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*Research article*

# Note on subdirect sums of $\{i_0\}$-Nekrasov matrices

**Jing Xia***

School of Mathematics and Information Science, Baoji University of Arts and Sciences, Baoji, Shaanxi, 721013, China

* **Correspondence:** Email: jingxiaxia888@163.com.

**Abstract:** The concept of $k$-subdirect sums of matrices, as a generalization of the usual sum and the direct sum, plays an important role in scientific computing. In this paper, we introduce a new subclass of $S$-Nekrasov matrices, called $\{i_0\}$-Nekrasov matrices, and some sufficient conditions are given which guarantee that the $k$-subdirect sum $A \bigoplus_k B$ is an $\{i_0\}$-Nekrasov matrix, where $A$ is an $\{i_0\}$-Nekrasov matrix and $B$ is a Nekrasov matrix. Numerical examples are reported to illustrate the conditions presented.

**Keywords:** subdirect sum; $\{i_0\}$-Nekrasov matrices; $S$-Nekrasov matrices
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## 1. Introduction

In 1999, Fallat and Johnson [1] introduced the concept of $k$-subdirect sums of square matrices, which generalizes the usual sum and the direct sum of matrices [2], and has potential applications in several contexts such as matrix completion problems [3–5], overlapping subdomains in domain decomposition methods [6–8], and global stiffness matrices in finite elements [7, 9], etc.

**Definition 1.1.** [1] *Let $A \in \mathbb{C}^{n_1\times n_1}$ and $B \in \mathbb{C}^{n_2\times n_2}$, and $k$ be an integer such that $1 \le k \le \min\{n_1, n_2\}$. Suppose that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{1.1}$$

*where $A_{22}$ and $B_{11}$ are square matrices of order $k$. Then*

$$C = \begin{bmatrix} A_{11} & A_{12} & \mathbf{0} \\ A_{21} & A_{22} + B_{11} & B_{12} \\ \mathbf{0} & B_{21} & B_{22} \end{bmatrix}$$

*is called the k-subdirect sum of A and B and is denoted by* $C = A \bigoplus_k B$.

For the $k$-subdirect sums of matrices, one of the important problems is that if $A$ and $B$ lie in a certain subclass of $H$-matrices must a $k$-subdirect sum $C$ lie in this class, since it can be used to analyze the convergence of Jacobi and Gauss-Seidel methods in solving the linearized system of nonlinear equations [10]. Here, a square matrix $A$ is called an $H$-matrix if there exists a positive diagonal matrix $X$ such that $AX$ is a strictly diagonally dominant matrix [11]. To answer this question, several results about subdirect sum problems for $H$-matrices and some subclasses of $H$-matrices have been obtained, such as $S$-strictly diagonally dominant matrices [12], doubly diagonally dominant matrices [13], $\Sigma$-strictly diagonally dominant matrices [14], $\alpha_1$ and $\alpha_2$-matrices [15], Nekrasov matrices [16], weakly chained diagonally dominant matrices [17], $QN$-(quasi-Nekrasov) matrices [18], SDD($p$)-matrices [10], and $H$-matrices [19]. Besides, the subdirect sum problems for some other structure matrices, including $B$-matrices, $B_\pi^R$-matrices, $P$-matrices, doubly non-negative matrices, completely positive matrices, and totally non-negative matrices, were also studied; for details, see [1, 20–22] and references therein.

In 2009, Cvetković, Kostić, and Rauški [23] introduced a new subclass of $H$-matrices called $S$-Nekrasov matrices.

**Definition 1.2.** [23] *Given any nonempty proper subset* $S$ *of* $N := \{1, 2, \ldots, n\}$ *and* $\overline{S} = N \setminus S$. *A matrix* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ *is called an* $S$*-Nekrasov matrix if* $|a_{ii}| > h_i^S(A)$ *for all* $i \in S$, *and*

$$\left(|a_{ii}| - h_i^S(A)\right)\left(|a_{jj}| - h_j^{\overline{S}}(A)\right) > h_i^{\overline{S}}(A) h_j^S(A), \quad for\ all\ i \in S,\ j \in \overline{S},$$

*where* $h_1^S(A) = \sum_{j\in S\setminus\{1\}} |a_{1j}|$ *and*

$$h_i^S(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j^S(A) + \sum_{j=i+1, j\in S}^{n} |a_{ij}|,\ i = 2, 3, \ldots, n. \tag{1.2}$$

Specially, if $S = N$, then Definition 1.2 coincides with the definition of Nekrasov matrices [23], that is, a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called a Nekrasov matrix if $|a_{ii}| > h_i(A)$ for all $i \in N$, where $h_i(A) := h_i^N(A)$. It is worth noticing that the class of $S$-Nekrasov matrices has many potential applications in scientific computing, such as estimating the infinity norm for the inverse of $S$-Nekrasov matrices [24], estimating error bounds for linear complementarity problems [25–27], and identifying nonsingular $H$-tensors [28], etc. However, to the best of the author's knowledge, the subdirect sum problem for $S$-Nekrasov matrices remains unclear. In this paper, we introduce the class of $\{i_0\}$-Nekrasov matrices and prove that it is a subclass of $S$-Nekrasov matrices, and then we focus on the subdirect sum problem of $\{i_0\}$-Nekrasov matrices. We provide some sufficient conditions such that the $k$-subdirect sum of $\{i_0\}$-Nekrasov matrices and Nekrasov matrices belongs to the class of $\{i_0\}$-Nekrasov matrices. Numerical examples are presented to illustrate the corresponding results.

