(\mathbb{F}_q)) \mid \dim(Z \cap Y) = 1\}$. It is clear that $Y \in A$ and $X \in B$. By Theorem 2.1, $|A| = |B| = q^{2\nu-1} + q^{2\nu-2} - q - 1$ and $|A \cap B| = q^{2\nu-2} + q^2 - q - 2$. Thus the inclusion-exclusion principle gives the number of common neighbors of X and Y in $\Gamma_2(\mathbb{F}_q)$ which is the number of vertices Z such that $\dim(Z \cap X) = \dim(Z \cap Y) = 0$, and it equals

$$\begin{aligned}
d_1 &= n - |A \cup B|\\
&= \frac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 2(q^{2\nu-1} + q^{2\nu-2} - q - 1) + (q^{2\nu-2} + q^2 - q - 2).
\end{aligned}$$

Finally, let X and Y be two adjacent vertices in $\Gamma_2(\mathbb{F}_q)$. Then X and Y are nonadjacent in $\Gamma_1(\mathbb{F}_q)$. By Theorem 2.2, the number of common neighbors of X and Y in $\Gamma_1(\mathbb{F}_q)$ which is the number of vertices Z such that $\dim(Z \cap X) = 1$ and $\dim(Z \cap Y) = 1$ is $q^2 + q$ if $\dim(X^\perp \cap Y) = 0$, is q^2 if $\dim(X^\perp \cap Y) = 1$ and is 0 if $\dim(X^\perp \cap Y) = 2$. Therefore, the number of common neighbors of X and Y in $\Gamma_2(\mathbb{F}_q)$ which is the number of vertices Z such that $\dim(Z \cap X) = 0$ and $\dim(Z \cap Y) = 0$ is

$$\begin{cases}
d_2 = \dfrac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 2 - 2(q^{2\nu-1} + q^{2\nu-2} - q - 1) + (q^2 + q) & \text{if } \dim(X^\perp \cap Y) = 0,\\
d_3 = \dfrac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 2 - 2(q^{2\nu-1} + q^{2\nu-2} - q - 1) + q^2 & \text{if } \dim(X^\perp \cap Y) = 1,\\
d_4 = \dfrac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1} - 2 - 2(q^{2\nu-1} + q^{2\nu-2} - q - 1) & \text{if } \dim(X^\perp \cap Y) = 2,
\end{cases}$$

by the inclusion-exclusion principle. Note that the term -2 is due to the fact that $Z \neq X$ and $Z \neq Y$. □

Remark. *The complete graph K_n where $n = \dfrac{q^{2\nu-2}(q^{2\nu}-1)}{q^2-1}$ can be decomposed into two 4-Deza graphs $\Gamma_1(\mathbb{F}_q)$ and $\Gamma_2(\mathbb{F}_q)$.*

Since the complement graph of a vertex transitive graph is also vertex transitive, we immediately have the following theorem.

Theorem 2.4. *The graph $\Gamma_2(\mathbb{F}_q)$ is vertex transitive.*

Theorem 2.3 shows that any two vertices of $\Gamma_2(\mathbb{F}_q)$ always have a common neighbor. So we obtain the diameter of $\Gamma_2(\mathbb{F}_q)$.

Corollary 2.5. *The diameter of the graph $\Gamma_2(\mathbb{F}_q)$ is 2.*

2.2. Over finite local rings

Let R be a finite local ring with the nontrivial maximal ideal M, i.e. R is not a finite field. Then $R \setminus M$ is the group of units of R and the quotient ring $\mathbf{k} = R/M$ is the field which we call the *residue fields*. Moreover, we have the canonical map $\pi : R \to R/M$ given by $\pi(r) = r + M$ for all $r \in R$. The rank of matrices over R can be obtained by the following lemma.

Lemma 2.6. [16] *If A is a matrix over R, then the rank of A equals the rank of $\pi(A)$ over its residue field $\mathbf{k} = R/M$.*

Lemma 2.7. *For $s = 1, 2$,*

1. *if X is a vertex in $\Gamma_s(R)$, then there are $|M|^{4(\nu-1)}$ many vertices in $\Gamma_s(R)$ which are lifted from the vertex $\pi(X)$ of $\Gamma_s(k)$, i.e.*

$$|\{Y \in \mathcal{V}(\Gamma_s(R)) \mid \pi(Y) = \pi(X)\}| = |M|^{4(\nu-1)},$$

2. *X is adjacent to Y in $\Gamma_s(R)$ if and only if $\pi(X)$ is adjacent to $\pi(Y)$ in $\Gamma_s(\mathbf{k})$,*
3. *if $\pi(X) = \mathbf{k}\pi(\vec{x}_1) \oplus \mathbf{k}\pi(\vec{x}_2)$ is adjacent to $\pi(Y) = \mathbf{k}\pi(\vec{y}_1) \oplus \mathbf{k}\pi(\vec{y}_2)$ in $\Gamma_s(\mathbf{k})$, then $X = R(\vec{x}_1 + \vec{m}_1) \oplus R(\vec{x}_2 + \vec{m}_2)$ is adjacent to $Y = R(\vec{y}_1 + \vec{n}_1) \oplus R(\vec{y}_2 + \vec{n}_2)$ in $\Gamma_s(R)$ for all $\vec{m}_j, \vec{n}_j \in M^{2\nu}$.*

Proof. To show (1), let X be a vertex in $\Gamma_s(R)$. Then $X = R\vec{x}_1 \oplus R\vec{x}_2$ is a free submodule of V of rank 2 in which $\det(XKX^T)$ is a unit in R. By Theorem 1.8 of [11], we have $\pi(X) = \mathbf{k}\pi(\vec{x}_1) \oplus \mathbf{k}\pi(\vec{x}_2)$ is a subspace of $V' = \mathbf{k}^{2\nu}$ of dimension 2. Since $\det(XKX^T)$ is a unit in R, it follows that $\pi(X)K\pi(X)^T = \pi(XKX^T)$ is nonsingular. So $\pi(X)$ is a vertex of $\Gamma_s(\mathbf{k})$. On the other hand, Theorem 1.8 of [11] implies that the subspace $\pi(X)$ can be lifted to $|M|^{4(\nu-1)}$ free submodules of V of rank 2 which are of the form $Y = R(\vec{x}_1 + \vec{m}_1) \oplus R(\vec{x}_2 + \vec{m}_2)$ where $\vec{m}_1, \vec{m}_2 \in M^{2\nu}$ and $\pi(Y) = \pi(X)$. Since $\det \pi(YKY^T) = \det \pi(XKX^T) \neq 0$, it implies that $\det(YKY^T)$ is a unit in R. Thus, Y is a vertex of $\Gamma_s(R)$. Therefore, all such $|M|^{4(\nu-1)}$ free submodules are lifts of the vertex $\pi(X)$. For (2) we apply Lemma 2.6 to obtain that X is adjacent to Y in $\Gamma_s(R)$ if and only if rank $\begin{pmatrix} X \\ Y \end{pmatrix} = 2 + s$ if and only if rank $\begin{pmatrix} \pi(X) \\ \pi(Y) \end{pmatrix} = 2 + s$ if and only if $\dim(\pi(X) \cap \pi(Y)) = 2 - s$ if and only if $\pi(X)$ is adjacent to $\pi(Y)$ in $\Gamma_s(\mathbf{k})$. Finally, (3) follows from (2). □

Now, we can classify and find the parameters of $\Gamma_1(R)$ and $\Gamma_2(R)$.

