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Research article

Overtaking optimality in a discrete-time advertising game

Jewaidu Rilwan[1,2,4], **Poom Kumam**[2,3,*] **and Idris Ahmed**[1,2,5]

[1] Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
[2] Center of Excellence in Theoretical and Computational Science (TaCS-CoE),Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT),126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
[3] Departments of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[4] Department of Mathematical Sciences, Bayero University, Kano 700231, Nigeria
[5] Department of Mathematics and Computer Science, Sule Lamido University, P.M.B 048 Kafin-Hausa, Jigawa State, Nigeria

* **Correspondence:** Email: poom.kum@kmutt.ac.th; Tel: +6624708994; Fax: +6624284025.

Abstract: In this paper, advertising competition among m firms is studied in a discrete-time dynamic game framework. Firms maximize the present value of their profits which depends on their advertising strategy and their market share. The evolution of market shares is determined by the firms' advertising activities. By employing the concept of the discrete-time potential games of González-Sánchez and Hernández-Lerma (2013), we derived an explicit formula for the Nash equilibrium (NE) of the game and obtained conditions for which the NE is an overtaking optimal. Moreover, we analyze the asymptotic behavior of the overtaking NE where the convergence towards a unique steady state (turnpike) is established.

Keywords: overtaking optimality; advertising game; Nash equilibrium
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1. Introduction

This paper is concerned with the problem of finding conditions for overtaking optimality and asymptotic behavior of the Nash equilibrium solution of a discrete-time advertising game problem. The study of advertising competition in a differential game framework has received a good deal of attention in the last three decades. Most of the studies which involve numerical, analytical, and qualitative analysis are devoted to the continuous-time model (see e.g. [12, 14, 19, 29] and the survey paper of Huang et al. [18]). Articles on the discrete-time model of advertising games are rare because of the difficulty in obtaining an explicit solution. Among the few papers [1, 8, 19, 25, 30] on discrete-time models, the authors [1, 8, 25, 30] literally addressed the problem of finding, characterising or comparing the optimal advertising strategies in duopolistic scenarios. Park and Hahn [25] investigated the effects of pulsing for a firm responding to its competitor's even or pulsed advertising using the discrete version of the Lanchester model. In [8], Chintagunta and Vilcassim derived an equlibrium profit-maximizing advertising policies for firms operating in a dynamic duopoly. The authors [1] studied the optimal advertising problem of a monopolistic seller using the Bellman method of dynamic programming and obtained an explicit recurrence relations for the optimal control and the market share up to the step t. With the exception of the early work of Sorger [30] on convergence to steady-state, asymptotic behavior of the Nash equilibrium advertising strategies in the discrete-time case has remained unexplored to the best of our knowledge.

The concept of overtaking optimality criterion was first studied in economics in the framework of capital accumulation models (see Ramsey [26], vonWeizsäcker [31], Gale [15], Brock [4]). Thereafter it was considered in many papers on Markov decision processes and optimal control problems (OCP) (see e.g. the survey paper of Carlson and Haurie [6], Nowak and Vega Amaya [24], McKenzie [21], Dana and Le Van [9]). In differential games, Brock [5] first considered this concept with the classical open-loop information structure of the equilibrium advertising strategies. This is followed by few other papers (e.g. Rubinstein [28], Carlson and Haurie [6] and Nowak [23]). In [23], Nowak studied a class of discrete-time symmetric games of capital accumulation and obtained the equilibrium properties using some functional characterization of the overtaking optimality in dynamic programming. We have found Nowak's approach in [23] very useful to study overtaking optimality in advertising game because it concerns the closed-loop information structures which is the case in advertising games.

The problem of finding the overtaking optimality conditions and some properties of the Nash equilibrium (NE) solutions of the advertising game drew our attention to the *potential game* concept proposed in [16] (see also [17]). The concept associates an OCP to the original game, whose optimal solution is a Nash equilibrium for the game. Hence, finding these properties of the NE solution reduces to that of the optimal control of the associated OCP. An obvious question here will be, is the discrete-time advertising game a potential game? If this is the case, can we extend to advertising games some results (e.g. overtaking optimality and turnpike property) in the literature of OCPs? Finding answers to these questions is one of the key contributions of the present paper.

Summarizing, the main objective of this paper is to find conditions for overtaking optimality and turnpike property of Nash equilibrium for discrete-time advertising games of $m \geq 2$ competing firms. To this end, we employ the concept of the discrete-time potential games.

The rest of the paper is organized as follows. In Section 2, we present the general model of discrete time advertising model including the concept of potential games and overtaking optimality. Section 3 presents the main results of the paper which includes three subsections 3.1–3.3. Section 3.1 presents an explicit formula of the Nash equilibrium solution at any stage of the game, while in Sections 3.2 and 3.3, we provide conditions for overtaking optimality and asymptotic turnpike property of the Nash equilibrium solution, respectively. Section 4 concludes the paper.

2. Discrete-time advertising game model

Consider the advertising difference game of $m \geq 2$ firms competing for the same market share. For each firm $i = 1, 2, \cdots, m$, the state $x_t^i \in X^i := [0, 1]$ is firm i's market share at time period t, and the control variable $u_t^i \in U^i = [0, 1]$ is the rate advertising expenditure of firm i at time t. The set U^i denotes the set of all possible advertising strategies of firm i. The game's state space $X := X^1 \times \cdots \times X^m$ is the $m-$simplex in $\mathbb{R}^m$, that is, the set of vectors $x_t = (x_t^1, \cdots, x_t^m)$ with nonnegative components whose sum equals 1.

