Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 455–466.
DOI: 10.3934/math.2022029
Received: 26 July 2021
Accepted: 02 October 2021
Published: 13 October 2021

Research article

A D-N alternating algorithm for exterior 3-D problem with ellipsoidal artificial boundary

Xuqiong Luo[1,2,*]

[1] School of Mathematics and Statistics, Changsha University of Science and Technology, Hunan, Changsha 410114, China

[2] College of Liberal Arts and Sciences, National University of Defense Technology, Hunan, Changsha 410073, China

* **Correspondence:** Email: luoxuqiong@163.com; Tel: +8613973181521.

Abstract: In this study, based on a general ellipsoidal artificial boundary, we present a Dirichlet-Neumann (D-N) alternating algorithm for exterior three dimensional (3-D) Poisson problem. By using the series concerning the ellipsoidal harmonic functions, the exact artificial boundary condition is derived. The convergence analysis and the error estimation are carried out for the proposed algorithm. Finally, some numerical examples are given to show the effectiveness of this method.

Keywords: D-N alternating algorithm; exterior 3-D poisson problem; ellipsoidal artificial boundary; natural boundary reduction
Mathematics Subject Classification: 65N38, 65N55

1. Introduction

In this paper, we consider the following exterior Dirichlet problem:

$$\begin{cases} -\Delta u = f(\mathbf{x}), & \text{in } \Omega, \\ u = g, & \text{on } \Gamma_0, \\ u \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{1.1}$$

where $\Omega_0 \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and $\Gamma_0 = \partial\Omega_0$ be its boundary, $\Omega = R^3 \backslash \overline{\Omega}_0$. Δ is the Laplace operator. Assume that the given function g satisfies $g \in H^{\frac{1}{2}}(\Gamma_0)$ and $f(\mathbf{x}) \in L^2(\Omega)$ has the compact supported set.

It is well known that partial differential equation boundary value problems with bounded or unbounded domains widely arise in many fields of scientific and engineering. For partial differential equation boundary value problems with bounded domain, the commonly finite element method (FEM) [1] and finite difference method (FDM) [2] were used to solve these problems. Meanwhile, for unbounded domains partial differential equation, boundary element method (BEM) [3–5], natural boundary element method (NBEM) [6, 7], artificial boundary method [8–13], and the coupling of NBEM (or BEM) and FEM [14, 15] were all efficient numerical methods. Both the methods on bounded domains and the methods on unbounded domains, they all have their own advantages and disadvantages.

It should be pointed out that the coupling method of FEM and BEM proposed by Feng and Yu [6,7] is fully compatible with the FEM. It is easy to be implemented on the computer and can be coupled with FEM naturally and directly by natural boundary reduction (NBR). The coupling of FEM and NBEM was first applied to solve two dimensional (2-D) unbounded domain of the elliptic problems [16, 17]. A circle [11] or an ellipse [12, 16] was usually selected as the artificial boundary for exterior 2-D problems. For exterior 3-D problems, a sphere was usually selected as the artificial boundary [12, 18, 19]. For some special shapes, such as cigar-shaped, rotating ellipsoid boundary, some special artificial boundary methods [20–23] are given. Since it leads to a smaller computational domain and does not increase the computational cost of the stiffness matrix from boundary reduction, it shows that these methods are very efficient. In the last decade, mesh-less methods which involve the radial basis functions (RBF) are also used to solve higher dimensions exterior problems. The motivations is that such functions or kernels are easily applicable and implementable in higher order partial differential equations, independent from geometry, etc [24–26].

In this paper, based on FEM and NBEM, a Dirichlet-Neumann (D-N) alternating algorithm has been devised to solve exterior 3-D Poisson problem in an infinite region with an ellipsoidal boundary. The standard procedure of the method is described as follows: Firstly, let Ω be an unbounded domain with boundary Γ_0. Then we introduce an artificial boundary Γ_1, where Γ_0 is surrounded by Γ_1. We denote the bounded domain between Γ_0 and Γ_1 as Ω_1 , and Ω_2 is the unbounded domain with boundary Γ_1. Secondly, in Ω_2, we use the NBEM to solve this problem. Furthermore, we use FEM to solve this problem in Ω_1. Finally, on the interface between Ω_1 and Ω_2, a sequence of boundary conditions is generated iteratively until convergence to the solution of the original problem.

The outline of the paper is as follows: In Section 2, the original domain Ω is divide two non-overlapping regions by an ellipsoidal artificial boundary Γ_1 and the corresponding D-N alternating algorithm of the problem is constructed. In Section 3 and Section 4, the convergence analysis of the algorithm and the error estimates were given, respectively. In Section 5, two numerical examples are carried out to demonstrate the effectiveness and accuracy of this method. Finally, we state the conclusions in Section 6.