## 2. Subdirect sums of $\{i_0\}$-Nekrasov matrices

We start with some notations and definitions. For a non-zero complex number $z$, we define $\arg(z) = \{\theta : z = |z|\exp(i\theta), -\pi < \theta \leq \pi\}$. As is shown in [12], if we let $C = A \bigoplus_k B = [c_{ij}]$, where $A = [a_{ij}] \in \mathbb{C}^{n_1\times n_1}$ and $B = [b_{ij}] \in \mathbb{C}^{n_2\times n_2}$, then

$$c_{ij} = \begin{cases} a_{ij}, & i \in S_1,\ j \in S_1 \bigcup S_2, \\ 0, & i \in S_1,\ j \in S_3, \\ a_{ij}, & i \in S_2,\ j \in S_1, \\ a_{ij} + b_{i-t,j-t}, & i \in S_2,\ j \in S_2, \\ b_{i-t,j-t}, & i \in S_2,\ j \in S_3, \\ 0, & i \in S_3,\ j \in S_1, \\ b_{i-t,j-t}, & i \in S_3,\ j \in S_2 \bigcup S_3, \end{cases}$$

where $t = n_1 - k$ and

$$S_1 = \{1, 2, \ldots, n_1 - k\},\ S_2 = \{n_1 - k + 1, \ldots, n_1\},\ S_3 = \{n_1 + 1, \ldots, n\}, \tag{2.1}$$

with $n = n_1 + n_2 - k$. Obviously, $S_1 \bigcup S_2 \bigcup S_3 = N$.

We introduce the following subclass of $S$-Nekrasov matrices by requiring $S$ is a singleton.

**Definition 2.1.** *A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is called an $\{i_0\}$-Nekrasov matrix if there exists $i_0 \in N$ such that $|a_{i_0,i_0}| > \eta_{i_0}(A)$, and that for all $j \in N \setminus \{i_0\}$,*

$$(|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \left(|a_{jj}| - h_j(A) + \eta_j(A)\right) > (h_{i_0}(A) - \eta_{i_0}(A)) \cdot \eta_j(A),$$

*where $\eta_i(A) = 0$ for all $i \in N$ if $i_0 = 1$, otherwise, $\eta_1(A) = |a_{i,i_0}|$ and*

$$\eta_i(A) = \begin{cases} \sum\limits_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}\eta_j(A) + |a_{i,i_0}|, & i = 2, \ldots, i_0 - 1, \\ \sum\limits_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}\eta_j(A), & i = i_0, i_0 + 1, \ldots, n. \end{cases} \tag{2.2}$$

**Remark 2.1.** *(i) If $A$ is an $\{i_0\}$-Nekrasov matrix, then $A$ is an $S$-Nekrasov matrix for $S = \{i_0\}$. In fact, using recursive relations (1.2) and (2.2), it follows that $h_{i_0}^S(A) = 0 = \eta_{i_0}(A)$ if $i_0 = 1$, otherwise, $h_1^S(A) = \eta_1(A)$ and*

$$\begin{aligned} h_i^S(A) &= \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j^S(A) + \sum_{j=i+1, j\in S}^{n} |a_{ij}| \\ &= \begin{cases} \sum\limits_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}\eta_j(A) + |a_{i,i_0}|, & i = 2, \ldots, i_0 - 1, \\ \sum\limits_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}\eta_j(A), & i = i_0, i_0 + 1, \ldots, n. \end{cases} \\ &= \eta_i(A). \end{aligned}$$

*In addition,* $h_i^{\overline{S}}(A) = h_i(A) - \eta_i(A)$ *follows from the fact that* $h_i(A) = h_i^S(A) + h_i^{\overline{S}}(A)$ *for each* $i \in N$. *These imply that an* $\{i_0\}$*-Nekrasov matrix is an* $S$*-Nekrasov matrix for* $S = \{i_0\}$.

*(ii) Since a Nekrasov matrix is an* $S$*-Nekrasov matrices for any* $S$*, it follows that a Nekrasov matrix is an* $\{i_0\}$*-Nekrasov matrices.*

The following example shows that the $k$-subdirect sum of two $\{i_0\}$-Nekrasov matrices may not be an $\{i_0\}$-Nekrasov matrix in general.

**Example 2.1.** *Consider the* $\{i_0\}$*-Nekrasov matries* $A$ *and* $B$ *for* $i_0 = 2$*, where*

$$A = \left[\begin{array}{c|ccc} 9 & -4 & -1 & -2 \\ \hline -1 & 10 & -1 & -2 \\ -1 & -2 & 4 & -2 \\ -2 & -1 & -3 & 7 \end{array}\right] \text{ and } B = \left[\begin{array}{ccc|c} 1 & -\frac{3}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{4} & -1 & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{5} & -\frac{2}{5} & -1 & -\frac{2}{5} \\ \hline -\frac{9}{7} & 0 & 0 & 1 \end{array}\right].$$

Then, the 3-subdirect sum $C = A \bigoplus_3 B$ gives

$$C = \left[\begin{array}{c|ccc|c} 9 & -4 & -1 & -2 & 0 \\ \hline -1 & 11 & -\frac{8}{5} & -\frac{11}{5} & 0 \\ -1 & -\frac{9}{4} & 3 & -\frac{9}{4} & -\frac{1}{2} \\ -2 & -\frac{6}{5} & -\frac{17}{5} & 6 & -\frac{2}{5} \\ \hline 0 & -\frac{9}{7} & 0 & 0 & 1 \end{array}\right].$$

It is easy to check that $C = A \bigoplus_3 B$ is not an $\{i_0\}$-Nekrasov matrix for neither one index $i_0$. This motivates us to seek some simple conditions such that $C = A \bigoplus_k B$ for any $k$ is an $\{i_0\}$-Nekrasov matrix. First, we provide the following conditions such that $A \bigoplus_1 B$ is an $\{i_0\}$-Nekrasov matrix, where $A$ is an $\{i_0\}$-Nekrasov matrix and $B$ is a Nekrasov matrix.