An extension of the discrete-time duopolistic Lanchester model of advertising games proposed in [8, 25, 30] to m competing firms is given for each firm $i = 1, \cdots, m$ by

$$x_{t+1}^i = f_t^i(x_t^i, u_t^1, \cdots, u_t^m), \tag{2.1}$$

where

$$f_t^i(x_t^i, u_t^1, \cdots, u_t^m) = \left(1 - u_t^i - \sum_{j \neq i}^{m} u_t^j\right) x_t^i + u_t^i$$

is a given function which decomposes the market share of firm i into that carried over from the previous period and that gained by advertising.

The Markov strategy for firm i is a sequence $\mu^i = \{u_0^i, u_1^i(x_1^i), u_2^i(x_2^i), \cdots\}$ that specifies the rate of advertising expenditure u_t^i of firm i in every period t as a function of its market share. Let the strategy profile of all firms $\mu = (\mu^1, \cdots, \mu^m)$ determine the sequence $(u_0^i, u_1^i, u_2^i, u_3^i, \cdots)$ of advertising expenditure rate for every firm i. The goal of each firm i is to maximize the following payoff functions

$$J^i(x_0^i, \mu) = \sum_{t=0}^{\infty} g_t^i(x_{t+1}^i, u_t^i) \tag{2.2}$$

subject to (2.1), where $g_t^i(x_{t+1}^i, u_t^i) := \beta_i^t(q_i x_{t+1}^i - \frac{c_i}{2}(u_t^i)^2)$ denotes the running payoff of player i at stage t, $\beta_i > 0$ represents firm i's discount rate, $q_i > 0$ is firm i's revenue per unit of market share and c_i is the advertising cost function of firm i. Note that the function g_t^i can also be expressed as a function of x_t and u_t for all i.

Let $\sigma^i \in U^i$, and write (μ^{-i}, σ^i) to denote μ with μ^i replaced by σ^i.

Definition 2.1. A profile of strategies $\mu^* = (\mu^{1*}, \cdots, \mu^{m*})$ is a Nash equilibrium for the noncooperative advertising difference game (ADG) problems (2.1) and (2.2) if

$$J^i(x_0^i, \mu^*) - J^i(x_0^i, (\mu^{-i*}, \sigma^i)) \geq 0 \tag{2.3}$$

for every firm i and $\sigma^i \in U^i$, $x_0^i \in (0, 1]$.

Since the series (2.2) might fail to converge, we introduce the finite horizon payoff function

$$J_T^i(x_0^i, \mu) = \sum_{t=0}^{T} g_t^i(x_{t+1}^i, u_t^i),$$

for any $T \in \mathbb{N}$.

Definition 2.2. A profile $\mu^* = (\mu^{1*}, \cdots, \mu^{m*})$ is said to be an overtaking Nash equilibrium if, for every $i \in \{1, ..., m\}$ and $\sigma^i \in U^i$, we have

$$\liminf_{T\to\infty}[J^i_T(x^i_0, \mu^*) - J^i_T(x^i_0, (\mu^{-i*}, \sigma^i))] \geq 0. \tag{2.4}$$

The inequality (2.4) is equivalent to the following. For any given $\varepsilon > 0$: For every $i = 1, ..., m$ and $\sigma^i \in U^i$, there exists $T^* = T^*(\varepsilon, \sigma^i, i = 1, ..., m)$ such that

$$J^i_T(\cdot, \mu^*) \geq J^i_T(\cdot, (\mu^{-i*}, \sigma^i)) + \varepsilon \tag{2.5}$$

for all $T > T^*$. In other words, (2.5) states that, for any given $\varepsilon > 0$, the left-hand side of (2.5) overtakes (or "catches up") the right-hand side for all T "sufficiently large" (i.e., for all $T > T^*$).

Definition 2.3 (Definition 5.1, [16])**.** The noncooperative ADGs (2.1) and (2.2) is said to be a potential difference game (PDG) if there exists an OCP such that a solution to the OCP is also a Nash equilibrium for the game. A PDG is called a Markov potential game if only Markov multistrategies are considered for the game as well as in the corresponding OCP.

Remark 1. Given a real valued function ρ_t on $X \times U$, where $U = \prod_{i=1}^m U^i$, we can define an objective function

$$J(x_0, u_t) := \sum_{t=0}^{\infty} \rho_t(x_{t+1}, u_t), \tag{2.6}$$

which together with the system (2.1) describes an OCP.

From [16], if the games (2.1) and (2.2) is a PDG with an associated OCP as in Remark 1, then ρ_t is called a potential function for the noncooperative difference games (2.1) and (2.2).

Research problem. The research problem is to use the concept of potential difference game to

1. prove the existence of overtaking equilibria for the discrete-time ADGs (2.1) and (2.2);
2. analyze the asymptotic behaviour of the overtaking equilibria in (1).

3. Main results

To address the research problems 1 and 2, we first verify if the noncooperative ADGs (2.1) and (2.2) is indeed a PDG. To this end, we prove the discrete time version of the potential games proposed in [13].

Theorem 3.1. *Let G= $(\{U^i\}_{i=1}^m, \{J^i\}_{i=1}^m, \{f^i_t\}_{i=1}^m)$ be the compact form of the games (2.1), (2.2) and $P_t : X \times U \to \mathbb{R}$ a certain function. Then G is a PDG with potential function P_t if, for every $i = 1, \cdots, m$, there exists a function $h^i_t : X^{-i} \times U^{-i} \to \mathbb{R}$ where $U^{-i} := \prod_{j\neq i} U^j$ and $X^{-i} := \prod_{j\neq i} X^j$ such that*

$$g^i_t(x_{t+1}, u_t) = P_t(x_{t+1}, u_t) + h^i_t(x^{-i}_t, u^{-i}_t), \tag{3.1}$$

for all $u_t \in U$.