2. A D-N alternating algorithm based on NBR

Let $\Gamma_1 = \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, a > b > c > 0\}$ denote an ellipsoid. Then an unbounded domain outside the boundary Γ_1 is Ω_2.

Let Γ_0 be a surface inside the Γ_1, and $\text{dist}(\Gamma_1, \Gamma_0) > 0$. Then for problem (1.1), Ω is divided into two non-overlapping subdomains Ω_1 and Ω_2.

Let Ω_1 be the bounded domain among Γ_0 and Γ_1, and Ω_2 be the unbounded domain outside Γ_1. Then the original problem (1.1) is decomposed into two subproblems over subdomains Ω_1 and Ω_2.

Firstly, based on ellipsoidal system of coordinates $(\lambda_1, \lambda_2, \lambda_3)$ which is defined in [27] and Γ_1 coincides with the ellipsoid $\lambda_1 = a$, the Cartesian coordinates (x, y, z) related to the ellipsoidal coordinates $(\lambda_1, \lambda_2, \lambda_3)$ can be expressed as follows:

$$\begin{cases} x^2 = \frac{\lambda_1^2\lambda_2^2\lambda_3^2}{h^2k^2}, \\ y^2 = \frac{(\lambda_1^2-h^2)(\lambda_2^2-h^2)(h^2-\lambda_3^2)}{h^2(k^2-h^2)}, \\ z^2 = \frac{(\lambda_1^2-k^2)(k^2-\lambda_2^2)(k^2-\lambda_3^2)}{k^2(k^2-h^2)}, \end{cases} \tag{2.1}$$

where $h^2 = a^2 - b^2$ and $k^2 = a^2 - c^2$.

Secondly, the Laplace's equation $\Delta u = 0$ in the following problem:

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega_2, \\ u = u_1(\mathbf{x}), & \text{on } \Gamma_1, \\ u(\mathbf{x}) \to 0 & \text{as } |\mathbf{x}| \to \infty \end{cases} \tag{2.2}$$

can be expressed in terms by the ellipsoidal coordinates $(\lambda_1, \lambda_2, \lambda_3)$. In the orthogonal set of coordinates $(\lambda_1, \lambda_2, \lambda_3)$, the Laplace's equation is expressed as in [27]

$$\begin{aligned} &(\lambda_2^2 - \lambda_3^2)\sqrt{(\lambda_1^2 - h^2)(\lambda_1^2 - k^2)}\frac{\partial}{\partial\lambda_1}(\sqrt{(\lambda_1^2 - h^2)(\lambda_1^2 - k^2)}\frac{\partial u}{\partial\lambda_1}) \\ &+(\lambda_1^2 - \lambda_3^2)\sqrt{(\lambda_2^2 - h^2)(k^2 - \lambda_1^2)}\frac{\partial}{\partial\lambda_2}(\sqrt{(\lambda_2^2 - h^2)(k^2 - \lambda_1^2)}\frac{\partial u}{\partial\lambda_2}) \\ &+(\lambda_1^2 - \lambda_2^2)\sqrt{(h^2 - \lambda_3^2)(k^2 - \lambda_1^2)}\frac{\partial}{\partial\lambda_3}(\sqrt{(h^2 - \lambda_3^2)(k^2 - \lambda_1^2)}\frac{\partial u}{\partial\lambda_2}) = 0. \end{aligned}$$

In exterior Ω_2, the normal solution of the Laplace's equation [27] is

$$u(\lambda_1, \lambda_2, \lambda_3) = E_1(\lambda_1)E_2(\lambda_2)E_3(\lambda_3).$$

As we know, the functions E_1–E_3 must satisfy Lamé's equation

$$(\lambda_i^2 - h^2)(\lambda_i^2 - k^2)\frac{\mathrm{d}^2E(\lambda_i)}{\mathrm{d}\lambda_i^2} + \lambda_i(2\lambda_i^2 - h^2 - k^2)\frac{\mathrm{d}E(\lambda_i)}{\mathrm{d}\lambda_i} + (p - q\lambda_i^2)E(\lambda_i) = 0,$$

where p and q are called the separation parameters.

Then, the Laplace equation's analytic solution over the domain Ω_2 is

$$u(\lambda_1, \lambda_2, \lambda_3) = \mathcal{P}u_1 \doteq \sum_{n=0}^{+\infty}\sum_{p=1}^{2n+1}\frac{U_n^p}{\sqrt{\gamma_n^p}}\frac{F_n^p(\lambda_1)}{F_n^p(a)}E_n^p(\lambda_2)E_n^p(\lambda_3), \quad \lambda_1 \geq a, \tag{2.3}$$

