Mathematics

AIMS Mathematics, 7(1): 398–424.
DOI: 10.3934/math.2022027
Received: 19 July 2021
Accepted: 22 September 2021
Published: 13 October 2021

http://www.aimspress.com/journal/Math

Research article

Stochastic pricing formulation for hybrid equity warrants

Teh Raihana Nazirah Roslan[1,2,*], Sharmila Karim[3], Siti Zulaiha Ibrahim[3], Ali Fareed Jameel[3] and Zainor Ridzuan Yahya[4]

[1] Othman Yeop Abdullah Graduate School of Business, Universiti Utara Malaysia, 50300 Kuala Lumpur, Malaysia
[2] Institute of Strategic Industrial Decision Modelling (ISIDM), Universiti Utara Malaysia, UUM Sintok, 06010 Kedah, Malaysia
[3] School of Quantitative Sciences, Universiti Utara Malaysia, 06010 Sintok, Kedah, Malaysia
[4] Institute of Engineering Mathematics, Faculty of Applied Sciences and Humanities, Pauh Putra Campus, Universiti Malaysia Perlis (UniMAP), 02600 Arau, Perlis, Malaysia

* **Correspondence:** Email: raihana@uum.edu.my; Tel: +60125141407; Fax: +6049287160.

Abstract: A warrant is a financial agreement that gives the right but not the responsibility, to buy or sell a security at a specific price prior to expiration. Many researchers inadvertently utilize call option pricing models to price equity warrants, such as the Black Scholes model which had been found to hold many shortcomings. This paper investigates the pricing of equity warrants under a hybrid model of Heston stochastic volatility together with stochastic interest rates from Cox-Ingersoll-Ross model. This work contributes to exploration of the combined effects of stochastic volatility and stochastic interest rates on pricing equity warrants which fills the gap in the current literature. Analytical pricing formulas for hybrid equity warrants are firstly derived using partial differential equation approaches. Further, to implement the pricing formula to realistic contexts, a calibration procedure is performed using local optimization method to estimate all parameters involved. We then conducted an empirical application of our pricing formula, the Black Scholes model, and the Noreen Wolfson model against the real market data. The comparison between these models is presented along with the investigation of the models' accuracy using statistical error measurements. The outcomes revealed that our proposed model gives the best performance which highlights the crucial elements of both stochastic volatility and stochastic interest rates in valuation of equity warrants. We also examine the warrants' moneyness and found that 96.875% of the warrants are in-the-money which gives positive returns to investors. Thus, it is beneficial for warrant holders concerned in purchasing warrants to elect the best warrant with the most profitable and more benefits at a future date.

Keywords: equity warrants; stochastic interest rate; stochastic volatility; Heston-CIR model; hybrid model
Mathematics Subject Classification: 91B70, 91G20, 91G39

1. Introduction

Derivatives are products having payoffs with prices which depend on the stochastic development of underlying financial variables. Recently, the authors of [1] emphasized that warrants have been regarded as a leveraging mechanism for investment, hence warrants are broadly recognized as a high-return investment tool. A warrant is a contract between a buyer and a seller. As claimed by Sae-ue [2], this contract provides rights for investors to buy or sell the underlying shares in future at a fixed price, quantity and time as stipulated by issuers. Warrants are issued by third parties, normally the financial institutions which are not affiliated to the listed securities-issuing companies. There are two primary types of warrants known as company warrants and structured warrants. Company warrants are issued by a company which provide the holders the right but not an obligation, during a specified period of time, to subscribe new ordinary shares at a specified price. Meanwhile, structured warrants are issued by a third-party issuer, which allow the holders the right to purchase or sell the instrument for a fixed price in the future. The general formula for equity warrants was given by Xiao et al. [3] as

$$W(t) = \frac{1}{N + Mk}(kS(t) - NG)^{+}, \tag{1}$$

where W is the equity warrant price, N is the quantity of common stocks, M is the number of equity warrants outstanding, and S is the value of underlying shares. When G is paid, the warrant holder shall obtain k shares for each warrant at time t. The symbol $+$ marks the role of the warrant as a call or put option. A call (put) warrant gives the holder a right, but not the obligation, to buy from (sell to) the issuer the underlying asset at a predetermined price, also known as the exercise price, on or before the expiry date. It is crucial for investors to differentiate these two types of put or call warrant based on their expectations of the underlying asset price. Basically, if a rise is expected, it will be a call warrant, whereas if adjustments are expected, then it will be categorized as a put warrant.

In Malaysia, both types of warrants are products under securities in Bursa Malaysia which is the primary regulator for Malaysia's market in securities and derivatives. The work of [4] informed that the CEO of Bursa Malaysia asserted that the development in derivative sectors and warrants listed induced Bursa Malaysia to a healthy performance in year 2012. However, a comparatively low annual sales rate for warrants was still recorded for the nation and was found less profitable than other markets. Further study by [5] divulged that market disparities were discerned in valuing warrants for a span of 100 days, and thus declared that pricing efficiency allegedly occurred in the Malaysian market. These highlight the importance of conducting a research to improve the pricing of warrants by specifically investigating the revolving issues.

At first glance, Abbasi [6] pointed out that warrants and options may elicit confusion as they share the similar concepts of underlying asset, strike price and expiry date. However, there are two important discrepancies where warrants can endure up to seven years to expire, while an option generally will not surpass a few months. Moreover, warrants are issued by the company itself, whereas options are issued by individuals that leads to the transition from one operator to another of the underlying assets.

In the literature, some papers such as [1, 6, 7] had used the Black Scholes Option Pricing formula as an alternative to evaluate warrant prices. Being one of the successful pricing models used in most markets, Arulanandam, Sin and Muita [8] claimed that this model is also easy to implement. Nevertheless, this model had some presumptions that other researchers criticized. These presumptions will be discussed below.

In a recent research of [9], the authors mentioned that the issues in warrant pricing have been thoroughly discussed in theory and practice, but some concerns remain. First, the paper [10] revealed that the maturity period is substantially linked to warrant mispricing, as longer maturities had positive impacts on the increase of warrant pricing errors. In fact, they reported that the pricing errors of the Black Scholes model might increase as the maturity period increased. Haron [5] applied the Black Scholes model to price Malaysia's warrant for six years period and found that the warrants were significantly mispriced. Moreover, possible dilution effect appears when new stocks are released, and influence the value of warrants. This effect leads to an issue with the prediction of warrant values. At the earliest, [10] established the principle of dilution effect in warrant pricing that was distribution-free in the Black Scholes model. This study indicated that the improvement in the firm's value was eliminated by dilution factor once warrants are exercised. Then, the academics scholars of [11–14] generalized the dilution adjustment based on option pricing model. However, Abínzano and Navas [15] reported that these papers involved the knowledge of the firm as a function of the warrant value which was commonly unobservable. For example, as stated by Bhat and Arekar [16], the underlying share's volatility is the unobservable parameter in the Black Scholes formula, and maximum likelihood estimation is one way of estimating this parameter by deriving the volatility from the returns of physical assets using economic instruments. In addition, the Black Scholes model is commonly used to price all sorts of warrants and the value of the firm is considered to follow a stochastic differential equation. However, recent work by [17] claimed that this notion did not work well, as the principle of uncertainty managed to reduce the overweight small probability to guard against losses, and underweight large probability with reference to a low probability of worse outcome. Responding to this, recent research by [18] first studied equity warrants in an uncertain environment and presented formulas based on Liu's stock model [19]. On the other hand, several authors investigated the problem of warrant pricing on the presumption that underlying share prices relied on diffusion process. In a latter study [20] analyzed warrant prices with jump diffusion that presumed the stock price adopts a model with default intensity and mean-reversion of the interest rate. They concluded that the warrant price decreased as the conversion price of warrant bonds increased, and the prices increased with the values of recovery rate. Alongside with this, fractional Brownian motion (fBm) also contributed to the issues in warrant pricing. fBm is a Gaussian family process characterized by the Hurst parameter in the interval [0,1]. Due to its non-semi-martingale property, Zili [21] asserted that the mixed fractional Brownian motion (mfBm) should be used by presenting the random aspects of financial derivatives. Some researchers had applied fBm to evade independence on warrant pricing. For instance, preliminary study by [22] investigated the fair value of warrants using fBm to investigate the analogy between warrant prices and capital stocks on Changdian warrants. At the end of the study, the author observed that the warrant values using fBm was close to the values of European call option.

Among all these issues, the assumptions of constant volatility and constant interest rate in the Black Scholes model steal the limelight due to the failure of the model to represent important characteristics of financial markets. Formerly, early study by [23] found that market observations demonstrated that constant interest rates could not justify interest rate shifts over time. In agreement, the authors of [3] revealed that it is inappropriate to assert the interest rate to be constant for long-term warrants as the rates evolved in the course of time. At the earliest, the scholars of [24] studied the Black Scholes model for valuing warrants, and they observed that volatility of stock warrants is non-stationary. This was supported by [25] who investigated that the Black Scholes model will lead to pricing errors if there exist stochastic gestures in volatility. To defeat this predicament, the authors of [26] deemed the Black Scholes model under stochastic volatility and solved the pricing formula problem in series form. They explained that the volatility and asset prices were negatively correlated and found that the outcomes were responsive to the stochastic parameters. On the contrary, the Black Scholes model also restricted on constant risk-free interest rate which seems unrealistic to market trend. Yet, interest rates in real market is not constant but moves stochastically. Since then, many methodologies for warrant pricing have been developed by employing stochastic interest rates. For example, prior study by [27] was among the earliest who demonstrated that the Black Scholes model could be modified to accommodate stochastic interest rate. This study utilized the maturity rate for default-free bond which matures at the expiry date of the option to depict interest rates. Later, a work by [28] created the generalizations of Black Scholes pricing model, which considered short-term interest rate that follows stochastic Gaussian process. Their results supported the fact that models with stochastic interest rate produced better estimation compared to the Black Scholes model, and produced small pricing errors relative to market prices. Moreover, the paper [3] presented the fractional Vasicek model by applying a hybrid intelligent algorithm to explain the short rate dynamics for pricing equity warrants under stochastic interest rates. They indicated that stochastic interest rate with long memory property outperformed the existing traditional models. In addition, recent studies such as [29] and [30] readdressed the problems of financial models by incorporating stochastic interest rates. Hereby, these empirical findings revealed that by allowing the volatility or interest rate to be stochastic could enhance pricing efficiency of some financial derivatives. Motivated by these researches, the notion of stochastic volatility and stochastic interest rate has become increasingly important to demonstrate financial uncertainty.