**Theorem 2.1.** *Let* $A = [a_{ij}] \in \mathbb{C}^{n_1 \times n_1}$ *be an* $\{i_0\}$*-Nekrasov matrix with* $i_0 \in S_1$ *and* $B = [b_{ij}] \in \mathbb{C}^{n_2 \times n_2}$ *be a Nekrasov matrix, partitioned as in (1.1), which defines the sets* $S_1$*,* $S_2$ *and* $S_3$ *as in (2.1), and let* $t = n_1 - 1$*. If arg(*$a_{ii}$*)=arg(*$b_{i-t,i-t}$*) for all* $i \in S_2$ *and* $B_{21} = 0$*, then the* 1*-subdirect sum* $C = A \bigoplus_1 B$ *is an* $\{i_0\}$*-Nekrasov matrix.*

*Proof.* Since $A$ is an $\{i_0\}$-Nekrasov matrix and $i_0 \in S_1$, it follows that if $i_0 = 1$ then $|c_{11}| = |a_{11}| > \eta_1(A) = 0 = \eta_1(C)$, otherwise,

$$|c_{i_0,i_0}| = |a_{i_0,i_0}| > \eta_{i_0}(A) = \sum_{j=1}^{i_0-1} \frac{|a_{i_0,j}|}{|a_{jj}|}\eta_j(A) = \sum_{j=1}^{i_0-1} \frac{|c_{i_0,j}|}{|c_{jj}|}\eta_j(C) = \eta_{i_0}(C). \tag{2.3}$$

Case 1: For $i \in S_1$, we have

$$h_i(C) = \sum_{j=1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|}h_j(C) + \sum_{j=i+1}^{n} |c_{ij}| = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}h_j(A) + \sum_{j=i+1}^{n_1} |a_{ij}| = h_i(A),$$

and if $i_0 = 1$, then $\eta_i(C) = 0 = \eta_i(A)$, and if $i_0 \neq 1$, then $\eta_1(C) = |c_{i,i_0}| = |a_{i,i_0}| = \eta_1(A)$ and

$$
\begin{aligned}
\eta_i(C) &= \begin{cases} \sum_{j=1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|}\eta_j(C) + |c_{i,i_0}|, & i = 2, \ldots, i_0 - 1, \\ \sum_{j=1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|}\eta_j(C), & i = i_0, \ldots, n_1 - k. \end{cases} \\
&= \begin{cases} \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}\eta_j(A) + |a_{i,i_0}|, & i = 1, 2, \ldots, i_0 - 1, \\ \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}\eta_j(A), & i = i_0, \ldots, n_1 - k. \end{cases} \\
&= \eta_i(A).
\end{aligned}
$$

Hence, for all $j \in S_1 \setminus \{i_0\}$,

$$
\begin{aligned}
(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \left(|c_{jj}| - h_j(C) + \eta_j(C)\right) &= (|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \left(|a_{jj}| - h_j(A) + \eta_j(A)\right) \\
&> (h_{i_0}(A) - \eta_{i_0}(A)) \cdot \eta_j(A) \\
&= (h_{i_0}(C) - \eta_{i_0}(C)) \cdot \eta_j(C).
\end{aligned}
$$

Case 2: For $i \in S_2 = \{n_1\}$, we have

$$
\begin{aligned}
h_{n_1}(C) &= \sum_{j=1}^{n_1-1} \frac{|c_{n_1,j}|}{|c_{jj}|} h_j(C) + \sum_{j=n_1+1}^{n} |c_{n_1,j}| \\
&= \sum_{j=1}^{n_1-1} \frac{|a_{n_1,j}|}{|a_{jj}|} h_j(A) + \sum_{j=n_1+1}^{n} |b_{n_1-t,j-t}| \\
&= h_{n_1}(A) + h_{n_1-t}(B),
\end{aligned}
$$

and

$$
\eta_{n_1}(C) = \sum_{j=1}^{n_1-1} \frac{|c_{n_1,j}|}{|c_{jj}|}\eta_j(C) = \sum_{j=1}^{n_1-1} \frac{|a_{n_1,j}|}{|a_{jj}|}\eta_j(A) = \eta_{n_1}(A).
$$

So,

$$
\begin{aligned}
(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot (|c_{n_1,n_1}| - h_{n_1}(C) + \eta_{n_1}(C)) &= (|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \left(|a_{n_1,n_1} + b_{11}| - (h_{n_1}(A) + h_1(B)) + \eta_{n_1}(A)\right) \\
&= (|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \left(|a_{n_1,n_1}| - h_{n_1}(A) + |b_{11}| - h_1(B) + \eta_{n_1}(A)\right) \\
&> (|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \left(|a_{n_1,n_1}| - h_{n_1}(A) + \eta_{n_1}(A)\right) \\
&> (h_{i_0}(A) - \eta_{i_0}(A)) \cdot \eta_{n_1}(A) \\
&= (h_{i_0}(C) - \eta_{i_0}(C)) \cdot \eta_{n_1}(C).
\end{aligned}
$$

Case 3: For $i \in S_3 = \{n_1 + 1, \ldots, n\}$, we have

$$
h_i(C) = \sum_{j=1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \sum_{j=i+1}^{n} |c_{ij}|
$$

$$
\begin{aligned}
&= \sum_{j=1}^{n_1-1} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \frac{|c_{i,n_1}|}{|c_{n_1,n_1}|} h_{n_1}(C) + \sum_{j=n_1+1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \sum_{j=i+1}^{n} |c_{ij}| \\
&= \frac{|b_{i-t,n_1-t}|}{|a_{n_1,n_1} + b_{n_1-t,n_1-t}|} (h_{n_1}(A) + h_{n_1-t}(B)) + \sum_{j=n_1+1}^{i-1} \frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|} h_{j-t}(B) + \sum_{j=i+1}^{n} |b_{i-t,j-t}| \\
&= \frac{|b_{i-t,n_1-t}|}{|b_{n_1-t,n_1-t}|} \cdot h_{n_1-t}(B) + \sum_{j=n_1+1}^{i-1} \frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|} h_{j-t}(B) + \sum_{j=i+1}^{n} |b_{i-t,j-t}| \ (by\ B_{21} = 0) \\
&= h_{i-t}(B).
\end{aligned}
$$

It follows from $B_{21} = 0$ that

$$
\eta_{n_1+1}(C) = \sum_{j=1}^{n_1} \frac{|c_{n_1+1,j}|}{|c_{jj}|} \eta_j(C) = \sum_{j=1}^{n_1-1} \frac{|c_{n_1+1,j}|}{|c_{jj}|} \eta_j(C) + \frac{|c_{n_1+1,n_1}|}{|c_{n_1,n_1}|} \eta_{n_1}(C) = \frac{|b_{n_1+1-t,n_1-t}|}{|b_{n_1-t,n_1-t}|} \eta_{n_1}(C) = 0,
$$

and for each $i = n_1 + 2, \dots, n$,

$$
\eta_i(C) = \sum_{j=1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|} \eta_j(C) = \frac{|c_{i,n_1}|}{|c_{n_1,n_1}|} \eta_{n_1}(C) + \sum_{j=n_1+1}^{i-1} \frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|} \eta_j(C) = 0.
$$

