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Research article

The common Re-nonnegative definite and Re-positive definite solutions to the matrix equations $A_1XA_1^* = C_1$ and $A_2XA_2^* = C_2$

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Abstract: In this paper, we consider the common Re-nonnegative definite (Re-nnd) and Re-positive definite (Re-pd) solutions to a pair of linear matrix equations $A_1XA_1^* = C_1,\ A_2XA_2^* = C_2$ and present some necessary and sufficient conditions for their solvability as well as the explicit expressions for the general common Re-nnd and Re-pd solutions when the consistent conditions are satisfied.

Keywords: matrix equation; Re-nonnegative definite solution; Re-positive definite solution; Moore-Penrose inverse; generalized singular value decomposition
Mathematics Subject Classification: 15A24, 15A57

1. Introduction

For convenience, the notations used in this paper are given in Table 1.

A lot of papers have been published for solving the Re-nnd and Re-pd solutions to some matrix equations. For example, the Re-nnd and Re-pd solutions to $AX = C$ have been considered by Wu [1], Wu and Cain [2] and Groß [3]. The Re-nnd and Re-pd solutions to $AXB = C$ have been discussed by Wang and Yang [4], Cvetković-Ilíc [5], Tian [6] and Yuan and Zuo [7]. The common Re-nnd and Re-pd solutions to $AX = C$ and $XB = D$ have been investigated by Liu [8] and Yuan et al., [9]. Although there are some results [10–12] concerning the Hermitian nonnegative definite and positive definite solutions of the following matrix equations

$$A_1XA_1^* = C_1,\ \ A_2XA_2^* = C_2, \tag{1}$$

where $A_1 \in \mathbb{C}^{m\times n}, A_2 \in \mathbb{C}^{p\times n}, C_1 \in \mathbb{C}^{m\times m}$ and $C_2 \in \mathbb{C}^{p\times p}$, the research results on the Re-nnd and Re-pd solutions to the Eq (1) are quite limited so far. Recently, determinantal representations of solutions and Hermitian solutions to Eq (1) have been considered by Kyrchei [13]. Song and Yu [14] studied the common Re-nnd and Re-pd solutions of (1) firstly by utilizing the maximal and minimal inertias of the linear Hermitian matrix function and the generalized inverses of matrices, and established some necessary and sufficient conditions for the existence of Re-nnd and Re-pd solutions of Eq (1).

The purpose of this paper is to provide an alternative approach to solve Eq (1). The necessary and sufficient conditions for the solvability along with the expressions for the Re-nnd and Re-pd solutions to the Eq (1) are presented with the help of the Moore-Penrose inverses and the spectral decompositions of matrices.

Table 1. Table of notations.

Notations	Meaning
$A^{\dagger}$	Moore-Penrose inverse of the matrix A
A^*	conjugate transpose of the matrix A
$A \geq 0$	A is Hermitian nonnegative definite
$A > 0$	A is Hermitian positive definite
$\frac{1}{2}(A + A^*) \geq 0$	A is Re-nonnegative definite (Re-nnd)
$\frac{1}{2}(A + A^*) > 0$	A is Re-positive definite (Re-pd)
$\mathbb{C}^{m\times n}$	set of all $m \times n$ complex matrices
$P_{\mathcal{L}}$	orthogonal projector on the subspace $\mathcal{L}$
$\mathcal{R}(A)$	range space of the complex matrix A
$\mathcal{N}(A)$	null space of the complex matrix A
I_n	$n \times n$ identity matrix
E_A	$= I_m - AA^{\dagger}, \forall\, A \in \mathbb{C}^{m\times n}$
F_A	$= I_n - A^{\dagger}A, \forall\, A \in \mathbb{C}^{m\times n}$

2. Some lemmas

Some lemmas are needed in the following.

Lemma 1. *[15, 16] Let $A_1 \in \mathbb{C}^{m\times n}, B_1 \in \mathbb{C}^{p\times q}, A_2 \in \mathbb{C}^{l\times n}, B_2 \in \mathbb{C}^{p\times k}$ and $C_1 \in \mathbb{C}^{m\times q}, C_2 \in \mathbb{C}^{l\times k}$. Then the pair of equations $A_1XB_1 = C_1, A_2XB_2 = C_2$ have a common solution X if and only if*

$$A_1A_1^{\dagger}C_1B_1^{\dagger}B_1 = C_1,\ \ A_2A_2^{\dagger}C_2B_2^{\dagger}B_2 = C_2,\ \ P_{\mathcal{T}}(A_1^{\dagger}C_1B_1^{\dagger} - A_2^{\dagger}C_2B_2^{\dagger})P_{\mathcal{S}} = 0,$$

where $\mathcal{T} = \mathcal{R}(A_1^*) \cap \mathcal{R}(A_2^*)$, $\mathcal{S} = \mathcal{R}(B_1) \cap \mathcal{R}(B_2)$. In this case, the general common solution to the equations can be expressed as

$$X = X_0 + F_AV_1 + V_2E_B + F_{A_1}V_3E_{B_2} + F_{A_2}V_4E_{B_1},$$

where

$$X_0 = A_2^{\dagger}C_2B_2^{\dagger} - F_{A_2}\Psi^{\dagger}A_1^{\dagger}A_1D - E_{\Psi}A_1^{\dagger}A_1DB_1B_1^{\dagger}\Xi^{\dagger}E_{B_2},$$

$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, B = [B_1, B_2], \Psi = A_1^{\dagger}A_1F_{A_2}, \Xi = E_{B_2}B_1B_1^{\dagger}, D = A_2^{\dagger}C_2B_2^{\dagger} - A_1^{\dagger}C_1B_1^{\dagger}$, and V_1, V_2, V_3, V_4 are arbitrary matrices.

Lemma 2. *[17, 18] Let $B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{p\times n}$ and $A = A^* \in \mathbb{C}^{n\times n}$. Then the matrix equation*

$$BXC + (BXC)^* = A, \tag{2}$$

has a solution $X \in \mathbb{C}^{m\times p}$ *if and only if*

$$E_BAE_B = 0,\ \ F_CAF_C = 0,\ \ [B, C^*][B, C^*]^{\dagger}A = A. \tag{3}$$

In this case, all the solutions $X \in \mathbb{C}^{m\times p}$ *satisfying Eq* (2) *are given by*

$$X = B^{\dagger}\left(\Theta + F_LS_XF_LBB^{\dagger}\right)C^{\dagger} + M - B^{\dagger}BMCC^{\dagger},$$

where $L = F_CBB^{\dagger}$, $M \in \mathbb{C}^{m\times p}, S_X \in \mathbb{C}^{n\times n}$ *are arbitrary matrices with* $S_X^* = -S_X$ *and* Θ *is given by* $\Theta = \frac{1}{2}A(2I_n - BB^{\dagger}) + \frac{1}{2}(\Phi - \Phi^*)BB^{\dagger}$ *with* $\Phi = 2L^{\dagger}F_CA + \left(I_n - L^{\dagger}F_C\right)AL^{\dagger}L$.