A robust justification exists on both theoretical and empirical grounds for employing the hybrid models of stochastic volatility and stochastic interest rate for pricing warrants. Even though many models have been proposed in the warrants pricing literature, the incorporation of stochastic volatility with stochastic interest rates into warrant pricing models in both theoretical and empirical contexts has not been proposed yet. Former work by [31] examined two hybrid models of Heston-Hull-White and Heston-Cox-Ingersoll-Ross (CIR) to value European-style options. In their paper, they intended to establish hybrid stochastic differential equation models that fit in the class of affine processes. Apart from that, Shen and Siu [32] investigated the Schöbel-Zhu-Hull-White hybrid model under Markovian regime-switching. They created an integral expression for variance swap pricing and further investigated the impact of the involved parameters towards the variance swap prices. Additionally, a hybrid model of the Heston-CIR has been employed in [33], where closed-form solutions for FX rate and interest rate products pricing were presented. In light of this, they utilized the Wishart process to construct a multi-factor model for specifying the currency return dynamics. Anew, we presented in [30] the characteristic function for warrant pricing under Heston and CIR models by performing the change of measure technique and Fourier transform of generalized function approaches. Another research of ours in [34] derived the warrant pricing formula under a different technique using the partial differential equation (PDE) approaches. Apart from the utilization of the Heston-CIR hybrid model, Recchioni, Sun, and Tedeschi [35] adopted the Heston-Hull-White model to estimate the call and put options prices on the US S&P 500 index and Eurodollar futures. In their study, they allowed the interest rates to be negative. Thereof, in such situations, they explored the effect of negative rates towards index options as well as Eurodollar futures pricing. Another recent contribution in the literature of the Vasicek-Heston hybrid model was attained by [36]. This paper incorporated the stochastic property of the short rate and volatility into the option pricing model. The author presented a closed-form pricing formula and compared the prices computed from the proposed hybrid model and the Heston model. The results indicated that the prices of European call option under the Vasicek-Heston model were better than the Heston model.

In this paper, we derive analytical pricing formulas for hybrid equity warrants using a combination of the Heston stochastic volatility model along with the CIR stochastic interest rate model and tested the formula both in theoretical and empirical contexts. It is imperative to model interest rates in market conditions via the CIR model since historic interest rates correspond to the CIR model even with negative rates [37,38]. In their study, the dataset from EURO interest rates with different maturities were adopted. They calibrated the CIR model parameters to market interest rates that moved randomly and employed a Monte Carlo scheme to simulate the expected value of interest rates. Aside from this, we particularly employ the Cauchy problem for heat equations and subsequently work it out using PDE techniques. This hybrid model extends the work of Xiao et al. [3] where stochastic volatility was ignored and unrealistic over time. Apart from that, the Vasicek interest rate model was adopted in their study to explain the short rate fluctuations in pricing equity warrants. However, the Vasicek interest rate model theoretically permits the interest rate to become negative, which is an unwanted characteristic in finance and economics. In periods of severe financial crisis, the negative rates are being used by central banks and it was regarded as extremely improbable. Additionally, the Vasicek interest rate model is a single-factor model where the more subtle structure shifts cannot be captured. Further, the authors of [3] used the data of China equity warrants to evaluate their model's performance consisting of 29 equity warrants traded from 2005 to 2011. However, finally they only focused on a single most actively traded warrant, and this made their data analysis may not be rigorous enough for validation purposes.

The contributions of this paper are as follows. First, we exhibit our pricing formula derived from the Heston-CIR model and implement it to realistic contexts via calibration and parameter estimation techniques. Second, to validate and evaluate the effectiveness of our pricing formula, we carry out empirical studies using data of equity warrants in Malaysia and compare the pricing results with the real market data and existing models. Finally, we also conduct a study on warrant moneyness to investigate the best warrant with the most profitable and more benefits at a future date. The remainder of this paper is structured as follows. Section 2 presents the model setup, while Section 3 deals with the derivation of pricing model for equity warrants. Section 4 presents the parameter estimation method for our proposed pricing formula, followed by section 5 which implements the proposed formula to validate and evaluate its efficiency. Here, the performance of our pricing formula is assessed against the Black Scholes model [39] and the Noreen Wolfson model of [24]. We also discuss about the warrants' moneyness to describe the intrinsic value of a warrant in its current state in Section 6. Finally, Section 7 concludes with a brief summary of the paper.

2. Model setup

In this section, we introduce the outline of our hybrid model namely the Heston-CIR model, along with the expressions for the bond price used as the numeraire in the model.

2.1. The Heston-CIR model

The structure of the model consists of a hybridization between the Heston stochastic volatility model [40] and the Cox-Ingersoll-Ross (CIR) stochastic interest rate model [41]. The following represents the Heston-CIR hybrid model in this paper

$$
\begin{aligned}
dS(t) &= r(t)S(t)dt + \sqrt{v(t)}S(t)d\widetilde{w}_1(t), \\
dv(t) &= k^*(\theta^* - v(t))dt + \sigma\sqrt{v(t)}d\widetilde{w}_2(t), \qquad (2) \\
dr(t) &= \alpha^*(\beta^* - r(t))dt + \eta\sqrt{r(t)}d\widetilde{w}_3(t),
\end{aligned}
$$

where $S(t)$, $v(t)$ and $r(t)$ represent the spot price of underlying asset, volatility, and interest rate respectively. We indicate $S(t)$ as the asset price driven by the drift $r(t)$, and $v(t)$ is its volatility. In the stochastic variance process of $v(t)$, the parameter k^* refers to the mean reversion process, θ^* is the long-term mean and σ is the volatility. Moreover, $r(t)$ is characterized as the instantaneous interest rate, whereupon α^* typify its mean-reversion speed, β^* is the long-term mean of the interest rate, and η observes the volatility of the interest rate. The return-to-the-mean structure in the interest rate model has several important features, such as providing a steady-state distribution for the interest rates, ensuring positive interest rates if zero values are attained, and increasing the absolute variance of the interest rate as the interest rates increase. Additionally, the Brownian motions $\big(d\widetilde{w}_1(t), d\widetilde{w}_2(t)\big) = \rho dt$, $\big(d\widetilde{w}_1(t), d\widetilde{w}_3(t)\big) = 0$, and $\big(d\widetilde{w}_2(t), d\widetilde{w}_3(t)\big) = 0$ are associated with the correlation coefficient of $-1 < \rho < 1$ and $0 \leq t \leq T$. Also, the conditions $2k^*\theta^* \geq \sigma^2$ and $2\alpha^*\beta^* \geq \eta^2$ are required to be positive real constants. The parameters $v(t)$ and $r(t)$ follow a square root process, also known as the Feller process with parameters that meet the following requirements

$$2(k^* - \rho\sigma)\frac{k^*\theta^*}{k^* - \rho\sigma} > \sigma^2 \Leftrightarrow 2k^*\theta^* \geq \sigma^2,$$

and

$$2(\alpha^* - \rho\eta)\frac{\alpha^*\beta^*}{\alpha^* - \rho\eta} > \eta^2 \Leftrightarrow 2\alpha^*\beta^* \geq \eta^2.$$

Besides that, to return to the limit of fixed interest rates, paper [42] primarily emphasized on the almost everywhere convergence of the long-term return $t^{-1}\int_0^t r(u)\,du$. The authors intrigued in the limit $\left(e^{\int_0^t r(u)du}\right)^{\frac{1}{t}}$, the accumulated factor average, which is beneficial in defining the participation models for saving products with a guaranteed minimum return. By adopting the results in [43], they discovered that $\left(e^{\int_0^t r(u)du}\right)^{\frac{1}{t}}$ converges almost everywhere in most extant models of interest rates to a constant independent of the current market environment, as the observation span approaches infinity. The authors further claimed that the model possessed the “strong convergence property”, while when the returns converge to a constant, it may be advert to models with the “weak convergence property”.

In such situations, it typically will rely upon the current economic conditions and may change stochastically. Aside from that, paper [44] showed that the non-arbitrage assumption implies that the rate of asymptotic zero-coupon and long forward will never be lessened. Besides, they demonstrated that almost all models had unexpected implication that both zero-coupon and long run forward rates converge to a constant, and independent of the current economic environment.

2.2. The CIR bond price

Due to the incorporation of stochastic interest rate in this study, we utilized the bond price as the numeraire. For the CIR model, the price of a zero-coupon bond, $P(r,t,T)$ with maturity T at time $t \in [0,T]$ is described by

$$P(r,t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{3}$$

where $A(t,T) = \left(\frac{2\left(e^{\left(\alpha^*+\sqrt{(\alpha^*)^2+2\eta^2}\right)\frac{(T-t)}{2}}\right)\left(\sqrt{(\alpha^*)^2+2\eta^2}\right)}{\left(2\sqrt{(\alpha^*)^2+2\eta^2}\right)+\left(\alpha^*+\sqrt{(\alpha^*)^2+2\eta^2}\right)\left(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}}-1\right)} \right)^{\frac{2\alpha^*\beta^*}{\eta^2}}$ and

$$B(t,T) = \frac{2\left(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}}-1\right)}{\left(2\sqrt{(\alpha^*)^2+2\eta^2}\right)+\left(\alpha^*+\sqrt{(\alpha^*)^2+2\eta^2}\right)\left(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}}-1\right)}.$$

3. Pricing equity warrants under the Heston-CIR model

In this section, we will derive a closed-form formula to price equity warrants using the Heston-CIR model. We imply $W(t)$ as the valuation of the equity warrant. At time $t \in [0,T]$, the valuation of the equity warrant complies with the following partial differential equation (PDE) which corresponds to the derivative value

$$\begin{aligned} \frac{\partial W}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 W}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 W}{\partial v^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 W}{\partial r^2} + rS\frac{\partial W}{\partial S} + k^*(\theta^* - v)\frac{\partial W}{\partial v} \\ + [\alpha^*\beta^* - (\alpha^* + B(t,T)\eta^2)r]\frac{\partial W}{\partial r} + \rho\sigma vS\frac{\partial^2 W}{\partial S\partial v} = 0 \end{aligned} \tag{4}$$

with the terminal condition

$$W(t) = \frac{1}{N+Mk}(kS(t) - NG)^+. \tag{5}$$

The proof of (4) was adopted from [45,46] which derived (4) from (2) using Cholesky decomposition, change of measure techniques, as well as the Feynman Kac theorem on stochastic differential equations.