Theorem 2.8. *The graph* $\Gamma_1(R)$ *is a quasi-strongly regular graph with parameters* $(n, k, \lambda; c_1, c_2, c_3, c_4)$ *where*

$$\begin{aligned}
n &= \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2},\\
k &= |M|^{2\nu-3}(|R| + |M|)(|R|^{2\nu-2} - |M|^{2\nu-2}),\\
\lambda &= (|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 - |M|^{4\nu-5}|R| - 2|M|^{4\nu-4},\\
c_1 &= |M|^{4\nu-6}|R|^2 + |M|^{4\nu-5}|R|,\\
c_2 &= |M|^{4\nu-6}|R|^2,\\
c_3 &= 0,\\
c_4 &= |M|^{2\nu-3}(|R| + |M|)(|R|^{2\nu-2} - |M|^{2\nu-2}).
\end{aligned}$$

Proof. From Theorem 2.1, the graph $\Gamma_1(\mathbf{k})$ is a quasi-strongly regular with parameters

$$\begin{aligned}
n &= \frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1},\\
k &= |\mathbf{k}|^{2\nu-1} + |\mathbf{k}|^{2\nu-2} - |\mathbf{k}| - 1,\\
\lambda &= |\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 - |\mathbf{k}| - 2,\\
c_1 &= |\mathbf{k}|^2 + |\mathbf{k}|,\\
c_2 &= |\mathbf{k}|^2,\\
c_3 &= 0.
\end{aligned}$$

By Lemma 2.7 (1), each vertex of $\Gamma_1(\mathbf{k})$ can be lifted to $|M|^{4(\nu-1)}$ vertices of $\Gamma_1(R)$. Thus, the number of vertices of $\Gamma_1(R)$ is

$$(|M|^{4(\nu-1)})\frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} = \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2}.$$

Let X be any vertex of $\Gamma_1(R)$. Then $\pi(X)$ is a vertex of $\Gamma_1(\mathbf{k})$. Since the graph $\Gamma_1(\mathbf{k})$ is regular, it follows from Lemma 2.7 that X has degree

$$(|M|^{4(\nu-1)})(|\mathbf{k}|^{2\nu-1} + |\mathbf{k}|^{2\nu-2} - |\mathbf{k}| - 1) = |M|^{2\nu-3}(|R| + |M|)(|R|^{2\nu-2} - |M|^{2\nu-2}).$$

This implies that the graph $\Gamma_1(R)$ is regular. Next, the parameter λ of $\Gamma_1(\mathbf{k})$ together with Lemma 2.7 (3) implies that for any two adjacent vertices of $\Gamma_1(R)$, there are

$$\begin{aligned}
&(|M|^{4(\nu-1)})(|\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 - |\mathbf{k}| - 2)\\
&= (|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 - |M|^{4\nu-5}|R| - 2|M|^{4\nu-4}
\end{aligned}$$

common neighbors.

We next consider two nonadjacent vertices of $\Gamma_1(R)$. Suppose that X and Y are two nonadjacent vertices of $\Gamma_1(R)$. We divide this into 2 cases. The first case is that $\pi(X) \neq \pi(Y)$. This can be divided into three subcases according to Theorem 2.2 . We also apply Lemma 2.7 (3) to show the results of these subcases.

(i) If $\dim(\pi(X)^{\perp} \cap \pi(Y)) = 0$, then there are $|\mathbf{k}|^2 + |\mathbf{k}|$ common neighbors in $\Gamma_1(\mathbf{k})$. Thus, the number of common neighbors of X and Y is $(|M|^{4(\nu-1)})(|\mathbf{k}|^2 + |\mathbf{k}|) = |M|^{4\nu-6}|R|^2 + |M|^{4\nu-5}|R|$.

(ii) If $\dim(\pi(X)^{\perp} \cap \pi(Y)) = 1$, then there are $|\mathbf{k}|^2$ common neighbors in $\Gamma_1(\mathbf{k})$. Thus, the number of common neighbors of X and Y is $(|M|^{4(\nu-1)})|\mathbf{k}|^2 = |M|^{4\nu-6}|R|^2$.

(iii) If $\dim(\pi(X)^{\perp} \cap \pi(Y)) = 2$, then there are no common neighbors in $\Gamma_1(\mathbf{k})$. Thus, there are no common neighbors of X and Y in this case.

For the second case which is $\pi(X) = \pi(Y)$, Lemma 2.7 implies that all lifts of the vertices in $\Gamma_1(\mathbf{k})$ adjacent to $\pi(X)$ are common neighbors of X and Y. Thus there are $(|M|^{4(\nu-1)})(|\mathbf{k}|^{2\nu-1} + |\mathbf{k}|^{2\nu-2} - |\mathbf{k}| - 1) = |M|^{2\nu-3}(|R| + |M|)(|R|^{2\nu-2} - |M|^{2\nu-2})$ common neighbors in this case. □

Example 1. *The graph* $\Gamma_1(\mathbb{Z}_4)$ *is a quasi-strongly regular graph with parameters* $(320, 144, 64; 32, 64, 0, 144)$

Note that by Lemma 2.7 the diameter of $\Gamma_1(\mathbf{k})$ and the diameter of $\Gamma_1(R)$ are identical.

Corollary 2.9. *The diameter of the graph* $\Gamma_1(R)$ *is* 3.

Remark. *The distance between any two distinct vertices X and Y in* $\Gamma_s(R)$, $s = 1, 2$ *such that* $\pi(X) = \pi(Y)$ *is equal to* 2.

Theorem 2.10. *The graph* $\Gamma_2(R)$ *is a* 5*-Deza graph with parameters* $(n, k, \{d_1, d_2, d_3, d_4, d_5\})$ *where*

$$\begin{aligned}
n &= \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2},\\
k &= \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2} - |M|^{2\nu-3}|R|^{2\nu-1} - (|M||R|)^{2\nu-2} + |M|^{4\nu-5}|R|,\\
d_1 &= \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2} - 2|M|^{2\nu-3}|R|^{2\nu-1} - (|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 + |M|^{4\nu-5}|R|,\\
d_2 &= \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2} - 2|M|^{2\nu-3}|R|^{2\nu-1} - 2(|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 + 3|M|^{4\nu-5}|R|,\\
d_3 &= \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2} - 2|M|^{2\nu-3}|R|^{2\nu-1} - 2(|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 + 2|M|^{4\nu-5}|R|,\\
d_4 &= \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2} - 2|M|^{2\nu-3}|R|^{2\nu-1} - 2(|M||R|)^{2\nu-2} + 2|M|^{4\nu-5}|R|,\\
d_5 &= \frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2} - |M|^{2\nu-3}|R|^{2\nu-1} - (|M||R|)^{2\nu-2} + |M|^{4\nu-5}|R|.
\end{aligned}$$

