Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 632–650.
DOI: 10.3934/math.2022040
Received: 01 August 2021
Accepted: 07 October 2021
Published: 14 October 2021

Research article

On a boundary value problem for fractional Hahn integro-difference equations with four-point fractional integral boundary conditions

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Abstract: In this paper, we study a boundary value problem consisting of Hahn integro-difference equation supplemented with four-point fractional Hahn integral boundary conditions. The novelty of this problem lies in the fact that it contains two fractional Hahn difference operators and three fractional Hahn integrals with different quantum numbers and orders. Firstly, we convert the given nonlinear problem into a fixed point problem, by considering a linear variant of the problem at hand. Once the fixed point operator is available, we make use the classical Banach's and Schauder's fixed point theorems to establish existence and uniqueness results. An example is also constructed to illustrate the main results. Several properties of fractional Hahn integral that will be used in our study are also discussed.

Keywords: fractional Hahn integral; fractional Hahn difference; boundary value problems; existence
Mathematics Subject Classification: 39A10, 39A13, 39A70

1. Introduction

The topic of fractional differential equations has gained considerable attention and has evolved as an interesting field of research, mainly due to the fact that the tools of fractional calculus are found to be more practical and effective than the corresponding ones of classical calculus in the mathematical modeling real wold problems. In fact, fractional calculus has numerous applications in various disciplines of science and engineering such as mechanics, chemistry, biology, economics, electricity, control theory, signal and image processing, regular variation in thermodynamics, biophysics, aerodynamics, viscoelasticity and damping, etc. For the basic theory and applications of fractional calculus, as well as for some recent developments in the field we refer to [1]– [9] and the references cited therein.

Many physical phenomena are described by equations involving non-differentiable functions, e.g., generic trajectories of quantum mechanics [10]. The substitution of the classical derivative by a difference operator, which allows to deal with sets of non-differentiable functions, give rise to the so-called quantum calculus. Many different types of quantum difference operators appeared in the literature, for example h-calculus, q-calculus, Hahn's calculus, forward quantum calculus and backward quantum calculus. These operators have applications in orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, particle physics, quantum mechanics and the theory of relativity (see [11]–[24] and the references therein for some applications and new results of the quantum calculus).

Recently, many researchers have extensively studied calculus without limit that deals with a set of non-differentiable functions, the so-called quantum calculus. Many types of quantum difference operators are employed in several applications of mathematical areas such as the calculus of variations, particle physics, quantum mechanics and theory of relativity (see [12]–[24] and the references therein for some applications and new results of the quatum calculus).

In this paper, we study the Hahn quantum calculus that is one type of quantum calculus. W. Hahn [25] introduced the Hahn difference operator $D_{q,\omega}$ in 1949 as follow:

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, \quad t \neq \omega_0 := \frac{\omega}{1-q}.$$

The Hahn difference operator is a combination of two well-known difference operators, the forward difference operator and the Jackson q-difference operator. Notice that

$$\begin{aligned} D_{q,\omega}f(t) &= \Delta_\omega f(t) \quad \text{whenever } q = 1, \\ D_{q,\omega}f(t) &= D_q f(t) \quad \text{whenever } \omega = 0, \\ D_{q,\omega}f(t) &= f'(t) \quad \text{whenever } q = 1, \omega \to 0. \end{aligned}$$

The Hahn difference operator has been employed to construct families of orthogonal polynomials and investigate some approximation problems (see [26]–[28] and the references therein).

In 2009, K. A. Aldwoah [29,30] defined the right inverse of $D_{q,\omega}$ in the terms of both the Jackson q-integral containing the right inverse of D_q [31] and Nörlund sum contaning the right inverse of Δ_ω [31].

In 2010, A. B. Malinowska and D. F. M. Torres [32, 33] introduced the Hahn quantum variational calculus. In 2013, A. B. Malinowska and N. Martins [34] studied the generalized transversality conditions for the Hahn quantum variational calculus. Later, A. E. Hamza and S. M. Ahmed [35, 36] studied the theory of linear Hahn difference equations, and investigated the existence and uniqueness results for the initial value problems for Hahn difference equations by using the method of successive approximations. Moreover, they proved Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator and established the mean value theorems for this calculus. In 2016, A. E. Hamza and S. D. Makharesh [37] investigated the Leibniz's rule and Fubini's theorem associated with Hahn difference operator. In the same year, T. Sitthiwirattham [38] considered a nonlinear Hahn difference equation with nonlocal boundary value conditions. In 2017, U. Sriphanomwan *et al.* [39]

considered a nonlocal boundary value problem for second-order nonlinear Hahn integro-difference equation with integral boundary condition.

In 2010, J. Čermák and L. Nechvátal [40] proposed the fractional (q,h)-difference operator and the fractional (q,h)-integral for $q > 1$. In 2011, Čermák *et al.* [41] studied discrete Mittag-Leffler functions in linear fractional difference equations for $q > 1$, and M. R. S. Rahmat [42, 43] studied the (q,h)-Laplace transform and some (q,h)-analogues of integral inequalities on discrete time scales for $q > 1$. In 2016, F. Du *et al.* [44] presented the monotonicity and convexity for nabla fractional (q,h)-difference for $q > 0$, $q \neq 1$. However, we realize that Hahn difference operator requires the condition $0 < q < 1$. Therefore, to fill the gap, T. Brikshavana and T. Sitthiwirattham [45] have introduced the fractional Hahn difference operators for $0 < q < 1$.

In quantum calculus, there are apparently few research works related to boundary value problems of fractional Hahn difference equations (see [46]–[48]). Motivated by the above discussion, to fill the gap on contributions concerning boundary value problems of fractional Hahn difference equations, the goal of this paper is to enrich this new research area. So, in this paper, we introduce and study a four-point fractional Hahn integral boundary value problems for fractional Hahn integrodifference equation of the form

$$\begin{aligned} D^{\alpha}_{q,\omega}u(t) &= F\left[t, u(t), \Psi^{\gamma}_{r,\rho}u(t), \Upsilon^{\nu}_{m,\chi}u(t)\right], \quad t \in I^T_{q,\omega}, \\ u(\xi) &= \phi_1(u) + \lambda_1 \mathcal{I}^{\beta_1}_{p_1,\theta_1} g_1(\eta_1)u(\eta_1), \quad \xi, \eta_1 \in I^T_{q,\omega} - \{\omega_0, T\}, \ \xi > \eta_1, \\ u(T) &= \phi_2(u) + \lambda_2 \mathcal{I}^{\beta_2}_{p_2,\theta_2} g_2(\eta_2)u(\eta_2), \quad \eta_2 \in I^T_{q,\omega} - \{\omega_0, T\}, \end{aligned} \tag{1.1}$$

where $I^T_{q,\omega} := \{q^k T + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$; $\alpha \in (1,2]$, $\beta_1, \beta_2, \gamma, \nu \in (0,1]$, $\omega > 0$, $p_1, p_2, q, r, m \in (0,1)$, $p_1 = q^a$, $p_2 = q^b$, $r = q^c$, $m = q^d$, $a, b, c, d \in \mathbb{N}$, $\theta_1 = \omega\left(\frac{1-p_1}{1-q}\right)$, $\theta_2 = \omega\left(\frac{1-p_2}{1-q}\right)$, $\rho = \omega\left(\frac{1-r}{1-q}\right)$, $\chi = \omega\left(\frac{1-m}{1-q}\right)$, $\lambda_1, \lambda_2 \in \mathbb{R}^+$, $F \in C(I^T_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g_1, g_2 \in C(I^T_{q,\omega}, \mathbb{R}^+)$ and given functions, $\phi_1, \phi_2 : C(I^T_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ are given functionals, and for $\varphi \in C(I^T_{r,\rho} \times I^T_{r,\rho}, [0,\infty))$ and $\psi \in C(I^T_{m,\chi} \times I^T_{m,\chi}, [0,\infty))$, we define

$$\begin{aligned} \Psi^{\gamma}_{r,\rho}u(t) &:= \left(\mathcal{I}^{\gamma}_{r,\rho}\varphi\, u\right)(t) = \frac{1}{\Gamma_r(\gamma)} \int_{\omega_0}^{t} \left(t - \sigma_{r,\rho}(s)\right)^{\underline{\gamma-1}}_{r,\rho} \varphi(t,s)\, u(s)\, d_{r,\rho}s, \\ \Upsilon^{\nu}_{m,\chi}u(t) &:= \left(D^{\nu}_{m,\chi}\psi\, u\right)(t) = \frac{1}{\Gamma_m(-\nu)} \int_{\omega_0}^{t} \left(t - \sigma_{m,\chi}(s)\right)^{\underline{-\nu-1}}_{m,\chi} \psi(t,s)\, u(s)\, d_{m,\chi}s. \end{aligned}$$

We emphasize that our problem contains two fractional Hahn difference operators and three fractional Hahn integrals with different quantum numbers and orders. To the authors' best knowledge, this is the new development on the topic as the quantum number and order of the problems studied in the literature are the same.