where

$$U_n^p = \int_{\Gamma_1} u(a, \lambda_2, \lambda_3)\frac{E_n^p(\lambda_2)E_n^p(\lambda_3)}{\sqrt{(a^2 - \lambda_2^2)(a^2 - \lambda_3^2)}}\frac{\mathrm{d}s}{\sqrt{\gamma_n^p}}, \quad \gamma_n^p = \int_{\Gamma_1}\frac{(E_n^p(\lambda_2)E_n^p(\lambda_3))^2}{\sqrt{(a^2 - \lambda_2^2)(a^2 - \lambda_3^2)}}\,\mathrm{d}s,$$

and $\mathrm{d}s$ is the surface element about the Cartesian coordinates on Γ_1. Let $E_n^p(\lambda)$ denote n order Lamé functions of the first kind with the eigenvalue p. As suggested by the Lamé function of the first kind, there exist the Lamé functions of the second kind. The Lamé's functions of the second kind are then defined by

$$F_n^p(\lambda_1) = (2n+1)E_n^p(\lambda_1)\int_{\lambda_1}^{\infty} \frac{\mathrm{d}s}{(E_n^p(s))^2\sqrt{(s^2-k^2)(s^2-h^2)}}$$

such that $F_n^p(\lambda_1) \to 0$ when $\lambda_1 \to \infty$ and they have the same eigenvalue p.

The expression (2.3) is called the Poisson integral formula. From (2.3) and the Lemma 1 of Huang [21], an expression of the normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ can be obtained. If we set $\lambda_1 = a$, the exact artificial boundary condition is

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\Gamma_1} = \mathcal{K}u_1 \doteq -bc\sum_{n=0}^{+\infty}\sum_{p=1}^{2n+1}\frac{U_n^p}{\sqrt{\gamma_n^p}}\frac{\frac{\mathrm{d}F_n^p(a)}{\mathrm{d}\lambda_1}}{F_n^p(a)}\frac{E_n^p(\lambda_2)E_n^p(\lambda_3)}{\sqrt{(a^2-\lambda_2^2)(a^2-\lambda_3^2)}}. \tag{2.4}$$

Here, the expression (2.4) is also called the natural integral equation. So we can obtain the solution of problem (2.2) directly from (2.3).

Based on (2.3) and (2.4), we construct a D-N alternating algorithm as follows:

Step 1. Put $k = 0$, and pick an initial value $\lambda^0 \in H^{\frac{1}{2}}(\Gamma_1)$.

Step 2. In exterior domain Ω_2, solve a Dirichlet boundary value problem:

$$\begin{cases} -\Delta u_2^k = f(\mathbf{x}), & \text{in } \Omega_2, \\ u_2^k = \lambda^k, & \text{on } \Gamma_1. \end{cases} \tag{2.5}$$

Step 3. In interior domain Ω_1, solve a mixed boundary value problem:

$$\begin{cases} -\Delta u_1^k = f(\mathbf{x}), & \text{in } \Omega_1, \\ \frac{\partial u_1^k}{\partial \mathbf{n}} = -\frac{\partial u_2^k}{\partial \mathbf{n}}, & \text{on } \Gamma_1, \\ u_1^k = g(\mathbf{x}), & \text{on } \Gamma_0. \end{cases} \tag{2.6}$$

Step 4. On Γ_1, update the boundary value by

$$\lambda^{k+1} = \theta_k u_1^k + (1-\theta_k)\lambda^k, \quad 0 < \theta_k < 1.$$

Step 5. Put $k = k + 1$, go to Step 2.

For the D-N alternating algorithm, we need know the relaxation factor θ_k, which is called a suitable real number. The algorithm's main work is on the Step 3, where a mixed boundary value problem in bounded domain Ω_1 is solved by standard finite element method. In order to solve the Dirichlet problem with exterior domain Ω_2 in the Step 2, we can use the normal derivative of the solution of the problem (2.5) on Γ_1. Similarly, $\frac{\partial u_2^k}{\partial \mathbf{n}}$ can be found from λ^k by using the natural integral equation (2.4).

3. Analysis of convergence

Let

$$V_g(\Omega_1) = \{v \mid v \in H^1(\Omega_1), v|_{\Gamma_0} = g(\mathbf{x})\}, \quad V_0(\Omega_1) = \{v \mid v \in H^1(\Omega_1), v|_{\Gamma_0} = 0\}.$$

Then the following coupled variational problem is equivalent to the problem (1.1).

$$\begin{cases} \texttt{Find } u \in V_g(\Omega_1) \texttt{ such that,} \\ D_1(u,v) + \widehat{D}_2(u,v) = \int_{\Omega_1} fv\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \forall\, v \in V_0(\Omega_1), \end{cases} \tag{3.1}$$

where $D_1(u,v) = \int_{\Omega_1} \nabla u \cdot \nabla v \mathrm{d}x\mathrm{d}y\mathrm{d}z$. From [20], we know

$$\widehat{D}_2(u,v) = \int_{\Omega_2} \nabla u \cdot \nabla v \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{\Gamma_1} v\frac{\partial u}{\partial \mathbf{n}}\mathrm{d}s = -bc\sum_{n=0}^{\infty}\sum_{p=1}^{2n+1} \frac{\frac{\mathrm{d}F_n^p(a)}{\mathrm{d}\lambda}}{F_n^p(a)} U_n^p V_n^p, \tag{3.2}$$

where

$$V_n^p = \int_{\Gamma_1} v(a,\lambda_2,\lambda_3)\frac{E_n^p(\lambda_2)E_n^p(\lambda_3)}{\sqrt{(a^2-\lambda_2^2)(a^2-\lambda_3^2)}}\frac{\mathrm{d}s}{\sqrt{\gamma_n^p}}.$$