Lemma 3. *[19,20] Let* $A \in \mathbb{C}^{m\times n}$ *and* $D \in \mathbb{C}^{m\times n}$, *and let the singular value decomposition of A be* $A = \tilde{U}\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}\tilde{V}^*$, *where* $\Sigma = diag(\sigma_1, ..., \sigma_r) > 0, r = rank(A)$, $\tilde{U} = [\tilde{U}_1, \tilde{U}_2] \in \mathbb{C}^{m\times m}, \tilde{V} = [\tilde{V}_1, \tilde{V}_2] \in \mathbb{C}^{n\times n}$ *are unitary matrices with* $\tilde{U}_1 \in \mathbb{C}^{m\times r}, \tilde{V}_1 \in \mathbb{C}^{n\times r}$. *Then:*
(a) *The matrix equation* $AY = D$ *has a Hermitian nonnegative definite solution* $Y \in \mathbb{C}^{n\times n}$ *if and only if* $DA^* \geq 0,\ \mathcal{R}(D) = \mathcal{R}(DA^*)$, *in this case, the general Hermitian nonnegative definite solution is*

$$Y = Y_0 + F_AHF_A,$$

where $Y_0 = A^{\dagger}D + F_A(A^{\dagger}D)^* + F_AD^*(DA^*)^{\dagger}DF_A$, *and* $H \in \mathbb{C}^{n\times n}$ *is an arbitrary Hermitian nonnegative definite matrix.*
(b) *The matrix equation* $AY = D$ *has a Hermitian positive definite solution* $Y \in \mathbb{C}^{n\times n}$ *if and only if* $AA^{\dagger}D = D,\ \tilde{U}_1^*DA^*\tilde{U}_1 > 0$, *in this case, the general Hermitian positive definite solution is*

$$Y = Y_0 + F_AHF_A,$$

where $Y_0 = A^{\dagger}D + F_A(A^{\dagger}D)^* + F_AD^*(DA^*)^{\dagger}DF_A$, *and* $H \in \mathbb{C}^{n\times n}$ *is an arbitrary Hermitian positive definite matrix.*

3. Re-nnd and Re-pd solutions to Eq (1)

In this section we propose a general theory elaborating how to solve the common Re-nnd and Re-pd solutions of Eq (1). From Lemma 1, Eq (1) has a common solution $X \in \mathbb{C}^{n\times n}$ if and only if

$$A_1A_1^{\dagger}C_1A_1A_1^{\dagger} = C_1,\ \ A_2A_2^{\dagger}C_2A_2A_2^{\dagger} = C_2,\ \ P_{\mathcal{T}}\left(A_1^{\dagger}C_1(A_1^*)^{\dagger} - A_2^{\dagger}C_2(A_2^*)^{\dagger}\right)P_{\mathcal{T}} = 0, \tag{4}$$

where $\mathcal{T} = \mathcal{R}(A_1^*) \cap \mathcal{R}(A_2^*)$. In this case, the general common solution of Eq (1) is

$$X = X_0 + F_AV_1 + V_2F_A + F_{A_1}V_3F_{A_2} + F_{A_2}V_4F_{A_1}, \tag{5}$$

where X_0 is a particular common solution of Eq (1), which is given by

$$X_0 = A_2^{\dagger}C_2(A_2^*)^{\dagger} - F_{A_2}\Psi^{\dagger}A_1^{\dagger}A_1D - E_{\Psi}A_1^{\dagger}A_1DA_1^{\dagger}A_1(\Psi^*)^{\dagger}F_{A_2}, \tag{6}$$

$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \Psi = A_1^{\dagger}A_1F_{A_2}, D = A_2^{\dagger}C_2(A_2^*)^{\dagger} - A_1^{\dagger}C_1(A_1^*)^{\dagger}$, and V_1, V_2, V_3, V_4 are arbitrary matrices. By Eq (5), we have

$$X + X^* = X_0 + X_0^* + F_AV_{12} + V_{12}^*F_A + F_{A_1}V_{34}F_{A_2} + F_{A_2}V_{34}^*F_{A_1} \triangleq H, \tag{7}$$

where $V_{12} = V_1 + V_2^*, V_{34} = V_3 + V_4^*$. Clearly, X is Re-nnd (Re-pd) if and only if $H \geq 0$ ($H > 0$). By applying Lemma 2, we know that Eq (7) with respect to V_{12} is solvable if and only if

$$A^\dagger A F_{A_1} V_{34} F_{A_2} A^\dagger A + A^\dagger A F_{A_2} V_{34}^* F_{A_1} A^\dagger A = A^\dagger A (H - X_0 - X_0^*) A^\dagger A. \tag{8}$$

In this case, the general solution is

$$V_{12} = \frac{1}{2} F_A \left(H - X_0 - X_0^* - F_{A_1} V_{34} F_{A_2} - F_{A_2} V_{34}^* F_{A_1}\right)(I_n + A^\dagger A) + F_A S_{12} F_A + A^\dagger A M_{12}, \tag{9}$$

where $M_{12} \in \mathbb{C}^{n\times n}$ and $S_{12} \in \mathbb{C}^{n\times n}$ are arbitrary matrices with $S_{12}^* = -S_{12}$.