Proof. In the difference game G, observe that the running payoff

$$\begin{aligned} g^i_t(x_{t+1}, u_t) &= \beta^t_i(q_i x^i_{t+1} - \frac{c_i}{2}(u^i_t)^2) \\ &= \sum_{i=1}^m \beta^t_i\Big(q_i x^i_{t+1} - \frac{c_i}{2}(u^i_t)^2\Big) + \sum_{j\neq i}\beta^t_j\Big(\frac{c_j}{2}(u^j_t)^2 - q_j x^j_{t+1}\Big). \end{aligned} \tag{3.2}$$

Let the functions P_t and h_t^i be defined by

$$P_t(x_{t+1}, u_t) := \rho_t(x_t, u_t) := \sum_{i=1}^{m} \beta_i^t (q_i x_{t+1}^i - \frac{c_i}{2}(u_t^i)^2) \tag{3.3}$$

and

$$h_t^i(x_t^{-i}, u_t^{-i}) := \sum_{j \neq i} \beta_j^t \left(\frac{c_j}{2}(u_t^j)^2 - q_j x_{t+1}^j \right), \; i = 1, 2, \cdots, m. \tag{3.4}$$

It follows from (3.2) that

$$g_t^i(x_{t+1}, u_t) = P_t(x_{t+1}, u_t) + h_t^i(x_t^{-i}, u_t^{-i})$$

for all $i = 1, \cdots, m$. That is, the expression (3.1) holds for i. In view of *Remark 1*, consider the OCP defined by the function P_t as follows:

$$\max_{(x_t, \mu) \in X \times U} \sum_{t=0}^{\infty} P_t(x_{t+1}, u_t) \tag{3.5}$$

subject to

$$x_{t+1} = \left(1 - \sum_{i=1}^{m} u_t^i \right) x_t + u_t, \tag{3.6}$$

where $x_{t+1} = (x_{t+1}^1, \cdots, x_{t+1}^m)$ and $u_t = (u_t^1, \cdots, u_t^m)$.

Let $u_t^* = (u_t^{1*}, \cdots, u_t^{m*}) \in U$ be an optimal solution (that will be characterized in the next section) of the associated OCPs (3.5)–(3.6) at stage (or step) t with corresponding optimal state $x_{t+1}^* = (x_{t+1}^{1*}, \cdots, x_{t+1}^{m*}) \in X$.

This implies

$$\sum_{t=0}^{\infty} P_t(x_{t+1}^*, u_t^*) \geq \sum_{t=0}^{\infty} P_t(x_{t+1}, u_t).$$

For an arbitrary i fixed, let $u_t^i \neq u_t^{i*}$ be the advertising strategy of firm i. Let $x_{t+1} = (x_{t+1}^1, \cdots, x_{t+1}^m)$ be the current market share given by (3.6) corresponding to (u_t^{-i*}, u_t^i).

The optimality of u_t^* and x_{t+1}^* implies that

$$\sum_{t=0}^{\infty} P_t(x_{t+1}^*, u_t^*) \geq \sum_{t=0}^{\infty} P_t(x_{t+1}, (u_t^i, u_t^{-i*})). \tag{3.7}$$

By adding $\sum_{t=0}^{\infty} h_t^i(x_t^{-i*}, u_t^{-i*})$ to both sides of the inequality (3.7), we obtain

$$\sum_{t=0}^{\infty} g_t^i(x_{t+1}^*, u_t^*) \geq \sum_{t=0}^{\infty} g_t^i(x_{t+1}, (u_t^{-i*}, u_t^i)) \tag{3.8}$$

for all $i = 1, \cdots, m$.

Since (3.8) holds for arbitrary i, we conclude from Definition 2.1 that u_t^* is a Nash equilibrium for the games (2.1) and (2.2).

Consequently the research problems 1 and 2 are now simplified in the sense that, rather than dealing with the complex cases (2.1) and (2.2), we solve both problems in the framework of the associated OCPs (3.5) and (3.6). To this end, we first derive a formula for the optimal solution of the associated OCPs (3.5) and (3.6) at any stage k (as in Nowak's papers [22, 23]).

3.1. The k-step nonsymmetric Nash equilibrium

To simplify notations, we let

$$A_t := 1 - \sum_{i=1}^{m} u_t^i \text{ and } f_t(x_t, u_t) := A_t x_t + u_t, \tag{3.9}$$

so that (3.6) becomes

$$x_{t+1} = f_t(x_t, u_t). \tag{3.10}$$

Observe that for almost all t, the functions P_t and f_t are both continuously differentiable in the interior of $X \times U$, and also have defined and bounded partial derivatives. Moreover, P_t can be expressed in terms of the vectors x_t and u_t since

$$\begin{aligned} P_t(x_{t+1}, u_t) &= \sum_{i=1}^{m} \beta_i^t \left(q_i x_{t+1}^i - \frac{c_i}{2}(u_t^i)^2 \right) \\ &= \sum_{i=1}^{m} \beta_i^t \left(q_i (A_t x_t^i + u_t^i) - \frac{c_i}{2}(u_t^i)^2 \right). \end{aligned} \tag{3.11}$$

Assumption 1. For all $t \in \mathbb{N}$, $A_t \in (0, 1]$.

Lemma 3.2. *Under Assumption 1, the series*

$$\sum_{r=0}^{\infty} \beta_i^r \prod_{s=r}^{1} A_{t+s} \tag{3.12}$$

converges uniformly for all $i = 1, 2, \cdots, m$*, where*

$$\prod_{p=r}^{r_0} A_p := \begin{cases} A_r A_{r-1} A_{r-2} \cdots A_{r_0} & \textit{if } r \geq r_0, \\ 1 & \textit{if } \text{ otherwise.} \end{cases}$$

Proof. For each fixed $t \in \mathbb{N}$, Assumption 1 implies that

$$\prod_{s=r}^{1} A_{t+s} \leq 1$$

for all $r \in \mathbb{N} \cup \{0\}$. This further implies

$$\beta_i^r \prod_{s=r}^{1} A_{t+s} \leq \beta_i^r$$

for all $r \in \mathbb{N} \cup \{0\}$ and $i = 1, 2, \cdots, m$.