We aim to show the existence and uniqueness of a solution to the problem (1.1) by using the Banach fixed point theorem, and the existence of at least one solution by using the Schauder's fixed point theorem. In addition, an example is provided to illustrate our results in the last section.

The rest of this paper is organized as follows: We present our existence and uniqueness result in Section 3, and our existence result in Section 4, while Section 2 contains some preliminary concepts related to our problem. An example is constructed to illustrate the main results in Section 5. Finally, Section 6 is a conclusion section.

2. Preliminaries

In this section, we briefly recall some definitions and lemmas used in this research work. In this work, we use the Banach space $C = C\left(I^T_{q,\omega}, \mathbb{R}\right)$ of all function u with the norm defined as

$$\|u\|_C = \|u\| + \|D^{\nu}_{m,\chi}u\|,$$

where $\|u\| = \max_{t\in I^T_{q,\omega}} \{|u(t)|\}$ and $\|D^{\nu}_{m,\chi}u\| = \max_{t\in I^T_{m,\chi}} \{ \left|D^{\nu}_{m,\chi}u(t)\right| \}$.

Let $q \in (0, 1)$, $\omega > 0$. We define the notations

$$[n]_q := \frac{1-q^n}{1-q} = q^{n-1} + \ldots + q + 1 \qquad \text{and} \qquad [n]_q! := \prod_{k=1}^{n} \frac{1-q^k}{1-q}, \qquad n \in \mathbb{R}.$$

The forward jump operator and the backward jump operator are defined as

$$\sigma^k_{q,\omega}(t) := q^k t + \omega[k]_q \quad \text{and} \quad \rho^k_{q,\omega}(t) := \frac{t-\omega[k]_q}{q^k} \quad \text{for} \ k \in \mathbb{N}.$$

The q-analogue of the power function $(a-b)^{\underline{n}}_q$ and the q,ω-analogue of the power function $(a-b)^{\underline{n}}_{q,\omega}$ with $n \in \mathbb{N}_0 := [0, 1, 2, \ldots]$, $a, b \in \mathbb{R}$ are defined as

$$(a-b)^{\underline{0}}_q := 1, \qquad (a-b)^{\underline{n}}_q := \prod_{k=0}^{n-1}(a - bq^k),$$

$$(a-b)^{\underline{0}}_{q,\omega} := 1, \qquad (a-b)^{\underline{n}}_{q,\omega} := \prod_{k=0}^{n-1}\left[a - (bq^k + \omega[k]_q)\right],$$

respectively.

In generally, if $\alpha \in \mathbb{R}$, we get

$$(a-b)^{\underline{\alpha}}_q = a^{\alpha} \prod_{n=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^n}{1-\left(\frac{b}{a}\right)q^{\alpha+n}}, \ a \neq 0,$$

$$(a-b)^{\underline{\alpha}}_{q,\omega} = (a-\omega_0)^{\alpha} \prod_{n=0}^{\infty} \frac{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^n}{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{\alpha+n}} = \left((a-\omega_0)-(b-\omega_0)\right)^{\underline{\alpha}}_q, \ a \neq \omega_0.$$

Note that $a^{\underline{\alpha}}_q = a^{\alpha}$, $(a-\omega_0)^{\underline{\alpha}}_{q,\omega} = (a-\omega_0)^{\alpha}$, and $(0)^{\underline{\alpha}}_q = (\omega_0)^{\underline{\alpha}}_{q,\omega} = 0$ for $\alpha > 0$. The q-gamma and q-beta functions are defined as

$$\Gamma_q(x) := \frac{(1-q)^{\underline{x-1}}_q}{(1-q)^{x-1}}, \qquad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

$$B_q(x,s) := \int_0^1 t^{x-1}(1-qt)^{\underline{s-1}}_q d_qt = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)},$$

respectively.

Definition 2.1. *For* $q \in (0,1)$*,* $\omega > 0$ *and* f *defined on an interval* $I \subseteq \mathbb{R}$ *which containing* $\omega_0 := \frac{\omega}{1-q}$*, the Hahn difference of* f *is defined as*

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega} \quad \text{for } t \neq \omega_0,$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$*, provided that* f *is differentiable at* ω_0*. We call* $D_{q,\omega}f$ *the* q,ω*-derivative of* f*, and say that* f *is* q,ω*-differentiable on* I*.*

The Hahn difference operator has the following properties:

Lemma 2.1. *[30] Let* $f, g : I \to \mathbb{R}$ *are* q,ω*-differentiable on* I*. Then we have:*

(1) $D_{q,\omega}[f(t)+g(t)] = D_{q,\omega}f(t) + D_{q,\omega}g(t)$.
(2) $D_{q,\omega}[\alpha f(t)] = \alpha D_{q,\omega}f(t)$.
(3) $D_{q,\omega}[f(t)g(t)] = f(t)D_{q,\omega}g(t) + g(qt+\omega)D_{q,\omega}f(t)$.
(4) $D_{q,\omega}\left[\frac{f(t)}{g(t)}\right] = \frac{g(t)D_{q,\omega}f(t) - f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)}$.

Definition 2.2. *Let* I *be any closed interval of* $\mathbb{R}$ *that contains* a, b *and* ω_0*. If* $f : I \to \mathbb{R}$ *is a given function, we define the* q,ω*-integral of* f *from* a *to* b *by*

$$\int_a^b f(t)d_{q,\omega}t := \int_{\omega_0}^b f(t)d_{q,\omega}t - \int_{\omega_0}^a f(t)d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)d_{q,\omega}t := [x(1-q)-\omega]\sum_{k=0}^{\infty} q^k f\left(xq^k + \omega[k]_q\right), \quad x \in I,$$

and the series converges at $x = a$ *and* $x = b$*. We say* f *is* q,ω*-integrable on* $[a,b]$ *and the sum to the right hand side of this equation is called the Jackson-Nörlund sum.*

Notice that the actual domain of function f is defined on $[a,b]_{q,\omega} \subset I$.

Next, we introduce the fundamental theorem of Hahn calculus.

Lemma 2.2. *[29] Let* $f : I \to \mathbb{R}$ *be continuous at* ω_0 *and define*

$$F(x) := \int_{\omega_0}^x f(t)d_{q,\omega}t, \quad x \in I.$$

Then, F *is continuous at* ω_0*. In addition,* $D_{q,\omega_0}F(x)$ *exists for every* $x \in I$ *and*

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega}F(t)d_{q,\omega}t = F(b) - F(a) \text{ for all } a, b \in I.$$

Lemma 2.3. *[38] Let* $q \in (0,1)$*,* $\omega > 0$ *and* $f : I \to \mathbb{R}$ *be continuous at* ω_0*. Then*

$$\int_{\omega_0}^t \int_{\omega_0}^r x(s)\, d_{q,\omega}s\, d_{q,\omega}r = \int_{\omega_0}^t \int_{qs+\omega}^t x(s)\, d_{q,\omega}r\, d_{q,\omega}s.$$

Lemma 2.4. *[38] Let $q \in (0,1)$ and $\omega > 0$. Then*

$$\int_{\omega_0}^{t} d_{q,\omega}s = t - \omega_0 \quad and \quad \int_{\omega_0}^{t} [t - \sigma_{q,\omega}(s)]\, d_{q,\omega}s = \frac{(t-\omega_0)^2}{1+q}.$$

Now, we give the definitions of fractioanal Hahn integral and fractional Hahn difference of Riemann-Liouville type, as follows:

Definition 2.3. *For $\alpha, \omega > 0,\ q \in (0,1)$ and $f : I_{q,\omega}^{T} \to \mathbb{R}$, the fractional Hahn integral is defined by*

$$\begin{aligned} \mathcal{I}_{q,\omega}^{\alpha} f(t) &:= \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}} f(s) d_{q,\omega}s \\ &= \frac{[t(1-q) - \omega]}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n \Big(t - \sigma_{q,\omega}^{n+1}(t)\Big)_{q,\omega}^{\underline{\alpha-1}} f\Big(\sigma_{q,\omega}^{n}(t)\Big), \end{aligned}$$

and $(\mathcal{I}_{q,\omega}^{0} f)(t) = f(t)$.