Let $S_h(\Omega_1) \subset V_0(\Omega_1)$ be an linear finite element space of $V_0(\Omega_1)$, and subdivide Ω_1 into hexahedrons or tetrahedrons. Then, the approximation variational problem of (3.1) can be written as:

$$\begin{cases} \texttt{Find } u_h \in S_h(\Omega_1) \texttt{ such that,} \\ D_1(u_h,v_h) + \widehat{D}_2(u_h,v_h) = \int_{\Omega_1} fv_h\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \forall\, v_h \in S_h(\Omega_1). \end{cases} \tag{3.3}$$

From the problem (3.3), we can obtain the algebraic equations as follow:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} + K_h \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{3.4}$$

where U and V are vectors whose components are function values at interior nodes of Ω_1 and nodes on Γ_1, respectively.

From FEM in Ω_1, we can get the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

which is a stiffness matrix. From the NBEM on Γ_1, we can obtain the K_h. The expression (3.4) can also be rewritten as follow:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - K_h V \end{pmatrix}. \tag{3.5}$$

Then, the iterative algorithm can be given as follow:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} U_k \\ V_k \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - K_h\Lambda_k \end{pmatrix} \tag{3.6}$$

with

$$\Lambda_{k+1} = \theta_k V_k + (1-\theta_k)\Lambda_k, \;\; k = 0, 1, \cdots . \tag{3.7}$$

Since A is a symmetric and positive definite matrix, we know that A_{11}^{-1} is nonsingular. From the expression (3.4), we know

$$(A_{22} - A_{21}A_{11}^{-1}A_{12} + K_h)V = b_2 - A_{21}A_{11}^{-1}b_1. \tag{3.8}$$

Let

$$\overline{b}_2 = b_2 - A_{21}A_{11}^{-1}b_1, \;\; S_h^{(1)} = A_{22} - A_{21}A_{11}^{-1}A_{12}, \;\; S_h = S_h^{(1)} + K_h,$$

then the expression (3.8) can also be rewritten $S_h \cdot V = \overline{b}_2$, where S_h is the discrete analogue of the Steklov-Poincaré operator on Γ_1. Therefore, preconditioned Richardson iteration is constructed as follow:

$$S_h^{(1)}(\Lambda_{k+1} - \Lambda_k) = \theta_k(\overline{b}_2 - S_h\Lambda_k). \tag{3.9}$$

Theorem 3.1. *[23] The discrete D-N alternating algorithms (3.6), (3.7) and the iteration (3.9) are equivalent.*

Theorem 3.2. *[23] The condition number of the iterative matrix $(S_h^{(1)})^{-1}S_h$ of the discrete D-N alternating algorithm is independent of the finite element mesh size h.*

Theorem 3.3. *[23] If $0 < \min\theta_k \leq \max\theta_k < 1$, then the D-N alternating methods (3.6) and (3.7) are convergent, and the convergence rate is independent of the finite element mesh size h.*

4. Error estimates

Let $V_h(\Omega_1)$ be a piecewise linear finite element space and $\mathcal{N}_h$ be the node set in $\Omega_1 \cup \Omega_0 \cup \Gamma_1$. We denote

$$\begin{aligned} V_g^h(\Omega_1) &= \{v \mid v \in V_h, v(\mathbf{a}) = g(\mathbf{a}), \mathbf{a} \in \mathcal{N}_h \cap \Gamma_0\}, \\ V_0^h(\Omega_1) &= \{v \mid v \in V_h, v(\mathbf{a}) = 0, \mathbf{a} \in \mathcal{N}_h \cap \Gamma_0\}. \end{aligned}$$

The discrete variational problems of the variational problems (3.1) and (3.3) are the followings respectively:

$$\begin{cases} \texttt{Find } u \in V_g^h(\Omega_1) \texttt{ such that,} \\ D_1(u,v) + \widehat{D}_2(u,v) = \int_{\Omega_1} fv\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \;\; \forall\, v \in V_0(\Omega_1), \end{cases} \tag{4.1}$$

and

$$\begin{cases} \texttt{Find } u_h^N \in V_h^N(\Omega_1) \texttt{ such that,} \\ D_1(u_h,v_h) + \widehat{D}_2(u_h,v_h) = \int_{\Omega_1} fv_h\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \;\; \forall\, v_h \in S_h(\Omega_1). \end{cases} \tag{4.2}$$

Similarly, the following error estimation can be obtained from [21].