Since the geometric series $\sum_{r=0}^{\infty} \beta_i^r$ converges for each i, then the conclusion of the lemma follows from the Weierstrass $M-$test of series convergence.

□

Assumption 2. Let the compact set U be connected, convex, and

$$J(x_0, u_t) = \sum_{t=0}^{\infty} P_t(x_t, u_t) < \infty$$

for each $u_t \in U$.

Given the vector of initial market shares $x_0 \in X$ and $T \in \mathbb{N}$, let

$$J_T(x_0, u_t) := \sum_{t=0}^{T} P_t(x_t, u_t).$$

Theorem 3.3. *Let Assumptions 1 and 2 hold. Then the k-step Markov Nash equilibrium* $u_k^* = (u_k^{1*}, \cdots, u_k^{m*})$ *for the ADGs (2.1) and (2.2) is such that*

$$u_k^{i*} := \frac{q_i \eta_k^i}{c_i}(1 - x_k^{i*}), \ i = 1, 2, \cdots, m,$$

where $\eta_k^i := \sum_{r=0}^{\infty} \beta_i^r \prod_{s=r}^{1} A_{k+s}^*$.

Proof. Let the hypotheses of the theorem hold and $H_k : X \times U \times \mathbb{R}_+^m \to \mathbb{R}$ be the Hamiltonian function of the OCPs (3.5) and (3.6) at each step k given by

$$H_k(x_k, u_k, \lambda_{k+1}) = P_k(x_k, u_k) + \lambda_{k+1} \cdot f_k(x_k, u_k), \tag{3.13}$$

where $\lambda_k = (\lambda_k^1, \cdots, \lambda_k^m)$ is the adjoint sequence at stage $k = 1, 2, \cdots$.

Since Assumption 2 holds and the potential function P_k is concave, then in view of the maximum principle for discrete time OCPs proposed in [11] (see also [2]), the optimal advertising control u_k^* of the OCPs (3.5) and (3.6) must satisfy

$$\begin{aligned} \lambda_k &= \frac{\partial H_k^*}{\partial x} \\ 0 &= \frac{\partial H_k^*}{\partial y}, \end{aligned} \tag{3.14}$$

where

$$H_k^* := H_k(x_k^*, u_k^*, \lambda_{k+1}),$$

while $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ denote the gradients corresponding to the first and the second variables respectively. Note that the convexity of the set U in Assumption 2 is required to guarantee the Eq (3.14) in [11].

For any $k \in \mathbb{N}$, the trajectory x_k^* corresponding to u_k^* starting from ξ $(=x_0^*)$ can be obtained as follows:

$$\begin{aligned} x_1^* &= A_0^* \xi + u_0^* \\ x_2^* &= A_1^* x_1 + u_1^* = A_1^* A_0^* \xi + A_1^* u_0^* + u_1^* \\ &\vdots \\ x_{k+1}^* &= \prod_{p=k}^{0} A_p^* \xi + \sum_{s=0}^{k} \prod_{p=k}^{s+1} A_p^* u_s^*. \end{aligned} \tag{3.15}$$

That is, $f_k(\xi, u_k^*) = \prod_{p=k}^{0} A_p^* \xi + \sum_{s=0}^{k} \prod_{p=k}^{s+1} A_p^* u_s^*$.

Then it follows from (3.14) (using (3.11) and (3.15)) that for each $i = 1, 2, \cdots, m$, we have

$$\begin{aligned} \lambda_k^i &= \beta_i^k q_i \prod_{p=k}^{0} A_p^* + \lambda_{k+1}^i \prod_{p=k}^{0} A_p^* = \prod_{p=k}^{0} A_p^* (\beta_i^k q_i + \lambda_{k+1}^i) \\ u_k^{i*} &= (\beta_i^k c_i)^{-1} (1 - f_{k-1}^i(\xi^i, u_{k-1}^{i*}))(\beta_i^k q_i + \lambda_{k+1}^i), \end{aligned}$$

where $A^*_p = 1 - \sum_{i=1}^m u^{i*}_p$.

Thus

$$u^{i*}_k = \frac{1 - f^i_{k-1}(\xi^i, u^{i*}_{k-1})}{\beta^k_i c_i \prod_{p=k}^1 A^*_p} \lambda^i_k \tag{3.16}$$

for all $i = 1, 2, \cdots, m$.

The formula for the adjoint sequence $\{\lambda_k\}_{k=1}^\infty$ proposed in [2] suggests that

$$\lambda_k = \sum_{s=k}^\infty \frac{\partial}{\partial \xi} P_s(\xi, u^*_s)_{|\xi = x^*_0} = \sum_{s=k}^\infty (\beta^s_1 q_1, \cdots, \beta^s_m q_m) \prod_{p=s}^0 A^*_p. \tag{3.17}$$

Indeed the expression (3.17) satisfies the Eq (3.14) for all $k \in \mathbb{N}$ as shown in [11] using some concept of Gâteaux differentials.