So, for all $j \in S_3$,

$$
\begin{aligned}
(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \Big(|c_{jj}| - h_j(C) + \eta_j(C)\Big) &\geq (|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \Big(|b_{j-t,j-t}| - h_{j-t}(B)\Big) \\
&> 0 = (h_{i_0}(C) - \eta_{i_0}(C)) \cdot \eta_j(C).
\end{aligned}
$$

The conclusion follows from (2.3), Case 1–3. □

Next, we give some conditions such that $C = A \bigoplus_k B$ for any $k$ is an $\{i_0\}$-Nekrasov matrix, where $A$ is an $\{i_0\}$-Nekrasov matrix and $B$ is a Nekrasov matrix. First, a lemma is given which will be used in the sequel.

**Lemma 2.1.** *Let $A = [a_{ij}] \in \mathbb{C}^{n_1 \times n_1}$ be an $\{i_0\}$-Nekrasov matrix with $i_0 \in S_1 \cup S_2$ and $B = [b_{ij}] \in \mathbb{C}^{n_2 \times n_2}$ be a Nekrasov matrix, partitioned as in (1.1), k be an integer such that $1 \leq k \leq min\{n_1, n_2\}$, which defines the sets $S_1$, $S_2$ and $S_3$ as in (2.1), let $t = n_1 - k$ and $C = A \bigoplus_k B$. If $arg(a_{ii})=arg(b_{i-t,i-t})$ for all $i \in S_2$, $B_{12} = 0$, and $|a_{ij} + b_{i-t,j-t}| \leq |a_{ij}|$ for $i \neq j,\ i, j \in S_2$, then*

$$
h_{i_0}(C) - \eta_{i_0}(C) \leq h_{i_0}(A) - \eta_{i_0}(A).
$$

*Proof.* If $i_0 \in S_1$, then it follows from the proof of Case I in Theorem 2.1 that $h_i(C) - \eta_i(C) = h_i(A) - \eta_i(A)$ for all $i \in S_1$, and thus $h_{i_0}(C) - \eta_{i_0}(C) = h_{i_0}(A) - \eta_{i_0}(A)$.

If $i_0 \in S_2 = \{n_1 - k + 1, \dots, n_1\}$, then from the assumptions and $t = n_1 - k$ we have

$$
h_{t+1}(C) - \eta_{t+1}(C) = \begin{cases} \sum_{j=1}^{t} \frac{|c_{t+1,j}|}{|c_{jj}|}(h_j(C) - \eta_j(C)) + \sum_{j=t+2}^{n} |c_{t+1,j}| - |c_{t+1,i_0}| \text{ if } t + 1 < i_0, \\ \sum_{j=1}^{t} \frac{|c_{t+1,j}|}{|c_{jj}|}(h_j(C) - \eta_j(C)) + \sum_{j=t+2}^{n} |c_{t+1,j}| \text{ if } t + 1 = i_0, \end{cases}
$$

$$\leq \begin{cases} \sum_{j=1}^{t} \frac{|a_{t+1,j}|}{|a_{jj}|}(h_j(A) - \eta_j(A)) + \sum_{j=t+2}^{n_1} |a_{t+1,j}| - |a_{t+1,i_0}| \text{ if } t+1 < i_0, \\ \sum_{j=1}^{t} \frac{|a_{t+1,j}|}{|a_{jj}|}(h_j(A) - \eta_j(A)) + \sum_{j=t+2}^{n_1} |a_{t+1,j}| \text{ if } t+1 = i_0, \end{cases}$$
$$= h_{t+1}(A) - \eta_{t+1}(A).$$

Suppose that $h_i(C) - \eta_i(C) \leq h_i(A) - \eta_i(A)$ for all $i < t + m$, where $m$ is a positive integer and $1 < m \leq k$. We next prove that $h_{t+m}(C) - \eta_{t+m}(C) \leq h_{t+m}(A) - \eta_{t+m}(A)$. Since

$$\begin{aligned} h_{t+m}(C) - \eta_{t+m}(C) &= \begin{cases} \sum_{j<t+m} \frac{|c_{t+m,j}|}{|c_{jj}|}(h_j(C) - \eta_j(C)) + \sum_{j>t+m}^{n} |c_{t+m,j}| - |c_{t+m,i_0}|, & t+m < i_0, \\ \sum_{j<t+m} \frac{|c_{t+m,j}|}{|c_{jj}|}(h_j(C) - \eta_j(C)) + \sum_{j>t+m}^{n} |c_{t+m,j}|, & t+m \geq i_0, \end{cases} \\ &\leq \begin{cases} \sum_{j<t+m} \frac{|a_{t+m,j}|}{|a_{jj}|}(h_j(A) - \eta_j(A)) + \sum_{j>t+m}^{n} |a_{t+m,j}| - |a_{t+m,i_0}|, & t+m < i_0, \\ \sum_{j<t+m} \frac{|a_{t+m,j}|}{|a_{jj}|}(h_j(A) - \eta_j(A)) + \sum_{j>t+m}^{n} |a_{t+m,j}|, & t+m \geq i_0, \end{cases} \\ &= h_{t+m}(A) - \eta_{t+m}(A), \end{aligned}$$

it follows that $h_i(C) - \eta_i(C) \leq h_i(A) - \eta_i(A)$ for all $i \in S_2$. Hence, $h_{i_0}(C) - \eta_{i_0}(C) \leq h_{i_0}(A) - \eta_{i_0}(A)$. The proof is complete. □

**Theorem 2.2.** *Let $A = [a_{ij}] \in \mathbb{C}^{n_1 \times n_1}$ be an $\{i_0\}$-Nekrasov matrix with $i_0 \in S_1$ and $B = [b_{ij}] \in \mathbb{C}^{n_2 \times n_2}$ be a Nekrasov matrix, partitioned as in (1.1), $k$ be an integer such that $1 \leq k \leq min\{n_1, n_2\}$, which defines the sets $S_1$, $S_2$ and $S_3$ as in (2.1), and let $t = n_1 - k$. If $arg(a_{ii})=arg(b_{i-t,i-t})$ for all $i \in S_2$, $A_{21} = 0$, and $|a_{ij} + b_{i-t,j-t}| \leq |b_{i-t,j-t}|$ for $i \neq j$, $i, j \in S_2$, then the $k$-subdirect sum $C = A \bigoplus_k B$ is an $\{i_0\}$-Nekrasov matrix.*

*Proof.* Since $A$ is an $\{i_0\}$-Nekrasov matrix and $i_0 \in S_1$, it is obvious that $|c_{i_0,i_0}| > \eta_{i_0}(C)$.