Theorem 3.1. *Let* $W(t)$ *be the function of equity warrant price with the variables* $S(t), T, t, G, \eta, v, r, k, N$ *and* M *at time* $t \in [0,T]$ *given by*

$$W(t)(S(t),T,t,G,\eta,v,r,k,N,M)=\frac{1}{N+Mk}\left[kS(t)\phi(d_1)-NGe^{-r(T-t)}\phi(d_2)\right] \tag{6}$$

where

$$d_1=\frac{ln\frac{kS}{NG}-ln\,P(r,t,T)+\frac{1}{2}L(T-t)+\frac{1}{2}Q}{\sqrt{L(T-t)+Q}},$$

$$Q=\eta^2 r\int_t^T\left(\frac{2(e^{2R}-2e^R+1)}{2((a^*)^2+2\eta^2)+(e^R-1)(C)}\right)ds\ ,$$

$$R=\ (T-s)\sqrt{(a^*)^2+2\eta^2},$$

$$C\ =\left(a^*\sqrt{(a^*)^2+2\eta^2}+(a^*)^2+3\eta^2+(e^R)\left((a^*)^2+\ a^*\sqrt{(a^*)^2+2\eta^2}+\eta^2\right)\right),$$

$$d_2=d_1-\sqrt{L(T-t)+\eta^2 r\int_t^T B^2(s,T)ds}\,,$$

and $\Phi(\cdot)$ is the cumulative distribution function of Gaussian distribution.

Apparently, (4) is a parabolic PDE with variable coefficients. In order to find its solution, (4) requires Cauchy transformation. By performing on the succeeding coordinate transforms of $y=\frac{S}{P(r,t,T)}$, $\widehat{W}(y,t,L)=\frac{W(S,v,r,T)}{P(r,t,T)}$ and $L=v$, we obtain the following expressions

$$\begin{cases}\dfrac{\partial W}{\partial t}=\widehat{W}\dfrac{\partial P}{\partial t}+P\dfrac{\partial\widehat{W}}{\partial t}+P\dfrac{\partial\widehat{W}}{\partial y}\dfrac{\partial y}{\partial t}=\ \widehat{W}\dfrac{\partial P}{\partial t}+P\dfrac{\partial\widehat{W}}{\partial t}-y\dfrac{\partial\widehat{W}}{\partial y}\dfrac{\partial P}{\partial t},\\ \dfrac{\partial W}{\partial S}=P\dfrac{\partial\widehat{W}}{\partial y}\dfrac{\partial y}{\partial S}=\dfrac{\partial\widehat{W}}{\partial y},\\ \dfrac{\partial^2 W}{\partial S^2}=\dfrac{\partial^2\widehat{W}}{\partial y^2}\dfrac{\partial y}{\partial S}=\dfrac{\partial^2\widehat{W}}{\partial y^2}\left(\dfrac{1}{P}\right),\\ \dfrac{\partial W}{\partial v}=P\dfrac{\partial\widehat{W}}{\partial L}\dfrac{\partial L}{\partial v}=P\dfrac{\partial\widehat{W}}{\partial L},\\ \dfrac{\partial^2 W}{\partial v^2}=P\dfrac{\partial^2\widehat{W}}{\partial L^2}\dfrac{\partial L}{\partial v}=P\left(\dfrac{\partial^2\widehat{W}}{\partial L^2}\right),\\ \dfrac{\partial W}{\partial r}=\ \widehat{W}\dfrac{\partial P}{\partial r}+P\left(\dfrac{\partial\widehat{W}}{\partial y}\left(-\dfrac{y}{P}\dfrac{\partial P}{\partial r}\right)\right)=\ \widehat{W}\dfrac{\partial P}{\partial r}-y\dfrac{\partial\widehat{W}}{\partial y}\dfrac{\partial P}{\partial r},\\ \dfrac{\partial^2 W}{\partial r^2}=\widehat{W}\dfrac{\partial^2 P}{\partial r^2}-y\dfrac{\partial\widehat{W}}{\partial y}\dfrac{\partial^2 P}{\partial r^2}+\dfrac{y^2}{P}\dfrac{\partial^2\widehat{W}}{\partial y^2}\left(\dfrac{\partial P}{\partial r}\right)^2,\\ \dfrac{\partial^2 W}{\partial S\partial v}=\dfrac{\partial^2\widehat{W}}{\partial S\partial L}.\end{cases} \tag{7}$$

Note that $\frac{\partial P}{\partial r}=-P(r,t,T)B(t,T)$, $\frac{\partial^2 P}{\partial r^2}=P(r,t,T)B^2(t,T)$ and $S^2=y^2P^2$. It is easy to check that applying (7) into (4) brings us to the following expression

$$\frac{\partial \widehat{W}}{\partial t}+\frac{1}{2}y^2L\left(\frac{\partial^2\widehat{W}}{\partial y^2}\right)+\frac{1}{2}\sigma^2L\left(\frac{\partial^2\widehat{W}}{\partial L^2}\right)+\frac{1}{2}\eta^2r\left((yB(t,T))^2\frac{\partial^2\widehat{W}}{\partial y^2}\right)=0. \tag{8}$$

Next, let $x=\ln y$. This transforms (8) into

$$\begin{aligned}&\frac{\partial \widehat{W}}{\partial t}-\frac{1}{2}L\frac{\partial \widehat{W}}{\partial x}+\frac{1}{2}L\frac{\partial^2\widehat{W}}{\partial x^2}+\frac{1}{2}\sigma^2L\left(\frac{\partial^2\widehat{W}}{\partial L^2}\right)\\&-\frac{1}{2}\eta^2rB^2(t,T)\frac{\partial \widehat{W}}{\partial x}+\frac{1}{2}\eta^2rB^2(t,T)\frac{\partial^2\widehat{W}}{\partial x^2}=0.\end{aligned} \tag{9}$$

We now define the function $\widehat{W}(y,t,L)=u(\hat{\eta},\tau,\hat{\lambda})$ and $\hat{\eta}=x+\hat{\alpha}(t)$, where $\hat{\alpha}(T)=\omega(T)=0$, $\tau=\omega(t)$ and $\hat{\lambda}=L+h(t)$. In this case, we can derive

$$\frac{\partial \widehat{W}}{\partial t}=\frac{\partial u}{\partial \hat{\eta}}\hat{\alpha}'(t)+\frac{\partial u}{\partial \tau}\omega'(t)+\frac{\partial u}{\partial \hat{\lambda}}h'(t),$$

$$\begin{aligned}\frac{\partial \widehat{W}}{\partial x}&=\frac{\partial u}{\partial \hat{\eta}}\left(\frac{\partial \hat{\eta}}{\partial x}\right)=\frac{\partial u}{\partial \hat{\eta}},\\ \frac{\partial^2\widehat{W}}{\partial x^2}&=\frac{\partial^2 u}{\partial \hat{\eta}^2},\\ \frac{\partial \widehat{W}}{\partial L}&=\frac{\partial u}{\partial \hat{\lambda}},\end{aligned} \tag{10}$$

and

$$\frac{\partial^2\widehat{W}}{\partial L^2}=\frac{\partial^2 u}{\partial \hat{\lambda}^2}.$$

Based on (9) and (10), (9) can be simplified as

$$\begin{aligned}&\frac{\partial u}{\partial \hat{\eta}}\left[\hat{\alpha}'(t)-\frac{1}{2}L-\frac{1}{2}\eta^2rB^2(t,T)\right]+\frac{\partial^2 u}{\partial \hat{\eta}^2}\left[\frac{1}{2}L+\frac{1}{2}\eta^2rB^2(t,T)\right]+\frac{\partial u}{\partial \tau}\omega'(t)\\&+\frac{\partial u}{\partial \hat{\lambda}}h'(t)+\frac{1}{2}\sigma^2L\left(\frac{\partial^2 u}{\partial \hat{\lambda}^2}\right)=0.\end{aligned} \tag{11}$$

Furthermore, (11) can be further deduced to be of the following form

$$\frac{\partial^2 u}{\partial \hat{\eta}^2}=\frac{\partial u}{\partial \tau} \text{ and } \frac{\partial^2 u}{\partial \hat{\lambda}^2}=\frac{\partial u}{\partial \hat{\lambda}}, \tag{12}$$

which subsequently gives

$$\begin{cases} \hat{\alpha}'(t) = \frac{1}{2}L + \frac{1}{2}\eta^2 rB^2(t,T)\,, \\ \omega'(t) = -\frac{1}{2}L - \frac{1}{2}\eta^2 rB^2(t,T), \\ h'(t) = -\frac{1}{2}\sigma^2 L. \end{cases} \tag{13}$$

Performing integration on (13) with respect to the variable *s*, we acquire the following results

$$\begin{cases} \hat{\alpha}(t) = \int_t^T \left(\frac{1}{2}L + \frac{1}{2}\eta^2 rB^2(s,T)\right) ds, \\ \omega(t) = -\int_t^T \left(\frac{1}{2}L + \frac{1}{2}\eta^2 rB^2(s,T)\right) ds, \\ h(t) = -\int_t^T \left(\frac{1}{2}\sigma^2 L\right) ds. \end{cases} \tag{14}$$

It can be observed that (12) stands for an explicit solution of a one-dimensional heat equation presented as

$$\begin{aligned} \widehat{W} = u\big(\hat{\eta},\tau,\hat{\lambda}\big) &= \frac{1}{2\sqrt{\pi\tau}}\int_{ln\frac{NG}{k}}^{+\infty}\frac{1}{N+Mk}\big(ke^{\xi}-NG\big)e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}d\xi, \\ &= \frac{k}{2\sqrt{\pi\tau}}\int_{ln\frac{NG}{k}}^{+\infty}\frac{e^{\xi}e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}}{N+Mk}d\xi - \frac{NG}{2\sqrt{\pi\tau}}\int_{ln\frac{NG}{k}}^{+\infty}\frac{e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}}{N+Mk}d\xi, \\ &= I_1 - I_2, \end{aligned} \tag{15}$$

with the final condition $u\big(\hat{\eta},0,\hat{\lambda}\big) = \frac{1}{N+Mk}(ke^{\hat{\eta}} - NG)^+$.