We now apply Lemma 1 and (3.17) on (3.16) to obtain

$$\begin{aligned} u^{i*}_k &= \frac{1 - f^i_{k-1}(\xi^i, u^{i*}_{k-1})}{\beta^k_i c_i \prod_{p=k}^0 A^*_p} \sum_{s=k}^\infty \beta^s_i q_i \prod_{p=s}^0 A^*_p \\ &= \frac{q_i(1 - x^{i*}_{k-1})}{\beta^k_i c_i \prod_{p=k}^0 A^*_p} \left(\beta^k_i \prod_{p=k}^0 A^*_p + \beta^{k+1}_i \prod_{p=k+1}^0 A^*_p + \beta^{k+2}_i \prod_{p=k+2}^0 A^*_p + \cdots \right) \\ &= \frac{q_i}{c_i}(1 - x^{i*}_k)\left(1 + \beta_i A^*_{k+1} + \beta^2_i A^*_{k+2} A^*_{k+1} + \cdots\right) \\ &= \frac{q_i \eta^i_k}{c_i}(1 - x^{i*}_k). \end{aligned}$$

That is,

$$u^*_k = \left(q_1 \eta^1_k (1 - x^{1*}_k)/c_1, \cdots, q_m \eta^m_k (1 - x^{m*}_k)/c_m\right). \tag{3.18}$$

The expression (3.18) is the $k-$step optimal solution of the OCPs (3.5) and (3.6) which according to Definition 2.1, is also the $k-$step Markov Nash equilibrium for the ADGs (2.1) and (2.2).

Remark 2. It is worth noting that the closed-loop optimal solution (3.18) is comparable to the Markov strategies for continuous time duopolistic advertising game obtained in [10], Section 11.1.

3.2. Conditions for overtaking optimality (OO)

To establish these conditions, we first present the following concept of overtaking optimality (OO) with respect to the associated OCPs (3.5) and (3.6).

Definition 3.4. The sequence $\{u^*_t\}_{t=1}^\infty$ is said to be OO if for all $\varepsilon > 0$ and $\{u_t\}_{t=1}^\infty \in U$, there exists $T^* = T^*(\varepsilon, u_t) > 0$ such that for all $T > T^*$ holds

$$J_T(x_0, u_t) - J_T(x_0, u^*_t) \leq \varepsilon,$$

which is equivalent to saying that

$$\liminf_{T\to\infty} [J_T(x_0, u^*_t) - J_T(x_0, u_t)] \geq 0. \tag{3.19}$$

Assumption 3. For almost all $t > 0$, there exists $T^* = T^*(t) > 0$ such that the Hamiltonian H_τ is concave in $(x_{\tau+1}, u_\tau)$ for each $\tau \geq T^*$, where we take λ_τ as defined in (3.17).

Let the brackets $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors.

Theorem 3.5. *Under Assumptions 2 and 3, the optimal control u_t^* with corresponding trajectory x_{t+1}^* is an OO for the associated OCPs (3.5) and (3.6) if for each $x_{t+1} \in X$ the following holds:*

$$\liminf_{\tau\to\infty} \sum_{t=1}^{\tau} \langle \lambda_t - \lambda_{t+1}, x_{t+1}^* - x_{t+1} \rangle \geq 0. \tag{3.20}$$

Proof. Consider the optimal pair $(x_{t+1}^*, u_t^*) \in X \times U$ obtained in Theorem 3.3. The concavity of H_t in Assumption 3 implies

$$H_t - H_t^* \leq \left\langle \frac{\partial}{\partial x} H_t^*, x_{t+1} - x_{t+1}^* \right\rangle + \left\langle \frac{\partial}{\partial y} H_t^*, u_t - u_t^* \right\rangle \tag{3.21}$$

for all $(x_{t+1}, u_t) \in X \times U,\ t = 1, 2, \cdots, \tau$.

Then for any admissible pair $(x_{t+1}, u_t) \in X \times U$, we have from (3.13) that

$$\begin{aligned} J_\tau(x_0, u_t^*) - J_\tau(x_0, u_t) &= \sum_{t=1}^{\tau} [P_t(x_{t+1}^*, u_t^*) - P_t(x_{t+1}, u_t)] \\ &= \sum_{t=1}^{\tau} [H_t^* - H_t] + \sum_{t=1}^{\tau} \langle \lambda_{t+1}, f_t(x_t, u_t) - f_t(x_t^*, u_t^*) \rangle. \end{aligned} \tag{3.22}$$

Applying (3.14) and (3.21) on (3.22), and rearranging some terms yields

$$\begin{aligned} J_\tau(x_0, u_t^*) - J_\tau(x_0, u_t) &\geq \sum_{t=1}^{\tau} \left\langle \frac{\partial}{\partial x} H_t^*, x_{t+1}^* - x_{t+1} \right\rangle + \sum_{t=1}^{\tau} \left\langle \frac{\partial}{\partial y} H_t^*, u_t^* - u_t \right\rangle \\ &\quad + \sum_{t=1}^{\tau} \langle \lambda_{t+1}, f_t(x_t, u_t) - f_t(x_t^*, u_t^*) \rangle \\ &= \sum_{t=1}^{\tau} \langle \lambda_t, x_{t+1}^* - x_{t+1} \rangle + \sum_{t=0}^{\tau} \langle \lambda_{t+1}, x_{t+1} - x_{t+1}^* \rangle \\ &= \sum_{t=1}^{\tau} \langle \lambda_t - \lambda_{t+1}, x_{t+1}^* - x_{t+1} \rangle. \end{aligned} \tag{3.23}$$

Hence

$$\liminf_{\tau\to\infty} [J_\tau(x_0, u_t^*) - J_\tau(x_0, u_t)] \geq 0.$$

The last inequality follows from (3.20) and (3.23). Thus, the optimal solution $u_t^* = (u_t^{1*}, \cdots, u_t^{m*})$ (which is also a Markov Nash equilibrium) with corresponding optimal path x_{t+1}^* is an overtaking optimal for the associated OCPs (3.5) and (3.6) of the ADGs (2.1) and (2.2).