Definition 2.4. *For $\alpha, \omega > 0,\ q \in (0,1)$ and $f : I_{q,\omega}^{T} \to \mathbb{R}$, the fractional Hahn difference of the Riemann-Liouville type of order α is defined by*

$$\begin{aligned} D_{q,\omega}^{\alpha} f(t) &:= (D_{q,\omega}^{N} \mathcal{I}_{q,\omega}^{N-\alpha} f)(t) \\ &= \frac{1}{\Gamma_q(-\alpha)} \int_{\omega_0}^{t} \Big(t - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{-\alpha-1}} f(s) d_{q,\omega}s, \end{aligned}$$

and $D_{q,\omega}^{0} f(t) = f(t)$, where $N-1 \le \alpha \le N,\ N \in \mathbb{N}$.

Lemma 2.5. *[45] Letting $\alpha > 0, q \in (0,1), \omega > 0$ and $f : I_{q,\omega}^{T} \to \mathbb{R}$,*

$$\mathcal{I}_{q,\omega}^{\alpha} D_{q,\omega}^{\alpha} f(t) = f(t) + C_1 (t-\omega_0)^{\alpha-1} + \ldots + C_N (t-\omega_0)^{\alpha-N},$$

for some $C_i \in \mathbb{R},\ i = 1,2,\ldots,N$ and $N-1 < \alpha \le N, N \in \mathbb{N}$.

Next, we give some auxiliary lemmas to use in simplifying calculations.

Lemma 2.6. *[45] Let $\alpha, \beta > 0,\ p, q \in (0,1)$ and $\omega > 0$,*

$$(i) \quad \int_{\omega_0}^{t} \Big(t - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}} d_{q,\omega}s = \frac{(t-\omega_0)^{\alpha}}{[\alpha]_q},$$

$$(ii) \quad \int_{\omega_0}^{t} \Big(t - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}} (s-\omega_0)_{q,\omega}^{\underline{\beta}}\, d_{q,\omega}s = (t-\omega_0)^{\alpha+\beta} B_q(\beta+1, \alpha).$$

Lemma 2.7. *Let $\alpha, \beta, \nu > 0,\ n \in \mathbb{N},\ q, \chi \in (0,1),\ \omega, m > 0$ and $\chi = \omega\Big(\frac{1-m}{1-q}\Big)$. Then,*

$$(i) \quad \int_{\omega_0}^{t} \Big(t - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\beta-1}} (s-\omega_0)^{\alpha-n}\, d_{q,\omega}s = (t-\omega_0)^{\alpha+\beta-n} B_q(\alpha-n+1, \beta),$$

$$(ii)\quad \int_{\omega_0}^{t}\Big(t-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{-\nu-1}}(s-\omega_0)^{\alpha-n}\,d_{q,\omega}s=(t-\omega_0)^{\alpha-\nu-n}B_q(\alpha-n+1,-\nu),$$

$$(iii)\quad \int_{\omega_0}^{t}\int_{\omega_0}^{x}\Big(t-\sigma_{m,\chi}(x)\Big)_{m,\chi}^{\underline{-\nu-1}}\Big(x-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}}\,d_{q,\omega}s\,d_{m,\chi}x$$
$$=\frac{(t-\omega_0)^{\alpha-\nu}}{[\alpha]_q}B_m(\alpha+1,-\nu).$$

Proof. From the definition of q,ω-analogue of the power function and Definition 2.3, we obtain

$$\begin{aligned}(i)\quad &\int_{\omega_0}^{t}\Big(t-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\beta-1}}(s-\omega_0)^{\alpha-n}\,d_{q,\omega}s\\
&=(t-\omega_0)^{\beta}(1-q)\sum_{i=0}^{\infty}q^i\Big(1-q^{i+1}\Big)_q^{\underline{\beta-1}}\Big(q^i(t-\omega_0)\Big)^{\alpha-n}\\
&=(t-\omega_0)^{\alpha+\beta-n}(1-q)\sum_{i=0}^{\infty}q^i\,(1-q^{i+1})_q^{\underline{\beta-1}}\,q^{i(\alpha-n)}\\
&=(t-\omega_0)^{\alpha+\beta-n}B_q(\alpha-n+1,\beta).\end{aligned}$$

Similarly, we obtain (ii).

$$\begin{aligned}(iii)\quad &\int_{\omega_0}^{t}\int_{\omega_0}^{x}\Big(t-\sigma_{m,\chi}(x)\Big)_{m,\chi}^{\underline{-\nu-1}}\Big(x-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}}\,d_{q,\omega}s\,d_{m,\chi}x\\
&=\int_{\omega_0}^{t}\Big(t-\sigma_{m,\chi}(x)\Big)_{m,\chi}^{\underline{-\nu-1}}\left[\int_{\omega_0}^{x}\Big(x-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}}\,d_{q,\omega}s\right]d_{m,\chi}x\\
&=\frac{1}{[\alpha]_q}\int_{\omega_0}^{t}\Big(t-\sigma_{m,\chi}(x)\Big)_{m,\chi}^{\underline{-\nu-1}}(x-\omega_0)^{\alpha}\,d_{m,\chi}x\\
&=\frac{(t-\omega_0)^{\alpha-\nu}}{[\beta]_q}B_m(\alpha+1,-\nu).\end{aligned}$$

□

The following lemma, dealing with a linear variant of problem (1.1), plays an important role in the forthcoming analysis.

Lemma 2.8. *Let* $\Omega\neq 0$, $\omega>0$, $q\in(0,1)$, $\alpha\in(1,2]$, *and for* $i=1,2$, $\theta_i>0$, $\beta_i\in(0,1]$, $p_i\in(0,1)$, $p_i=q^{m_i}$, $m_i\in\mathbb{N}$, $\theta_i=\omega\left(\frac{1-p_i}{1-q}\right)$, $\lambda_i\in\mathbb{R}^+$; $h\in C(I_{q,\omega}^T,\mathbb{R})$, $g_1,g_2\in C(I_{q,\omega}^T,\mathbb{R}^+)$ *are given functions,* $\phi_1,\phi_2: C(I_{q,\omega}^T,\mathbb{R})\to\mathbb{R}$ *are given functionals. Then the problem*

$$\begin{aligned}D_{q,\omega}^{\alpha}u(t)&=h(t),\quad t\in I_{q,\omega}^T,\\
u(\xi)&=\phi_1(u)+\lambda_1\mathcal{I}_{p_1,\theta_1}^{\beta_1}g_1(\eta_1)u(\eta_1),\quad \xi,\eta_1\in I_{q,\omega}^T-\{\omega_0,T\},\ \xi>\eta_1,\\
u(T)&=\phi_2(u)+\lambda_2\mathcal{I}_{p_2,\theta_2}^{\beta_2}g_2(\eta_2)u(\eta_2),\quad \eta_2\in I_{q,\omega}^T-\{\omega_0,T\},\end{aligned}\tag{2.1}$$

has the unique solution

$$\begin{aligned} u(t) \;=\; & \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{t} \Big(t - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}} \, h(s)\, d_{q,\omega}s \\ & + \frac{(t-\omega_0)^{\alpha-1}}{\Omega}\Big(\boldsymbol{B}_{T,\eta_2}\mathcal{O}_{\xi,\eta_1}[\phi_1,h] - \boldsymbol{B}_{\xi,\eta_1}\mathcal{O}_{T,\eta_2}[\phi_2,h]\Big) \\ & - \frac{(t-\omega_0)^{\alpha-2}}{\Omega}\Big(\boldsymbol{A}_{T,\eta_2}\mathcal{O}_{\xi,\eta_1}[\phi_1,h] - \boldsymbol{A}_{\xi,\eta_1}\mathcal{O}_{T,\eta_2}[\phi_2,h]\Big), \end{aligned} \tag{2.2}$$

where the functionals $\mathcal{O}_{\xi,\eta_1}[\phi_1,h],\ \mathcal{O}_{T,\eta_2}[\phi_2,h]$ *are defined by*

$$\begin{aligned} \mathcal{O}_{\xi,\eta_1}[\phi_1,h] := & \,\phi_1(u) - \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{\xi}\Big(\xi-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}}\, h(s)\, d_{q,\omega}s + \frac{\lambda_1}{\Gamma_q(\alpha)\Gamma_{p_1}(\beta_1)}\times \\ & \int_{\omega_0}^{\eta_1}\int_{\omega_0}^{x}\Big(\eta_1-\sigma_{p_1,\theta_1}(s)\Big)_{p_1,\theta_1}^{\underline{\beta_1-1}}\Big(x-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}}\, g_1(x)h(s)\, d_{q,\omega}s\, d_{p_1,\theta_1}x, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathcal{O}_{T,\eta_2}[\phi_2,h] := & \,\phi_2(u) - \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}\Big(T-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}}\, h(s)\, d_{q,\omega}s + \frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_{p_2}(\beta_2)}\times \\ & \int_{\omega_0}^{\eta_2}\int_{\omega_0}^{x}\Big(\eta_2-\sigma_{p_2,\theta_2}(s)\Big)_{p_2,\theta_2}^{\underline{\beta_2-1}}\Big(x-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}}\, g_2(x)h(s)\, d_{q,\omega}s\, d_{p_2,\theta_2}x, \end{aligned} \tag{2.4}$$