Theorem 4.1. *Suppose that $u \in H^2(\Omega_1) \cap V_g(\Omega_1)$ and $u^{Nh} \in V_g(\Omega_1)$ are the solutions of problems (3.1) and (4.2) respectively, and $u|_{\Gamma_1} \in H^{\frac{3}{2}}(\Gamma_1)$. Then there exists a positive constant C independent of h and N such that*

$$\|u - u^{Nh}\|_{H^1(\Omega_1)} \leq C\Big\{h\|u\|_{H^2(\Omega_1)} + \frac{1}{N+2}\Big(\frac{a(a_1^2+k^2)}{a_1(a^2+k^2)}\Big)^{N+2}\|u\|_{H^{\frac{3}{2}}(\Gamma_1)}\Big\}.$$

Theorem 4.2. *Suppose that* $u \in H^2(\Omega_1) \cap V_g(\Omega_1)$ *and* $u^{Nh} \in V_g(\Omega_1)$ *are the solutions of problems (3.1) and (4.2) respectively, and* $u|_{\Gamma_1} \in H^{\frac{3}{2}}(\Gamma_1)$*. Then there exists a positive constant C independent of h and N such that*

$$\|u - u^{Nh}\|_{L^2(\Omega_1)} \le C\Big\{h\|u\|_{H^2(\Omega_1)} + \frac{1}{N+2}\Big(\frac{a(a_1^2 + k^2)}{a_1(a^2 + k^2)}\Big)^{N+2}\|u\|_{H^{\frac{3}{2}}(\Gamma_1)}\Big\},$$

where α *is related to the smoothness convexity of the domain* Ω_1 *and the differentiability of the solution of boundary value problem.*

5. Numerical experiments

In this section, two numerical examples are used to show the effectiveness of the D-N alternating algorithm, where the exact solutions of all two examples are known. Let

$$e(k) = \sup_{P_i \in \overline{\Omega}_1} \left| u(P_i) - u_{1h}^{(k)}(P_i) \right|$$

be the maximal error of all node functions on $\overline{\Omega}_1$,

$$e_h(k) = \sup_{P_i \in \overline{\Omega}_1} \left| u_{1h}^{(k+1)}(P_i) - u_{1h}^{(k)}(P_i) \right|$$

be the maximum node-error of the adjacent two-steps on nodes and

$$q_h(k) = \frac{e_h(k-1)}{e_h(k)}$$

be the approximation of the convergence rate. We use three meshes: Mesh I, Mesh II and Mesh III to express our computations. Mesh I is a coarse mesh by a lot of small tetrahedrons. Mesh II is refined from Mesh I in such a way what every tetrahedron of Mesh I is divided into eight "equal" tetrahedron. Mesh III is the refined mesh of Mesh II in the same way.

Example 5.1. *Let* $f(\boldsymbol{x}) = 0$*,* $\Gamma_0 = \{(x, y, z) : \frac{x^2}{6.25} + \frac{y^2}{4.0} + \frac{z^2}{2.25} = 1\}$*, the ellipsoidal artificial boundary* $\Gamma_1 = \{(x, y, z) : \frac{x^2}{6.25} + \frac{y^2}{4.0} + \frac{z^2}{2.25} = a^2, a > 1\}$ *and the exact solution of problem (1.1) is*

$$u(\boldsymbol{x}) = \frac{x}{x^2 + y^2 + z^2}.$$

Taking $g(\boldsymbol{x}) = u(\boldsymbol{x})|_{\Gamma_0}$*. The correspondent results are shown in Table 1, Table 2 and Figure 1.*

Example 5.2. *Let* $f(\boldsymbol{x}) = 0$*,* $\Omega = \{(x, y, z) : |x| \ge 2.5, |y| \ge 2.0, |z| \ge 1.5\}$ *and the ellipsoidal artificial boundary* $\Gamma_1 = \{(x, y, z) : \frac{x^2}{6.25} + \frac{y^2}{4.0} + \frac{z^2}{2.25} = a^2, a > \sqrt{3}\}$*. The exact solution of problem (1.1) be*

$$u(\boldsymbol{x}) = \frac{x}{x^2 + y^2 + z^2}.$$

Taking $g(\boldsymbol{x}) = u(\boldsymbol{x})|_{\Gamma_0}$*. The correspondent results is shown by Table 3, Table 4 and Figure 2.*

In the numerical experiments, we use the $\Gamma_1 = \{(x, y, z) : \frac{x^2}{6.25} + \frac{y^2}{4.0} + \frac{z^2}{2.25} = 4, \}$ as the numerical example 1 and example 2. The radius $a = 2$ just represents the shape of the Γ_1. The larger a is, the larger domain of Γ_1 and Γ_0 enclosed will be. So the more computation time will be spent. On the contrary, The smaller a is, the smaller area of Γ_1 and Γ_0 enclosed will be. So The less computation time is spent for the computer.