Proof. Clearly, the number of vertices of $\Gamma_2(R)$ equals that of $\Gamma_1(R)$. We know from Theorem 2.3 that the graph $\Gamma_2(\mathbf{k})$ is a 4-Deza graph with parameters

$$\begin{aligned}
k &= \frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - |\mathbf{k}|^{2\nu-1} - |\mathbf{k}|^{2\nu-2} + |\mathbf{k}|\\
d_1 &= \frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - 2|\mathbf{k}|^{2\nu-1} - |\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 + |\mathbf{k}|,\\
d_2 &= \frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - 2|\mathbf{k}|^{2\nu-1} - 2|\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 + 3|\mathbf{k}|,
\end{aligned}$$

$$d_3 = \frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - 2|\mathbf{k}|^{2\nu-1} - 2|\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 + 2|\mathbf{k}|,$$
$$d_4 = \frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - 2|\mathbf{k}|^{2\nu-1} - 2|\mathbf{k}|^{2\nu-2} + 2|\mathbf{k}|.$$

Thus, if X is any vertex of $\Gamma_2(R)$, then the degree of $\pi(X)$ is k. Moreover, for any two vertices X and Y in $\Gamma_2(R)$, if $\pi(X) \neq \pi(Y)$, then the number of common neighbors of $\pi(X)$ and $\pi(Y)$ in $\Gamma_2(\mathbf{k})$ can possibly be d_1, d_2, d_3 or d_4. By Lemma 2.7, each vertex $\pi(X)$ of $\Gamma_2(\mathbf{k})$ can be lifted to $|M|^{4(\nu-1)}$ vertices of $\Gamma_2(R)$ with preserving the adjacency. Thus, we obtain the desired parameters k, d_1, d_2, d_3 and d_4 of $\Gamma_2(R)$ as follows:

$$k = |M|^{4(\nu-1)}(\frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - |\mathbf{k}|^{2\nu-1} - |\mathbf{k}|^{2\nu-2} + |\mathbf{k}|),$$
$$d_1 = |M|^{4(\nu-1)}(\frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - 2|\mathbf{k}|^{2\nu-1} - |\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 + |\mathbf{k}|),$$
$$d_2 = |M|^{4(\nu-1)}(\frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - 2|\mathbf{k}|^{2\nu-1} - 2|\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 + 3|\mathbf{k}|),$$
$$d_3 = |M|^{4(\nu-1)}(\frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - 2|\mathbf{k}|^{2\nu-1} - 2|\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 + 2|\mathbf{k}|),$$
$$d_4 = |M|^{4(\nu-1)}(\frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - 2|\mathbf{k}|^{2\nu-1} - 2|\mathbf{k}|^{2\nu-2} + 2|\mathbf{k}|).$$

Substituting $|\mathbf{k}| = \frac{|R|}{|M|}$, these parameters become the numbers in the statement.

Next, let X and Y be two vertices in $\Gamma_2(R)$ such that $\pi(X) = \pi(Y)$. By the same argument of the last paragraph in the proof of Theorem 2.8, Lemma 2.7 gives the parameter

$$d_5 = |M|^{4(\nu-1)}(\frac{|\mathbf{k}|^{2\nu-2}(|\mathbf{k}|^{2\nu} - 1)}{|\mathbf{k}|^2 - 1} - |\mathbf{k}|^{2\nu-1} - |\mathbf{k}|^{2\nu-2} + |\mathbf{k}|)$$

This completes the proof. □

We have seen that the numbers of common neighbors of any two vertices of $\Gamma_2(R)$ are not zero. This implies the diameter of the graph.

Corollary 2.11. *The diameter of* $\Gamma_2(R)$ *is* 2.

Moreover, we have vertex transitivity for $\Gamma_s(R)$, $s = 1, 2$.

Theorem 2.12. *The graph* $\Gamma_s(R)$ *is vertex transitive for all* $s = 1, 2$.

Proof. Note that a permutation of vertices can be regarded as an automorphism of $\Gamma_s(R)$. By the description in [14] and Theorem 2.4, the graph $\Gamma_s(\mathbf{k})$ is vertex transitive. The composition of an automorphism of $\Gamma_s(\mathbf{k})$ and permutations give vertex transitivity for $\Gamma_s(R)$. □

2.3. *Over finite commutative rings*

Let R be a finite commutative ring. It is well-known that R can be decomposed as

$$R \cong R_1 \times R_2 \times \cdots \times R_t$$

where R_i is a finite local ring with unique maximal ideal M_i for all $i = 1, 2, \ldots, t$. Moreover, we have the projection map

$$\mathcal{P}_i : r = (r_1, r_2, \ldots, r_t) \mapsto r_i$$

for all $i \in \{1, 2, \ldots, t\}$. Furthermore, the group of units of R (denoted by $R^\times$) is the direct product of groups of units of R_i's, that is,

$$R^\times \cong R_1^\times \times R_2^\times \cdots \times R_t^\times.$$

Next, let A be a matrix over R. We can consider A as

$$\mathcal{P}_1(A) \times \mathcal{P}_2(A) \times \cdots \times \mathcal{P}_t(A)$$

where $\mathcal{P}_i(A)$ is a matrix over R_i for all $i = 1, 2, \ldots, t$. If A is a square matrix, you can write $\det A = (\det \mathcal{P}_1(A), \det \mathcal{P}_2(A), \ldots, \det \mathcal{P}_t(A))$. To compute the rank of matrices over R, we use the following lemma.

Lemma 2.13. [17] *If A is an $m \times n$ matrix over R, then*

$$\operatorname{rank} A = \min_{1 \le i \le t} \{\operatorname{rank} \mathcal{P}_t(A)\}.$$

Let V be a symplectic space over R of rank 2ν. Then V induces the symplectic space V_i over R_i of rank 2ν. Note that X is a free submodule of V over R if and only if $\mathcal{P}(X)$ is a free submodule of V_i over R_i. Indeed, $X \cong \mathcal{P}_1(X) \times \mathcal{P}_2(X) \times \cdots \times \mathcal{P}_t(X)$. Moreover, $\det(XKX^T)$ is a unit in R if and only if $\det\left(\mathcal{P}_i(X)K\mathcal{P}_i(X)^T\right)$ is a unit in R_i for all $i = 1, 2, \ldots, t$. This implies that X is a nonisotropic free submodule of V of rank 2 over R if and only if $\mathcal{P}_i(X)$ is a nonisotropic free submodule of V_i of rank 2 over R_i for all $i = 1, 2, \ldots, t$. This allows us to consider a vertex X of the graph $\Gamma_s(R)$ as

$$(\mathcal{P}_1(X), \mathcal{P}_2(X), \ldots, \mathcal{P}_t(X))$$

where $\mathcal{P}_i(X)$ is a vertex of the graph $\Gamma_s(R_i)$. In other words,

$$\mathcal{V}(\Gamma_s(R)) = \mathcal{V}(\Gamma_s(R_1)) \times \mathcal{V}(\Gamma_s(R_2)) \times \cdots \times \mathcal{V}(\Gamma_s(R_t)).$$

Thus, we have the following theorem.