and the constants $\boldsymbol{A}_{\xi,\eta_1}, \boldsymbol{A}_{T,\eta_2}, \boldsymbol{B}_{\xi,\eta_1}, \boldsymbol{B}_{T,\eta_2}$ *and* Ω *are defined by*

$$\boldsymbol{A}_{\xi,\eta_1} := (\xi-\omega_0)^{\alpha-1} - \frac{\lambda_1}{\Gamma_{p_1}(\beta_1)}\int_{\omega_0}^{\eta_1}\Big(\eta_1-\sigma_{p_1,\theta_1}(s)\Big)_{p_1,\theta_1}^{\underline{\beta_1-1}} g_1(s)(s-\omega_0)^{\alpha-1}\, d_{p_1,\theta_1}s, \tag{2.5}$$

$$\boldsymbol{A}_{T,\eta_2} := (T-\omega_0)^{\alpha-1} - \frac{\lambda_2}{\Gamma_{p_2}(\beta_2)}\int_{\omega_0}^{\eta_2}\Big(\eta_2-\sigma_{p_2,\theta_2}(s)\Big)_{p_2,\theta_2}^{\underline{\beta_2-1}} g_2(s)(s-\omega_0)^{\alpha-1}\, d_{p_2,\theta_2}s, \tag{2.6}$$

$$\boldsymbol{B}_{\xi,\eta_1} := (\xi-\omega_0)^{\alpha-2} - \frac{\lambda_1}{\Gamma_{p_1}(\beta_1)}\int_{\omega_0}^{\eta_1}\Big(\eta_1-\sigma_{p_1,\theta_1}(s)\Big)_{p_1,\theta_1}^{\underline{\beta_1-1}} g_1(s)(s-\omega_0)^{\alpha-2}\, d_{p_1,\theta_1}s, \tag{2.7}$$

$$\boldsymbol{B}_{T,\eta_2} := (T-\omega_0)^{\alpha-2} - \frac{\lambda_2}{\Gamma_{p_2}(\beta_2)}\int_{\omega_0}^{\eta_2}\Big(\eta_2-\sigma_{p_2,\theta_2}(s)\Big)_{p_2,\theta_2}^{\underline{\beta_2-1}} g_2(s)(s-\omega_0)^{\alpha-2}\, d_{p_2,\theta_2}s, \tag{2.8}$$

$$\Omega := \boldsymbol{A}_{\xi,\eta_1}\boldsymbol{B}_{T,\eta_2} - \boldsymbol{A}_{T,\eta_2}\boldsymbol{B}_{\xi,\eta_1}. \tag{2.9}$$

Proof. Taking fractional Hahn q,ω-integral of order α to (2.1), we obtain

$$u(t) = C_1(t-\omega_0)^{\alpha-1} + C_2(t-\omega_0)^{\alpha-2} + \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}\, h(x)\, d_{q,\omega}s. \tag{2.10}$$

Next, we take fractional Hahn p_i,θ_i-integral of order $\beta_i,\ i=1,2$ to (2.10) to get

$$\begin{aligned} \mathcal{I}_{p_i,\theta_i}^{\beta_i}u(t) = & \frac{1}{\Gamma_{p_i}(\beta_i)}\int_{\omega_0}^{t}(t-\sigma_{p_i,\theta_i}(s))_{p_i,\theta_i}^{\underline{\beta_i-1}}\left[C_1(s-\omega_0)^{\alpha-1}+C_2(s-\omega_0)^{\alpha-2}\right] d_{p_i,\theta_i}s \\ & + \frac{1}{\Gamma_q(\alpha)\Gamma_{p_i}(\beta_i)}\int_{\omega_0}^{t}\int_{\omega_0}^{x}(t-\sigma_{p_i,\theta_i}(x))_{p_i,\theta_i}^{\underline{\beta_i-1}}(x-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha-1}}\, h(s)\, d_{q,\omega}s\, d_{p_i,\theta_i}x. \end{aligned} \tag{2.11}$$

Substituting $i=1$ into (2.11) and employing the first condition of (2.1), we have

$$\mathbf{A}_{\xi,\eta_1}C_1 + \mathbf{B}_{\xi,\eta_1}C_2 \;=\; \mathcal{O}_{\xi,\eta_1}[\phi_1,h]. \tag{2.12}$$

Taking $i = 2$ into (2.11) and employing the second condition of (2.1), we have

$$\mathbf{A}_{T,\eta_2} C_1 + \mathbf{B}_{T,\eta_2} C_2 \ = \ O_{T,\eta_2}[\phi_2, h], \tag{2.13}$$

where $O_{\xi,\eta_1}, [\phi_1, h], O_{T,\eta_2}[\phi_2, h], \mathbf{A}_{\xi,\eta_1}, \mathbf{A}_{T,\eta_2}, \mathbf{B}_{\xi,\eta_1}, \mathbf{B}_{T,\eta_2}$ and Ω are defined as (2.3)–(2.9), respectively. The constants C_1 and C_2 are revealed from solving the system of equations (2.12) and (2.13) as

$$C_1 = \frac{\mathbf{B}_{T,\eta_2} O_{\xi,\eta_1}, [\phi_1, h] - \mathbf{B}_{\xi,\eta_1} O_{T,\eta_2}, [\phi_2, h]}{\Omega}$$
$$\text{and} \quad C_2 = \frac{\mathbf{A}_{\xi,\eta_1} O_{T,\eta_2}, [\phi_2, h] - \mathbf{A}_{T,\eta_2} O_{\xi,\eta_1}, [\phi_1, h]}{\Omega}.$$

Substituting the constants C_1, C_2 into (2.10), we obtain (2.2).

On the other hand, it's easy to show that (2.2) is the solution of problem (2.1), by taking fractional Hahn q, ω-difference of order α to (2.2), we get (2.1). □

3. Existence and uniqueness result

In this section, we show the existence and uniqueness result for problem (1.1). In view of Lemma 2.8 we define an operator $\mathcal{A} : C \to C$ as

$$\begin{aligned} (\mathcal{A}u)(t) \ := \ & \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{t} \left(t - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} \ F\left[s, u(s), \Psi^{\gamma}_{r,\rho} u(s), \Upsilon^{\nu}_{m,\chi} u(s)\right] \, d_{q,\omega} s \\ & + \frac{(t-\omega_0)^{\alpha-1}}{\Omega} \Big(\mathbf{B}_{T,\eta_2} O^{*}_{\xi,\eta_1}[\phi_1, F_u] - \mathbf{B}_{\xi,\eta_1} O^{*}_{T,\eta_2}[\phi_2, F_u] \Big) \\ & - \frac{(t-\omega_0)^{\alpha-2}}{\Omega} \Big(\mathbf{A}_{T,\eta_2} O^{*}_{\xi,\eta_1}[\phi_1, F_u] - \mathbf{A}_{\xi,\eta_1} O^{*}_{T,\eta_2}[\phi_2, F_u] \Big), \end{aligned} \tag{3.1}$$

where the functionals $O^{*}_{\xi,\eta_1}[\phi_1, F_u]$, $O^{*}_{T,\eta_2}[\phi_2, F_u]$ are defined by

$$\begin{aligned} O^{*}_{\xi,\eta_1}[\phi_1, F_u] := & \, \phi_1(u) - \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{\xi} \left(\xi - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} \ F\left[s, u(s), \Psi^{\gamma}_{r,\rho} u(s), \Upsilon^{\nu}_{m,\chi} u(s)\right] \, d_{q,\omega} s \\ & + \frac{\lambda_1}{\Gamma_q(\alpha)\Gamma_{p_1}(\beta_1)} \int_{\omega_0}^{\eta_1} \int_{\omega_0}^{x} \left(\eta_1 - \sigma_{p_1,\theta_1}(s)\right)_{p_1,\theta_1}^{\underline{\beta_1-1}} \left(x - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} g_1(x) \times \\ & F\left[s, u(s), \Psi^{\gamma}_{r,\rho} u(s), \Upsilon^{\nu}_{m,\chi} u(s)\right] \, d_{q,\omega} s \, d_{p_1,\theta_1} x, \end{aligned} \tag{3.2}$$