Here, we first derive the formulation of I_2 by assuming $z_2 = \frac{\hat{\eta}-\xi}{\sqrt{2\tau}}$. Then $-\sqrt{2\tau}dz_2 = d\xi$. We are able to obtain

$$\begin{aligned} I_2 &= \frac{NG}{2\sqrt{\pi\tau}}\int_{ln\frac{NG}{k}}^{+\infty}\frac{e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}}}{N+Mk}d\xi, \\ &= \frac{NG}{(N+Mk)}\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\frac{\hat{\eta}-ln\frac{NG}{k}}{\sqrt{2\tau}}} e^{-\frac{z_2^2}{2}}(-\sqrt{2\tau})dz_2\,, \\ &= \frac{NG}{N+Mk}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\hat{\eta}-ln\frac{NG}{k}}{\sqrt{2\tau}}} e^{-\frac{z_2^2}{2}}\,dz_2\,, \\ &= \frac{NG}{N+Mk}\phi(d_2), \end{aligned} \tag{16}$$

where

$$d_2 = \frac{ln\frac{kS}{NG} - ln\,P(t,T) - \frac{1}{2}L(T-t) - \frac{1}{2}Q}{\sqrt{L(T-t)+Q}},$$

$$Q = \eta^2 r \int_t^T \left(\frac{2(e^{2R} - 2e^R + 1)}{2((a^*)^2 + 2\eta^2) + (e^R - 1)(C)}\right) ds\,,$$

$$R = (T-s)\sqrt{(a^*)^2 + 2\eta^2}\ ,$$

and

$$C = \left(a^*\sqrt{(a^*)^2 + 2\eta^2} + (a^*)^2 + 3\eta^2 + (e^R)\left((a^*)^2 + a^*\sqrt{(a^*)^2 + 2\eta^2} + \eta^2\right)\right).$$

Obtaining the above, we proceed with the derivations of I_1. We let $z_1 = \frac{\hat{\eta}-\xi+2\tau}{\sqrt{2\tau}}$ and $-\sqrt{2\tau}dz_1 = d\xi$. Accordingly, we obtain the following expressions by using the same procedure as in I_2.

$$\begin{aligned} I_1 &= \frac{1}{2\sqrt{\pi\tau}} \int_{ln\frac{NG}{k}}^{+\infty} \frac{ke^{\xi}}{N+Mk} e^{-\frac{(\hat{\eta}-\xi)^2}{4\tau}} d\xi, \\ &= \frac{ke^{\hat{\eta}+\tau}}{(N+Mk)2\sqrt{\pi\tau}} \int_{-\infty}^{\frac{-ln\frac{NG}{k}+\hat{\eta}+2\tau}{\sqrt{2\tau}}} e^{-\frac{z_1^2}{2}}\left(-\sqrt{2\tau}\right) dz_1, \\ &= \frac{kS(T)}{N+Mk}\frac{1}{P(r,t,T)}\phi(d_1), \end{aligned} \tag{17}$$

where

$$d_1 = \frac{ln\frac{kS}{NG} - ln\,P(t,T) + \frac{1}{2}L(T-t) + \frac{1}{2}Q}{\sqrt{L(T-t)+Q}},$$

$$Q = \eta^2 r \int_t^T \left(\frac{2(e^{2R} - 2e^R + 1)}{2((a^*)^2 + 2\eta^2) + (e^R - 1)(C)}\right) ds\,,$$

$$R = (T-s)\sqrt{(a^*)^2 + 2\eta^2},$$

and

$$C = \left(a^*\sqrt{(a^*)^2 + 2\eta^2} + (a^*)^2 + 3\eta^2 + (e^R)\left((a^*)^2 + a^*\sqrt{(a^*)^2 + 2\eta^2} + \eta^2\right)\right).$$

Finally, the required result as in Theorem 1 is obtained by the integration of (15)–(17), along with the relationship $W(S,v,r,t) = \widehat{W}(y,t,L)P(r,t,T)$.

4. Parameter estimation technique

In the previous section, we discussed about the model and formula for pricing hybrid equity warrants under stochastic volatility and stochastic interest rates. In light of this, certain parameters which are not available from the market data have to be estimated in order to implement our formula in a realistic context. Consequently, we present here the parameter estimation method for the pricing model in (6) to attain values for the required parameters.

4.1. Data description

Some of the data used in this study is obtained from www.bursamalaysia.com which is the website of Bursa Malaysia Berhad, whereas other inaccessible data are purchased from the company and external sources. From the data obtained, there exist 7974 data from 1329 companies in Bursa Malaysia from December 2015 to December 2019. However, we eliminate the companies with inadequate information, for example incomplete number of shares outstanding and mid-price values. Finally, 160 warrants data are involved in this study following the time considered which is 5 years. Due to the data availability restriction made by Bursa Malaysia, we display in Table 1 below the example of the real data considered for year 2019.

Table 1. List of selected equity warrants in year 2019.

Number of Warrants	Equity Warrants	Warrants Outstanding (millions)	Shares Outstanding (millions)	Exercise Price	Share Per Warrant	Maturity Period
1	APPASIA-WA	135690400	341748000	0.13	1	10
2	AZRB-WA	116201952	596435000	0.63	1	10
3	BIMB-WA	426715078	1764283000	4.72	1	10
4	BTM-WB	26295146	141344000	0.2	1	10
5	DIGISTA-WB	74024334	650966000	0.26	1	10
6	DNONCE-WA	51920700	261296000	0.25	1	5
7	DOMINAN-WA	45643879	165240000	1.3	1	5
8	DPS-WB	194261746	587770000	0.1	1	10
9	ECOWLD-WA	525392340	2944368000	2.08	1	7
10	EG-WC	68963282	257423000	0.42	1	5
11	GPA-WA	490243800	980488000	0.1	1	10
12	GUNUNG-WB	62942500	236180000	0.4	1	10
13	INIX-WA	104317125	298255000	0.1	1	5
14	JIANKUN-WA	75586889	166845000	0.32	1	7
15	KIMLUN-WA	58954600	339801000	1.68	1	10
16	LBS-WB	99949262	1592579000	0.56	1	5
17	LEWEKO-WB	100181356	321893000	0.2	1	5
18	LUSTER-WB	216000000	2076035000	0.1	1	10
19	MAGNA-WB	164422270	332626000	0.9	1	5
20	MBL-WA	39913680	201941000	0.8	1	10

Continued on next page

Number of Warrants	Equity Warrants	Warrants Outstanding (millions)	Shares Outstanding (millions)	Exercise Price	Share Per Warrant	Maturity Period
21	MCLEAN-WB	23175996	178778000	0.25	1	5
22	OCK-WA	264072954	958573000	0.71	1	5
23	PENSONI-WB	64834000	129668000	0.6	1	10
24	POHUAT-WB	36829532	229979000	1	1	5
25	REACH-WA	1277822225	1096413000	0.75	1	8
26	SERSOL-WA	96151000	215349000	0.18	1	10
27	SOLID-WA	184227454	393272000	0.21	1	5
28	SRIDGE-WA	48033250	140354000	0.18	1	10
29	SYMLIFE-WB	149292368	600135000	0.79	1	7
30	THRIVEN-WB	248621305	546943000	0.48	1	5
31	WCT-WE	232484710	1424065000	2.08	1	5
32	WZSATU-WA	131440908	511665000	0.38	1	10

4.2. The method of parameter estimation

In this section, we adopt the market information of equity warrants from Bursa Malaysia to estimate the unknown parameters in our pricing formula. The process of fitting our formula to market data which is also referred as model calibration is conducted here. Many efficient calibration methods are presented to analyze a variety of stochastic models. According to [47], the methods of optimization-based calibration render more systematic and are essential to most financial models, such as the Heston model, Vasicek model and CIR model. Additionally, Zhang et al. [48] mentioned that these methods also benefit from minimal computational costs since they involve small samples. For example, the calibration of the Heston stochastic volatility model as a nonlinear least squares problem to estimate the model parameter values ensures that the model reproduces market prices as precisely as feasible [49]. Another technique for calibrating asset pricing models to market prices is the weighted Monte Carlo approach, which is based on more probability distortion schemes and more extensive kinds of utility, such as recursive utility [50]. Furthermore, to identify the optimal parameters, it is important to identify a measure to evaluate the distance between the model and market prices. We conduct a local optimization scheme to determine the parameter values that reduce the respective distance. The model calibration's objective is to minimize the absolute value of mean square error (MSE) between the market and model prices. The MSE can be specified as

$$\text{MSE} = \frac{1}{n}\left(\sum_{i=1}^{n} W_i - W_i^{\Omega}\right)^2$$

where n is the number of warrants, W_i is the market prices and W_i^{Ω} is the model prices with Ω needed for calibration. In particular, the parameters involved in the model calibration are represented by

$$\Omega = \theta^*, \sigma, \beta^*, \eta, r(0), v(0), \rho,\ k^*, \alpha^*,$$

where θ^* is the long term mean for variance process and σ is its volatility, β^* is the long-term mean of the interest rate, η observes the volatility of the interest rate, $r(0)$ is the initial interest rate, $v(0)$ is the initial volatility, ρ is the correlation coefficient, k^* is the mean-reversion speed parameter of instantaneous variance, and α^* determines the speed of mean-reversion for instantaneous rate process. From this perspective, it is obvious that there are nine parameters involved in the Heston-CIR model with four parameters accompanying the volatility process consisting of $\theta^*, \sigma, k^*, v(0)$, four parameters accompanying the interest rate comprising of $\alpha^*, \beta^*, \eta, r(0)$, and the correlation ρ.

The calibration procedure is performed using local optimization method, by applying the built-in function *lsqnonlin* in MATLAB to identify the optimal parameters of Ω. Remark that the *lsqnonlin* works for solutions that minimize the sum of the squared functions for all x values. In this case, we first declare the lower bounds (lb) and upper bounds (ub) to perform the model calibration. Below, we present the overall algorithm to obtain the valuation of the equity warrants.

Algorithm 1

Input: Stock Market data, upper bound and lower bound of parameters
Initiate parameter values
Repeat
Call Calibration Function
Evaluate Cost Function using Least Square Method
    Call Characteristic Function
    Call Price Function
    Update new cost function value
Until optimized parameter estimation
Print best solution
Print equity warrant value and best parameter value

Table 2 below displays the nine estimated parameter values for our pricing formula for the specified years. It can be observed that the long-term mean for variance process and its volatility, the long term-mean of the interest rate, the initial interest rate and volatility, the correlation coefficient and the mean-reversion speed of instantaneous variance converged to the same values over the years. The outputs shown in Table 2 will be employed in the following section to apply our pricing formula in the real context.

Table 2. Parameters estimated for our pricing formula according to years.

Parameters	2015	2016	2017	2018	2019
θ^*	0.5	0.5	0.5	0.5	0.5
σ	0.1	0.1	0.1	0.1	0.1
β^*	1	1	1	1	1
η	1	1	1	1	0.7477
$r(0)$	1	1	1	1	1
$v(0)$	1	1	1	1	1
ρ	0.5	0.5	0.5	0.5	0.5
k^*	4	4	4	4	4
α^*	0.1489	0.1349	0.15	0.1949	0.2403

5. Empirical results and analysis

5.1. Model comparison

In this subsection, a brief description regarding two existing pricing models namely the Black Scholes model [39] and the Noreen Wolfson model [24] is given for comparison purposes with our pricing formula.