Notice that

$$\begin{aligned} &\mathcal{R}(F_A) = \mathcal{R}(I_n - A^\dagger A) = \mathcal{N}(A^\dagger A) = \mathcal{N}(A) = \mathcal{N}(A_1) \cap \mathcal{N}(A_2), \\ &\mathcal{R}(F_{A_1}) = \mathcal{N}(A_1),\ \mathcal{R}(F_{A_2}) = \mathcal{N}(A_2). \end{aligned}$$

Therefore,

$$\begin{aligned} &F_{A_1} F_A = F_A = F_A F_{A_1},\ F_{A_2} F_A = F_A = F_A F_{A_2}, \\ &A^\dagger A F_{A_1} = (I_n - F_A) F_{A_1} = F_{A_1} - F_A,\ A^\dagger A F_{A_2} = (I_n - F_A) F_{A_2} = F_{A_2} - F_A, \end{aligned}$$

which implies that

$$\begin{aligned} &A^\dagger A F_{A_1} = F_{A_1} A^\dagger A,\ A^\dagger A F_{A_2} = F_{A_2} A^\dagger A, \\ &(F_{A_1} - F_A)^2 = F_{A_1} - F_A,\ (F_{A_2} - F_A)^2 = F_{A_2} - F_A. \end{aligned}$$

Namely, $A^\dagger A F_{A_1}$ and $A^\dagger A F_{A_2}$ are the orthogonal projectors (see [21, p.80, Ex.63]):

$$A^\dagger A F_{A_1} = P_{\mathcal{R}(A^*)\cap\mathcal{N}(A_1)} \triangleq P_{\mathcal{T}_1},\ A^\dagger A F_{A_2} = P_{\mathcal{R}(A^*)\cap\mathcal{N}(A_2)} \triangleq P_{\mathcal{S}_1}, \tag{10}$$

from which it is easily deduced that

$$A^\dagger A P_{\mathcal{T}_1} = P_{\mathcal{T}_1} = P_{\mathcal{T}_1} A^\dagger A,\ A^\dagger A P_{\mathcal{S}_1} = P_{\mathcal{S}_1} = P_{\mathcal{S}_1} A^\dagger A. \tag{11}$$

From Lemma 2, Eq (8) with respect to V_{34} is solvable if and only if the following three equations hold simultaneously:

$$(I_n - P_{\mathcal{T}_1}) A^\dagger A H A^\dagger A (I_n - P_{\mathcal{T}_1}) = (I_n - P_{\mathcal{T}_1}) A^\dagger A (X_0 + X_0^*) A^\dagger A (I_n - P_{\mathcal{T}_1}), \tag{12}$$

$$(I_n - P_{\mathcal{S}_1}) A^\dagger A H A^\dagger A (I_n - P_{\mathcal{S}_1}) = (I_n - P_{\mathcal{S}_1}) A^\dagger A (X_0 + X_0^*) A^\dagger A (I_n - P_{\mathcal{S}_1}), \tag{13}$$

$$[P_{\mathcal{T}_1}, P_{\mathcal{S}_1}][P_{\mathcal{T}_1}, P_{\mathcal{S}_1}]^\dagger A^\dagger A (H - X_0 - X_0^*) A^\dagger A = A^\dagger A (H - X_0 - X_0^*) A^\dagger A. \tag{14}$$

In the following, we will deduce the necessary and sufficient conditions on $H \geq 0$ ($H > 0$) such that Eqs (12)–(14) are solvable. Let

$$[P_{\mathcal{T}_1}, P_{\mathcal{S}_1}][P_{\mathcal{T}_1}, P_{\mathcal{S}_1}]^\dagger = P_{\mathcal{T}_1 + \mathcal{S}_1} \triangleq P_{\mathcal{L}}. \tag{15}$$

Then, by using the relations of (10) and (11) we get

$$A^\dagger A P_{\mathcal{L}} = P_{\mathcal{L}} = P_{\mathcal{L}} A^\dagger A, \tag{16}$$

$$P_{\mathcal{L}}P_{\mathcal{T}_1}=P_{\mathcal{T}_1}=P_{\mathcal{T}_1}P_{\mathcal{L}},\ P_{\mathcal{L}}P_{\mathcal{S}_1}=P_{\mathcal{S}_1}=P_{\mathcal{S}_1}P_{\mathcal{L}}. \tag{17}$$

By (15), the Eq (14) can be equivalently written as

$$(I_n-P_{\mathcal{L}})A^{\dagger}AHA^{\dagger}A=(I_n-P_{\mathcal{L}})A^{\dagger}A(X_0+X_0^*)A^{\dagger}A. \tag{18}$$

Now, let the spectral decomposition of $A^{\dagger}A$ be

$$A^{\dagger}A=U\begin{bmatrix} I_a & 0\\ 0 & 0\end{bmatrix}U^*=U_1U_1^*, \tag{19}$$

where $a=\text{rank}(A^{\dagger}A)=\text{rank}(A)$ and $U=[U_1,U_2]$ is a unitary matrix with $U_1\in\mathbb{C}^{n\times a}$. It follows from (11), (16), (17) and (19) that

$$P_{\mathcal{T}_1}=U\begin{bmatrix} P_1 & 0\\ 0 & 0\end{bmatrix}U^*, \tag{20}$$

$$P_{\mathcal{S}_1}=U\begin{bmatrix} P_2 & 0\\ 0 & 0\end{bmatrix}U^*, \tag{21}$$

$$P_{\mathcal{L}}=U\begin{bmatrix} P_0 & 0\\ 0 & 0\end{bmatrix}U^*, \tag{22}$$

where $P_1^2=P_1=P_1^*, P_2^2=P_2=P_2^*, P_0^2=P_0=P_0^*$ and $P_0P_1=P_1, P_0P_2=P_2$. By applying the relations of (19) and (22), Eq (18) can be further simplified as

$$(I_a-P_0)H_{11}=(I_a-P_0)U_1^*(X_0+X_0^*)U_1, \tag{23}$$

where

$$U^*HU=\begin{bmatrix} H_{11} & H_{12}\\ H_{12}^* & H_{22}\end{bmatrix}. \tag{24}$$

From Lemma 3, Eq (23) has a Hermitian nonnegative definite solution $H_{11}\in\mathbb{C}^{a\times a}$ if and only if

$$\begin{aligned}&(I_a-P_0)U_1^*(X_0+X_0^*)U_1(I_a-P_0)\geq 0,\\ &\mathcal{R}\Big((I_a-P_0)U_1^*(X_0+X_0^*)U_1\Big)=\mathcal{R}\Big((I_a-P_0)U_1^*(X_0+X_0^*)U_1(I_a-P_0)\Big),\end{aligned} \tag{25}$$

in this case, by simple calculations, we can obtain the general Hermitian nonnegative definite solution of Eq (23) is

$$H_{11}=H_{110}+P_0KP_0, \tag{26}$$

where $D_0=U_1^*(X_0+X_0^*)U_1$ and

$$H_{110}=D_0-P_0D_0P_0+P_0D_0(I_a-P_0)\left[(I_a-P_0)D_0(I_a-P_0)\right]^{\dagger}(I_a-P_0)D_0P_0, \tag{27}$$

and $K\in\mathbb{C}^{a\times a}$ is an arbitrary Hermitian nonnegative definite matrix.