Table 1. The relationship between meshes and convergence rate ($N = 20$, $\theta_k = 0.5$ and $a = 2.0$).

Mesh		Number of iteration and corresponding values					
		0	1	2	3	4	5
I	e	1.537E-1	3.762E-2	2.512E-2	2.320E-2	2.290E-2	2.285E-2
	e_h	–	1.161E-1	1.250E-2	1.920E-3	2.958E-4	4.627E-5
	q_h	–	–	8.129	6.512	6.490	6.393
II	e	1.074E-1	1.836E-2	7.971E-3	6.397E-3	6.157E-3	6.120E-3
	e_h	–	8.904E-2	1.039E-2	1.574E-3	2.399E-4	3.670E-5
	q_h	–	–	8.571	6.602	6.561	6.537
III	e	6.107E-2	8.287E-3	2.262E-3	1.354E-3	1.215E-3	1.194E-3
	e_h	–	5.278E-2	6.025E-3	9.075E-4	1.388E-4	2.126E-5
	q_h	–	–	8.760	6.639	6.536	6.529

Table 2. The relationship between θ and convergence rate.

θ		Number of iteration and corresponding values					
		0	1	2	3	4	5
0.2	e	1.074E-1	4.863E-2	3.748E-2	3.466E-2	3.393E-2	3.374E-2
	e_h	–	5.877E-2	1.115E-2	2.821E-3	7.289E-4	1.864E-4
	q_h	–	–	5.270	3.953	3.870	3.910
0.3	e	1.074E-1	3.965E-2	2.891E-2	2.639E-2	2.580E-2	2.566E-2
	e_h	–	6.775E-2	1.074E-2	2.515E-3	5.936E-4	1.414E-4
	q_h	–	–	6.308	4.270	4.237	4.197
0.5	e	1.074E-1	1.836E-2	7.971E-3	6.397E-3	6.157E-3	6.120E-3
	e_h	–	8.904E-2	1.039E-2	1.574E-3	2.399E-4	3.670E-5
	q_h	–	–	8.571	6.602	6.561	6.537
0.6	e	1.074E-1	3.038E-2	1.959E-2	1.765E-2	1.730E-2	1.723E-2
	e_h	–	7.702E-2	1.079E-2	1.936E-3	3.545E-4	6.529E-4
	q_h	–	–	7.136	5.573	5.461	5.430
0.7	e	1.074E-1	4.529E-2	3.504E-2	3.255E-2	3.194E-2	3.179E-2
	e_h	–	6.211E-2	1.025E-2	2.495E-3	6.112E-4	1.520E-4
	q_h	–	–	6.058	4.109	4.082	4.020

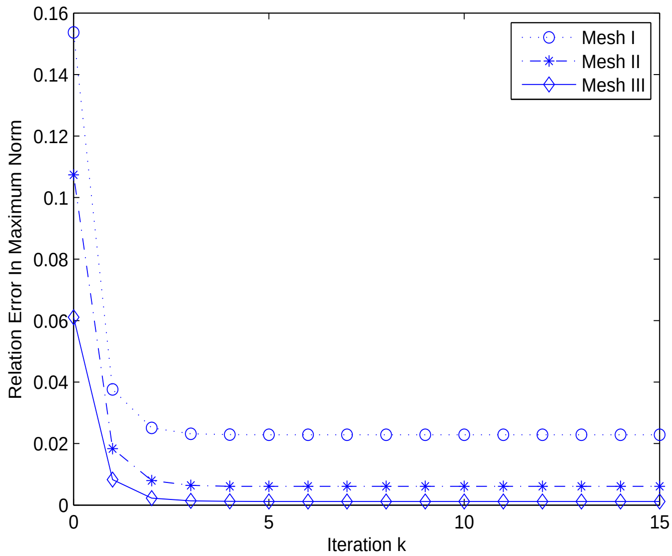

Figure 1. The relationship between maximal errors in relative maximum norm and iteration k about different mesh for Example 5.1.

Table 3. The relationship between meshes and convergence rate ($N = 20$, $\theta_k = 0.5$ and $a = 2.0$).

Mesh		Number of iteration and corresponding values					
		0	1	2	3	4	5
I	e	1.182E-1	2.590E-2	1.241E-2	1.049E-2	1.021E-2	1.017E-2
	e_h	–	9.230E-2	1.349E-2	1.927E-3	2.768E-4	4.015E-5
	q_h	–	–	6.842	7.001	6.962	6.895
II	e	5.626E-2	9.528E-3	3.131E-3	2.231E-3	2.103E-3	2.085E-3
	e_h	–	4.673E-2	6.397E-3	9.004E-4	1.279E-4	1.846E-5
	q_h	–	–	7.305	7.105	7.041	6.927
III	e	1.072E-2	1.875E-3	7.583E-4	6.087E-4	5.884E-4	5.856E-4
	e_h	–	8.845E-3	1.117E-3	1.496E-4	2.028E-5	2.836E-6
	q_h	–	–	7.918	7.468	7.375	7.150

Table 4. The relationship between θ and convergence rate.