5.1.1. The Black Scholes model

The Black Scholes model is an evaluation model used to ascertain the prices for a call or put options formulated on five variables involving volatility, stock price, strike price, time to maturity and risk-free interest rate. It is commonly used to price European options. In this matter, the warrant value is comparable to the call option value, coupled with the same strike price and duration up to the expiry date. As pointed out by [1], the Black Scholes model can provide some degree of precision in estimating warrant price. The pricing formula given by Black Scholes model is as follows

$$P_{BS} = S(t)\Phi(d_1^{BS}) - Ge^{-r(T-t)}\Phi(d_2^{BS}), \tag{18}$$

where $d_1^{BS} = \frac{\ln\left(\frac{S(t)}{G}\right)+(r+\frac{\sigma_S^2}{2})(T-t)}{\sigma_S\sqrt{T-t}}$, $d_2^{BS} = d_1^{BS} - \sigma_S\sqrt{T-t}$, Φ is the cumulative Gaussian distribution function, $S(t)$ is the stock price at time t, G is the strike price, r is the rate of interest, σ_S is the standard deviation of the log returns, and $T-t$ is the time expiration. Additionally, this model does not take into account the effects of dilution in pricing warrant. Contrarily, the transaction of equity warrants grants the company to issue shares which leads to stock dilution.

5.1.2. The Noreen Wolfson model

According to [39], most papers dealing with warrants ignored the possible dilution effects on the firm's equity. The paper [24] generalized the Black Scholes model predicated on constant elasticity variance (CEV) model, adapted for the dilution factor related to the valuation of executive stock options. The Noreen Wolfson model utilizes the Black Scholes European call option model by providing the preceding modification to the predicted call price

$$P_{NW} = \frac{N}{N+Mk}\left(S(t)\phi\,(d_1^{NW}) - Ge^{-r(T-t)}\phi\,(d_2^{NW})\right), \tag{19}$$

where $d_1^{NW} = \frac{\ln\left(\frac{S(t)}{G}\right)+\left(r+\frac{\sigma_S^2}{2}\right)(T-t)}{\sigma_S\sqrt{T-t}}$, $d_2^{NW} = d_1^{NW} - \sigma_S\sqrt{T-t}$, N is the amount of outstanding shares of common stock, M is the amount of outstanding warrants, and k is the conversion ratio. Here, the d_1^{NW} and d_2^{NW} expressions are similar to the expressions of d_1^{BS} and d_2^{BS} in the Black Scholes model respectively. Further, the remaining symbols are the same as those in the Black Scholes model. Moreover, the Noreen Wolfson model involved eight parameters, where five of it could be procured using the same approaches as in the Black Scholes model. The remaining parameters comprising of the amount of shares outstanding, the amount of outstanding warrants and the conversion ratio could also be extracted from the data.

5.2. Results and analysis

In this subsection, our pricing formula is numerically examined with the Black Scholes model and the Noreen Wolfson model in terms of how these models fit the observed market data, and whether these models led to the prediction improvements respectively. In the field of financial mathematics, once analytical pricing formulas are obtained for a particular model, the model's performance will be tested for its accuracy and efficiency via numerical experiments. The objective of conducting such experiments is to test and validate which pricing model offers the best performance compared to existing models.

5.2.1. Models' performance and market fitting

Here, we compute the equity warrant prices using our pricing formula and compare the acquired results with the existing models presented in Section 5.1. The graph plots of the three pricing models alongside the market prices are displayed in Figures 1–5, and the results attained according to yearly basis are shown in Tables 3–7.

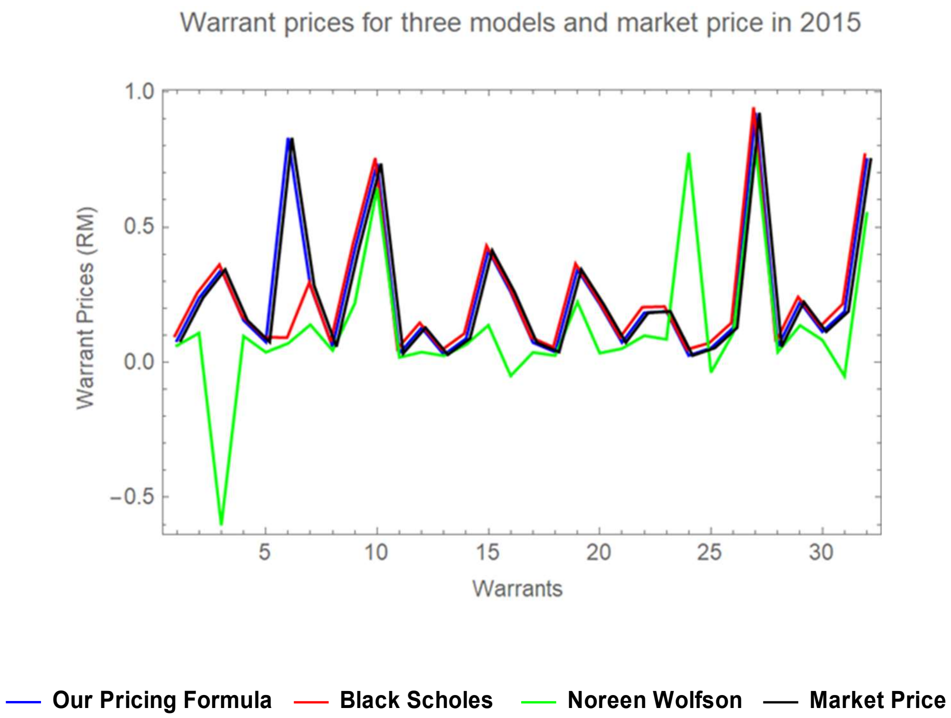

Figure 1. The comparison of three models and market price in year 2015.

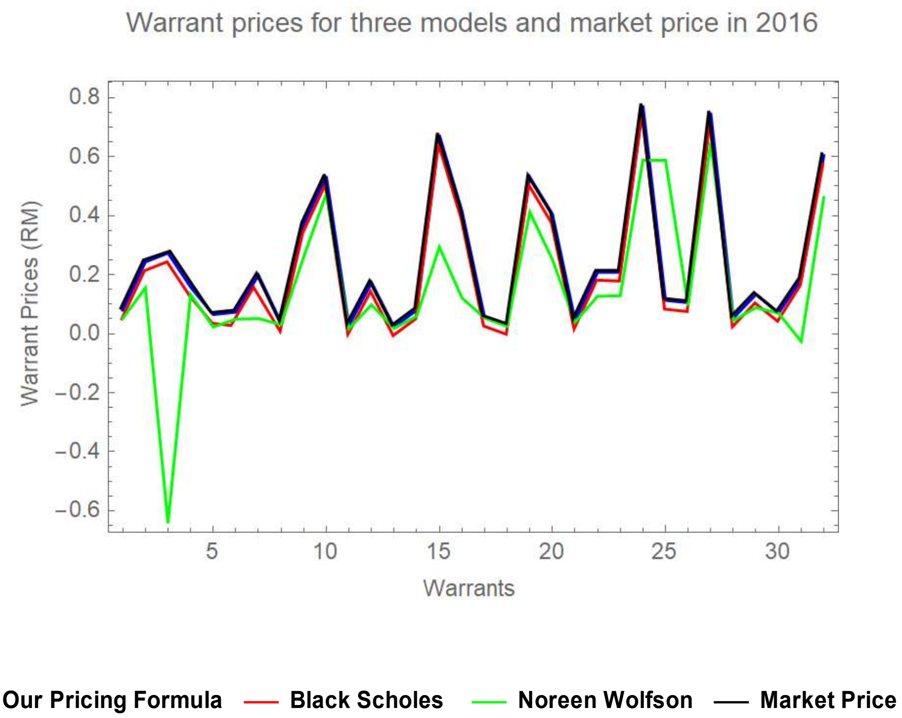

Figure 2. The comparison of three models and market price in year 2016.

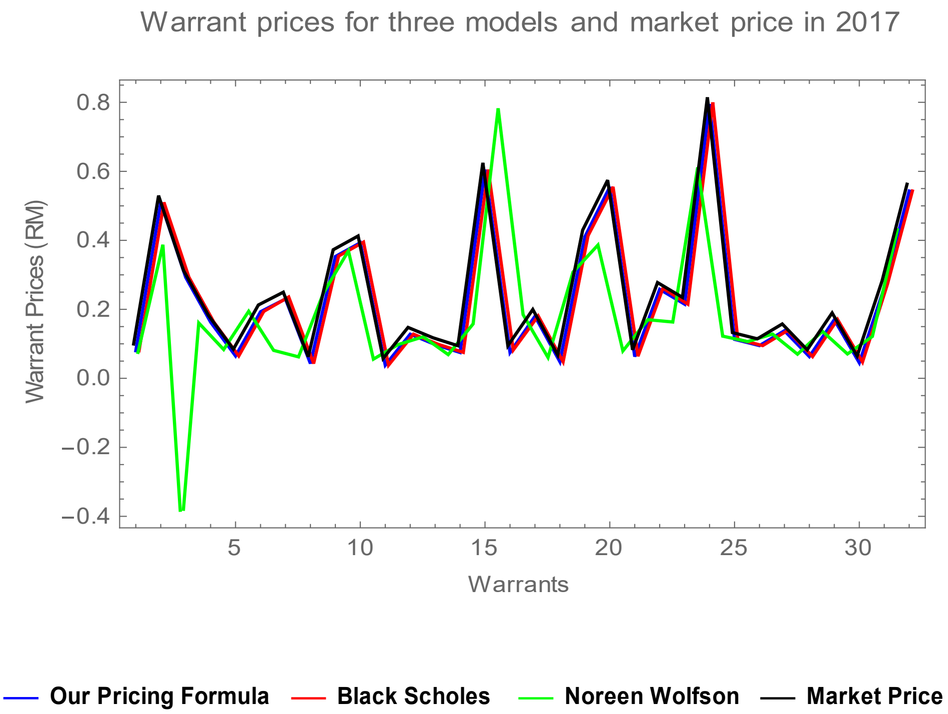

Figure 3. The comparison of three models and market price in year 2017.