Theorem 2.14. *The number of vertices of the graph $\Gamma_s(R)$ is*

$$\prod_{i=1}^{t} \frac{|R_i|^{2\nu-2}(|R_i|^{2\nu} - |M_i|^{2\nu})}{|R_i|^2 - |M_i|^2}.$$

Since the decomposition of vertices of $\Gamma_s(R)$, more properties of $\Gamma_s(R)$ can be determined by the tensor product and a decomposition of graphs. For two graphs G and H with vertex sets $\mathcal{V}(G)$ and $\mathcal{V}(H)$, respectively, the *tensor product of G and H*, denoted by $G \otimes H$, is the graph whose vertex set is $\mathcal{V}(G) \times \mathcal{V}(H)$ and two vertices (g, h) and (g', h') are adjacent if g is adjacent to g' in G and h is adjacent to h' in H. A *decomposition of G* is a family of subgraphs $H_1, H_2, \ldots, H_l$ that partition the edges of G with $\mathcal{V}(G) = \mathcal{V}(H_i), i = 1, 2, \ldots, l$.

We first focus on the graph $\Gamma_1(R)$. Let $X = (\mathcal{P}_1(X), \mathcal{P}_2(X), \ldots, \mathcal{P}_t(X))$ and $Y = (\mathcal{P}_1(Y), \mathcal{P}_2(Y), \ldots, \mathcal{P}_t(Y))$ be two vertices of $\Gamma_1(R)$. Then

$$X \text{ is adjacent to } Y \text{ in } \Gamma_1(R) \Longleftrightarrow \operatorname{rank}\begin{pmatrix} X \\ Y \end{pmatrix} = 3 \Longleftrightarrow \min_{1\le i\le t}\left\{ \operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} \right\} = 3.$$

With this relation, we prove the following decomposition of $\Gamma_1(R)$.

Theorem 2.15. *The graph $\Gamma_1(R)$ can be decomposed into a family of vertex transitive subgraphs*

$$\Gamma_{s_1}(R_1) \otimes \Gamma_{s_2}(R_2) \otimes \cdots \otimes \Gamma_{s_t}(R_t)$$

where $s_i = 1$ or 2 for all $i = 1, 2, \ldots, t$ but $(s_1, s_2, \ldots, s_t) \neq (2, 2, \ldots, 2)$.

Proof. Let $s_i = 1$ or 2 for all $i = 1, 2, \ldots, t$ but $s_1 = s_2 = \cdots = s_t = 2$. We first consider the graph $\mathcal{G} = \Gamma_{s_1}(R_1) \otimes \Gamma_{s_2}(R_2) \otimes \cdots \otimes \Gamma_{s_t}(R_t)$. Clearly, the vertex set of this graph is that of the graph $\Gamma_1(R)$. Let $X = (\mathcal{P}_1(X), \mathcal{P}_2(X), \ldots, \mathcal{P}_t(X))$ and $Y = (\mathcal{P}_1(Y), \mathcal{P}_2(Y), \ldots, \mathcal{P}_t(Y))$ be two adjacent vertices of $\mathcal{G}$. Then $\mathcal{P}_i(X)$ is adjacent to $\mathcal{P}_i(Y)$ in $\Gamma_{s_i}(R_i)$ for all $i = 1, 2, \ldots, t$. This implies $\operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} = 2 + s_i$ for all $i = 1, 2, \ldots, t$. Since there is no the case $s_1 = s_2 = \cdots = s_t = 2$, it follows that $\operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} = 2 + 1 = 3$ for some $i = 1, 2, \ldots, t$. Thus, $\min_{1\le i\le t}\left\{ \operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} \right\} = 3$. So X is adjacent to Y in $\Gamma_1(R)$. This implies that $\mathcal{G}$ is a subgraph of $\Gamma_1(R)$. Moreover, it is easy to see that

$$\operatorname{Aut}(\Gamma_{s_1}(R_1)) \times \operatorname{Aut}(\Gamma_{s_2}(R_2)) \times \cdots \times \operatorname{Aut}(\Gamma_{s_t}(R_t)) \subseteq \operatorname{Aut}(\mathcal{G}).$$

By Theorem 2.12, the graph $\Gamma_{s_i}(R_i)$ is vertex transitive for all $i = 1, 2, \ldots, t$. Thus, $\mathcal{G}$ is a vertex transitive subgraph of $\Gamma_1(R)$.

Next, let $\mathcal{G} = \Gamma_{s_1}(R_1) \otimes \Gamma_{s_2}(R_2) \otimes \cdots \otimes \Gamma_{s_t}(R_t)$ and $\mathcal{G}' = \Gamma_{s_1'}(R_1) \otimes \Gamma_{s_2'}(R_2) \otimes \cdots \otimes \Gamma_{s_t'}(R_t)$ be two distinct subgraphs in the family. Suppose that

$$X = (\mathcal{P}_1(X), \mathcal{P}_2(X), \ldots, \mathcal{P}_t(X)) \text{ and } Y = (\mathcal{P}_1(Y), \mathcal{P}_2(Y), \ldots, \mathcal{P}_t(Y))$$

are adjacent in both $\mathcal{G}$ and $\mathcal{G}'$. Then for each $i = 1, 2, \ldots, t$, we have $2 + s_i = \operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} = 2 + s_i'$, so that $s_i = s_i'$. Thus, $\mathcal{G} = \mathcal{G}'$ which is a contradiction. Therefore, $\mathcal{G}$ and $\mathcal{G}'$ have disjoint edge sets.

Next, let $X = (\mathcal{P}_1(X), \mathcal{P}_2(X), \ldots, \mathcal{P}_t(X))$ and $Y = (\mathcal{P}_1(Y), \mathcal{P}_2(Y), \ldots, \mathcal{P}_t(Y))$ be two adjacent vertices of $\Gamma_1(R)$. Then for each $i = 1, 2, \ldots, t$, we have

$$\min_{1\le i\le t}\left\{ \operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} \right\} = 3.$$

So $\operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} = 2 + s_i$ where $s_i = 1$ or 2 but $(s_1, s_2, \ldots, s_t) \neq (2, 2, \ldots, 2)$. Thus, X and Y are adjacent in some subgraphs in the family. This follows that the graph $\Gamma_1(R)$ can be decomposed into the family of tensor products of graphs. □

Once we can decompose $\Gamma_1(R)$, we can compute the degree of a vertex of the graph. It suffices to show this for the case $R \cong R_1 \times R_2$ where R_1 and R_2 are finite local rings. From Theorem 2.15, the graph $\Gamma_1(R)$ can be decomposed into 3 subgraphs

$$\Gamma_1(R_1) \otimes \Gamma_1(R_2), \quad \Gamma_1(R_1) \otimes \Gamma_2(R_2) \quad \text{and} \quad \Gamma_2(R_1) \otimes \Gamma_1(R_2).$$