Suppose that the spectral decomposition of I_a-P_0 is

$$I_a-P_0=W\begin{bmatrix} I_g & 0\\ 0 & 0\end{bmatrix}W^*=W_1W_1^*, \tag{28}$$

where $g = \mathrm{rank}(I_a - P_0)$ and $W = [W_1, W_2] \in \mathbb{C}^{a\times a}$ is a unitary matrix with $W_1 \in \mathbb{C}^{a\times g}$. From Lemma 3, Eq (23) has a Hermitian positive definite solution $H_{11} \in \mathbb{C}^{a\times a}$ if and only if

$$W_1^* U_1^*(X_0 + X_0^*)U_1 W_1 > 0, \tag{29}$$

in this case, the general Hermitian positive definite solution is

$$H_{11} = H_{110} + P_0 K P_0, \tag{30}$$

where H_{110} is given by (27) and $K \in \mathbb{C}^{a\times a}$ is an arbitrary Hermitian positive definite matrix.

By the ralations of (19)–(21) and (24), the Eqs (12) and (13) can be equivalently written as

$$(I_a - P_1)H_{11}(I_a - P_1) = (I_a - P_1)U_1^*(X_0 + X_0^*)U_1(I_a - P_1), \tag{31}$$
$$(I_a - P_2)H_{11}(I_a - P_2) = (I_a - P_2)U_1^*(X_0 + X_0^*)U_1(I_a - P_2). \tag{32}$$

Substituting (30) into (31) and (32), and noting that $P_0P_1 = P_1 = P_1P_0, P_0P_2 = P_2 = P_2P_0$, we obtain that

$$(P_0 - P_1)K(P_0 - P_1) = (P_0 - P_1)B(P_0 - P_1), \tag{33}$$
$$(P_0 - P_2)K(P_0 - P_2) = (P_0 - P_2)B(P_0 - P_2), \tag{34}$$

where

$$B = D_0 - D_0(I_a - P_0)\left[(I_a - P_0)D_0(I_a - P_0)\right]^\dagger (I_a - P_0)D_0. \tag{35}$$

Direct verifications shows that $P_0 - P_1$ and $P_0 - P_2$ are orthogonal projectors. Hence, there exist unitary matrices $G \in \mathbb{C}^{a\times a}$ and $Q \in \mathbb{C}^{a\times a}$ such that

$$P_0 - P_1 = G\begin{bmatrix} I_e & 0 \\ 0 & 0 \end{bmatrix}G^* = G_1G_1^*, \tag{36}$$
$$P_0 - P_2 = Q\begin{bmatrix} I_f & 0 \\ 0 & 0 \end{bmatrix}Q^* = Q_1Q_1^*, \tag{37}$$

where $e = \mathrm{rank}(P_0 - P_1), f = \mathrm{rank}(P_0 - P_2)$, and $G_1 \in \mathbb{C}^{a\times e}, Q_1 \in \mathbb{C}^{a\times f}$ are column unitary matrices. By substituting the relations of (36) and (37) into (33) and (34), we arrive at the following equations:

$$G_1^*KG_1 = G_1^*BG_1, \quad Q_1^*KQ_1 = Q_1^*BQ_1. \tag{38}$$

Note that $G_1 \in \mathbb{C}^{a\times e}$ and $Q_1 \in \mathbb{C}^{a\times f}$ are column unitary matrices, it follows from [22] that the generalized singular value decomposition of the matrix pair $[G_1, Q_1]$ is of the following form:

$$G_1 = M\Sigma_1E^*, \quad Q_1 = M\Sigma_2F^*, \tag{39}$$

where $M \in \mathbb{C}^{a\times a}$ is a nonsingular matrix and $E \in \mathbb{C}^{e\times e}, F \in \mathbb{C}^{f\times f}$ are unitary matrices, and

$$\Sigma_1 = \begin{array}{c} \begin{bmatrix} I & 0 \\ 0 & \Gamma \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{matrix} e-s & s \end{matrix} \end{array} \begin{array}{c} e-s \\ s \\ k-e \\ a-k \end{array}, \qquad \Sigma_2 = \begin{array}{c} \begin{bmatrix} 0 & 0 \\ \Delta & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \\ \begin{matrix} s & f-s \end{matrix} \end{array} \begin{array}{c} e-s \\ s \\ k-e \\ a-k \end{array},$$

$k = \text{rank}([G_1, Q_1]) = e + f - s$, and

$$\Gamma = \text{diag}(\gamma_1, \cdots, \gamma_s), \quad \Delta = \text{diag}(\delta_1, \cdots, \delta_s)$$

with

$$1 > \gamma_1 \geq \cdots \geq \gamma_s > 0, \;\; 0 < \delta_1 \leq \cdots \leq \delta_s < 1, \; \gamma_i^2 + \delta_i^2 = 1, \; i = 1, \cdots, s.$$

Substituting (39) into (38) and Partitioning M^*BM into the following form:

$$M^*BM = \begin{array}{c} \left[\begin{array}{cccc} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{12}^* & B_{22} & B_{23} & B_{24} \\ B_{13}^* & B_{23}^* & B_{33} & B_{34} \\ B_{14}^* & B_{24}^* & B_{34}^* & B_{44} \end{array} \right] \begin{array}{c} e-s \\ s \\ k-e \\ a-k \end{array} \\ \begin{array}{cccc} e-s & s & k-e & a-k \end{array} \end{array}. \tag{40}$$

Then, by applying an established result in [9], we obtain

(a) Eq (38) has a common Hermitian nonnegative definite solution K if and only if

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} \geq 0, \;\; \begin{bmatrix} B_{22} & B_{23} \\ B_{23}^* & B_{33} \end{bmatrix} \geq 0.$$

In this case, the general Hermitian nonnegative definite solution of (38) can be expressed as

$$K = (M^*)^{-1} \begin{bmatrix} F(K_{13}) & F(K_{13})S \\ S^*F(K_{13}) & T + S^*F(K_{13})S \end{bmatrix} M^{-1}, \tag{41}$$

where

$$F(K_{13}) \triangleq \begin{bmatrix} B_{11} & B_{12} & K_{13} \\ B_{12}^* & B_{22} & B_{23} \\ K_{13}^* & B_{23}^* & B_{33} \end{bmatrix}, \tag{42}$$

with

$$K_{13} = B_{12}B_{22}^{\dagger}B_{23} + \left(B_{11} - B_{12}B_{22}^{\dagger}B_{12}^*\right)^{\frac{1}{2}} N \left(B_{33} - B_{23}^*B_{22}^{\dagger}B_{23}\right)^{\frac{1}{2}}, \tag{43}$$

and $S \in \mathbb{C}^{k\times(a-k)}$ is an arbitrary matrix, $T \in \mathbb{C}^{(a-k)\times(a-k)}$ is an arbitrary Hermitian nonnegative definite matrix and $N \in \mathbb{C}^{(e-s)\times(k-e)}$ is an arbitrary contraction matrix (i.e., the largest singular value of the matrix N is not greater than 1).

