



*Research article*

## Some fuzzy-interval integral inequalities for harmonically convex fuzzy-interval-valued functions

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**Abstract:** It is well-known fact that fuzzy interval-valued functions (F-I-V-Fs) are generalizations of interval-valued functions (I-V-Fs), and inclusion relation and fuzzy order relation on interval space and fuzzy space are two different concepts. Therefore, by using fuzzy order relation (FOR), we derive inequalities of Hermite-Hadamard ( $H \cdot H$ ) and Hermite-Hadamard Fejér ( $H \cdot H$  Fejér) like for harmonically convex fuzzy interval-valued functions by applying fuzzy Riemann integrals. Moreover, we establish the relation between fuzzy integral inequalities and fuzzy products of harmonically convex fuzzy interval-valued functions. The outcomes of this study are generalizations of many known results which can be viewed as an application of a defined new version of inequalities.

**Keywords:** Harmonically convex fuzzy interval-valued function; fuzzy-interval integrals; Hermite-Hadamard inequality; Hermite-Hadamard Fejér inequality

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## 1. Introduction

Hermite and Hadamard's inequality [1,2] is one of the most well-known inequalities in convex function theory, with a geometrical interpretation and numerous applications. The  $H \cdot H$  inequality is defined as follows for the convex function  $\Psi: K \rightarrow \mathbb{R}$  on an interval  $K = [u, v]$ :

$$\Psi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v \Psi(x) dx \leq \frac{\Psi(u)+\Psi(v)}{2}, \quad (1)$$

for all  $u, v \in K$ .

If  $f$  is concave, the inequalities in (1) hold in the reversed direction. We should point out that Hermite-Hadamard inequality is a refinement of the concept of convexity, and it follows naturally from Jensen's inequality. In recent years, the Hermite-Hadamard inequality for convex functions has gotten a lot of attention, and a lot of improvements and generalizations have been examined; see [3–12] and the references therein.

Interval analysis, on the other hand, is a subset of set-valued analysis, which is the study of sets in the context of mathematical analysis and topology. It was created as a way to deal with interval uncertainty, which can be found in many mathematical or computer models of deterministic real-world phenomena. Archimedes' method, which is used to calculate the circumference of a circle, is a historical example of an interval enclosure. Moore, who is credited with being the first user of intervals in computer mathematics, published the first book on interval analysis in 1966, see [13]. Following the publication of his book, a number of scientists began to research the theory and applications of interval arithmetic. Interval analysis is now a useful technique in a variety of fields that are interested in ambiguous data because of its applications. Computer graphics, experimental and computational physics, error analysis, robotics, and many other fields have applications.

In recent years, several major inequalities (Hermite-Hadamard, Ostrowski, etc.) for interval-valued functions have been studied. Chalco-Cano et al. used the Hukuhara derivative for interval-valued functions to construct Ostrowski type inequalities for interval-valued functions in [14,15]. For interval-valued functions, Román-Flores et al. established Minkowski and Beckenbach's inequalities in [18]. For the rest, see [16–20]. Inequalities, on the other hand, were investigated for the more generic set-valued maps. Sadowska, for example, presented the Hermite-Hadamard inequality in [21]. Other investigations can be found at [22,23].

Recently, Khan et al. [24] introduced the new class of convex fuzzy mappings is known as  $(h_1, h_2)$ -convex F-I-V-Fs by means of FOR such that:

Let  $h_1, h_2: [0, 1] \subseteq K = [u, v] \rightarrow \mathbb{R}^+$  such that  $h_1, h_2 \not\equiv 0$ . Then F-I-V-F  $\tilde{\Psi}: K = [u, v] \rightarrow \mathbb{F}_C(\mathbb{R})$  is said to be  $(h_1, h_2)$ -convex on  $[u, v]$  if

$$\tilde{\Psi}(\xi w + (1 - \xi)y) \preceq h_1(\xi) h_2(1 - \xi) \tilde{\Psi}(w) \tilde{+} h_1(1 - \xi) h_2(\xi) \tilde{\Psi}(y), \quad (2)$$

for all  $w, y \in [u, v]$ ,  $\xi \in [0, 1]$ .