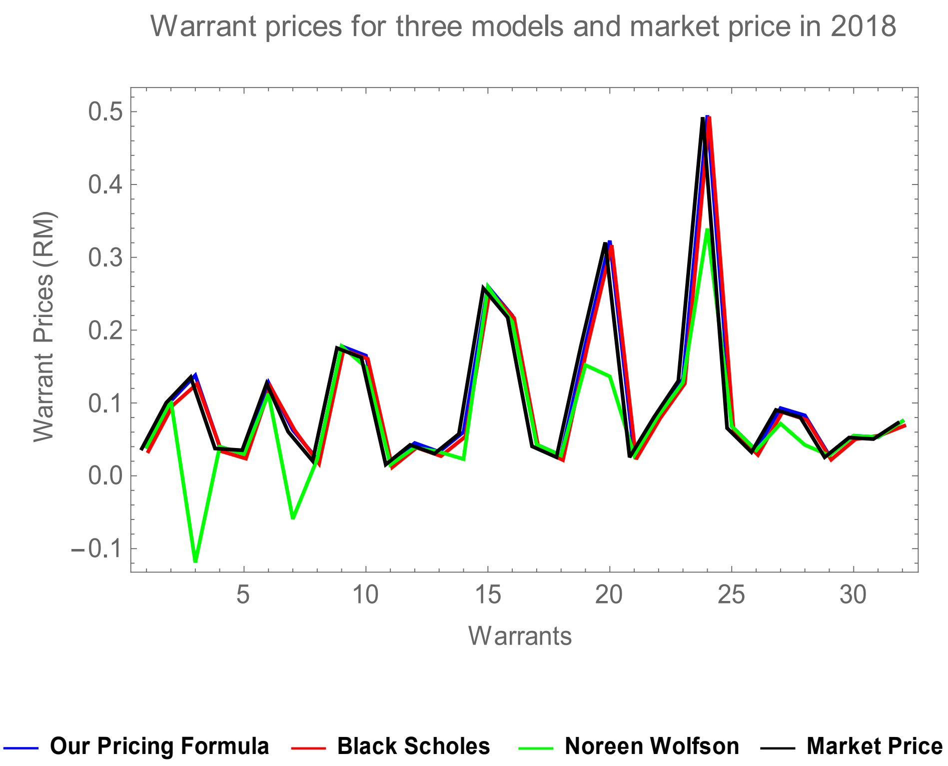

Figure 4. The comparison of three models and market price in year 2018.

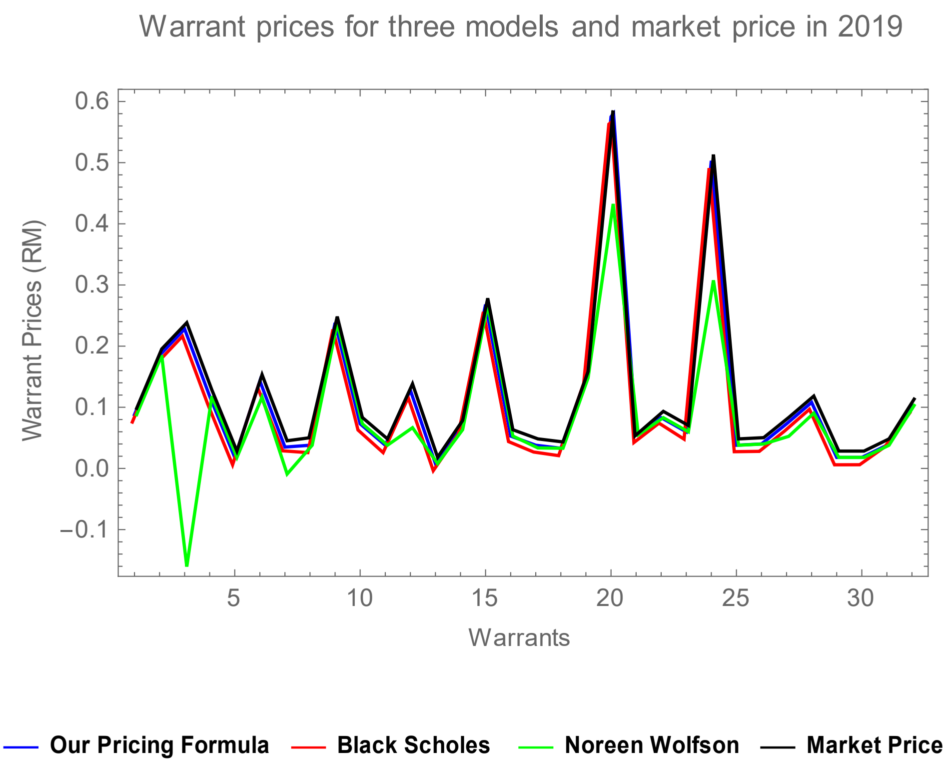

Figure 5. The comparison of three models and market price in year 2019.

Figures 1–5 illustrate the comparison between the pricing models and market prices in the years 2015 to 2019. Depicted in figures above, our pricing formula gives a good fit to the market prices for the specified years. Besides that, prices calculated from the Black Scholes model are point-wisely close and match market prices for most of the warrants. We can see that the difference between the Black Scholes model and the market price is small, and still follows the pattern of the market price. Moreover, observe that the evaluated warrant prices in both our pricing formula and the Black Scholes model for certain warrants seem to have very small differences as the plot lines of the two warrant prices align together. In contrast, the Noreen Wolfson model obviously shows to be slightly distant from the market prices. This model acquired negative values in its estimated warrant prices which is inappropriate since warrant values should never be negative.

As a whole, there is very minor difference between our pricing model and market prices, as shown in Figures 1–5. In addition, this implies that our pricing formula is accurate and can be safely applied in practice. The valuation of five selected warrants per year can be referred in Tables 3–7 below for further analysis.

From Tables 3–7, the results indicate that our pricing formula outperforms the other two models. It is observed that the two existing valuation models tend to overprice and underprice the warrants, while most of the pricing results calculated by our pricing formula are consistent with market prices. Specifically, our pricing formula works well throughout the 5 years since the warrants' predicted prices are closely similar with the warrants' market prices. For example, in Table 3, we obtained the predicted prices of 0.2830, 0.0730, 0.3430 and 0.1130 respectively for DOMINAN-WA, LEWEKO-WB, MAGNA-WB and THRIVEN-WB, which are exactly equal to the market prices in the same year. There is no doubt that our pricing formula easily defeated other two models in this case. Apart from that, observe that the Black Scholes model tend to underestimate and overestimate equity warrants prices. Tables 5 and 6 reveals that DNONCE-WB, INIX-WA, MCLEAN-WB and REACH-WA overestimated the market prices by 0.000960829943652985, 0.000392526154014999, 0.00115399424313 and 0.000099196562548989 respectively. These findings verify that the assumptions of constant volatility and constant interest rate in the Black Scholes model contradict to the financial phenomenon. Otherwise stated, ignoring these two assumptions may lead to significant underestimation or overestimation of the warrant prices. Nevertheless, the Black Scholes model performed better than the Noreen Wolfson model. The Noreen Wolfson model performed very poorly from 2015 to 2017. In contrast, for the years 2018 and 2019, this model improved significantly as the predicted prices obtained are similar and closer to market prices for certain warrants. For instance in Table 7, only the price of equity warrant JIANKUN-WA was underestimated by the Noreen Wolfson model.

Table 3. Warrant prices for three models and the market price in year 2015.

Warrants	Our pricing formula	Black Scholes	Noreen Wolfson	Market price
APPASIA-WA	0.080000000574791	0.080003664824940	0.061549782101358	0.0800
DOMINAN-WA	0.283000000000000	0.288717639718196	0.138933493031117	0.2830
LEWEKO-WB	0.073000000000000	0.288717639718196	0.138933493031117	0.0730
MAGNA-WB	0.343000000000000	0.346434767708419	0.222849636704769	0.3430
THRIVEN-WB	0.113000000000000	0.115626296918640	0.082471369943301	0.1130

Table 4. Warrant prices for three models and the market price in year 2016.

Warrants	Our pricing formula	Black Scholes	Noreen Wolfson	Market price
DIGISTA-WB	0.065000000471245	0.065004672265969	0.022636426429228	0.0650
EG-WC	0.533000000000000	0.535248899020325	0.473592435261704	0.5330
LUSTER-WB	0.028000000015157	0.028003262019334	0.024650390874453	0.0280
OCK-WA	0.208000000000000	0.211117377728254	0.126691199602151	0.2080
SRIDGE-WA	0.053000001185461	0.053005118963523	0.044698129225113	0.0530

Table 5. Warrant prices for three models and the market price in year 2017.

Warrants	Our pricing formula	Black Scholes	Noreen Wolfson	Market price
AZRB-WA	0.510000001959723	0.510025354703685	0.361592796601072	0.5100
DNONCE-WA	0.193000000000000	0.193960829943653	0.166235417342544	0.1930
GUNUNG-WB	0.128000002816721	0.128013335565788	0.069760151685293	0.1280
INIX-WA	0.098000000000000	0.098392526154015	0.092394185898286	0.0980
WZATU-WA	0.543000002306095	0.543017646593045	0.402166492759913	0.5430

Table 6. Warrant prices for three models and the market price in year 2018.

Warrants	Our pricing formula	Black Scholes	Noreen Wolfson	Market price
DPS-WB	0.023000000267042	0.023003109631697	0.022999916591967	0.0230
GPA-WA	0.018000000196653	0.018002810962309	0.015346085719489	0.0180
MBL-WA	0.323000019842525	0.323028117261753	0.136465949783279	0.3230
MCLEAN-WB	0.028000000000000	0.029153994724313	0.028000000000610	0.0280
REACH-WA	0.068000000001872	0.068099196562549	0.068000000000000	0.0680

Table 7. Warrant prices for three models and the market price in year 2019.

Warrants	Our pricing formula	Black Scholes	Noreen Wolfson	Market price
BTM-WB	0.118000000811604	0.118007073166777	0.118000000265438	0.1180
JIANKUN-WA	0.068000000001654	0.068180165926507	0.063899230814451	0.0680
LBS-WB	0.053000000000000	0.056001902808156	0.052996951364994	0.0530
PENSONI-WB	0.060000001570295	0.060016564798600	0.059999987905250	0.0600
SERSOL-WA	0.040000000003767	0.040005117238332	0.039999995411248	0.0400

In general, our pricing formula offers outstanding results compared to the Black Scholes model and the Noreen Wolfson model. Undoubtedly, as illustrated previously in from Figures 1–5, the prices calculated by our pricing formula matches the warrants market prices. Additionally, to further illustrate the accuracy of our model, we compare the pricing errors and measure the warrants' moneyness of the proposed model with those calculated from the Black Scholes model and Noreen Wolfson model. The details for each scenario will be discussed in the next subsection.