(b) Eq (38) has a common Hermitian positive definite solution K if and only if

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} > 0, \;\; \begin{bmatrix} B_{22} & B_{23} \\ B_{23}^* & B_{33} \end{bmatrix} > 0.$$

In this case, the general Hermitian positive definite solution of (38) can be expressed as

$$K = (M^*)^{-1} \begin{bmatrix} F(K_{13}) & S \\ S^* & T + S^*\left(F(K_{13})\right)^{-1} S \end{bmatrix} M^{-1}, \tag{44}$$

where $F(K_{13})$ is defined by (42) with

$$K_{13} = B_{12}B_{22}^{-1}B_{23} + \left(B_{11} - B_{12}B_{22}^{-1}B_{12}^*\right)^{\frac{1}{2}} N \left(B_{33} - B_{23}^*B_{22}^{-1}B_{23}\right)^{\frac{1}{2}}, \tag{45}$$

and $S \in \mathbb{C}^{k\times(a-k)}$ is an arbitrary matrix, $T \in \mathbb{C}^{(a-k)\times(a-k)}$ is an arbitrary Hermitian positive definite matrix and $N \in \mathbb{C}^{(e-s)\times(k-e)}$ is an arbitrary strict contraction matrix (i.e., the largest singular value of the matrix N is less than 1).

Once we achieve the Hermitian nonnegative definite (positive definite) solution K of Eq (38), the matrix H_{11} in (30) is completely specified. Also, when $H_{11} \geq 0$, by (24) and Theorem 1 of [23], we can determine the expression of the matrix $H \geq 0$ by

$$H = U \begin{bmatrix} H_{11} & H_{11}E \\ E^*H_{11} & E^*H_{11}E + F \end{bmatrix} U^*, \tag{46}$$

and when $H_{11} > 0$, by (24) and Theorem 1 of [23], we can determine the expression of the matrix $H > 0$ by

$$H = U \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{12}^*H_{11}^{-1}H_{12} + R \end{bmatrix} U^*, \tag{47}$$

where E, H_{12}, F and R are arbitrary matrices with $F \geq 0$ and $R > 0$.

By using (19)–(21), the Eq (8) can be simplified as

$$P_1V_{11}^{(34)}P_2 + P_2\left(V_{11}^{(34)}\right)^* P_1 = U_1^*(H - X_0 - X_0^*)U_1, \tag{48}$$

where

$$U^*V_{34}U = \begin{bmatrix} V_{11}^{(34)} & V_{12}^{(34)} \\ V_{21}^{(34)} & V_{22}^{(34)} \end{bmatrix}.$$

When the conditions (12)–(14) hold, From Lemma 2, all the solutions $V_{11}^{(34)} \in \mathbb{C}^{a\times a}$ satisfying Eq (48) are given by

$$V_{11}^{(34)} = P_1\left[\Theta + F_LS_{34}F_LP_1\right]P_2 + M_{34} - P_1M_{34}P_2, \tag{49}$$

where $L = P_1 - P_2P_1$, $M_{34} \in \mathbb{C}^{a\times a}, S_{34} \in \mathbb{C}^{a\times a}$ are arbitrary matrices with $S_{34}^* = -S_{34}$ and Θ is given by

$$\Theta = \frac{1}{2}U_1^*(H - X_0 - X_0^*)U_1(2I_a - P_1) + \frac{1}{2}(\Phi - \Phi^*)P_1,$$

with

$$\Phi = 2L^\dagger(I_a - P_2)U_1^*(H - X_0 - X_0^*)U_1 + \left(I_n - L^\dagger(I_a - P_2)\right)U_1^*(H - X_0 - X_0^*)U_1L^\dagger L,$$

and H being given by (47).

In summary of the discussion above, we have proved the following results.

Theorem 1. *For given matrices $A_1 \in \mathbb{C}^{m\times n}, A_2 \in \mathbb{C}^{p\times n}, C_1 \in \mathbb{C}^{m\times m}$ and $C_2 \in \mathbb{C}^{p\times p}$, let $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \Psi = A_1^\dagger A_1F_{A_2}, D = A_2^\dagger C_2(A_2^*)^\dagger - A_1^\dagger C_1(A_1^*)^\dagger, \mathcal{T} = \mathcal{R}(A_1^*) \cap \mathcal{R}(A_2^*)$ and let $X_0, P_{\mathcal{T}_1}, P_{\mathcal{S}_1}$ and $P_{\mathcal{L}}$ be given by* (6), (10) *and* (15)*, respectively. Suppose that the spectral decompositions of $A^\dagger A, I_a - P_0, P_0 - P_1$ and $P_0 - P_2$ are respectively given by* (19), (28), (36) *and* (37) *with P_1, P_2 and P_0 being given by* (20)–(22)*. Let $D_0 = U_1^*(X_0 + X_0^*)U_1$ and B be given by* (35)*. Furthermore, assume that the generalized singular value decomposition of the matrix pair $[G_1, Q_1]$ is given by* (39)*, and the partition of the matrix M^*BM is given by* (40)*. Then:*

(a) *The Eq* (1) *has a common Re-nnd solution if and only if*

$$A_1A_1^\dagger C_1A_1A_1^\dagger = C_1,\ \ A_2A_2^\dagger C_2A_2A_2^\dagger = C_2,\ \ P_{\mathcal{T}}\left(A_1^\dagger C_1(A_1^*)^\dagger - A_2^\dagger C_2(A_2^*)^\dagger\right)P_{\mathcal{T}} = 0, \tag{4}$$

$$
\begin{aligned}
&(I_a - P_0)U_1^*(X_0 + X_0^*)U_1(I_a - P_0) \geq 0, \\
&\mathcal{R}\Big((I_a - P_0)U_1^*(X_0 + X_0^*)U_1\Big) = \mathcal{R}\Big((I_a - P_0)U_1^*(X_0 + X_0^*)U_1(I_a - P_0)\Big),
\end{aligned} \tag{25}
$$

$$
\begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} \geq 0, \ \begin{bmatrix} B_{22} & B_{23} \\ B_{23}^* & B_{33} \end{bmatrix} \geq 0. \tag{50}
$$