θ		Number of iteration and corresponding values					
		0	1	2	3	4	5
0.2	e	5.626E-2	3.429E-2	2.817E-2	2.647E-2	2.599E-2	2.586E-2
	e_h	–	2.197E-2	6.125E-3	1.698E-3	4.751E-4	1.316E-4
	q_h	–	–	3.587	3.608	3.574	3.610
0.3	e	5.626E-2	2.530E-2	1.931E-2	1.816E-2	1.794E-2	1.780E-2
	e_h	–	3.096E-2	5.994E-3	1.151E-3	2.232E-4	4.345E-5
	q_h	–	–	5.165	5.208	5.157	5.137
0.5	e	5.626E-2	9.528E-3	3.131E-3	2.231E-3	2.103E-3	2.085E-3
	e_h	–	4.673E-2	6.397E-3	9.004E-4	1.279E-4	1.846E-5
	q_h	–	–	7.305	7.105	7.041	6.927
0.6	e	5.626E-2	1.227E-2	5.248E-3	4.108E-3	3.921E-3	3.890E-3
	e_h	–	4.399E-2	7.022E-3	1.140E-3	1.871E-4	3.061E-5
	q_h	–	–	6.265	6.159	6.093	6.112
0.7	e	5.626E-2	2.881E-2	2.316E-2	2.201E-2	2.177E-2	2.172E-2
	e_h	–	2.745E-2	5.646E-3	1.146E-3	2.372E-4	4.880E-
	q_h	–	–	4.862	4.926	4.832	4.861

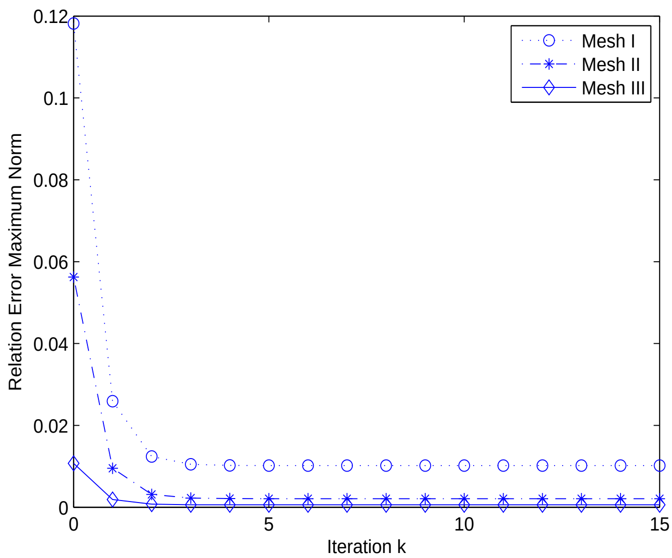

Figure 2. The relationship between maximal errors in relative maximum norm and iteration k about different mesh for Example 5.2.

6. Conclusions

In this paper, we have proposed a D-N alternating algorithm based on the NBR for exterior 3-D Poisson problem with ellipsoidal artificial boundary. The maximal error of the approximate solution decreases with the mesh refining from the numerical results Table 1 and Table 3. When we fixed convergence factor θ_k, the convergence rate can be basically considered to maintain invariable in a same mesh. Table 2 and Table 4 have demonstrated that the convergence of the algorithm is the best as θ_k approaches to 0.5. Figure 1 and Figure 2 have shown that maximal errors in relative maximum norm will quickly dwindle until approach to stable states. The numerical results agree with the theoretic results.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 11401054, 61972055), the Hunan Provincial Natural Science Foundation of China (No. 2018JJ3520), and the general Project of Hunan provincial education department (No. 18C0220).

Conflict of interest

We declare that we have no conflict of interest.