$$\begin{aligned} O^{*}_{T,\eta_2}[\phi_2, F_u] := & \, \phi_2(u) - \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{T} \left(T - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} \ F\left[s, u(s), \Psi^{\gamma}_{r,\rho} u(s), \Upsilon^{\nu}_{m,\chi} u(s)\right] \, d_{q,\omega} s \\ & + \frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_{p_2}(\beta_2)} \int_{\omega_0}^{\eta_2} \int_{\omega_0}^{x} \left(\eta_2 - \sigma_{p_2,\theta_2}(s)\right)_{p_2,\theta_2}^{\underline{\beta_2-1}} \left(x - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} g_2(x) \times \\ & F\left[s, u(s), \Psi^{\gamma}_{r,\rho} u(s), \Upsilon^{\nu}_{m,\chi} u(s)\right] \, d_{q,\omega} s \, d_{p_2,\theta_2} x, \end{aligned} \tag{3.3}$$

and the constants $\mathbf{A}_{\xi,\eta_1}, \mathbf{A}_{T,\eta_2}, \mathbf{B}_{\xi,\eta_1}, \mathbf{B}_{T,\eta_2}$ and Ω are defined by (2.5)–(2.9), respectively.

Obviously the problem (1.1) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

Theorem 3.1. *Assume that* $F : I^T_{q,\omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous,* $\varphi : I^T_{r,\rho} \times I^T_{r,\rho} \to [0,\infty)$, $\psi : I^T_{m,\chi} \times I^T_{m,\chi} \to [0,\infty)$ *are continuous with* $\varphi_0 = \max\{\varphi(t,s) : (t,s) \in I^T_{r,\rho} \times I^T_{r,\rho}\}$ *and* $\psi_0 = \max\{\psi(t,s) : (t,s) \in I^T_{m,\chi} \times I^T_{m,\chi}\}$. *In addition, assume that the following conditions hold:*

(H_1) *There exist positive constants* ℓ_1, ℓ_2, ℓ_3, *such that for each* $t \in I^T_{q,\omega}$ *and* $u_i, v_i \in \mathbb{R},\ i = 1,2,3,$

$$\left|F[t,u_1,u_2,u_3] - F[t,v_1,v_2,v_3]\right| \leq \ell_1\left|u_1 - v_1\right| + \ell_2\left|u_2 - v_2\right| + \ell_3\left|u_3 - v_3\right|.$$

(H_2) *There exist positive constants* τ_1, τ_2, *such that for each* $u, v \in C$,

$$|\phi_1(u) - \phi_1(v)| \leq \tau_1\|u - v\|_C \quad \text{and} \quad |\phi_2(u) - \phi_2(v)| \leq \tau_2\|u - v\|_C.$$

(H_3) *There exist positive constants* $g_i, G_i,\ i = 1,2$, *such that for each* $t \in I^T_{q,\omega}$,

$$g_i < g_i(t) < G_i.$$

(H_4) $\Xi := \mathcal{L}\mathcal{X} + \tau_1\Theta^*_{T,\eta_2} + \tau_2\Theta^*_{\xi,\eta_1} < 1,$

where

$$\mathcal{L} := \ell_1 + \ell_2\varphi_0\frac{(T-\omega_0)^\gamma}{\Gamma_r(\gamma+1)} + \ell_3\psi_0\frac{(T-\omega_0)^{-\nu}}{\Gamma_m(1-\nu)}, \tag{3.4}$$

$$\mathcal{X} := \left[\frac{(T-\omega_0)^\alpha}{\Gamma_q(\alpha+1)} + \frac{(T-\omega_0)^{\alpha-\nu}}{\Gamma_m(\alpha-\nu+1)} + \Phi_{\xi,\eta_1}\Theta^*_{T,\eta_2} + \Phi_{T,\eta_2}\Theta^*_{\xi,\eta_1}\right], \tag{3.5}$$

$$\Phi_{\xi,\eta_1} := \frac{1}{\Gamma_q(\alpha+1)}\left[(\xi-\omega_0)^\alpha + \lambda_1 G_1\frac{(\eta_1-\omega_0)^{\alpha+\beta_1}\Gamma_{p_1}(\alpha+1)}{\Gamma_{p_1}(\alpha+\beta_1+1)}\right], \tag{3.6}$$

$$\Phi_{T,\eta_2} := \frac{1}{\Gamma_q(\alpha+1)}\left[(T-\omega_0)^\alpha + \lambda_2 G_2\frac{(\eta_2-\omega_0)^{\alpha+\beta_2}\Gamma_{p_2}(\alpha+1)}{\Gamma_{p_2}(\alpha+\beta_2+1)}\right], \tag{3.7}$$

$$\Theta^*_{\xi,\eta_1} := \Theta_{\xi,\eta_1} + \overline{\Theta}_{\xi,\eta_1}, \tag{3.8}$$

$$\Theta^*_{T,\eta_2} := \Theta_{T,\eta_2} + \overline{\Theta}_{T,\eta_2}, \tag{3.9}$$

$$\Theta_{\xi,\eta_1} := \frac{1}{\min|\Omega|}\left[(T-\omega_0)^{\alpha-1}\max|\boldsymbol{B}_{\xi,\eta_1}| + (T-\omega_0)^{\alpha-2}\max|\boldsymbol{A}_{\xi,\eta_1}|\right], \tag{3.10}$$

$$\overline{\Theta}_{\xi,\eta_1} := \frac{1}{\min|\Omega|}\left[\frac{(T-\omega_0)^{\alpha-\nu-1}\Gamma_m(\alpha)}{\Gamma_m(\alpha-\nu)}\max|\boldsymbol{B}_{\xi,\eta_1}| + \frac{(T-\omega_0)^{\alpha-\nu-2}\Gamma_m(\alpha-1)}{\Gamma_m(\alpha-\nu-1)}\max|\boldsymbol{A}_{\xi,\eta_1}|\right], \tag{3.11}$$

$$\Theta_{T,\eta_2} := \frac{1}{\min|\Omega|}\left[(T-\omega_0)^{\alpha-1}\max|\boldsymbol{B}_{T,\eta_2}| + (T-\omega_0)^{\alpha-2}\max|\boldsymbol{A}_{T,\eta_2}|\right], \tag{3.12}$$

$$\overline{\Theta}_{T,\eta_2} := \frac{1}{\min|\Omega|}\left[\frac{(T-\omega_0)^{\alpha-\nu-1}\Gamma_m(\alpha)}{\Gamma_m(\alpha-\nu)}\max|\boldsymbol{B}_{T,\eta_2}| + \frac{(T-\omega_0)^{\alpha-\nu-2}\Gamma_m(\alpha-1)}{\Gamma_m(\alpha-\nu-1)}\max|\boldsymbol{A}_{T,\eta_2}|\right]. \tag{3.13}$$

Then problem (1.1) *has a unique solution in* $I^T_{q,\omega}$.

Proof. For each $t \in I^T_{r,\rho}$, we have

$$\begin{aligned}\left|\Psi^{\gamma}_{r,\rho}u(t) - \Psi^{\gamma}_{r,\rho}v(t)\right| &\leq \frac{\phi_0}{\Gamma_r(\gamma)}\int_{\omega_0}^{t}\left(t-\sigma_{r,\rho}(s)\right)^{\gamma-1}_{r,\rho}|u(s)-v(s)|\,d_{r,\rho}s \\ &\leq \frac{\phi_0}{\Gamma_r(\gamma)}\left\|u-v\right\|\int_{\omega_0}^{T}\left(T-\sigma_{r,\rho}(s)\right)^{\gamma-1}_{r,\rho}\,d_{r,\rho}s \\ &= \frac{\phi_0\,(T-\omega_0)^{\gamma}}{\Gamma_r(\gamma+1)}\left\|u-v\right\|.\end{aligned}$$

Similarly, for each $t \in I^T_{m,\chi}$, we obtain

$$\left|\Upsilon^{\nu}_{m,\chi}u(t) - \Upsilon^{\nu}_{m,\chi}v(t)\right| \leq \frac{\psi_0\,(T-\omega_0)^{-\nu}}{\Gamma_m(1-\nu)}\left\|u-v\right\|.$$