From Theorem 2.8 and 2.10, we have the valencies $k(1, R_1)$ of $\Gamma_1(R_1)$, $k(2, R_1)$ of $\Gamma_2(R_1)$, $k(1, R_2)$ of $\Gamma_1(R_2)$ and $k(2, R_2)$ of $\Gamma_2(R_2)$. Let $X = (\mathcal{P}_1(X), \mathcal{P}_2(X))$ be any vertex of $\Gamma_1(R)$. If $Y = (\mathcal{P}_1(Y), \mathcal{P}_2(Y))$ is a vertex adjacent to X in $\Gamma_1(R)$, then Y is adjacent to X in one of the three subgraphs. This implies that the degree of X in $\Gamma_1(R)$ is equal to the sum of the degree of X in each subgraph. Since the degree of $X = (\mathcal{P}_1(X), \mathcal{P}_2(X))$ in $\Gamma_{s_1}(R_1) \otimes \Gamma_{s_2}(R_2)$ is the product of the degree of $\mathcal{P}_1(X)$ in $\Gamma_{s_1}(R_1)$ and the degree of $\mathcal{P}_2(X)$ in $\Gamma_{s_2}(R_2)$, the degree of X in $\Gamma_1(R)$ is

$$k(1, R_1)k(1, R_2) + k(1, R_1)k(2, R_2) + k(2, R_1)k(1, R_2).$$

Similarly, if $R \cong R_1 \times R_2 \times \cdots \times R_t$ where R_i is a finite local ring for all i, then the degree of a vertex in $\Gamma_1(R)$ is

$$\sum_{\substack{s_1, s_2, \ldots, s_t \in \{1,2\} \\ (s_1, s_2, \ldots, s_t) \neq (2,2,\ldots,2)}} \prod_{i=1}^{t} k(s_i, R_i)$$

where $k(s_i, R_i)$ is the degree of a vertex in $\Gamma_{s_i}(R_i)$. Indeed,

$$k(s_i, R_i) = \begin{cases} |M_i|^{2\nu-3}(|R_i| + |M_i|)(|R_i|^{2\nu-2} - |M_i|^{2\nu-2}) & \text{if } s_i = 1 \\ \dfrac{|R_i|^{2\nu-2}(|R_i|^{2\nu} - |M_i|^{2\nu})}{|R_i|^2 - |M_i|^2} - |M_i|^{2\nu-3}|R_i|^{2\nu-1} & \\ \qquad -(|M_i||R_i|)^{2\nu-2} + |M_i|^{4\nu-5}|R_i| & \text{if } s_i = 2. \end{cases}$$

Moreover, the parameters of common neighbors can also be computed from the decomposition in a similar way.

For the graph $\Gamma_2(R)$, we observe that for two vertices X, Y of $\Gamma_1(R)$ where $X = (\mathcal{P}_1(X), \mathcal{P}_2(X), \ldots, \mathcal{P}_t(X))$ and $Y = (\mathcal{P}_1(Y), \mathcal{P}_2(Y), \ldots, \mathcal{P}_t(Y))$, we have

$$\begin{aligned} & X \text{ is adjacent to } Y \text{ in } \Gamma_2(R) \\ & \Longleftrightarrow \operatorname{rank}\begin{pmatrix} X \\ Y \end{pmatrix} = 4 \\ & \Longleftrightarrow \min_{1 \le i \le t} \left\{ \operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} \right\} = 4 \\ & \Longleftrightarrow \operatorname{rank}\begin{pmatrix} \mathcal{P}_i(X) \\ \mathcal{P}_i(Y) \end{pmatrix} = 4 \text{ for all } i = 1, 2, \ldots, t \\ & \Longleftrightarrow \mathcal{P}_i(X) \text{ is adjacent to } \mathcal{P}_i(Y) \text{ in } \Gamma_2(R_i) \text{ for all } i = 1, 2, \ldots, t. \end{aligned}$$

With this relation, we prove the next theorem.

Theorem 2.16. *Let R be a finite commutative ring decomposed as $R = R_1 \times R_2 \times \cdots \times R_t$ where R_i is a finite local ring for all $i = 1, 2, \ldots, t$. Then*

$$\Gamma_2(R) = \Gamma_2(R_1) \otimes \Gamma_2(R_2) \otimes \cdots \otimes \Gamma_2(R_t).$$

Moreover, $\Gamma_2(R)$ is vertex transitive and so it is regular of degree

$$\prod_{i=1}^{t} \Big(\frac{|R_i|^{2\nu-2}(|R_i|^{2\nu} - |M_i|^{2\nu})}{|R_i|^2 - |M_i|^2} - |M_i|^{2\nu-3}|R_i|^{2\nu-1} - (|M_i||R_i|)^{2\nu-2} + |M_i|^{4\nu-5}|R_i| \Big).$$

Proof. By the above discussion, it implies that the graph $\Gamma_2(R)$ is the tensor product of the graphs $\Gamma_2(R_1), \Gamma_2(R_2), \ldots, \Gamma_2(R_t)$, i.e.,

$$\Gamma_2(R) = \Gamma_2(R_1) \otimes \Gamma_2(R_2) \otimes \cdots \otimes \Gamma_2(R_t).$$

Since $\Gamma_2(R_i)$ is vertex transitive for all $i = 1, 2, \ldots, t$ by Theorem 2.12, it follows that $\Gamma_2(R)$ is vertex transitive and so regular. Moreover, this tensor product of graphs gives the degree of a vertex of $\Gamma_2(R)$ which is equal to the product of the degrees of a vertex of $\Gamma_2(R_i)$ for all $i = 1, 2, \ldots, t$. This completes the proof. □

Remark. *From Theorem 2.15 and 2.16, the tensor product*

$$\Gamma_{s_1}(R_1) \otimes \Gamma_{s_2}(R_2) \otimes \cdots \otimes \Gamma_{s_t}(R_t)$$

of nonsymplectic graphs over finite local rings is either a subgraph of $\Gamma_1(R)$ or a certain graph $\Gamma_2(R)$ depending on each s_i where $i = 1, 2, \ldots, t$. Indeed, if $(s_1, s_2, \ldots, s_t) \neq (2, 2, \ldots, 2)$, then it is a subgraph of $\Gamma_1(R)$. On the other hand, if $(s_1, s_2, \ldots, s_t) = (2, 2, \ldots, 2)$, then it actually is $\Gamma_2(R)$. Furthermore, we can say that

$$\Gamma_1(R) = \bigcup_{\substack{s_1, s_2, \ldots, s_t \in \{1,2\} \\ (s_1, s_2, \ldots, s_t) \neq (2,2,\ldots,2)}} \Gamma_{s_1}(R_1) \otimes \Gamma_{s_2}(R_2) \otimes \cdots \otimes \Gamma_{s_t}(R_t)$$

$$\Gamma_2(R) = \Gamma_2(R) = \Gamma_2(R_1) \otimes \Gamma_2(R_2) \otimes \cdots \otimes \Gamma_2(R_t).$$

3. Subconstituents

Let R be a finite local ring with the maximal ideal M and its residue field $\mathbf{k} = R/M$ and let $X_0 = R\vec{e}_1 \oplus R\vec{e}_{\nu+1}$ where $\vec{e}_i$ is the standard vector having 1 in the ith position and 0 elsewhere. From Corollary 2.9 and 2.11, the distance $d(X, Y)$ between any two vertices X and Y of $\Gamma_1(R)$ and $\Gamma_2(R)$ are at most 3 and 2, respectively. In this section, we work on the *subconstituents* $\Gamma_s^{(i)}(R)$ for $i = 1, 2, 3$ if $s = 1$ and $i = 1, 2$ if $s = 2$ which are defined to be the induced subgraphs of $\Gamma_s(R)$ on the vertex sets

$$\mathcal{V}(\Gamma_s^{(i)}(R)) := \{X \in \mathcal{V}(\Gamma_s(R)) : d(X, X_0) = i\}.$$

We observe that it is possible to define another subconstituents associated with other vertices. However, our graph $\Gamma_s(R)$ is vertex transitive. Therefore, it suffices to consider only the ones associated with X_0. To study these subconstituents, we require an analog version of Lemma 2.7 as follows.