Remark 3. Conditions for OO similar to our Theorem 3.5 is obtained for continuous-time OCPs in [3] where the author considered a Cauchy-type formula for the adjoint function $\lambda(\cdot)$. Here, we obtain the same result in discrete-time settings. More precisely, we obtain a discrete-time version of the adjoint formula (3.17) using the concept of Gâteaux differentials and obtain the conditions for OO using the concavity of the Hamiltonian function H_t and the discrete-time maximum principle (3.14).

3.3. Asymptotic turnpike property of the overtaking NE

To study the asymptotic behaviour of the overtaking optimal pair (x^*_{t+1}, u^*_t), we transform the OCP to a dynamic programming problem (described in terms of the state evolution) that is consistent with the standard approach to overtaking optimality as follows.

From the OCPs (3.5) and (3.6), observe that for all $t = 1, 2, \cdots$,

$$u^i_t = \frac{x^i_{t+1} + (\Sigma_{-i_t} - 1)x^i_t}{1 - x^i_t} \quad \text{for all } i = 1, 2, \cdots, m, \tag{3.24}$$

where $\Sigma_{-i_t} := \sum_{j \neq i} u^j_t$.

Then the potential function P_t can be rewritten as

$$P_t(x_t, x_{t+1}) = \sum_{i=1}^{m} \beta^t_i \left(q_i x^i_{t+1} - \frac{c_i}{2} \left(\frac{x^i_{t+1} + (\Sigma_{-i_t} - 1)x^i_t}{1 - x^i_t} \right)^2 \right). \tag{3.25}$$

Consequently, the OCPs (3.5) and (3.6) becomes

$$\max_{(x_t, x_{t+1}) \in \Omega} \sum_{t=1}^{\infty} P_t(x_t, x_{t+1}), \tag{3.26}$$

where Ω is a closed subset of $X \times X$. In the terminology of Gale [15] (see also [33]), the sequence of market share $\{x_t\}_{t=1}^{\infty}$ is said to be a *program* for Ω if $(x_t, x_{t+1}) \in \Omega$ for all $t = 1, 2, \cdots$.

In the sequel, we assume equal discount rate $\beta_1 = \cdots = \beta_m = \beta$ and refer to the potential function P_t as in (3.25).

Definition 3.6. The program $\{\bar{x}_t\}_{t=1}^{\infty}$ is called:

i. A stationary (constant) program if $\bar{x}_{t+1} = \bar{x}_t = \bar{x}$ for all t;

ii. An optimal stationary program (OSP) if (i) holds and

$$\bar{x} = \underset{x \in X}{\operatorname{argmax}}\ P_t(x, x) \tag{3.27}$$

for all $t = 1, 2, \cdots$.

Let $\bar{x}$ be a unique solution of the maximization problem (3.27).

Definition 3.7. A program $\{x_t\}_{t=1}^{T}$ is said to be good if the sequence

$$\left\{ \sum_{t=1}^{T} [P_t(x_t, x_{t+1}) - P_t(\bar{x}, \bar{x})] \right\}_{T=1}^{\infty} \tag{3.28}$$

is bounded or if

$$\liminf_{T \to \infty} \sum_{t=1}^{T} [P_t(x_t, x_{t+1}) - P_t(x_t', x_{t+1}')] > -\infty \tag{3.29}$$

for any other program $\{x_t'\}_{t=1}^{T}$.

In the classical turnpike theory for OCPs, the objective function P_t is said to possess the *turnpike property* if there exists a point $\bar{x} \in X$ (a turnpike) such that for each positive number ε there exists an integer $L \geq 1$ such that for each integer $T \geq 2L$ and each solution $\{x_t\}_{t=0}^T$ of the OCPs (3.5) and (3.6), the inequality $\rho(x_t, \bar{x}) \leq \varepsilon$ is true for all $t = L, \cdots, T-L$ (where (X, ρ) is a compact metric space, and L does not depend on x_0 and x_T) [32]. Moreover, if the turnpike $\bar{x}$ is a singleton (unique) then it is calculated as a solution of the maximization problem (3.27).

As noted in [33], some of the basic tools required to establish this property are: the compactness of the state space X, convexity of Ω and the (strict) concavity of the the objective function P_t (see e.g. [15, 20, 27, 31]). The author [33] further described the asymptotic turnpike property of a program $\{x_t\}$ using the notion of a "good" program. That is, the asymptotic turnpike property is established if the program $\{x_t\}$ is good and converges to the turnpike $\bar{x}$ as T tends to infinity (see [33] Section 1.2, property (iii)). Brock [4] also established this property for the average utility functions using the concept of good programmes and termed it "average turnpike property" (see [4], Lemma 4).

Equipped with the aforementioned tools and the fact that the OO program $\{x_t^*\}_{t=1}^T$ is a good program (in view of (3.19)), to establish the asymptotic turnpike property we show the existence of the turnpike $\bar{x}$ and the convergence of the sequence $\{x_t^*\}_{t=1}^T$ to $\bar{x}$ as T tends to infinity.

Note that $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_m)$ solves (3.27) if and only if $\bar{x}_i$ solves

$$x_i^3 - 3x_i^2 + (3 + \Sigma_{-i_t}^2 c_i/q_i)x_i - 1 = 0, \quad i = 1, 2, \cdots, m. \tag{3.30}$$

The cubic equation (3.30) is obtained by applying the first-order conditions for maximum on (3.27).