In this case, the general Re-nnd solution of (1) *can be expressed as*

$$
X = X_0 + F_A V_1 + (V_{12} - V_1^*)F_A + F_{A_1} V_3 F_{A_2} + F_{A_2}(V_{34} - V_3^*)F_{A_1}, \tag{51}
$$

where

$$
\begin{aligned}
H &= U \begin{bmatrix} H_{11} & H_{11}E \\ E^* H_{11} & E^* H_{11} E + F \end{bmatrix} U^*, \\
V_{34} &= U \begin{bmatrix} V_{11}^{(34)} & V_{12}^{(34)} \\ V_{21}^{(34)} & V_{22}^{(34)} \end{bmatrix} U^*,
\end{aligned}
$$

$V_{12}, V_{11}^{(34)}$ *and* H_{11} *are respectively given by* (9), (49) *and* (26) *with* K *being given by* (41), *and* $V_1, V_3, V_{12}^{(34)}, V_{21}^{(34)}, V_{22}^{(34)}, E$ *and* F *are arbitrary matrices with* $F \geq 0$.
(*b*) *The Eq* (1) *has a common Re-pd solution if and only if*

$$
A_1 A_1^\dagger C_1 A_1 A_1^\dagger = C_1, \ \ A_2 A_2^\dagger C_2 A_2 A_2^\dagger = C_2, \ \ P_{\mathcal{T}} \left(A_1^\dagger C_1 (A_1^*)^\dagger - A_2^\dagger C_2 (A_2^*)^\dagger\right) P_{\mathcal{T}} = 0, \tag{4}
$$

$$
W_1^* U_1^* (X_0 + X_0^*) U_1 W_1 > 0, \tag{29}
$$

$$
\begin{bmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{bmatrix} > 0, \ \begin{bmatrix} B_{22} & B_{23} \\ B_{23}^* & B_{33} \end{bmatrix} > 0. \tag{52}
$$

In this case, the general Re-pd solution of (1) *can be expressed as*

$$
X = X_0 + F_A V_1 + (V_{12} - V_1^*)F_A + F_{A_1} V_3 F_{A_2} + F_{A_2}(V_{34} - V_3^*)F_{A_1}, \tag{53}
$$

where

$$
\begin{aligned}
H &= U \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{12}^* H_{11}^{-1} H_{12} + R \end{bmatrix} U^*, \\
V_{34} &= U \begin{bmatrix} V_{11}^{(34)} & V_{12}^{(34)} \\ V_{21}^{(34)} & V_{22}^{(34)} \end{bmatrix} U^*,
\end{aligned}
$$

$V_{12}, V_{11}^{(34)}$ *and* H_{11} *are respectively given by* (9), (49) *and* (30) *with* K *being given by* (44), *and* $V_1, V_3, V_{12}^{(34)}, V_{21}^{(34)}, V_{22}^{(34)}, H_{12}$ *and* R *are arbitrary matrices with* $R > 0$.

4. Numerical algorithm and numerical examples

According to Theorem 1, we can describe a numerical algorithm to solve Eq (1).

Algorithm 1.

1). Input A_1, A_2, C_1, C_2.
2). If the conditions (4) are satisfied, go to 3; otherwise, Eq (1) has no solution, and stop.
3). Compute X_0 by (6).
4). Compute $P_{\mathcal{T}_1}, P_{\mathcal{S}_1}$ and $P_{\mathcal{L}}$ by (10) and (15).
5). Compute the spectral decomposition of the matrix $A^{\dagger}A$ by (19).
6). Compute P_1, P_2, P_0 by (20) − (22), respectively.
7). Compute the spectral decomposition of the matrix $I_a - P_0$ by (28).
8). (a) If the conditions (25) are satisfied, go to 9; otherwise, Eq (1) has no Re-nnd solutions, and stop. (b) If the condition (29) is satisfied, go to 9; otherwise, Eq (1) has no Re-pd solutions, and stop.
9). Compute B by (35).
10). Compute the spectral decomposition of the matrices P_0-P_1, P_0-P_2 by (36) and (37), respectively.
11). Compute $B_{ij}, i, j = 1, 2, 3, 4$ by (40).
12). (a) If the conditions (50) are satisfied, go to 13; otherwise, Eq (1) has no Re-nnd solutions, and stop. (b) If the conditions (52) are satisfied, go to 13; otherwise, Eq (1) has no Re-pd solutions, and stop.
13). Choosee matrices M_{12}, S_{12} and M_{34}, S_{34}, and compute V_{12} and $V_{11}^{(34)}$ by (9) and (49), respectively.
14). (a) Choose matrix S, Hermitian nonnegative definite matrix T and contraction matrix N, and compute H_{11}, K by (26) and (41), respectively.
(b) Choose matrix S, Hermitian positive definite matrix T and strict contraction matrix N, and compute H_{11}, K by (30) and (44), respectively.
15). (a) Choose matrices $V_1, V_3, V_{12}^{(34)}, V_{21}^{(34)}, V_{22}^{(34)}, E$ and F, and compute Re-nnd solutions by (51).
(b) Choose matrices $V_1, V_3, V_{12}^{(34)}, V_{21}^{(34)}, V_{22}^{(34)}, H_{12}$ and R, and compute Re-pd solutions by (53).

Example 1. Let $m = 8, n = 7, p = 6$, and the matrices A_1, A_2, C_1, C_2 be given by