Case 1: For $i \in S_1$, since $h_i(C) = h_i(A)$ and $\eta_i(C) = \eta_i(A)$, it holds that for all $j \in S_1 \setminus \{i_0\}$,

$$(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \left(|c_{jj}| - h_j(C) + \eta_j(C)\right) > (h_{i_0}(C) - \eta_{i_0}(C)) \cdot \eta_j(C).$$

Case 2: For $i \in S_2$, by the assumptions, we have

$$h_{n_1-k+1}(C) = \sum_{j=1}^{n_1-k} \frac{|c_{n_1-k+1,j}|}{|c_{jj}|} h_j(C) + \sum_{j=n_1-k+2}^{n} |c_{n_1-k+1,j}| \leq \sum_{j=n_1-k+2}^{n} |b_{n_1-k+1-t,j-t}| = h_1(B).$$

Similarly, for $i = n_1 - k + 2, \ldots, n_1$,

$$\begin{aligned} h_i(C) &= \sum_{j=1}^{n_1-k} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \sum_{j=n_1-k+1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \sum_{j=i+1}^{n} |c_{ij}| \\ &\leq \sum_{j=n_1-k+1}^{i-1} \frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|} h_{j-t}(B) + \sum_{j=i+1}^{n} |b_{i-t,j-t}| \end{aligned}$$

$$= \quad h_{i-t}(B).$$

And for $i = n_1 - k + 1$, by $A_{21} = 0$,

$$\eta_{n_1-k+1}(C) = \sum_{j=1}^{n_1-k} \frac{|c_{n_1-k+1,j}|}{|c_{jj}|}\eta_j(C) = 0,$$

implying that for all $i \in S_2$,

$$\eta_i(C) = \sum_{j=1}^{n_1-k} \frac{|c_{ij}|}{|c_{jj}|}\eta_j(C) + \sum_{j=n_1-k+1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|}\eta_j(C) = 0.$$

So, for all $j \in S_2$,

$$\begin{aligned}(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \left(|c_{jj}| - h_j(C) + \eta_j(C)\right) &> (|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \left(|b_{j-t,j-t}| - h_{j-t}(B)\right) \\ &> 0 \\ &= (h_{i_0}(C) - \eta_{i_0}(C)) \cdot \eta_j(C). \end{aligned} \tag{2.4}$$

Analogously to the proof of Case 2, we can easily obtain that (2.4) holds for all $j \in S_3$. Combining with Case 1 and Case 2, the conclusion follows. □

**Example 2.2.** *Consider the following matrices:*

$$A = \left[\begin{array}{cc:cc} 3 & 1 & 1 & 1 \\ 1 & 20 & 0 & -2 \\ \hdashline 0 & 0 & 10 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right] \text{ and } B = \left[\begin{array}{cc:cc} 60 & -15 & -15 & -15 \\ -75 & 105 & -45 & 0 \\ \hdashline -60 & -60 & 120 & -15 \\ -15 & -15 & -15 & 45 \end{array}\right].$$

It is easy to verify that $A$ is an $\{i_0\}$-Nekrasov matrix for $i_0 \in S_1 = \{1, 2\}$ and $B$ is a Nekrasov matrix, which satisfy the hypotheses of Theorem 2.2. So, by Theorem 2.2, $A \bigoplus_2 B$ is an $\{i_0\}$-Nekrasov matrix for $i_0 \in S_1 = \{1, 2\}$. In fact, let $C = A \bigoplus_2 B$. Then,

$$C = \left[\begin{array}{cc:cc:cc} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 20 & 0 & -2 & 0 & 0 \\ \hdashline 0 & 0 & 70 & -13 & -15 & -15 \\ 0 & 0 & -73 & 109 & -45 & 0 \\ \hdashline 0 & 0 & -60 & -60 & 120 & -15 \\ 0 & 0 & -15 & -15 & -15 & 45 \end{array}\right],$$

and from Definition 2.1, one can verify that $C$ is an $\{i_0\}$-Nekrasov matrix for $i_0 \in S_1 = \{1, 2\}$.

**Theorem 2.3.** *Let $A = [a_{ij}] \in \mathbb{C}^{n_1\times n_1}$ be an $\{i_0\}$-Nekrasov matrix with $i_0 \in S_1 \cup S_2$ and $B = [b_{ij}] \in \mathbb{C}^{n_2\times n_2}$ be a Nekrasov matrix, partitioned as in (1.1), $k$ be an integer such that $1 \le k \le min\{n_1, n_2\}$, which defines the sets $S_1$, $S_2$ and $S_3$ as in (2.1), and let $t = n_1 - k$. If $arg(a_{ii})$=$arg(b_{i-t,i-t})$ for all $i \in S_2$, $B_{12} = B_{21} = 0$, and $|a_{ij} + b_{i-t,j-t}| \le |a_{ij}|$ for $i \ne j$, $i, j \in S_2$, then the $k$-subdirect sum $C = A \bigoplus_k B$ is an $\{i_0\}$-Nekrasov matrix.*