To show that F is contraction, we denote that

$$\mathcal{F}|u-v|(t) := \left|F\left[t,u(t),\Psi^{\gamma}_{r,\rho}u(t),\Upsilon^{\nu}_{m,\chi}u(t)\right] - F\left[t,v(t),\Psi^{\gamma}_{r,\rho}v(t),\Upsilon^{\nu}_{m,\chi}v(t)\right]\right|,$$

for each $t \in I^T_{q,\omega}$ and $u, v \in C$. We find that

$$\begin{aligned}&\left|O^{*}_{\xi,\eta_1}[\phi_1,F_u] - O^{*}_{\xi,\eta_1}[\phi_1,F_v]\right| \\ &\leq |\phi_1(u)-\phi_1(v)| + \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{\xi}\left(\xi-\sigma_{q,\omega}(s)\right)^{\alpha-1}_{q,\omega}\mathcal{F}|u-v|(s)\,d_{q,\omega}s + \frac{\lambda_1}{\Gamma_q(\alpha)\Gamma_{p_1}(\beta_1)}\times \\ &\quad\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{x}\left(\eta_1-\sigma_{p_1,\theta_1}(s)\right)^{\beta_1-1}_{p_1,\theta_1}\left(x-\sigma_{q,\omega}(s)\right)^{\alpha-1}_{q,\omega}g_1(x)\mathcal{F}|u-v|(s)\,d_{q,\omega}s\,d_{p_1,\theta_1}x \\ &\leq \tau_1\|u-v\|_C + \left(\ell_1\left|u-v\right| + \ell_2\left|\Psi^{\gamma}_{r,\rho}u - \Psi^{\gamma}_{r,\rho}\right| + \ell_3\left|D^{\nu}_{m,\chi}u - D^{\nu}_{m,\chi}v\right|\right)\frac{1}{\Gamma_q(\alpha+1)}\times \\ &\quad\left[(\xi-\omega_0)^{\alpha} + \lambda_1 G_1\frac{(\eta_1-\omega_0)^{\alpha+\beta_1}\Gamma_{p_1}(\alpha+1)}{\Gamma_{p_1}(\alpha+\beta_1+1)}\right] \\ &\leq \tau_1\|u-v\|_C + \left[\ell_1 + \ell_2\varphi_0\frac{(T-\omega_0)^{\gamma}}{\Gamma_r(\gamma+1)} + \ell_3\psi_0\frac{(T-\omega_0)^{-\nu}}{\Gamma_r(1-\nu)}\right]\left\|u-v\right\|\Phi_{\xi,\eta_1} \\ &\leq [\tau_1 + \mathcal{L}\Phi_{\xi,\eta_1}]\|u-v\|_C. \qquad (3.14)\end{aligned}$$

Similarly, we get

$$\left|O^{*}_{T,\eta_2}[\phi_2,F_u] - O^{*}_{T,\eta_2}[\phi_2,F_v]\right| \leq [\tau_2 + \mathcal{L}\Phi_{T,\eta_2}]\|u-v\|_C. \qquad (3.15)$$

Next, we have

$$\begin{aligned}&\left|(\mathcal{A}u)(t) - (\mathcal{A}v)(t)\right| \\ &\leq \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{T}\left(T-\sigma_{q,\omega}(s)\right)^{\alpha-1}_{q,\omega}\mathcal{F}|u-v|(s)\,d_{q,\omega}s + \frac{(T-\omega_0)^{\alpha-1}}{|\Omega|}\times \\ &\quad + \left\{|\mathbf{B}_{T,\eta_2}|\left|O^{*}_{\xi,\eta_1}[\phi_1,F_u] - O^{*}_{\xi,\eta_1}[\phi_1,F_v]\right| + |\mathbf{B}_{\xi,\eta_1}|\left|O^{*}_{T,\eta_2}[\phi_2,F_u] - O^{*}_{T,\eta_2}[\phi_2,F_v]\right|\right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{(T-\omega_0)^{\alpha-2}}{|\Omega|}\Big\{|\mathbf{A}_{T,\eta_2}|\Big|O^*_{\xi,\eta_1}[\phi_1,F_u]-O^*_{\xi,\eta_1}[\phi_1,F_v]\Big|+|\mathbf{A}_{\xi,\eta_1}|\times\\
&\Big|O^*_{T,\eta_2}[\phi_2,F_u]-O^*_{T,\eta_2}[\phi_2,F_v]\Big|\Big\}\\
\leq\ &\Bigg[\frac{\mathcal{L}(T-\omega_0)^{\alpha}}{\Gamma_q(\alpha+1)}+\frac{\big[\tau_1+\mathcal{L}\,\Phi_{\xi,\eta_1}\big]}{\min|\Omega|}\Big\{\max|\mathbf{B}_{T,\eta_2}|(T-\omega_0)^{\alpha-1}+\max|\mathbf{A}_{T,\eta_2}|(T-\omega_0)^{\alpha-2}\Big\}\\
&+\frac{\big[\tau_2+\mathcal{L}\,\Phi_{T,\eta_2}\big]}{\min|\Omega|}\Big\{\max|\mathbf{B}_{\xi,\eta_1}|(T-\omega_0)^{\alpha-1}+\max|\mathbf{A}_{\xi,\eta_1}|(T-\omega_0)^{\alpha-2}\Big\}\Bigg]\|u-v\|_C\\
=\ &\Bigg\{\mathcal{L}\Bigg[\frac{(T-\omega_0)^{\alpha}}{\Gamma_q(\alpha+1)}+\Phi_{\xi,\eta_1}\Theta_{T,\eta_2}+\Phi_{T,\eta_2}\Theta_{\xi,\eta_1}\Bigg]+\tau_1\Theta_{T,\eta_2}+\tau_2\Theta_{\xi,\eta_1}\Bigg\}\|u-v\|_C. \qquad (3.16)
\end{aligned}
$$

Taking fractional Hahn m,χ-difference of order ν to (3.1), we obtain

$$
\begin{aligned}
&(D^{\nu}_{m,\chi}\mathcal{A}u)(t)\\
&=\frac{1}{\Gamma_q(\alpha)\Gamma_m(-\nu)}\int_{\omega_0}^{t}\int_{\omega_0}^{x}\Big(t-\sigma_{m,\chi}(x)\Big)_{m,\chi}^{\underline{-\nu-1}}\Big(x-\sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}}\times\\
&\quad F\Big[s,u(s),\Psi^{\gamma}_{r,\rho}u(s),\Upsilon^{\nu}_{m,\chi}u(s)\Big]\,d_{q,\omega}s\,d_{m,\chi}x+\frac{1}{\Omega\Gamma_m(-\nu)}\times\\
&\quad\Big(\mathbf{B}_{T,\eta_2}O^*_{\xi,\eta_1}[\phi_1,F_u]-\mathbf{B}_{\xi,\eta_1}O^*_{T,\eta_2}[\phi_2,F_u]\Big)\int_{\omega_0}^{t}\Big(t-\sigma_{m,\chi}(s)\Big)_{m,\chi}^{\underline{-\nu-1}}(s-\omega_0)^{\alpha-1}\,d_{m,\chi}s\\
&\quad-\frac{1}{\Omega\Gamma_m(-\nu)}\Big(\mathbf{A}_{T,\eta_2}O^*_{\xi,\eta_1}[\phi_1,F_u]-\mathbf{A}_{\xi,\eta_1}O^*_{T,\eta_2}[\phi_2,F_u]\Big)\times\\
&\quad\int_{\omega_0}^{t}\Big(t-\sigma_{m,\chi}(s)\Big)_{m,\chi}^{\underline{-\nu-1}}(s-\omega_0)^{\alpha-2}\,d_{m,\chi}s. \qquad (3.17)
\end{aligned}
$$

By the same expression as above, we obtain

$$
\begin{aligned}
&\Big|(D_{m,\chi}{}^{\nu}\mathcal{A}u)(t)-(D^{\nu}_{m,\chi}\mathcal{A}v)(t)\Big|\\
&<\Bigg\{\mathcal{L}\Bigg[\frac{(T-\omega_0)^{\alpha-\nu}\Gamma_m(\alpha+1)}{\Gamma_m(\alpha-\nu+1)\Gamma_q(\alpha+1)}+\Phi_{\xi,\eta_1}\overline{\Theta}_{T,\eta_2}+\Phi_{T,\eta_2}\overline{\Theta}_{\xi,\eta_1}\Bigg]+\tau_1\overline{\Theta}_{T,\eta_2}+\tau_2\overline{\Theta}_{\xi,\eta_1}\Bigg\} \qquad (3.18)\\
&\quad\times\|u-v\|_C.
\end{aligned}
$$

From (3.16) and (3.18), we get

$$
\|\mathcal{A}u-\mathcal{A}v\|_C\leq\Big[\mathcal{L}\mathcal{X}+\tau_1\Theta^*_{T,\eta_2}+\tau_2\Theta^*_{\xi,\eta_1}\Big]\|u-v\|_C=\Xi\,\|u-v\|_C.
$$

By (H_4) and Banach fixed point theorem, we get that $\mathcal{A}$ is a contraction and hence $\mathcal{A}$ has a fixed point. Consequently problem (1.1) has a unique solution of on $I^T_{q,\omega}$. □

4. Existence of at least one solution

In this section, we prove an existence result for the problem (1.1) via Schauder's fixed point theorem.