Lemma 3.1. *Under the above description, we have the following statements.*

1. *If X is a vertex in $\Gamma_s^{(i)}(R)$, then there are $|M|^{4(\nu-1)}$ many vertices which are lifts of the vertex $\pi(X)$ of $\Gamma_s^{(i)}(\mathbf{k})$, i.e. $|\{Y \in \mathcal{V}(\Gamma_s^{(i)}(R)) | \pi(Y) = \pi(X)\}| = |M|^{4(\nu-1)}$.*
2. *X is adjacent to Y in $\Gamma_s^{(i)}(R)$ if and only if $\pi(X)$ is adjacent to $\pi(Y)$ in $\Gamma_s^{(i)}(\mathbf{k})$.*
3. *If $\pi(X) = \mathbf{k}\pi(\vec{x}_1) \oplus \mathbf{k}\pi(\vec{x}_2)$ is adjacent to $\pi(Y) = \mathbf{k}\pi(\vec{y}_1) \oplus \mathbf{k}\pi(\vec{y}_2)$ in $\Gamma_s^{(i)}(\mathbf{k})$, then $X = R(\vec{x}_1 + \vec{m}_1) \oplus R(\vec{x}_2 + \vec{m}_2)$ is adjacent to $Y = R(\vec{y}_1 + \vec{n}_1) \oplus R(\vec{y}_2 + \vec{n}_2)$ in $\Gamma_s^{(i)}(R)$ for all $\vec{m}_i, \vec{n}_i \in M^{2\nu}$.*

Proof. The proof is analogous to the proof of Lemma 2.7 since X is adjacent to X_0 in $\Gamma_s(R)$ if and only if $\pi(X)$ is adjacent to $\pi(X_0)$ in $\Gamma_s(\mathbf{k})$. □

Note that the number of vertices of $\Gamma_s^{(1)}(R)$ is the valency of $\Gamma_s(R)$ and the valency of $\Gamma_s^{(1)}(R)$ is the number of common neighbors of any two vertices in $\Gamma_s(R)$.

3.1. Subconstituents of $\Gamma_1(R)$

By Proposition 3.1 and 3.2 of [14], for any two vertices $\pi(X)$ and $\pi(Y)$ in $\Gamma_1^{(1)}(\mathbf{k})$,

1. if $\pi(X)$ and $\pi(Y)$ are adjacent, then the number of their common neighbors in $\Gamma_1^{(1)}(\mathbf{k})$ is
$$|\mathbf{k}|^{2\nu-2} - 3, |\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^2 - |\mathbf{k}| - 3 \text{ or } |\mathbf{k}|^2 - 3,$$
2. if $\pi(X)$ and $\pi(Y)$ are nonadjacent, then the number of their common neighbors in $\Gamma_1^{(1)}(\mathbf{k})$ is
$$2|\mathbf{k}| - 2.$$

We apply Lemma 3.1 to find all parameters of $\Gamma_1^{(1)}(R)$, so we have the following theorem.

Theorem 3.2. *Assume that R is not a field.*

1. *If $\nu \geq 3$, then the number of common neighbors of any two adjacent vertices of $\Gamma_1^{(1)}(R)$ is*
$$(|M||R|)^{2\nu-2} - 3|M|^{4\nu-4},$$
$$(|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 - |M|^{4\nu-5}|R| - 3|M|^{4\nu-4} \text{ or}$$
$$|M|^{4\nu-6} - 3|M|^{4\nu-4}.$$
If $\nu = 2$, then the number of common neighbors of any two adjacent vertices of $\Gamma_1^{(1)}(R)$ is $2(|M||R|)^2 - |M|^3|R| - 3|M|^4$ or $(|M||R|)^2 - 3|M|^4$.
2. *The number of common neighbors of any two nonadjacent vertices X, Y of $\Gamma_1^{(1)}(R)$ is*
$$2|M|^{4\nu-5}|R| - 2|M|^{4\nu-4} \text{ if } \pi(X) \neq \pi(Y) \text{ and}$$
$$(|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 - |M|^{4\nu-6}|R|^2 - |M|^{4\nu-5}|R| - 2|M|^{4\nu-4} \text{ if } \pi(X) = \pi(Y).$$

Proof. It remains to show the last part of 2. For any two nonadjacent vertices X and Y such that $\pi(X) = \pi(Y)$, the number of their common neighbors is equal to their degree in $\Gamma_1^{(1)}(R)$. □

We conclude here about the graph $\Gamma_1^{(1)}(R)$.

Theorem 3.3. *Assume that R is not a field.*

1. *If $\nu \geq 3$, then the graph $\Gamma_1^{(1)}(R)$ is a 5-Deza graph with parameters $(n, k, \{d_1, d_2, d_3, d_4, d_5\})$ where*

$$\begin{aligned} n &= |M|^{2\nu-3}(|R| + |M|)(|R|^{2\nu-2} - |M|^{2\nu-2}), \\ k &= (|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 - |M|^{4\nu-5}|R| - 2|M|^{4\nu-4}, \\ d_1 &= (|M||R|)^{2\nu-2} - 3|M|^{4\nu-4}, \\ d_2 &= (|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 - |M|^{4\nu-5}|R| - 3|M|^{4\nu-4}, \\ d_3 &= |M|^{4\nu-6} - 3|M|^{4\nu-4}, \\ d_4 &= 2|M|^{4\nu-5}|R| - 2|M|^{4\nu-4}, \\ d_5 &= (|M||R|)^{2\nu-2} + |M|^{4\nu-6}|R|^2 - |M|^{4\nu-5}|R| - 2|M|^{4\nu-4}. \end{aligned}$$

2. *If $\nu = 2$, then then the graph $\Gamma_1^{(1)}(R)$ is a 4-Deza graph with parameters $(n, k, \{d_1, d_2, d_3, d_4\})$ where*

$$\begin{aligned} n &= |M|(|R| + |M|)(|R|^2 - |M|^2), \\ k &= 2(|M||R|)^2 - |M|^3|R| - 2|M|^4, \\ d_1 &= (|M||R|)^2 - 3|M|^4, \\ d_2 &= 2(|M||R|)^2 - |M|^3|R| - 3|M|^4, \\ d_3 &= 2|M|^3|R| - 2|M|^4, \\ d_4 &= 2(|M||R|)^2 - |M|^3|R| - 2|M|^4. \end{aligned}$$

According to [14], the number of vertices of $\Gamma_1^{(2)}(\mathbf{k})$ is $(|\mathbf{k}|^{2\nu-2} - 1)(|\mathbf{k}|^{2\nu-2} + |\mathbf{k}|^{2\nu-4} - |\mathbf{k}|)$. By Lemma 3.1, the lifts of these vertices are the vertices of $\Gamma_1^{(2)}(R)$. The space $\pi(X_0)$ is not the vertex of $\Gamma_1^{(2)}(\mathbf{k})$, however, any lift X of $\pi(X_0)$ in $\Gamma_1(R)$, $d(X, X_0) = 2$ by Theorem 2.8. Let X be a vertex of $\Gamma_1^{(2)}(R)$. Then by [14], the number of neighbors of $\pi(X)$ in $\Gamma_1^{(2)}(\mathbf{k})$ is $|\mathbf{k}|^{2\nu-1} + |\mathbf{k}|^{2\nu-2} - |\mathbf{k}|^{2\nu-4} - |\mathbf{k}|^2 - |\mathbf{k}| - 1$ or $|\mathbf{k}|^{2\nu-1} + |\mathbf{k}|^{2\nu-2} - |\mathbf{k}|^2 - 2|\mathbf{k}| - 1$. Therefore,