*Proof.* Since $A$ is an $\{i_0\}$-Nekrasov matrix, it follows that if $i_0 \in S_1$, then $|c_{i_0,i_0}| > \eta_{i_0}(C)$, and if $i_0 \in S_2$, then

$$\begin{aligned}
\eta_{n_1-k+1}(C) &= \begin{cases} \sum\limits_{j=1}^{n_1-k} \frac{|c_{n_1-k+1,j}|}{|c_{jj}|}\eta_j(C) + |c_{n_1-k+1,i_0}|, & n_1-k+1 \neq i_0, \\ \sum\limits_{j=1}^{n_1-k} \frac{|c_{n_1-k+1,j}|}{|c_{jj}|}\eta_j(C), & n_1-k+1 = i_0. \end{cases} \\
&\leq \begin{cases} \sum\limits_{j=1}^{n_1-k} \frac{|a_{n_1-k+1,j}|}{|a_{jj}|}\eta_j(A) + |a_{n_1-k+1,i_0}|, & n_1-k+1 \neq i_0, \\ \sum\limits_{j=1}^{n_1-k} \frac{|a_{n_1-k+1,j}|}{|a_{jj}|}\eta_j(A), & n_1-k+1 = i_0. \end{cases} \\
&= \eta_{n_1-k+1}(A).
\end{aligned}$$

Similarly, we can obtain that $\eta_j(C) \leq \eta_j(A)$ for all $j \in \{n_1-k+2, \ldots, n_1\}$. Therefore,

$$\begin{aligned}
\eta_{i_0}(C) = \sum_{j=1}^{i_0-1} \frac{|c_{i_0,j}|}{|c_{jj}|}\eta_j(C) &= \sum_{j=1}^{n_1-k} \frac{|c_{i_0,j}|}{|c_{jj}|}\eta_j(C) + \sum_{j=n_1-k+1}^{i_0-1} \frac{|c_{i_0,j}|}{|c_{jj}|}\eta_j(C) \\
&\leq \sum_{j=1}^{n_1-k} \frac{|a_{i_0,j}|}{|a_{jj}|}\eta_j(A) + \sum_{j=n_1-k+1}^{i_0-1} \frac{|a_{i_0,j}|}{|a_{jj}|}\eta_j(A) \\
&= \eta_{i_0}(A),
\end{aligned}$$

and

$$|c_{i_0,i_0}| = |a_{i_0,i_0}| + |b_{i_0-t,i_0-t}| > |a_{i_0,i_0}| > \eta_{i_0}(A) \geq \eta_{i_0}(C).$$

Case 1: For $i \in S_1$, proceeding as in the proof of Case 1 in Theorem 2.1, we have $h_i(C) = h_i(A)$ and $\eta_i(C) = \eta_i(A)$, which implies that for all $j \in S_1 \setminus \{i_0\}$,

$$(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \Big(|c_{jj}| - h_j(C) + \eta_j(C)\Big) > (h_{i_0}(C) - \eta_{i_0}(C)) \cdot \eta_j(C).$$

Case 2: For $i \in S_2$, by the assumptions, we have

$$\begin{aligned}
h_i(C) &= \sum_{j=1}^{n_1-k} \frac{|c_{ij}|}{|c_{jj}|}h_j(C) + \sum_{j=n_1-k+1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|}h_j(C) + \sum_{j=i+1}^{n_1} |c_{ij}| \\
&\leq \sum_{j=1}^{n_1-k} \frac{|a_{ij}|}{|a_{jj}|}h_j(A) + \sum_{j=n_1-k+1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}h_j(A) + \sum_{j=i+1}^{n_1} |a_{ij}| \\
&= h_i(A),
\end{aligned}$$

and

$$\eta_i(C) = \begin{cases} \sum\limits_{j=1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|}\eta_j(C) + |c_{i,i_0}|, & i = n_1-k+1, \ldots, i_0-1, \\ \sum\limits_{j=1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|}\eta_j(C), & i = i_0, \ldots, n_1. \end{cases}$$

$$
\begin{aligned}
&\leq \begin{cases} \sum\limits_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}\eta_j(A) + |a_{i,i_0}|, & i = n_1 - k + 1, \ldots, i_0 - 1, \\ \sum\limits_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|}\eta_j(A), & i = i_0, \ldots, n_1. \end{cases} \\
&= \eta_i(A).
\end{aligned}
$$

Hence, by Lemma 2.1, it follows that for all $j \in S_2 \setminus \{i_0\}$,

$$
\begin{aligned}
(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \left( \frac{|c_{jj}| - h_j(C)}{\eta_j(C)} + 1 \right) &> (|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \left( \frac{|a_{jj}| - h_j(A)}{\eta_j(A)} + 1 \right) \\
&> (h_{i_0}(A) - \eta_{i_0}(A)) \\
&\geq (h_{i_0}(C) - \eta_{i_0}(C)).
\end{aligned}
$$

Case 3: For $i \in S_3$, similarly to the proof of Case 3 in Theorem 2.1, we show that for all $i \in S_3$,

$$h_i(C) = h_{i-t}(B), \text{ and } \eta_i(C) = 0,$$

which implies that for all $j \in S_3$,

$$
\begin{aligned}
(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \Big( |c_{jj}| - h_j(C) + \eta_j(C) \Big) &> (|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \Big( |b_{j-t,j-t}| - h_{j-t}(B) \Big) \\
&> 0 \\
&= (h_{i_0}(C) - \eta_{i_0}(C))\eta_j(C).
\end{aligned}
$$

From the above three cases, the conclusion follows. □

**Example 2.3.** *Consider the following matrices:*

$$
A = \begin{bmatrix} 9 & -4 & -1 & -4 \\ -1 & 10 & -1 & -2 \\ -1 & -2 & 4 & -2 \\ -2 & -1 & -3 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 & 0.2 & 0 \\ 1 & 21 & 1 & 0 \\ 2 & 0.5 & 6.4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix},
$$

*where $A$ is an $\{i_0\}$-Nekrasov matrix for $i_0 \in S_1 \cup S_2 = \{1, 2, 3, 4\}$ and $B$ is a Nekrasov matrix, and they satisfy the hypotheses of Theorem 2.3. Then, from Theorem 2.3, we get that the 3-subdirect sum $C = A \bigoplus_3 B$ is also an $\{i_0\}$-Nekrasov matrix for $i_0 \in S_1 \cup S_2 = \{1, 2, 3, 4\}$. Actually, by Definition 2.1, one can check that*

$$
C = \begin{bmatrix} 9 & -4 & -1 & -4 & 0 \\ -1 & 15 & 0 & -1.8 & 0 \\ -1 & -1 & 25 & -1 & 0 \\ -2 & 1 & -2.5 & 13.4 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}
$$