Lemma 4.1. *[49] (Schauder's fixed point theorem) Let (D, d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

Theorem 4.1. *Suppose that* (H_1), (H_3) *and* (H_4) *hold. Then, problem* (1.1) *has at least one solution on* $I^T_{q,\omega}$.

Proof. The proof is organized into three steps as follows:

Step I. $\mathcal{A}$ maps bounded sets into bounded sets in $B_R = \{u \in C : \|u\|_C \le R\}$. Let $\max_{t \in I^T_{q,\omega}} |F(t, 0, 0, 0)| = M$, $\sup_{u \in C} |\phi_i(u)| = N_i,\ i = 1, 2$ and choose a constant

$$R \ge \frac{M\mathcal{X} + N_1\Theta^*_{T,\eta_2} + N_2\Theta^*_{\xi,\eta_1}}{1 - \mathcal{L}\mathcal{X}}. \tag{4.1}$$

Letting

$$\left|\mathcal{F}(t, u, 0)\right| = \left|F[t, u(t), \Psi^\gamma_{r,\rho}u(t), \Upsilon^\nu_{m,\chi}u(t)] - F[t, 0, 0, 0]\right| + \left|F[t, 0, 0, 0]\right|$$

for each $t \in I^T_{q,\omega}$ and $u \in B_R$, we obtain

$$\begin{aligned}
&\left|O^*_{\xi,\eta_1}[\phi_1, F_u]\right| \\
\le\ & N_1 + \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^{\xi} \left(\xi - \sigma_{q,\omega}(s)\right)^{\underline{\alpha-1}}_{q,\omega} \left|\mathcal{F}(s, u, 0)\right| d_{q,\omega}s \\
&+ \frac{\lambda_1}{\Gamma_q(\alpha)\Gamma_{p_1}(\beta_1)} \int_{\omega_0}^{\eta_1} \int_{\omega_0}^{x} \left(\eta_1 - \sigma_{p_1,\theta}(s)\right)^{\underline{\beta_1-1}}_{p_1,\theta} \left(x - \sigma_{q,\omega}(s)\right)^{\underline{\alpha-1}}_{q,\omega} g_1(x)\left|\mathcal{F}(s, u, 0)\right| d_{q,\omega}s\, d_{p_1,\theta}x \\
\le\ & N_1 + \left(\left[\ell_1 + \ell_2\varphi_0 \frac{(T - \omega_0)^\gamma}{\Gamma_r(\gamma + 1)} + \ell_3\psi_0 \frac{(T - \omega_0)^{-\nu}}{\Gamma_r(1 - \nu)}\right] \|u\| + M\right) \Phi_{\xi,\eta_1} \\
\le\ & N_1 + (\mathcal{L}\|u\|_C + M)\, \Phi_{\xi,\eta_1}.
\end{aligned} \tag{4.2}$$

Similarly,

$$\left|O^*_{T,\eta_2}[\phi_2, F_u]\right| \le N_2 + (\mathcal{L}\|u\|_C + M)\, \Phi_{T,\eta_2}. \tag{4.3}$$

Then, we have

$$\left|(\mathcal{A}u)(t)\right| \le\ (\mathcal{L}\|u\|_C + M) \left[\frac{(T - \omega_0)^\alpha}{\Gamma_q(\alpha + 1)} + \Phi_{\xi,\eta_1}\Theta_{T,\eta_2} + \Phi_{T,\eta_2}\Theta_{\xi,\eta_1}\right] + N_1\Theta_{T,\eta_2} + N_2\Theta_{\xi,\eta_1} \tag{4.4}$$

and

$$\begin{aligned}
\left|\left(D^\gamma_{m,\chi}\mathcal{A}u\right)(t)\right| \le\ & (\mathcal{L}\|u\|_C + M) \left[\frac{(T - \omega_0)^{\alpha-\nu}}{\Gamma_m(\alpha - \nu + 1)} + \Phi_{\xi,\eta_1}\overline{\Theta}_{T,\eta_2} + \Phi_{T,\eta_2}\overline{\Theta}_{\xi,\eta_1}\right] \\
&+ N_1\overline{\Theta}_{T,\eta_2} + N_2\overline{\Theta}_{\xi,\eta_1}.
\end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we obtain $\|\mathcal{A}u\|_C \le R$. Hence $\mathcal{A}$ is uniformly bounded.

Step II. By the continuity of F, the operator $\mathcal{A}$ is continuous on B_R.

Step III. Next, we show that $\mathcal{A}$ is equicontinuous on B_R. For any $t_1, t_2 \in I^T_{q,\omega}$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\left|(\mathcal{A}u)(t_2) - (\mathcal{A}u)(t_1)\right| \\
&\leq \frac{\|F\|}{\Gamma_q(\alpha+1)}\left|(t_2-\omega_0)^{\alpha} - (t_1-\omega_0)^{\alpha}\right| \\
&\quad + \frac{\left|(t_2-\omega_0)^{\alpha-1} - (t_1-\omega_0)^{\alpha-1}\right|}{|\Omega|}\left\{|\mathbf{B}_{T,\eta_2}|\left|O^*_{\xi,\eta_1}[\phi_1, F_u]\right| + |\mathbf{B}_{\xi,\eta_1}|\left|O^*_{T,\eta_2}[\phi_2, F]\right|\right\} \\
&\quad + \frac{\left|(t_2-\omega_0)^{\alpha-2} - (t_1-\omega_0)^{\alpha-2}\right|}{|\Omega|}\left\{|\mathbf{A}_{T,\eta_2}|\left|O^*_{\xi,\eta_1}[\phi_1, F_u]\right| + |\mathbf{A}_{\xi,\eta_1}|\left|O^*_{T,\eta_2}[\phi_2, F]\right|\right\}
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\left|(D_{m,\chi}{}^{\nu}\mathcal{F}u)(t_1) - (D^{\nu}_{m,\chi}\mathcal{F}u)(t_2)\right| \\
&\leq \frac{\|F\|\Gamma_q(-\nu)}{\Gamma_m(-\nu)\Gamma_q(\alpha-\nu+1)}\left|(t_2-\omega_0)^{\alpha-\nu} - (t_1-\omega_0)^{\alpha-\nu}\right| \\
&\quad + \frac{\Gamma_q(\alpha)\Gamma_q(-\nu)}{|\Omega|\Gamma_m(-\nu)\Gamma_q(\alpha-\nu)}\left\{|\mathbf{B}_{T,\eta_2}|\left|O^*_{\xi,\eta_1}[\phi_1, F_u]\right| + |\mathbf{B}_{\xi,\eta_1}|\left|O^*_{T,\eta_2}[\phi_2, F]\right|\right\}\times \\
&\quad \left|(t_2-\omega_0)^{\alpha-\nu-1} - (t_1-\omega_0)^{\alpha-\nu-1}\right| \\
&\quad + \frac{\Gamma_q(\alpha-1)\Gamma_q(-\nu)}{|\Omega|\Gamma_m(-\nu)\Gamma_q(\alpha-\nu-1)}\left\{|\mathbf{A}_{T,\eta_2}|\left|O^*_{\xi,\eta_1}[\phi_1, F_u]\right| + |\mathbf{A}_{\xi,\eta_1}|\left|O^*_{T,\eta_2}[\phi_2, F]\right|\right\}\times \\
&\quad \left|(t_2-\omega_0)^{\alpha-\nu-2} - (t_1-\omega_0)^{\alpha-\nu-2}\right|.
\end{aligned} \tag{4.7}$$

Clearly the right-hand side of (4.6) and (4.7) tend to be zero when $|t_2 - t_1| \to 0$. So $\mathcal{A}$ is relatively compact on B_R.