5.2.2. Pricing errors

Statistical error measurement is used as the comparison criterion for the accuracy and performance of the models involved in this study. The error measurements used consist of the mean absolute error (MAE), the mean absolute percentage error (MAPE), and the root mean squared error (RMSE), and are computed using the following

$$\text{MAE} = \frac{1}{\text{sample size}} \sum |\text{Market price} - \text{Model price}|,$$

$$\text{MAPE} = \frac{1}{\text{sample size}} \sum \left| \frac{\text{Market price} - \text{Model price}}{\text{Market price}} \right|,$$

$$\text{RMSE} = \sqrt{\frac{1}{\text{sample size}} \sum \left(|\text{Market price} - \text{Model price}|\right)^2}.$$

The accuracy analysis for our pricing formula, the Black Scholes model and the Noreen Wolfson model are exhibited in Tables 8–12. Observe from Tables 8–12 that the pricing errors of the Black Scholes model and the Noreen Wolfson model appear a bit higher compared to our pricing formula. Also, our pricing fomula obtain very small values for MAE, MAPE and RMSE among these comparative models, suggesting that it gives the most accurate results. Besides, the pricing errors of the Noreen Wolfson model might seem a bit higher compared to our pricing formula and the Black Scholes model. Further examination of the performance of these two pricing models revealed that the Black Scholes model performed better compared to the Noreen Wolfson model. Specifically, the Noreen Wolfson model gives the worst performance on MAE, MAPE and RMSE among the three models. By referring to Table 8, this model produces the highest MAPE of 54.69%, in contrast to our pricing model and the Black Scholes model. In addition to this, the MAPE for Noreen Wolfson model in 2015 is the highest among all models over all involved years. The same trend of the Noreen Wolfson model inducing a higher MAPE compared to the Black Scholes model was also justified in Xiao et al. [3] who tested their proposed model against the Black Scholes model, the Noreen Wolfson model, the Lauterbach Schultz model, and the Ukhov model. Their empirical findings revealed that the Black Scholes model and the Noreen Wolfson model generated MAPE of 35.26% and 37.67% respectively.

Table 8. The pricing errors for three models in year 2015.

Errors	Our pricing formula	Black Scholes	Noreen Wolfson
MAE	1.35319E-09	0.00144800973934	0.12492262037343
MAPE	7.22701E-09	0.00682421027709	0.54688518050022
RMSE	2.79527E-09	0.00272513337126	0.21131903099817

Table 9. The pricing errors for three models in year 2016.

Errors	Our pricing formula	Black Scholes	Noreen Wolfson
MAE	1.13043E-09	0.00146016727456	0.10759001778087
MAPE	4.77358E-09	0.00778280021382	0.41975189114179
RMSE	2.48916E-09	0.00277225510439	0.20121088479459

Table 10. The pricing errors for three models in year 2017.

Errors	Our pricing formula	Black Scholes	Noreen Wolfson
MAE	1.52831E-09	0.00142362754452	0.11950887047077
MAPE	6.17151E-09	0.00636789327638	0.40481695329482
RMSE	3.41041E-09	0.00276940210685	0.22872894531028

Table 11. The pricing errors for three models in year 2018.

Errors	Our pricing formula	Black Scholes	Noreen Wolfson
MAE	2.19473E-09	0.00122434688289	0.02818232309285
MAPE	1.92459E-08	0.01562445449550	0.21117442460036
RMSE	5.0946E-09	0.00235187076420	0.06726636305672

Table 12. The pricing errors for three models in year 2019.

Errors	Our pricing formula	Black Scholes	Noreen Wolfson
MAE	4.99892E-10	0.00125707060739	0.02464038076790
MAPE	3.12E-09	0.02534732795222	0.13907183879248
RMSE	1.14338E-09	0.00242815410303	0.07398294871201

It is also important to note that the authors of [51] mentioned that warrants generally have maturities ranging from 3 months to 15 years. In this research, as presented in Table 1, we have considered the warrants data with maturity from 5 to 10 years. Predicated on these available sample data as well as employing our derived analytic pricing formula, our approach is fundamentally appropriate to value warrants since our data covers the first 10 years of the warrants' lifespan.

Based on the results displayed in this section, it is evident that our model which consists of the hybridization of stochastic volatility and stochastic interest rate performs magnificently compared to existing pricing models. This is intuitive since both the Black Scholes model and the Noreen Wolfson model do not incorporate either the stochastic volatility or the stochastic interest rate elements which is against the real market phenomena. Moreover, our empirical study demonstrated that the prices calculated by our pricing formula are persistent with the market prices, despite other existing pricing models which appeared to underprice or overprice equity warrants. Our results support many empirical evidences in the literature which suggested that a hybridization of the Heston-CIR model may provide to improvement of model performance.

6. Moneyness

In essence, moneyness is the term used to specify the profitability of a warrant. It is the strike price's position in relation to the warrant value, either the warrant is a call warrant or a put warrant. Aziz et al. [7] reported that there are 3 categories driven by moneyness, namely at-the-money (ATM), in-the-money (ITM) and out-of-the-money (OTM). The applicability of these categories differs according to the nature of the warrant itself as a call or put warrant.

In relation to call warrants, ATM is a situation where the strike price is equal to the stock price of the underlying asset. Next is the ITM that have both positive intrinsic values and time values, with the strike price below the stock price. Meanwhile, OTM occurs if the strike price is above the stock price. The anticipated output is that if the warrants is ATM, the investors is break even, if the warrant is ITM, the investors will make money, while OTM means that the warrant shall expire or leave without value. However, for the case of put warrants, it should be emphasized that the definitions of ITM and OTM are conversely to the definition of call warrants. Note that the moneyness factor has been calculated in several ways in literature, but in this study we determined moneyness by comparing the strike price and stock price of warrants. Using the definition of call warrants stated above, we exhibit the results of warrant prices calculated from our pricing formula and define the warrants' moneyness for 32 equity warrants of year 2019 in Table 13.

According to Table 13, we perceive that out of the 32 equity warrants, none of them are ATM warrants. Apart from that, there is only one warrant in 2019 that grasp the moneyness status of OTM which is APPASIA-WA. The warrant that holds status OTM are no longer profitable for the investors. In fact, the warrant is the right to purchase stocks which represent the expectation of investors of their underlying stock. Hence, from this perspective, we claim that the stock price fails to meet the investors' expectation of the future values of stock. In contrast, most of the warrants in Table 13 are ITM. These warrants are profitable by ensuring that the warrants are beneficial to the investors and exercising the warrants will make profit. Therefore, it is very crucial for the warrant holders concerned in purchasing warrant from a particular company to elect the best warrant with the most profitable and more benefits at a future date. On average, 3.125% of the warrants in year 2019 are OTM and 96.875% of them are ITM. Thus, we conclude that 96.875% of warrants illustrated in Table 13 give positive returns to the investors.

Table 13. Warrants moneyness for 32 equity warrants in 2019.

Warrants	Stock price	Strike price	Moneyness status
APPASIA-WA	0.0847	0.088000000148371	OTM
AZRB-WA	0.3052	0.185000000252882	ITM
BIMB-WA	4.1956	0.228000000898383	ITM
BTM-WB	0.1250	0.118000000811604	ITM
DIGISTA-WB	0.0371	0.018000000015144	ITM
DNONCE-WA	0.2533	0.143000000000000	ITM
DOMINAN-WA	1.2082	0.035000000000000	ITM
DPS-WB	0.0571	0.038000000046692	ITM
ECOWLD-WA	0.6694	0.238000000000001	ITM
EG-WC	0.3351	0.073000000000000	ITM
GPA-WA	0.0704	0.038000000026699	ITM
GUNUNG-WB	0.3924	0.128000001404027	ITM
INIX-WA	0.0455	0.008000000000000	ITM
JIANKUN-WA	0.2749	0.068000000001654	ITM
KIMLUN-WA	1.0579	0.268000002196129	ITM
LBS-WB	0.4471	0.053000000000000	ITM
LEWEKO-WB	0.1826	0.038000000000000	ITM
LUSTER-WB	0.0643	0.033000000010146	ITM
MAGNA-WB	0.7083	0.148000000000000	ITM

Continued on next page

Warrants	Stock price	Strike price	Moneyness status
MBL-WA	0.8000	0.575000003707543	ITM
MCLEAN-WB	0.1292	0.055000000000000	ITM
OCK-WA	0.5224	0.083000000000000	ITM
PENSONI-WB	0.2932	0.060000001570295	ITM
POHUAT-WB	1.0134	0.503000004130376	ITM
REACH-WA	0.1576	0.038000000000000	ITM
SERSOL-WA	0.0777	0.040000000003767	ITM
SOLID-WA	0.1953	0.073000000304881	ITM
SRIDGE-WA	0.1832	0.108000000000000	ITM
SYMLIFE-WB	0.3876	0.018000000306296	ITM
THRIVEN-WB	0.1776	0.018000000000017	ITM
WCT-WE	0.8536	0.038000000000000	ITM
WZSATU-WA	0.1702	0.103000000161639	ITM

7. Conclusions

In this paper, we study the evaluation of equity warrants using the Heston-CIR hybrid model both in the analytical and empirical contexts, by assimilating the element of stochastic volatility with stochastic interest rates. We apply the techniques of partial differential equation to solve Cauchy problems which results in analytical formulas for pricing equity warrants. Using the data of Malaysia warrant market, we compare the pricing performances of our pricing model with the Black Scholes model and the Noreen Wolfson model, and validate the models' accuracy using statistical error measurements. Moreover, we also examine the warrants' moneyness and observe that 96.875% of the warrants give positive returns to investors. Our findings indicate that the hybrid model of stochastic volatility and stochastic interest rate plays an important role in assessing equity warrants prices, since our pricing model performed magnificently compared to existing pricing models. Among the three models studied, the Noreen Wolfson model has the worst performance and should not be considered for pricing warrants in Malaysia. It can be affirmed that the rationale for the pricing accuracy of our pricing formula is due to the hybridization of the stochastic volatility and stochastic interest rate which establishes a new model for equity warrants and captures the real market phenomena. Besides that, our pricing formula also gives the smallest MAE, MAPE and RMSE among the three models in all considered years. These results render a great deal of support for the conclusion that our pricing formula can be considered as a benchmark pricing tools for equity warrants. Thus, we can infer that the prices calculated by our pricing formula are persistent with the market prices, despite other existing pricing models which appear to underprice or overprice equity warrants. For future research, it is suggested to include jumps in the pricing model's structure to illustrate occurrence of fluctuations in the financial market, as well as to consider longer data periods for investigating the specific effect of variables.