*is an $\{i_0\}$-Nekrasov matrix for $i_0 \in S_1 \cup S_2 = \{1, 2, 3, 4\}$.*

**Theorem 2.4.** *Let $A = [a_{ij}] \in \mathbb{C}^{n_1 \times n_1}$ be an $\{i_0\}$-Nekrasov matrix for some $i_0 \in S_2$ and $B = [b_{ij}] \in \mathbb{C}^{n_2 \times n_2}$ be a Nekrasov matrix, partitioned as in (1.1), $k$ be an integer such that $1 \leq k \leq min\{n_1, n_2\}$, which defines the sets $S_1$, $S_2$ and $S_3$ as in (2.1), and let $t = n_1 - k$. If* (*i*) *arg*($a_{ii}$)=*arg*($b_{i-t,i-t}$) *for all* $i \in S_2$,

(*ii*) $B_{12} = 0$, $h_i(A) \leq h_{i-t}(B)$, $\eta_i(A) \leq \eta_{i-t}(B)$, *and* $|a_{ij} + b_{i-t,j-t}| \leq |a_{ij}|$ *for* $i \neq j,\ i, j \in S_2$,

(*iii*) $(h_{i_0-t}(B) - \eta_{i_0-t}(B))(|a_{i_0,i_0}| - \eta_{i_0}(A)) \geq (|b_{i_0-t,i_0-t}| - \eta_{i_0-t}(B))(h_{i_0}(A) - \eta_{i_0}(A))$,

*then the k-subdirect sum* $C = A \bigoplus_k B$ *is an* $\{i_0\}$*-Nekrasov matrix.*

*Proof.* Due to $A$ is an $\{i_0\}$-Nekrasov matrix and $i_0 \in S_2$, it follows from the proof of Case 2 in Theorem 2.3 that $\eta_i(C) \leq \eta_i(A)$ for all $i \in S_2$, which leads to

$$|c_{i_0,i_0}| > |a_{i_0,i_0}| > \eta_{i_0}(A) \geq \eta_{i_0}(C).$$

Case 1: For $i \in S_1$, it is obvious that $h_i(C) = h_i(A)$ and $\eta_i(C) = \eta_i(A)$. Hence, for all $j \in S_1$,

$$\begin{aligned}(|c_{i_0,i_0}| - \eta_{i_0}(C)) \cdot \left(|c_{jj}| - h_j(C) + \eta_j(C)\right) &> (|a_{i_0,i_0}| - \eta_{i_0}(A)) \cdot \left(|a_{jj}| - h_j(A) + \eta_j(A)\right) \\ &> (h_{i_0}(A) - \eta_{i_0}(A)) \cdot \eta_j(A) \\ &\geq (h_{i_0}(C) - \eta_{i_0}(C)) \cdot \eta_j(C). \end{aligned} \tag{2.5}$$

Case 2: For $i \in S_2$, it follows from $B_{12} = 0$ and $|a_{ij} + b_{i-t,j-t}| \leq |a_{ij}|$ for $i \neq j,\ i, j \in S_2$ that $h_i(C) \leq h_i(A)$, and thus for all $j \in S_2 \setminus \{i_0\}$, (2.5) also holds.

Case 3: For $i = n_1 + 1 \in S_3$, by the assumption, it follows that

$$\begin{aligned} h_{n_1+1}(C) &= \sum_{j=1}^{n_1-k} \frac{|c_{n_1+1,j}|}{|c_{jj}|} h_j(C) + \sum_{j=n_1-k+1}^{n_1} \frac{|c_{n_1+1,j}|}{|c_{jj}|} h_j(C) + \sum_{j=n_1+2}^{n} |c_{n_1+1,j}| \\ &\leq \sum_{j=n_1-k+1}^{n_1} \frac{|b_{n_1+1-t,j-t}|}{|c_{j-t,j-t}|} h_j(A) + \sum_{j=n_1+2}^{n} |b_{n_1+1-t,j-t}| \\ &\leq \sum_{j=n_1-k+1}^{n_1} \frac{|b_{n_1+1-t,j-t}|}{|c_{j-t,j-t}|} h_{j-t}(B) + \sum_{j=n_1+2}^{n} |b_{n_1+1-t,j-t}|, \\ &= h_{n_1+1-t}(B), \end{aligned}$$

which recursively yields that for $i = n_1 + 2, \ldots, n$,

$$\begin{aligned} h_i(C) &= \sum_{j=1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \sum_{j=i+1}^{n} |c_{ij}| \\ &= \sum_{j=1}^{n_1-k} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \sum_{j=n_1-k+1}^{n_1} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \sum_{j=n_1+1}^{i-1} \frac{|c_{ij}|}{|c_{jj}|} h_j(C) + \sum_{j=i+1}^{n} |c_{ij}| \\ &\leq \sum_{j=n_1-k+1}^{n_1} \frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|} h_j(A) + \sum_{j=n_1+1}^{i-1} \frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|} h_{j-t}(B) + \sum_{j=i+1}^{n} |b_{i-t,j-t}| \\ &\leq \sum_{j=n_1-k+1}^{i-1} \frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|} h_{j-t}(B) + \sum_{j=i+1}^{n} |b_{i-t,j-t}| \\ &= h_{i-t}(B). \end{aligned}$$