$$A_1 = \begin{bmatrix} 0.1505 & 0.3784 & -0.5594 & 0.1939 & -0.2660 & 0.4028 & 0.2790 \\ 0.2672 & 0.0819 & 0.2731 & 0.2876 & -0.3302 & -0.4236 & 0.2238 \\ -0.3406 & -0.1090 & 0.0522 & 0.5436 & -0.1926 & 0.5860 & -0.1038 \\ 0.1986 & 0.1280 & 0.0197 & -0.4028 & 0.3272 & 0.3400 & 0.1071 \\ 0.1530 & 0.1361 & 0.3086 & 0.1080 & 0.1966 & -0.5378 & 0.0197 \\ -0.1595 & 0.3274 & -0.8908 & -0.0056 & 0.0308 & 1.1375 & 0.0823 \\ -0.2300 & -0.0446 & 0.5236 & 0.2363 & 0.5448 & -0.2771 & -0.3015 \\ 0.2282 & -0.2426 & 0.4956 & -0.3762 & -0.0149 & 0.0686 & 0.1080 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3437 & -0.1868 & 0.0463 & 0.1660 & 0.6676 & -0.1841 & 0.0578 \\ 0.5780 & -0.2612 & 0.0910 & 0.6817 & -0.6522 & 0.2167 & -0.3638 \\ 0.0433 & 1.0768 & -0.6168 & -0.6039 & 0.4963 & 0.1074 & -0.0272 \\ -0.0546 & 0.2299 & 0.5315 & 0.1299 & -0.4568 & -0.2744 & 0.2075 \\ -0.1846 & -0.1516 & 0.5293 & 0.1725 & -0.3328 & 0.2160 & 0.2213 \\ -0.3050 & 0.1542 & -0.1967 & -0.4381 & 0.7181 & 0.5132 & 0.2174 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1.6194 & 0.4017 & 1.1172 & 1.0461 & 0.1390 & 1.9358 & 0.1469 & -0.0251 \\ 0.3975 & 0.8697 & 0.2367 & 0.1655 & 0.4800 & -0.3411 & 0.1407 & 0.3210 \\ 1.1213 & 0.2385 & 0.8258 & 0.6943 & 0.0711 & 1.3894 & 0.1250 & -0.0688 \\ 1.0472 & 0.1680 & 0.6923 & 1.2075 & 0.3225 & 1.4312 & 0.5419 & 0.1833 \\ 0.1358 & 0.4778 & 0.0720 & 0.3212 & 0.4447 & -0.2584 & 0.3688 & 0.2969 \\ 1.9415 & -0.3354 & 1.3861 & 1.4331 & -0.2571 & 3.2019 & 0.1518 & -0.3429 \\ 0.1477 & 0.1369 & 0.1291 & 0.5433 & 0.3654 & 0.1582 & 0.5433 & 0.2055 \\ -0.0256 & 0.3222 & -0.0702 & 0.1828 & 0.2981 & -0.3453 & 0.2057 & 0.2709 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1.3478 & 0.5274 & 0.9819 & 0.2793 & 0.4614 & 0.9670 \\ 0.5263 & 1.8846 & -0.3873 & 0.6385 & 0.6056 & -0.6616 \\ 0.9819 & -0.3880 & 2.2990 & -0.1273 & 0.2424 & 2.0467 \\ 0.2798 & 0.6396 & -0.1276 & 0.8107 & 0.6086 & -0.2911 \\ 0.4625 & 0.6031 & 0.2418 & 0.6107 & 0.5554 & 0.1027 \\ 0.9684 & -0.6663 & 2.0459 & -0.2878 & 0.1040 & 2.0062 \end{bmatrix}.$$

It is easy to verify that the conditions (4), (25) and (50) hold. According to Algorithm 1, by choosing matrices $M_{12} = 0, S_{12} = 0, M_{34} = 0, S_{34} = 0, S = 0, V_1 = 0, V_3 = 0, N = I_3$ and $T = \mathrm{diag}(1, 1, 0)$. We can obtain a Re-nnd solution X of Eq (1) as follows:

$$X = \begin{bmatrix} 2.7294 & 2.2145 & 2.0416 & 2.1806 & 1.4082 & 2.1196 & 1.3577 \\ 2.2762 & 3.3339 & 2.2500 & 2.2724 & 2.0174 & 2.8962 & 1.8725 \\ 2.0656 & 2.0314 & 2.0039 & 1.7633 & 1.4075 & 1.6155 & 1.5488 \\ 2.1532 & 2.4969 & 1.7626 & 2.3181 & 1.2999 & 2.2186 & 1.5595 \\ 1.3659 & 2.0056 & 1.2573 & 1.4536 & 2.1437 & 1.9364 & 1.2512 \\ 2.1264 & 2.8759 & 1.6320 & 2.2022 & 1.9514 & 2.9011 & 1.5724 \\ 1.2936 & 2.1706 & 1.4247 & 1.6830 & 1.0376 & 1.5638 & 1.8132 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|A_1XA_1^* - C_1\| = 1.2097 \times 10^{-14}, \ \|A_2XA_2^* - C_2\| = 1.4756 \times 10^{-14},$$

and the eigenvalues of $\frac{1}{2}(X + X^*)$ are $(0.0000, 0.1176, 0.3090, 0.7064, 0.9076, 1.3411, 13.8617)$, which implies that X is a Re-nnd matrix.

Example 2. Let $m = 8, n = 7, p = 6$, and the matrices A_1, A_2 be given as in Example 1, and C_1, C_2 be given by

$$C_1 = \begin{bmatrix} 1.7242 & 0.4144 & 1.0792 & 1.0928 & 0.1592 & 2.0375 & 0.1091 & -0.0736 \\ 0.3775 & 0.8720 & 0.2338 & 0.1408 & 0.4908 & -0.3724 & 0.1604 & 0.3041 \\ 1.1186 & 0.2201 & 0.8853 & 0.6967 & 0.0491 & 1.4232 & 0.1448 & -0.0630 \\ 1.0714 & 0.1515 & 0.6933 & 1.2330 & 0.3025 & 1.4813 & 0.5153 & 0.1907 \\ 0.1416 & 0.4972 & 0.0488 & 0.3012 & 0.4774 & -0.2840 & 0.3880 & 0.2596 \\ 2.0815 & -0.3334 & 1.3653 & 1.5166 & -0.2578 & 3.3675 & 0.0893 & -0.3746 \\ 0.1344 & 0.1489 & 0.1476 & 0.5159 & 0.3870 & 0.1297 & 0.5848 & 0.1796 \\ -0.1206 & 0.2766 & -0.0162 & 0.1611 & 0.2384 & -0.3914 & 0.2131 & 0.3403 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1.3901 & 0.5725 & 0.7567 & 0.2303 & 0.4863 & 0.9310 \\ 0.5929 & 2.0041 & -0.7776 & 0.5634 & 0.6684 & -0.7256 \\ 0.8842 & -0.5659 & 3.1518 & 0.0164 & 0.0683 & 2.1386 \\ 0.2280 & 0.5627 & 0.0364 & 0.8619 & 0.5975 & -0.2604 \\ 0.4375 & 0.5973 & 0.1306 & 0.6040 & 0.6128 & 0.1264 \\ 0.9251 & -0.7174 & 2.2510 & -0.2616 & 0.0863 & 2.0681 \end{bmatrix}.$$

It is easy to verify that the conditions (4), (29), (52) hold. According to Algorithm 1, by choosing matrices $M_{12} = 0, S_{12} = 0, M_{34} = 0, S_{34} = 0, S = 0, V_1 = 0, V_3 = 0, T = I_3$ and $N = \mathrm{diag}(0.5, 0.9)$. We can obtain a Re-pd solution X of Eq (1) as follows:

$$X = \begin{bmatrix} 2.6683 & 2.0218 & 2.0422 & 2.1799 & 1.3933 & 2.1078 & 1.3243 \\ 2.3675 & 3.9741 & 2.2175 & 2.2237 & 2.0140 & 2.8619 & 1.9035 \\ 1.9666 & 2.1038 & 2.0869 & 1.6926 & 1.3193 & 1.6533 & 1.4578 \\ 2.1507 & 2.3774 & 1.7100 & 2.4628 & 1.4043 & 2.1940 & 1.6027 \\ 1.3413 & 2.0309 & 1.2486 & 1.4966 & 2.1548 & 1.9287 & 1.2183 \\ 2.1127 & 2.8483 & 1.6637 & 2.1831 & 1.9384 & 3.0000 & 1.5665 \\ 1.2164 & 2.0357 & 1.3582 & 1.7193 & 1.1233 & 1.5730 & 1.9679 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|A_1XA_1^* - C_1\| = 1.4493 \times 10^{-14}, \ \|A_2XA_2^* - C_2\| = 1.5563 \times 10^{-14},$$

and the eigenvalues of $\frac{1}{2}(X + X^*)$ are $(0.1237, 0.2834, 0.7033, 0.8109, 1.0267, 1.4143, 13.9510)$, which implies that X is a Re-pd matrix.