Hence the set $\mathcal{F}(B_R)$ is an equicontinuous set. As a result of Steps I to III and the Arzelá-Ascoli theorem, we can conclude that $\mathcal{A} : C \to C$ is completely continuous. By Schauder's fixed point theorem, we obtain that problem (1.1) has at least one solution. □

5. Example

Consider the following fractional Hahn integro-difference equation

$$\begin{aligned}
D^{\frac{4}{3}}_{\frac{1}{2},\frac{2}{3}}u(t) \;=\; & \frac{1}{(100\pi^2+t^3)(1+|u(t)|)}\Bigg[e^{-3t+1}\left(u^2+2|u|\right) + e^{-(\pi+\sin^2\pi t)}\left|\Psi^{\frac{1}{2}}_{\frac{1}{8},\frac{7}{6}}u(t)\right| \\
& + e^{-(2\pi+\cos^2\pi t)}\left|\Upsilon^{\frac{1}{4}}_{\frac{1}{4},1}u(t)\right|\Bigg], \quad t \in I^{10}_{\frac{1}{2},\frac{2}{3}}
\end{aligned} \tag{5.1}$$

with four-point fractional Hahn integral boundary condition

$$u\left(\frac{15}{8}\right) = 10e\,\mathcal{I}^{\frac{1}{3}}_{\frac{1}{16},\frac{5}{4}}u\left(2e+\sin\frac{699055}{524288}\right)^2 u\left(\frac{699055}{524288}\right) + \sum_{i=0}^{\infty}\frac{C_i|u(t_i)|}{1+|u(t_i)|}, \quad t_i \in \sigma^i_{\frac{1}{2},\frac{2}{3}}(10),$$

$$u(10)=\frac{1}{10\pi}\mathcal{I}^{\frac{2}{3}}_{\frac{1}{32},\frac{31}{24}}u\left(\pi+\cos\frac{65549}{49125}\right)^2u\left(\frac{65549}{49125}\right)+\sum_{i=0}^{\infty}\frac{D_i|u(t_i)|}{1+|u(t_i)|},\quad t_i\in\sigma^i_{\frac{1}{2},\frac{2}{3}}(10),$$

where $\varphi(t,s)=\frac{e^{-|t-s|}}{(t+20)^3}$, $\psi(t,s)=\frac{e^{-|t-s|}}{(t+30)^2}$ and C_i, D_i are given constants with $\frac{1}{100t^3}\le\sum_{i=0}^{\infty}C_i\le\frac{e}{100t^3}$ and $\frac{1}{200t^2}\le\sum_{i=0}^{\infty}D_i\le\frac{\pi}{200t^2}$.

Here $\alpha=\frac{4}{3}$, $\beta_1=\frac{1}{3}$, $\beta_2=\frac{2}{3}$, $\gamma=\frac{1}{2}$, $\nu=\frac{1}{4}$, $q=\frac{1}{2}$, $p_1=\frac{1}{16}$, $p_2=\frac{1}{32}$, $r=\frac{1}{8}$, $m=\frac{1}{4}$, $\omega=\frac{2}{3}$, $\theta_1=\frac{5}{4}$, $\theta_1=\frac{31}{24}$, $\rho=\frac{7}{6}$, $\chi=1$, $\omega_0=\frac{\omega}{1-q}=\frac{4}{3}$, $T=10$, $\xi=\sigma^4_{\frac{1}{2},\frac{2}{3}}(10)=\frac{15}{8}$, $\eta_1=\sigma^5_{\frac{1}{16},\frac{5}{4}}(10)=\frac{699055}{524288}$, $\eta_2=\sigma^3_{\frac{1}{32},\frac{31}{24}}(10)=\frac{65549}{49125}$, $\lambda_1=10e$, $\lambda_2=\frac{1}{10\pi}$, $\phi_1(u)=\sum_{i=0}^{\infty}\frac{C_i|u(t_i)|}{1+|u(t_i)|}$, $\phi_2=\sum_{i=0}^{\infty}\frac{D_i|u(t_i)|}{1+|u(t_i)|}$, $g_1(t)=(2e+\sin t)^2$, $g_2(t)=(\pi+\cos t)^2$, and $F\left(t,u(t),\Psi^{\gamma}_{r,\rho}u(t),\Upsilon^{\nu}_{m,\chi}u(t)\right)=\frac{1}{\left(100\pi^2+t^3\right)(1+|u(t)|)}\left[e^{-3t+1}\left(u^2+2|u|\right)+e^{-(\pi+\sin^2\pi t)}\left|\Psi^{\frac{1}{2}}_{\frac{1}{8},\frac{7}{6}}u(t)\right|+e^{-(2\pi+\cos^2\pi t)}\left|\Upsilon^{\frac{1}{4}}_{\frac{1}{4},1}u(t)\right|\right]$.

For all $t\in I^{10}_{\frac{1}{2},\frac{2}{3}}$ and $u,v\in\mathbb{R}$, we have

$$\begin{aligned}&\left|F\left(t,u,\Psi^{\gamma}_{r,\rho}u,\Upsilon^{\nu}_{m,\chi}u\right)-F\left(t,v,\Psi^{\gamma}_{r,\rho}v,\Upsilon^{\nu}_{m,\chi}v\right)\right|\\ \le\ &\frac{1}{e^3\left(\left(\frac{4}{3}\right)^3+100\pi^2\right)}|u-v|+\frac{1}{e^{\pi}\left(\left(\frac{4}{3}\right)^3+100\pi^2\right)}\left|\Psi^{\gamma}_{r,\rho}u-\Psi^{\gamma}_{r,\rho}v\right|\\ &+\frac{1}{e^{2\pi}\left(\left(\frac{4}{3}\right)^3+100\pi^2\right)}\left|D^{\nu}_{m,\chi}u-D^{\nu}_{m,\chi}v\right|.\end{aligned}$$

Thus, (H_1) holds with $\ell_1=0.0000503$, $\ell_2=0.00004367$ and $\ell_3=1.8876\times10^{-6}$.

For all $u,v\in\mathcal{C}$,

$$\begin{aligned}|\phi_1(u)-\phi_1(v)|&=\frac{e}{100\left(\frac{4}{3}\right)^3}\|u-v\|_{\mathcal{C}},\\ |\phi_2(u)-\phi_2(v)|&=\frac{\pi}{200\left(\frac{4}{3}\right)^2}\|u-v\|_{\mathcal{C}}.\end{aligned}$$

So, (H_2) holds with $\tau_1=0.011468$ and $\tau_2=0.0066268$.

Moreover, (H_3) holds with $g_1=19.6831, G_1=41.4294, g_2=4.5864$ and $G_2=17.1528$.

We can find that

$$|\mathbf{A}_{\xi,\eta_1}|\le1.27503,\quad|\mathbf{A}_{T,\eta_2}|\le2.05422,\quad|\mathbf{B}_{\xi,\eta_1}|\le61548.5314,\quad|\mathbf{B}_{T,\eta_2}|\le0.08082$$
$$\text{and}\quad|\Omega|\ge60067.4763.$$

Also, we can show that

$$\begin{gathered}\mathcal{L}=0.0000503,\quad\Phi_{\xi,\eta_1}=0.39607,\quad\Phi_{T,\eta_2}=15.96844,\\ \Theta_{\xi,\eta_1}=2.10473,\quad\overline{\Theta}_{\xi,\eta_1}=1.25309,\quad\Theta_{T,\eta_2}=0.0000357,\quad\overline{\Theta}_{T,\eta_2}=0.00001716,\\ \Theta^*_{\xi,\eta_1}=3.35782,\quad\Theta^*_{T,\eta_2}=0.00005286,\quad\text{and}\quad\mathcal{X}=79.7178.\end{gathered}$$

Hence, (H_4) holds with

$$\Xi \approx 0.02626 < 1.$$

Therefore, by Theorem 3.1, problem (5.1) has a unique solution. Moreover, by Theorem 4.1, this problem has at least one solution. □

6. Conclusions

In the present research we considered a boundary value problem for Hahn integro-difference equation subject to four-point fractional Hahn integral boundary conditions. Notice that the problem at hand contains two fractional Hahn difference operators and three fractional Hahn integrals with different quantum numbers and orders. We note that if we let $q = r = m = p_1 = p_2$ and $\omega = \rho = \theta_1 = \theta_2$, our results reduce to the results obtained in [46]– [48]. After proving an auxiliary result concerning a linear variant of the considered problem, the problem at hand is transformed into a fixed point problem. Existence and uniqueness results are established via Banach's and Schauder's fixed point theorems. The main results are illustrated by a numerical example. Some properties of fractional Hahn integral needed in our study are also discussed. The results of the paper are new and enrich the subject of boundary value problems for Hahn integro-difference equations. In the future work, we may extend this work by considering new boundary value problems.

Acknowledgments

This research was funded by the Science, Research and Innovation Promotion Fund under Basic Research Plan-Saun Dusit University.

Conflict of interest

The authors declare no conflict of interest.

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