Lemma 3.8. *There exists a unique real solution*

$$\bar{x}_i = 1 + \frac{\kappa_i}{\sqrt[3]{18}} - \frac{\sqrt[3]{\frac{2}{3}}\Sigma_{-i_t}^2 c_i}{\kappa_i q_i}, \quad i = 1, 2, \cdots, m, \tag{3.31}$$

where

$$\kappa_i := \sqrt[3]{\sqrt{3}\sqrt{4(\Sigma_{-i_t}^2 c_i/q_i)^3 + 27(\Sigma_{-i_t}^2 c_i/q_i)^2} - 9(\Sigma_{-i_t}^2 c_i/q_i)}$$

for all $t = 1, 2, \cdots,$ *to the cubic equation (3.30).*

Proof. The solution (3.31) can be analytically derived using the Cardano's method [7].

Let c_i and q_i be chosen such that $\bar{x}_i \in (0, 1)$ for all $i = 1, 2, \cdots, m$, then $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_m)$ is the unique solution of (3.27).

The next two lemmas are crucial in proving the convergence of the OO program. Results similar to Lemma 3.9 below is proved in McKenzie [21] and Dana and Le Van [9] for bounded objective functions, and in Nowak [23] for unbounded objective functions.

Assumption 4. The components of the vector

$$\frac{\partial}{\partial x} P_t(\bar{x}, \bar{x}) - \frac{\partial}{\partial y} P_t(\bar{x}, \bar{x})$$

are all positive.

Lemma 3.9. *Under Assumption 4, let*

$$\zeta := \frac{1}{2\beta^t}\left(\frac{\partial}{\partial x}P_t(\bar{x},\bar{x}) - \frac{\partial}{\partial y}P_t(\bar{x},\bar{x})\right). \tag{3.32}$$

Then for all $(x_t, x_{t+1}) \in \Omega$ *holds*

$$P_t(x_t, x_{t+1})/\beta^t + \langle \zeta, x_{t+1} - x_t \rangle \leq \bar{P}, \tag{3.33}$$

where $\bar{P} = P_t(\bar{x},\bar{x})/\beta^t$.

Proof. Define

$$\phi(x,y) := P_t(x,y)/\beta^t + \langle \zeta, y - x \rangle.$$

for all $(x,y) \in \Omega$.

Observe that

$$\begin{aligned}\frac{\partial}{\partial x}\phi(x,y) &= \frac{1}{\beta^t}\frac{\partial}{\partial x}P_t(x,y) - \zeta \\ &= \frac{1}{\beta^t}\left(\frac{\partial}{\partial x}P_t(x,y) - \frac{1}{2}\frac{\partial}{\partial x}P_t(\bar{x},\bar{x}) + \frac{1}{2}\frac{\partial}{\partial y}P_t(\bar{x},\bar{x})\right).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial}{\partial x}\phi(\bar{x},\bar{x}) &= \frac{1}{2\beta^t}\left(\frac{\partial}{\partial x}P_t(\bar{x},\bar{x}) + \frac{\partial}{\partial y}P_t(\bar{x},\bar{x})\right) \\ &= \frac{1}{\beta^t}\frac{\partial}{\partial x}P_t(\bar{x},\bar{x}) = 0.\end{aligned}$$

Similarly, $\frac{\partial}{\partial y}\phi(\bar{x},\bar{x}) = 0$.

This implies that $\bar{x}$ is also a candidate for the maximizer of ϕ. Since X is compact and ϕ is continuous (which implies boundedness of ϕ), then it follows from the concavity of ϕ that the maximizer is unique. Hence we must have

$$\phi(x,y) \leq \bar{P}$$

for all $(x,y) \in \Omega$. □

Lemma 3.10. *There exists a number* M *such that for any program* $\{x_t\}_{t=1}^T$ *starting from* x_0,

$$\sum_{t=1}^{T}[P_t(x_t, x_{t+1})/\beta^t - \bar{P}] \leq M$$

for all T.

Proof. The proof follows directly from Lemma 3.9. That is, for all T we have

$$\begin{aligned}\sum_{t=1}^{T}[P_t(x_t, x_{t+1})/\beta^t - \bar{P}] &\leq \left\langle \zeta, \sum_{t=1}^{T}[x_t - x_{t+1}]\right\rangle \\ &= \langle \zeta, x_0 - x_T \rangle \\ &\leq M,\end{aligned}$$

where $M = \langle \zeta, x_0 \rangle$.

□

Now, let P_t be concave at the interior point $(\bar{x}, \bar{x})$ and define

$$\delta_t := \bar{P} - P_t(x_t^*, x_{t+1}^*)/\beta^t + \langle \zeta, x_t^* - x_{t+1}^* \rangle.$$

From Lemma 3.8, we have $\delta_t \geq 0$ for all $t = 1, 2, \cdots, T$. Note that

$$P_t(x_t^*, x_{t+1}^*)/\beta^t - \bar{P} = \langle \zeta, x_t^* - x_{t+1}^* \rangle - \delta_t. \tag{3.34}$$

Summing (3.34) on t and using the fact that $\{x_t^*\}$ is a good program, then there exists a number N and a positive integer T_N such that for all $T \geq T_N$ we have

$$\begin{aligned} N &\leq \sum_{t=1}^{T} [P_t(x_t^*, x_{t+1}^*)/\beta^t - \bar{P}] \\ &= \langle \zeta, \sum_{t=1}^{T} [x_t^* - x_{t+1}^*] \rangle - \sum_{t=1}^{T} \delta_t \\ &\leq \langle \zeta, x_0^* \rangle - \sum_{t=1}^{T} \delta_t. \end{aligned}$$

This implies

$$0 \leq \sum_{t=1}^{T} \delta_t \leq \langle \zeta, x_0^* \rangle - N$$

for all $T \geq T_N$. That is, the series $\sum_{t=1}^{\infty} \delta_t$ is convergent (which implies $\lim_{t\to\infty} \delta_t = 0$).