5. Conclusions

In the previous sections, we studied the common Re-nonnegative definite and Re-positive definite solutions of linear matrix equations (1). We established a set of necessary and sufficient conditions for the existence of a general common Re-nonnegative definite solution, Re-positive definite solution of (1) respectively. Moreover, we gave the explicit expressions for these general common solutions when the consistent conditions are satisfied. At the end, we showed an algorithm and two examples to illustrate the main results of this paper.

References

1. L. Wu, The Re-positive definite solutions to the matrix inverse problem $AX = B$, *Linear Algebra Appl.*, **174** (1992), 145–151. doi: 10.1016/0024-3795(92)90048-F.
2. L. Wu, B. Cain, The Re-nonnegative definite solutions to the matrix inverse problem $AX = B$, *Linear Algebra Appl.*, **236** (1996), 137–146. doi: 10.1016/0024-3795(94)00142-1.
3. J. Groß, Explicit solutions to the matrix inverse problem $AX = B$, *Linear Algebra Appl.*, **289** (1999), 131–134. doi: 10.1016/S0024-3795(97)10008-8.
4. Q. W. Wang, C. L. Yang, The Re-nonnegative definite solutions to the matrix equation $AXB = C$, *Comment. Math. Univ. Ca.*, **39** (1998), 7–13.

5. D. S. Cvetković-Iliíc, Re-nnd solutions of the matrix equation $AXB = C$, *J. Aust. Math. Soc.*, **84** (2008), 63–72. doi: doi:10.1017/S1446788708000207.
6. Y. G. Tian, Maximization and minimization of the rank and inertia of the Hermitian matrix expression $A - BX - (BX)^*$ with applications, *Linear Algebra Appl.*, **434** (2011), 2109–2139. doi: 10.1016/j.laa.2010.12.010.
7. Y. X. Yuan, K. Z. Zuo, The Re-nonnegative definite and Re-positive definite solutions to the matrix equation $AXB = D$, *Appl. Math. Comput.*, **256** (2015), 905–912. doi: 10.1016/j.amc.2015.01.098.
8. X. F. Liu, Comments on "The common Re-nnd and Re-pd solutions to the matrix equations $AX = C$ and $XB = D$", *Appl. Math. Comput.*, **236** (2014), 663–668. doi: 10.1016/j.amc.2014.03.074.
9. Y. X. Yuan, H. T. Zhang, L. Liu, The Re-nnd and Re-pd solutions to the matrix equations $AX = C$, $XB = D$, *Linear and Multilinear Algebra,* **69** (2021), 1645–1656. doi: 10.1080/03081087.2020.1845596.
10. X. Zhang, The general common Hermitian nonnegative-definite solution to the matrix equations $AXA^* = B$ and $CXC^* = D$, *Linear Multilinear A.*, **52** (2004), 49–60. doi: 10.1080/0308108031000122498.
11. X. Zhang, M. Y. Cheng, The general common nonnegative definite and positive definite solutions to the matrix equations $AXA^* = BB^*$ and $CXC^* = DD^*$, *Appl. Math. Lett.*, **17** (2004), 543–547. doi: 10.1016/S0893-9659(04)90123-1.
12. X. Zhang, The general common Hermitian nonnegative-definite solution to the matrix equations $AXA^* = BB^*$ and $CXC^* = DD^*$ with applications in statistics, *J. Multivariate Anal.*, **93** (2005), 257–266. doi: 10.1016/j.jmva.2004.04.009.
13. I. I. Kyrchei, Determinantal representations of solutions and Hermitian solutions to some system of two-sided quaternion matrix equations. *J. Math.*, **2018**, (2018), 6294672. doi: 10.1155/2018/6294672.
14. G. J. Song, S. W. Yu, Nonnegative definite and Re-nonnegative definite solutions to a system of matrix equations with statistical applications, *AppL. Math. Comput.*, **338** (2018), 828–841. doi: 10.1016/j.amc.2018.06.045.
15. H. T. Zhang, H. R. Zhang, L. Liu, Y. X. Yuan, A simple method for solving matrix equations $AXB = D$ and $GXH = C$, *AIMS Mathematics*, **6** (2021), 2579–2589. doi: 10.3934/math.2021156.
16. Y. G. Tian, The solvability of two linear matrix equations, *Linear Multilinear A.*, **48** (2000), 123–147. doi: 10.1080/03081080008818664.
17. K. Yasuda, R. E. Skelton, Assigning controllability and observability Gramians in feedback control, *JGCD*, **14** (1991), 878–885. doi: 10.2514/3.20727.
18. Y. H. Liu, Y. G. Tian, Max-min problems on the ranks and inertias of the matrix expressions $A - BXC \pm (BXC)^*$ with applications, *J. Optim. Theory Appl.*, **148** (2011), 593–622. doi: 10.1007/s10957-010-9760-8.
19. L. Zhang, The solvability conditions for the inverse problem of symmetric nonnegative definite matrices, *Math. Numer. Sinica*, **11** (1989), 337–343.
20. H. Dai, The stability of solutions for two classes of inverse problems of matrices, *Numerical Mathematics–A Journal of Chinese Universities,* **16** (1994), 87–96.

21. A. Ben-Israel, T. N. E. Greville, *Generalized inverses: Theory and applications* New York: Springer, 2003.

22. C. C. Paige, M. A. Saunders, Towards a generalized singular value decompostion, *SIAM J. Numer. Anal.*, **18** (1981), 398–405. doi: 10.1137/0718026.

23. A. Albert, Condition for positive and nonnegative definite in terms of pseudoinverse, *SIAM J. Appl. Math.*, **17** (1969), 434–440. doi: 10.1137/0117041.

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