Acknowledgments

This research was supported by Ministry of Higher Education (MoHE) of Malaysia through Fundamental Research Grant Scheme (FRGS/1/2018/STG06/UUM/02/3).

Conflict of interest

The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. N. I. I. I. Gunawan, S. N. I. Ibrahim, N. A. Rahim, A review on Black-Scholes model in pricing warrants in Bursa Malaysia, In: *AIP Conference Proceedings*, **1795** (2017), 020013. doi: 10.1063/1.4972157.
2. M. D. Sae-ue, Derivative warrant market and implied volatility, Chulalongkorn University, 2015.
3. W. Xiao, W. Zhang, X. Zhang, X. Chen, The valuation of equity warrants under the fractional Vasicek process of the short-term interest rate, *Phys. A*, **394** (2014), 320–337. doi: 10.1016/j.physa.2013.09.033.
4. M. R. P. Sakti, A. Qoyum, Testing the warrants mispricing and their determinants: the panel data models, *Glob. Rev. Islamic Econ. Bus.*, **5** (2017), 118–129. doi: 10.14421/grieb.2017.052-05.
5. R. Haron, Derivatives, pricing efficiency and gharar: evidence on embedded options in Malaysia, *J. Islamic Financ.*, **3** (2014), 039–048.
6. W. Abbasi, Pricing warrants models: An empirical study of the Indonesian market, *Res. J. Econ. Bus. ICT*, **10** (2015), 1.
7. K. A. A. Aziz, M. F. I. M. Idris, R. Saian, W. S. W. Daud, Adaptation of warrant price with Black Scholes model and historical volatility, In: *AIP Proceedings of Mathematics*, **1660** (2015), 090042. doi: 10.1063/1.4915886.
8. D. B. V. Arulanandam, K. W. Sin, M. A. Muita, Relevance of Black Scholes model on Malaysia KLCI options: An empirical study, *Internat. J. Financ. Res. Rev.*, **5** (2017). 1–33.
9. W. Xiao, X. Zhang, Pricing equity warrants with a promised lowest price in Merton's jump-diffusion model, *Phys. A*, **458** (2016), 219–238. doi: 10.1016/j.physa.2016.03.100.
10. D. Galai, M. I. Schneller, Pricing of warrants and the value of the firm, *J. Financ.*, **33** (1978), 1333–1342. doi: 10.2307/2327269.
11. G. U. Schulz, S. Trautmann, Robustness of option-like warrant valuation, *J. Bank. Financ.*, **18** (1994), 841–859. doi: 10.1016/0378-4266(94)00030-1.
12. J. C. Handley, On the valuation of warrants, *J. Fut. Mkts.,* **22** (2002), 765–782. doi; 10.1002/fut.10032.
13. K. G. Lim, E. Terry, The valuation of multiple stock warrants, *J. Fut. Mkts.,* **23** (2003), 517–534. doi: 10.1002/fut.10079.
14. A. D. Ukhov, Warrant pricing using observable variables, *J. Financ. Res.*, **27** (2004), 329–339. doi: 10.1111/j.1475-6803.2004.00100.x.
15. I. Abínzano, J. F. Navas, Pricing levered warrants with dilution using observable variables, *Quant. Financ.*, **13** (2013), 1199–1209. doi: 10.1080/14697688.2013.771280.
16. A. Bhat, K. Arekar, Empirical performance of Black-Scholes and GARCH option pricing models during turbulent times: The Indian evidence, *Int. J. Econ. Financ.*, 8 (2016), 123–136. doi: 10.5539/ijef.v8n3p123.
17. M. Tian, X. Yang, S. Kar, Equity warrants pricing problem of mean-reverting model in uncertain

environment, *Phys. A*, **531** (2019), 121593. doi: 10.1016/j.physa.2019.121593.
18. F. Shokrollahi, Valuation of equity warrants for uncertain financial market, 2017, arXiv:1711.08356.
19. B. Liu, Some research problems in uncertainty theory, *J. Uncertain Syst.*, **3** (2009), 3–10.
20. X. N. Su, W. Wang, W. S. Wang, Pricing warrant bonds with credit risk under a jump diffusion process, *Discrete Dyn. Nat. Soc.*, **2018** (2018), 4601395. doi: 10.1155/2018/4601395.
21. M. Zili, On the mixed fractional Brownian motion, *Int. J. Stoc. Anal.*, **2006** (2006), 1–9. doi: 10.1155/JAMSA/2006/32435.
22. W. G. Zhang, W. L. Xiao, C. X. He, Equity warrants pricing model under Fractional Brownian motion and an empirical study, *Expert Syst. Appl.*, **36** (2009), 3056–3065. doi: 10.1016/j.eswa.2008.01.056.
23. J. Hull, *Option, futures and other derivatives*, 7 Eds., New Jersey: Pearson Education, 2009.
24. N. Noreen, M. Wolfson, Equilibrium warrant pricing models and accounting for executive stock options, *J. Account. Res.*, **19** (1981), 384–398. doi: 10.2307/2490872.
25. X. J. He, S. P. Zhu, A closed-form pricing formula for European options under the Heston model with stochastic interest rate, *J. Comput. Appl. Math.*, **335** (2018), 323–333. doi: 10.1016/j.cam.2017.12.011.
26. J. Hull, A. White, The pricing of options on assets with stochastic volatilities, *J. Financ.*, **42** (1987), 281–300. doi: 10.1111/j.1540-6261.1987.tb02568.x.
27. R. C. Merton, Theory of rational option pricing, *Bell J. Econ. Manage. Sci.*, **4** (1973), 141–183. doi: 10.2307/3003143.
28. M. Abudy, Y. Izhakian, Pricing stock options with stochastic interest rate, *Int. J. Portfolio Anal. Manage.*, **1** (2013), 250–277. doi: 10.1504/IJPAM.2013.054408.
29. X. J. He, S. P. Zhu, A closed-form pricing formula for European options under the Heston model with stochastic interest rate, *J. Comput. Appl. Math.*, **335** (2018), 323–333. doi: 10.1016/j.cam.2017.12.011.
30. T. R. N. Roslan, A. F. Jameel, S. Z. Ibrahim, Modeling the price of hybrid equity warrants under stochastic volatility and interest rate, *COMPUSOFT: Int. J. Adv. Comput. Technol.*, **9** (2020), 3586–3589.
31. L. A. Grzelak, C. W. Oosterlee, On the Heston model with stochastic interest rates, *SIAM J. Financ. Math.*, **2** (2011), 255–286. doi: 10.1137/090756119.
32. Y. Shen, T. K. Siu, Pricing variance swaps under a stochastic interest rate and volatility model with regime-switching, *Oper. Res. Lett.*, **41** (2013), 180–187. doi: 10.1016/j.orl.2012.12.008.
33. A. Gnoatto, M. Grasselli, An analytic multi-currency model with stochastic volatility and stochastic interest rates, 2013, arXiv:1302.7246.
34. T. R. N. Roslan, S. Z. Ibrahim, S. Karim, Hybrid equity warrants pricing formulation under stochastic dynamics, *Int. Scholarly Sci. Res. Innov.*, **14** (2020), 133–136.
35. M. C. Recchioni, Y. Sun, G. Tedeschi, Can negative interest rates really affect option pricing? empirical evidence from an explicitly solvable stochastic volatility model, *Quant. Financ.*, **17** (2017), 1257–1275. doi: 10.1080/14697688.2016.1272763.
36. Z. Guo, Option pricing under the Heston model where the interest rate follows the Vasicek model, *Commun. Stat. Theor. M.*, **50** (2021), 2930–2937. doi: 10.1080/03610926.2019.1678643.
37. G. Orlando, R. M. Mininni, M. Bufalo, Interest rates calibration with a CIR model, *J. Risk Financ.*, **20** (2019), 370–387.

38. G. Orlando, R. M. Mininni, M. Bufalo, Forecasting interest rates through Vasicek and CIR models: A partitioning approach, *J. Forecasting*, **39** (2020), 569–579. doi: 10.1002/for.2642.
39. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.*, **81** (1973), 637–654. doi: 10.1142/9789814759588_0001.
40. S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Financ. Stud.*, **6** (1993), 327–343.
41. J. Cox, J. E. Ingersoll, S. A. A. Ross, Theory of the term structure of interest, *Econometrica*, **53** (1985), 129–164.
42. G. Deelstra, Long-term returns in stochastic interest rate models: Applications, *Astin Bull.*, **30** (2000), 123–140. doi: 10.2143/AST.30.1.504629.
43. G. Deelstra, F. Delbaen, Long-term returns in stochastic interest rate models, *Insur. Math. Econ.*, **17** (1995), 163–169. doi: 10.1016/0167-6687(95)00018-N.
44. P. H. Dybvig, J. E. Ingersoll Jr, S. A. Ross, Long forward and zero-coupon rates can never fall, *J. Bus.*, **69** (1996), 1–25.
45. J. Cao, G. Lian, T. R. N. Roslan, Pricing variance swaps under stochastic volatility and stochastic interest rate, *Appl. Math. Comput.*, **277** (2016), 72–81. doi: 10.1016/j.amc.2015.12.027.
46. J. Cao, T. R. N. Roslan, W. Zhang, The valuation of variance swaps under stochastic volatility, stochastic interest rate and full correlation structure, *J. Korean Math. Soc.*, **57** (2020), 1167–1186. doi: 10.4134/JKMS.J190616.
47. A. Loerx, E. W. Sachs, Model calibration in option pricing, *SQUJS*, **17** (2012), 84–102.
48. T. Zhang, L. Li, Y. Lin, W. Xue, F. Xie, H. Xu, et al., An automatic and effective parameter optimization method for model tuning, *Geosci. Model Dev.*, **8** (2015), 3579–3591. doi: 10.5194/gmd-8-3579-2015.
49. Y. Cui, S. del Bano Rollin, G. Germano, Full and fast calibration of the *Heston* stochastic volatility model, *Eur. J. Oper. Res.*, **263** (2017), 625–638. doi: 10.1016/j.ejor.2017.05.018.
50. H. Gudmundsson, D. Vyncke, A generalized weighted Monte Carlo calibration method for derivative pricing, *Mathematics*, **9** (2021), 739. doi: 10.3390/math9070739.
51. Y. Yen, *The effects of structured warrants on firm performance in Malaysia: The role of institutional ownership*, 2017. Available from: https://core.ac.uk/download/pdf/231832379.pdf.

AIMS Press

© 2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)