Next we define

$$\Theta(x_t^*, x_{t+1}^*) := P_t(x_t^*, x_{t+1}^*)/\beta^t - \bar{P} + \langle \zeta, x_{t+1}^* - x_t^* \rangle.$$

It is easy to see from Lemma 3.9 that the maximum value of Θ is zero. Moreover, Θ is strictly concave (from concavity of P_t). Hence, $(\bar{x}, \bar{x})$ is the unique maximum point of Θ and $\Theta(\bar{x}, \bar{x}) = 0$.

Since

$$\lim_{t\to\infty} \Theta(x_t^*, x_{t+1}^*) = -\lim_{t\to\infty} \delta_t = 0.$$

Then it follows from the continuity of Θ that $\lim_{t\to\infty} x_t^* = \bar{x}$.

4. An example

Consider the ADGs (2.1) and (2.2) with $m = 3$. Let the firms discount rate $(\beta_1, \beta_2, \beta_3)$, revenue per unit market shares (q_1, q_2, q_3), and advertising costs (c_1, c_2, c_3) be given by $(0.1, 0.12, 0.13), (0.59, 0.6, 0.63)$, and $(1, 1, 1)$ respectively. Suppose the initial market shares $x_0 = (0.49, 0.5, 0.52)$, then the Theorem 3.3 suggests that the $k-$step Markov Nash equilibrium

$$(u_k^{1*}, u_k^{2*}, u_k^{3*}) = \left(0.59^k \eta_k^1 (1 - x_k^{1*}),\ 0.6^k \eta_k^2 (1 - x_k^{2*}),\ 0.63^k \eta_k^3 (1 - x_k^{3*})\right),$$

where η_k^i is the sum of the series

$$1 + \beta_i A_{k+1} + \beta_i^2 A_{k+2} A_{k+1} + \cdots,$$

for each $i = 1, 2, 3$.

Consequently, the corresponding market shares of the firms become

$$x_{k+1}^{1*} = 0.49 \prod_{p=k}^{0} A_p^* + \sum_{s=0}^{k} \prod_{p=k}^{s+1} A_p^* \eta_s^1 (1 - x_s^{1*}) 0.59^s;$$

$$x_{k+1}^{2*} = 0.5 \prod_{p=k}^{0} A_p^* + \sum_{s=0}^{k} \prod_{p=k}^{s+1} A_p^* \eta_s^2 (1 - x_s^{2*}) 0.6^s;$$

$$x_{k+1}^{3*} = 0.52 \prod_{p=k}^{0} A_p^* + \sum_{s=0}^{k} \prod_{p=k}^{s+1} A_p^* \eta_s^3 (1 - x_s^{3*}) 0.63^s.$$

In view of Theorem 3.1, the potential function here takes the form

$$\begin{aligned} P_t(x_{t+1}, u_t) = 0.1^t \left(0.59 x_{t+1}^1 - \frac{1}{2}(u_t^1)^2\right) + 0.12^t \left(0.6 x_{t+1}^2 - \frac{1}{2}(u_t^2)^2\right) \\ + 0.13^t \left(0.63 x_{t+1}^3 - \frac{1}{2}(u_t^3)^2\right). \end{aligned}$$

The associated OCP becomes

$$\max_{(x_t, \mu) \in X \times U} \sum_{t=0}^{\infty} P_t(x_{t+1}, u_t)$$

subject to

$$x_{t+1}^i = \left(1 - \sum_{i=1}^{3} u_t^i\right) x_t + u_t^i, \ i = 1, 2, 3.$$

The Hamiltonian function H_k can be obtained from the expression

$$H_k = P_k(x_k, u_k) + (\lambda_{k+1}^1, \lambda_{k+1}^2, \lambda_{k+1}^3)^T \cdot (x_{t+1}^1, x_{t+1}^2, x_{t+1}^3), \tag{4.1}$$

and the adjoint sequence

$$(\lambda_k^1, \lambda_k^2, \lambda_k^3) = \sum_{s=k}^{\infty} (0.1^s (0.59) \prod_{p=s}^{0} A_p^*, \ 0.12^s (0.6) \prod_{p=s}^{0} A_p^*, \ 0.13^s (0.63) \prod_{p=s}^{0} A_p^*).$$

Since H_k is concave in (x_{t+1}^*, u_t^*) with respect to the associated OCP above, then it follows from Theorem 3.5 that the Markov Nash equilibrium strategy $\{(u_t^{1*}, u_t^{2*}, u_t^{3*})\}_{t=1}^{\infty}$ is indeed an overtaking optimal provided the

$$\liminf_{\tau \to \infty} \sum_{t=1}^{\tau} \sum_{i=1}^{3} (\lambda_t^i - \lambda_{t+1}^i)(x_{t+1}^{i*} - x_{t+1}^i) \geq 0$$

for each $x_{t+1}^i \in X, \ i = 1, 2, 3$.

5. Conclusions

We have studied a discrete-time game of advertising competition among $m \geq 2$ firms where firms maximize the present value of their profits which depend on their advertising strategy and their market share. The evolution of market shares is determined by the firms' advertising activities. By first identifying the advertising game as a potential game, an optimal control problem is obtained whose optimal solution is a Nash equilibrium of the game. We derived an explicit formula for the Nash equilibrium (NE) in Theorem 3.3 and obtained conditions for which the NE is an overtaking optimal (see Theorem 3.5). Lastly, we analyze the asymptotic behavior (that is, the turnpike property) of the overtaking NE where the convergence towards a unique steady state (turnpike) is established.

Abbreviations

ADG, Advertising difference games; PDG, potential difference game; OO, overtaking optimality; NE, Nash equilibrium.

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Conflict of interests

The authors declare that they have no competing interests.

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