Theorem 3.4. *Assume that R is not a field. The second subconstituent $\Gamma_1^{(2)}(R)$ is not regular with the number of vertices*

$$(|R|^{2\nu-2} - |M|^{2\nu-2})(|R|^{2\nu-2}|M|^{2\nu-2} + |R|^{2\nu-4}|M|^{2\nu} - |R||M|^{4\nu-5}) + |M|^{4\nu-4} - 1$$

and the number of neighbors of any vertex is

$$\begin{aligned} &|R|^{2\nu-1}|M|^{2\nu-3} + |R|^{2\nu-2}|M|^{2\nu-2} - |R|^{2\nu-4}|M|^{2\nu} - |R|^2|M|^{4\nu-6} - |R||M|^{4\nu-4} - |M|^{4\nu-4} \\ &\text{or } |R|^{2\nu-1}|M|^{2\nu-3} + |R|^{2\nu-2}|M|^{2\nu-2} - |R|^2|M|^{4\nu-6} - 2|R||M|^{4\nu-4} - |M|^{4\nu-4}. \end{aligned}$$

For the third subconstituent $\Gamma_1^{(3)}(\mathbf{k})$, there are 2 distinct cases up to ν to be considered [14]. If $\nu = 2$, then $\Gamma_1^{(3)}(\mathbf{k})$ is an empty graph with one vertex. On the other hand, for $\nu \geq 3$, the graph $\Gamma_1^{(3)}(\mathbf{k})$ is a quasi-strongly regular graph with parameters

$$\Big(\frac{|\mathbf{k}|^{2\nu-4}(|\mathbf{k}|^{2\nu-2} - 1)}{|\mathbf{k}|^2 - 1}, (|\mathbf{k}| + 1)(|\mathbf{k}|^{2\nu-4} - 1), |\mathbf{k}|^{2\nu-4} + |\mathbf{k}|^2 - |\mathbf{k}| - 2; |\mathbf{k}|^2 + |\mathbf{k}|, |\mathbf{k}|^2, 0\Big).$$

By applying Lemma 3.1, we can prove the result for the graph $\Gamma_1^{(3)}(R)$ analogously to the case of $\Gamma_1(R)$ and we record this in the following theorem.

Theorem 3.5. *Assume that R is not a field.*

1. *If* $\nu = 2$*, then the third subconstituent* $\Gamma_1^{(3)}(R)$ *is an empty graph with* $|M|^4$ *vertices.*
2. *If* $\nu \geq 3$*, then the third subconstituent* $\Gamma_1^{(3)}(R)$ *is a quasi-strongly regular graph with parameters* $(n, k, \lambda; c_1, c_2, c_3, c_4)$ *where*

$$\begin{aligned}
n &= \frac{|M|^4|R|^{2\nu-4}(|R|^{2\nu-2} - |M|^{2\nu-2})}{|R|^2 - |M|^2},\\
k &= |M|^{2\nu-1}(|R| - |M|)(|R|^{2\nu-4} - |M|^{2\nu-4}),\\
\lambda &= |R|^{2\nu-4}|M|^{2\nu} + |R|^2|M|^{4\nu-6} - |R||M|^{4\nu-5} - 2|M|^{4\nu-4},\\
c_1 &= |R|^2|M|^{4\nu-6} + |R||M|^{4\nu-5},\\
c_2 &= |R|^2|M|^{4\nu-6},\\
c_3 &= 0,\\
c_4 &= |M|^{2\nu-1}(|R| - |M|)(|R|^{2\nu-4} - |M|^{2\nu-4}).
\end{aligned}$$

3.2. Subconstituents of $\Gamma_2(R)$

For these subconstituents, we first investigate the graphs over a finite field. By Corollary 2.11, we can consider $\Gamma_2^{(1)}(\mathbb{F}_q)$ and $\Gamma_2^{(2)}(\mathbb{F}_q)$.

Theorem 3.6. *The first subconstituent* $\Gamma_2^{(1)}(\mathbb{F}_q)$ *is not regular with*

$$\frac{q^{2\nu-2}(q^{2\nu} - 1)}{q^2 - 1} - q^{2\nu-1} - q^{2\nu-2} + q$$

vertices. Moreover, the degree of a vertex of $\Gamma_2^{(1)}(\mathbb{F}_q)$ *is*

$$\begin{aligned}
&\frac{q^{2\nu-2}(q^{2\nu} - 1)}{q^2 - 1} - 2q^{2\nu-1} - 2q^{2\nu-2} + q^2 + 3q,\\
&\frac{q^{2\nu-2}(q^{2\nu} - 1)}{q^2 - 1} - 2q^{2\nu-1} - 2q^{2\nu-2} + q^2 + 2q \text{ or}\\
&\frac{q^{2\nu-2}(q^{2\nu} - 1)}{q^2 - 1} - 2q^{2\nu-1} - 2q^{2\nu-2}.
\end{aligned}$$

Proof. As mentioned, the number of vertices of $\Gamma_2^{(1)}(\mathbb{F}_q)$ is the valency of $\Gamma_2(\mathbb{F}_q)$ and the valency of $\Gamma_2^{(1)}(\mathbb{F}_q)$ is the number of common neighbors of any two vertices in $\Gamma_2(\mathbb{F}_q)$. By Theorem 2.3, we have the theorem. □

Theorem 3.7. 1. *If* $\nu = 2$*, then the second subconstituent* $\Gamma_2^{(2)}(\mathbb{F}_q)$ *is a quasi-strongly regular graph with parameters*

$$(q^3 + q^2 - q - 1, q^3 - q^2 + 1, q^3 - 3q^2 + 3q - 1; q^3 - 2q^2 + q, q^3 - q^2).$$

2. *If* $\nu \geq 3$*, then the second subconstituent* $\Gamma_2^{(2)}(\mathbb{F}_q)$ *is a quasi-strongly regular graph with parameters* $(n, k, \lambda; c_1, c_2, c_3)$ *where*

$$n = q^{2\nu-1} + q^{2\nu-2} - q - 1,$$

$$
\begin{aligned}
k &= q^{2\nu-1} - q^2 + 1,\\
\lambda &= q^{2\nu-1} - q^{2\nu-2} - 2q^2 + 3q - 1,\\
c_1 &= q^{2\nu-1} - 2q^2 + q,\\
c_2 &= q^{2\nu-1} - q^2,\\
c_3 &= q^{2\nu-1} - q^{2\nu-2} - q^2 + q.
\end{aligned}
$$