References

1. K. Bathe, E. L. Wilson, *Numerical methods in finite element analysis*, Englewood Cliffs: Prentice-Hall, 1976. doi: 10.1016/0898-1221(77)90079-7.
2. D. J. Evans, Numerical solution of exterior problems by the peripheral block over-relaxation method, *IMA J. Appl. Math.*, **19** (1977), 399–405. doi: 10.1093/imamat/19.4.399.
3. C. A. Brebbia, *Boundary element method in engineering*, Berlin: Spring-Verlag, 1982.
4. D. Givoli, *Numerical methods for problems in infinite domains*, Amsterdam: Elsevier, 1992.
5. J. L. Zhu, *Boundary element analysis for elliptic boundary value problems*, Beijing: Science Press, 1992.
6. K. Feng, Finite element method and natural boundary reduction, In: *Proceedings of the international congress of mathematicians*, 1983, 1439–1453.
7. K. Feng, D. H. Yu, Canonical integral equations of elliptic boundary value problems and their numerical solutions, In: *Proceedings of China-France symposium on the finite element method*, Beijing: Science Press, 1983, 211–252.
8. J. B. Keller, D. Givoli, Exact non-reflecting boundary conditions, *J. Comput. Phys.*, **82** (1989), 172–192. doi: 10.1016/0021-9991(89)90041-7.
9. H. D. Han, X. N. Wu, Approximation of infinite boundary condition and its application to finite element method, *J. Comput. Math.*, **3** (1985), 179–192.
10. H. D. Han, X. N. Wu, The approximation of the exact boundary conditions at an artificial boundary for linear elastic equations and its applications, *Math. Comput.*, **59** (1992), 21–37. doi: 10.1090/S0025-5718-1992-1134732-0.

11. D. Givoli, J. B. Keller, A finite element method for large domains, *Comput. Methods Appl. M.*, **76** (1989), 41–66. doi: 10.1016/0045-7825(89)90140-0.
12. M. J. Grote, J. B. Keller, On nonreflecting boundary conditions, *J. Comput. Phys.*, **122** (1995), 231–243. doi: 10.1006/jcph.1995.1210.
13. D. Givoli, J. B. Keller, Special finite elements for use with high-order boundary conditions, *Comput. Methods Appl. M.*, **119** (1994), 119–213. doi: 10.1016/0045-7825(94)90089-2.
14. F. Brezzi, C. Johnson, On the coupling of boundary integral and finite element methods, *Calcolo*, **16** (1979), 189–201. doi: 10.1007/BF02575926.
15. C. Johnson, J. C. Nedelec, On the coupling of boundary integral and finite element methods, *Math. Comp.*, **35** (1980), 1063–1079. doi: 10.1090/S0025-5718-1980-0583487-9.
16. D. H. Yu, Discretization of non-overlapping domain decomposition method for unbounded domains and its convergence (Chinese), *Math. Numer. Sinica*, **18** (1996), 328–336. doi: 10.12286/jssx.1996.3.328.
17. D. H. Yu, *Natural boundary integral mehtod and its applications*, Dordrecht: Klumer Academic Publishers, 2002.
18. J. M. Wu, D. H. Yu, The natural integral equations of 3-D harmonic problems and their numerical solution (Chinese), *Math. Numer. Sinica*, **20** (1998), 419–430. doi: 10.12286/jssx.1998.4.419.
19. J. M. Wu, D. H. Yu, The overlapping domain decomposition method for harmonic equation over exterior three-dimensional domain, *J. Comput. Math.*, **18** (2000), 83–94.
20. H. Y. Huang, D. H. Yu, Natural boundary element method for three dimensional exterior harmonic problem with an inner prolate spheriod boundary, *J. Comput. Math.*, **24** (2006), 193–208.
21. H. Y. Huang, D. J. Liu, D. H. Yu, Solution of exterior problem using ellipsoidal artificial boundary, *J. Comput. Appl. Math.*, **231** (2009), 434–446. doi: 10.1016/j.cam.2009.03.009.
22. X. Q. Luo, Q. K. Du, H. Y. Huang, T. S. He, A Schwarz alternating algorithm for a three-dimensional exterior harmonic problem with prolate spheroid boundary, *Comput. Math. Appl.*, **65** (2013), 1129–1139. doi: 10.1016/j.camwa.2013.02.004.
23. X. Q. Luo, Q. K. Du, L. B. Liu, A D-N alternating algorithm for exterior 3-D poisson problem with prolate spheroid boundary, *Appl. Math. Comput.*, **269** (2015), 252–264. doi: 10.1016/j.amc.2015.07.063.
24. C. S. Chen, A. Karageorghis, L. Amuzu, Kansa RBF collocation method with auxiliary boundary centres for high order BVPs, *J. Comput. Appl. Math.*, **398** (2021), 113680. doi: 10.1016/j.cam.2021.113680.
25. R. Cavoretto, A. De Rossi, An adaptive LOOCV-based refinement scheme for RBF collocation methods over irregular domains, *Appl. Math. Lett.*, **103** (2020), 106178. doi: 10.1016/j.aml.2019.106178.
26. R. Cavoretto, A. De Rossi, A two-stage adaptive scheme based on RBF collocation for solving elliptic PDEs, *Comput. Math. Appl.*, **79** (2020), 3206–3222. doi: 10.1016/j.camwa.2020.01.018.
27. E. W. Hobson, *The theory of spherical and ellipsoidal harmonics (Bateman Manuscript Project)*, London: McGraw-Hill, 1955. doi: 10.2307/3607762.

AIMS Press