Similarly, we have

$$\begin{aligned}\eta_{n_1+1}(C) &= \sum_{j=1}^{n_1-k}\frac{|c_{n_1+1,j}|}{|c_{jj}|}\eta_j(C)+\sum_{j=n_1-k+1}^{n_1}\frac{|c_{n_1+1,j}|}{|c_{jj}|}\eta_j(C)\\ &= \sum_{j=n_1-k+1}^{n_1}\frac{|b_{n_1+1-t,j-t}|}{|a_{jj}+b_{j-t,j-t}|}\eta_j(A)\\ &\leq \sum_{j=n_1-k+1}^{n_1}\frac{|b_{n_1+1-t,j-t}|}{|b_{j-t,j-t}|}\eta_{j-t}(B)\\ &= \eta_{n_1+1-t}(B),\end{aligned}$$

and for all $i = n_1 + 2, \ldots, n$,

$$\begin{aligned}\eta_i(C) &= \sum_{j=1}^{i-1}\frac{|c_{ij}|}{|c_{jj}|}\eta_j(C)\\ &= \sum_{j=1}^{n_1-k}\frac{|c_{ij}|}{|c_{jj}|}\eta_j(C)+\sum_{j=n_1-k+1}^{n_1}\frac{|c_{ij}|}{|c_{jj}|}\eta_j(C)+\sum_{j=n_1+1}^{i-1}\frac{|c_{ij}|}{|c_{jj}|}\eta_j(C)\\ &\leq \sum_{j=n_1-k+1}^{n_1}\frac{|b_{i-t,j-t}|}{|a_{jj}+b_{j-t,j-t}|}\eta_j(A)+\sum_{j=n_1+1}^{i-1}\frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|}\eta_{j-t}(B)\\ &\leq \sum_{j=n_1-k+1}^{n_1}\frac{|b_{i-t,j-t}|}{|a_{jj}+b_{j-t,j-t}|}\eta_{j-t}(B)+\sum_{j=n_1+1}^{i-1}\frac{|b_{i-t,j-t}|}{|b_{j-t,j-t}|}\eta_{j-t}(B)\\ &= \eta_{i-t}(B).\end{aligned}$$

Hence, for all $j \in S_3$,

$$\begin{aligned}(|c_{i_0,i_0}|-\eta_{i_0}(C))\cdot\left(\frac{|c_{jj}|-h_j(C)}{\eta_j(C)}+1\right) &> (|a_{i_0,i_0}|-\eta_{i_0}(A))\cdot\left(\frac{|b_{j-t,j-t}|-h_{j-t}(B)}{\eta_j(C)}+1\right)\\ &\geq (|a_{i_0,i_0}|-\eta_{i_0}(A))\cdot\left(\frac{|b_{j-t,j-t}|-h_{j-t}(B)}{\eta_{j-t}(B)}+1\right)\\ &> (|a_{i_0,i_0}|-\eta_{i_0}(A))\cdot\frac{h_{i_0-t}(B)-\eta_{i_0-t}(B)}{|b_{i_0-t,i_0-t}|-\eta_{i_0-t}(B)}\\ &\geq (|a_{i_0,i_0}|-\eta_{i_0}(A))\cdot\frac{h_{i_0}(A)-\eta_{i_0}(A)}{|a_{i_0,i_0}|-\eta_{i_0}(A)}\\ &= h_{i_0}(C)-\eta_{i_0}(C).\end{aligned}$$

From Case 1, Case 2 and Case 3, we can conclude that $C = A \bigoplus_k B$ is an $\{i_0\}$-Nekrasov matrix. □

**Example 2.4.** *Consider the following matrices:*

$$A=\left[\begin{array}{c:ccc}3 & 1 & 1 & 1\\ \hdashline 1 & 20 & -2 & -2\\ 0 & 1 & 10 & -1\\ -0.5 & 0.5 & 1 & 4\end{array}\right] \text{ and } B=\left[\begin{array}{ccc:c}9 & 4 & 1 & 0\\ -1 & 10 & 1 & 0\\ -1 & -2 & 4 & 0\\ \hdashline 0 & 0 & 0 & 7\end{array}\right],$$

*where A is an* $\{i_0\}$*-Nekrasov matrix for* $i_0 = 2$ *and B is a Nekrasov matrix. By computation, we have* $h_1(A) = 3, h_2(A) = 5,$ $h_3(A) = 1.25, h_4(A) = 0.75,$ $h_1(B) = 5, h_2(B) = 1.5556, h_3(B) = 0.8667, h_4(B) = 0,$ $\eta_1(A) = 1, \eta_2(A) = 0.3333, \eta_3(A) = 0.0167,$ $\eta_4(A) = 0.1767,$ $\eta_1(B) = 4, \eta_2(B) = 0.4444, \eta_3(B) = 0.5333,$ *and* $\eta_4(B) = 0,$ *which satisfy the hypotheses of Theorem 2.4. Hence, from Theorem 2.4, we have that* $A \bigoplus_3 B$ *is also an* $\{i_0\}$*-Nekrasov matrix for* $i_0 = 2$*. In fact, let* $C = A \bigoplus_3 B$*. Then*

$$C = \left[\begin{array}{c:ccc:c} 3 & 1 & 1 & 1 & 0 \\ \hdashline 1 & 29 & 2 & -1 & 0 \\ 0 & 0 & 20 & 0 & 0 \\ -0.5 & -0.5 & -1 & 8 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 7 \end{array}\right],$$

*and one can verify that C is an* $\{i_0\}$*-Nekrasov matrix for* $i_0 = 2$ *from Definition 2.1.*

## 3. Conclusions

In this paper, for an $\{i_0\}$-Nekrasov matrix $A$ as a subclass of $S$-Nekrasov matrices and a Nekrasov matrix $B$, we provide some sufficient conditions such that the $k$-subdirect sum $A \bigoplus_k B$ lies in the class of $\{i_0\}$-Nekrasov matrices. Numerical examples are included to illustrate the advantages of the given conditions. The results obtained here have potential applications in some scientific computing problems such as matrix completion problem and the convergence of iterative methods for large sparse linear systems. For instance, consider large scale linear systems

$$C\mathbf{x} = \mathbf{b}. \tag{3.1}$$

Note that if the coefficient matrix $C$ in (3.1) is an $H$-matrix, then the iterative methods of Jacobi and Gauss-Seidel associated with (3.1) are both convergent [29], but it is not easy to determine $C$ as an $H$-matrix in general. However, if $C$ is exactly the subdirect sum of matrices $A$ and $B$, i.e., $C = A \bigoplus_k B$, where $A$ and $B$ satisfy the sufficient conditions given here, then it is easy to see that $C$ is an $\{i_0\}$-Nekrasov matrix, and thus an $H$-matrix.

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## Conflict of interest

The author declares no conflict of interest.

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