Proof. By (2.1), a vertex of $\Gamma_2^{(2)}(\mathbb{F}_q)$ is a vertex in $\Gamma_1^{(1)}$. It is immediate from Theorem 3.3 of [14] that $n = (q+1)(q^{2\nu-2} - 1) = q^{2\nu-1} + q^{2\nu-2} - q - 1$. Note that any neighbor of a vertex X in $\Gamma_2^{(2)}(\mathbb{F}_q)$ is a vertex in $\Gamma_1^{(1)}(\mathbb{F}_q)$ which is not adjacent to X. Thus, by Theorem 3.3 of [14],

$$
\begin{aligned}
k &= (q+1)(q^{2\nu-2} - 1) - (q^{2\nu-2} + q^2 - q - 2)\\
&= q^{2\nu-1} - q^2 + 1.
\end{aligned}
$$

Let X and Y be two adjacent vertices in $\Gamma_2^{(2)}(\mathbb{F}_q)$. Suppose that Z is a vertex in $\Gamma_2^{(2)}(\mathbb{F}_q)$ such that Z is a common neighbor of X and Y. Then, by (2.1), X, Y are nonadjacent vertices in $\Gamma_1^{(1)}(\mathbb{F}_q)$ and Z is a vertex in $\Gamma_1^{(1)}(\mathbb{F}_q)$ such that $Z \neq X, Z \neq Y$ and is not adjacent to X or Y. The inclusion-exclusion principle and Theorem 3.3 of [14] give

$$
\begin{aligned}
\lambda &= (q+1)(q^{2\nu-2} - 1) - (2(q^{2\nu-2} + q^2 - q - 2) - (2q - 2)) - 2\\
&= q^{2\nu-1} - q^{2\nu-2} - 2q^2 + 3q - 1.
\end{aligned}
$$

The similar argument to the previous paragraph applies to the numbers of common neighbors of nonadjacent vertices of $\Gamma_2^{(2)}(\mathbb{F}_q)$. Thus, they are

$$
\begin{cases}
c_1 &= (q+1)(q^{2\nu-2} - 1) - (2(q^{2\nu-2} + q^2 - q - 2) - (q^{2\nu-2} - 3))\\
&= q^{2\nu-1} - 2q^2 + q,\\
c_2 &= (q+1)(q^{2\nu-2} - (2(q^{2\nu-2} + q^2 - q - 2) - (q^{2\nu-2} + q^2 - q - 3))\\
&= q^{2\nu-1} - q^2,\\
c_3 &= (q+1)(q^{2\nu-2} - (2(q^{2\nu-2} + q^2 - q - 2) - (q^2 - 3))\\
&= q^{2\nu-1} - q^{2\nu-2} - q^2 + q.
\end{cases}
$$

For $\nu = 2$, $c_1 = c_3$. Therefore, we have proved the theorem. □

Finally, we extend the results to the subconstituents of $\Gamma_2^{(2)}(R)$ where R is a not a field using Theorem 3.1 and the similar argument described in Subsection 3.1.

Theorem 3.8. *Assume that R is not a field. The first subconstituent $\Gamma_2^{(1)}(R)$ is not regular with*

$$
\frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2} - |R|^{2\nu-1}|M|^{2\nu-3} - |R|^{2\nu-2}|M|^{2\nu-2} + |R||M|^{4\nu-5}
$$

vertices. Moreover, the degree of a vertex of $\Gamma_2^{(1)}(R)$ is

$$
\frac{|R|^{2\nu-2}(|R|^{2\nu} - |M|^{2\nu})}{|R|^2 - |M|^2} - 2|R|^{2\nu-1}|M|^{2\nu-3} - 2|R|^{2\nu-2}|M|^{2\nu-2} + |R|^2|M|^{4\nu-6} + 3|R||M|^{4\nu-5},
$$

$$\frac{|R|^{2\nu-2}(|R|^{2\nu}-|M|^{2\nu})}{|R|^2-|M|^2}-2|R|^{2\nu-1}|M|^{2\nu-3}-2|R|^{2\nu-2}|M|^{2\nu-2}+|R|^2|M|^{4\nu-6}+2|R||M|^{4\nu-5} \text{ or}$$
$$\frac{|R|^{2\nu-2}(|R|^{2\nu}-|M|^{2\nu})}{|R|^2-|M|^2}-2|R|^{2\nu-1}|M|^{2\nu-3}-2|R|^{2\nu-2}|M|^{2\nu-2}.$$

Theorem 3.9. *Assume that R is not a field.*

1. *If $\nu = 2$, then the second subconstituent $\Gamma_2^{(2)}(R)$ is a quasi-strongly regular graph with parameters $(n, k, \lambda; c_1, c_2, c_3)$ where*

$$\begin{aligned} n &= |R|^3|M| + |R|^2|M|^2 - |R||M|^3 - |M|^4, \\ k &= |R|^3|M| - |R|^2|M|^2 + |M|^4, \\ \lambda &= |R|^3|M| - 3|R|^2|M|^2 + 3|R||M|^3 - |M|^4, \\ c_1 &= |R|^3|M| - 2|R|^2|M|^2 + |R||M|^3, \\ c_2 &= |R|^3|M| - |R|^2|M|^2, \\ c_3 &= |R|^3|M| - |R|^2|M|^2 + |M|^4. \end{aligned}$$

2. *If $\nu \geq 3$, then the second subconstituent $\Gamma_2^{(2)}(R)$ is a quasi-strongly regular graph with parameters $(n, k, \lambda; c_1, c_2, c_3, c_4)$ where*

$$\begin{aligned} n &= |R|^{2\nu-1}|M|^{2\nu-3} + |R|^{2\nu-2}|M|^{2\nu-2} - |R||M|^{4\nu-3} - |M|^{4\nu-4}, \\ k &= |R|^{2\nu-1}|M|^{2\nu-3} - |R|^2|M|^{2\nu-6} + |M|^{4\nu-4}, \\ \lambda &= |R|^{2\nu-1}|M|^{2\nu-3} - |R|^{2\nu-2}|M|^{2\nu-2} - 2|R|^2|M|^{4\nu-6} + 3|R||M|^{4\nu-5} - |M|^{4\nu-4}, \\ c_1 &= |R|^{2\nu-1}|M|^{2\nu-3} - 2|R|^2|M|^{4\nu-6} + |R||M|^{4\nu-5}, \\ c_2 &= |R|^{2\nu-1}|M|^{2\nu-3} - |R|^2|M|^{4\nu-6}, \\ c_3 &= |R|^{2\nu-1}|M|^{2\nu-3} - |R|^{2\nu-2}|M|^{2\nu-2} - |R|^2|M|^{4\nu-6} + |R||M|^{4\nu-5}, \\ c_4 &= |R|^{2\nu-1}|M|^{2\nu-3} - |R|^2|M|^{2\nu-6} + |M|^{4\nu-4}. \end{aligned}$$

4. Conclusions

The nonisotropic symplectic graphs of type 1 over finite fields were studied in [14]. The graphs were defined by 2-dimensional nonisotropic subspaces and their intersection for the adjacency condition. We considered this type of graphs and their complements over general finite commutative rings by using McCoy ranks of matrices over rings. All parameters were computed by using combinatorial approach and some graphs properties as in [14] were also analyzed. We also studied their subconstituents. These results may have some impacts to the theory of quasi-strongly regular graphs. Exploring the possible applications of these graphs requires further attention.

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Conflict of interest

Authors do not have any conflict of interests.

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