And they also presented the following new version of  $H \cdot H$  type inequality for  $(h_1, h_2)$ -convex F-I-V-F involving fuzzy-interval Riemann integrals:

Let  $\tilde{\Psi}: [u, v] \rightarrow \mathbb{F}_0$  be a  $(h_1, h_2)$ -convex F-I-V-F with  $h_1, h_2: [0, 1] \rightarrow \mathbb{R}^+$  and  $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$ . Then, from  $\theta$ -levels, we get the collection of I-V-Fs  $\Psi_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$  are given by  $\Psi_\theta(\omega) = [\Psi_*(\omega, \theta), \Psi^*(\omega, \theta)]$  for all  $\omega \in [u, v]$  and for all  $\theta \in [0, 1]$ . If  $\tilde{\Psi}$  is fuzzy-interval Riemann integrable (in short, *FR*-integrable), then

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} \tilde{\Psi}\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} (FR) \int_u^v \tilde{\Psi}(\omega) d\omega \leq [\tilde{\Psi}(u) \tilde{\Psi}(v)] \int_0^1 h_1(\xi) h_2(1-\xi) d\xi. \quad (3)$$

If  $h_1(\xi) = \xi$  and  $h_2(\xi) \equiv 1$ , then from (3), we get following the result for convex F-I-V-F:

$$\tilde{\Psi}\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} (FR) \int_u^v \tilde{\Psi}(\omega) d\omega \leq \frac{\tilde{\Psi}(u) \tilde{\Psi}(v)}{2}$$

A one step forward, Khan et al. introduced new classes of convex and generalized convex F-I-V-F, and derived new fractional  $H \cdot H$  type and  $H \cdot H$  type inequalities for convex F-I-V-F [25],  $h$ -convex F-I-V-F [26],  $(h_1, h_2)$ -preinvex F-I-V-F [27], log-s-convex F-I-V-Fs in the second sense [28], LR-log- $h$ -convex I-V-Fs [29], harmonically convex F-I-V-Fs [30], coordinated convex F-I-V-Fs [31] and the references therein. We refer to the readers for further analysis of literature on the applications and properties of fuzzy-interval, and inequalities and generalized convex fuzzy mappings, see [32–45] and the references therein.

The goal of this study is to complete the fuzzy Riemann integrals for interval-valued functions and use these integrals to get the Hermite-Hadamard inequality. These integrals are also used to derive Hermite-Hadamard type inequalities for harmonically convex F-I-V-Fs.

## 2. Preliminaries

In this section, we recall some basic preliminary notions, definitions and results. With the help of these results, some new basic definitions and results are also discussed.

We begin by recalling the basic notations and definitions. We define interval as,

$$[\omega_*, \omega^*] = \{\omega \in \mathbb{R}: \omega_* \leq \omega \leq \omega^* \text{ and } \omega_*, \omega^* \in \mathbb{R}\}, \text{ where } \omega_* \leq \omega^*.$$

We write  $\text{len}[\omega_*, \omega^*] = \omega^* - \omega_*$ , If  $\text{len}[\omega_*, \omega^*] = 0$  then,  $[\omega_*, \omega^*]$  is called degenerate. In this article, all intervals will be non-degenerate intervals. The collection of all closed and bounded intervals of  $\mathbb{R}$  is denoted and defined as  $\mathcal{K}_C = \{[\omega_*, \omega^*]: \omega_*, \omega^* \in \mathbb{R} \text{ and } \omega_* \leq \omega^*\}$ . If  $\omega_* \geq 0$  then,  $[\omega_*, \omega^*]$  is called positive interval. The set of all positive interval is denoted by  $\mathcal{K}_C^+$  and defined as  $\mathcal{K}_C^+ = \{[\omega_*, \omega^*]: [\omega_*, \omega^*] \in \mathcal{K}_C \text{ and } \omega_* \geq 0\}$ .

We'll now look at some of the properties of intervals using arithmetic operations. Let  $[q_*, q^*], [s_*, s^*] \in \mathcal{K}_C$  and  $\rho \in \mathbb{R}$ , then we have

$$\begin{aligned} [q_*, q^*] + [s_*, s^*] &= [q_* + s_*, q^* + s^*], \\ [q_*, q^*] \times [s_*, s^*] &= \left[ \min\{q_*s_*, q^*s_*, q_*s^*, q^*s^*\}, \right. \\ &\quad \left. \max\{q_*s_*, q^*s_*, q_*s^*, q^*s^*\} \right], \\ \rho \cdot [q_*, q^*] &= \begin{cases} [\rho q_*, \rho q^*] & \text{if } \rho > 0 \\ \{0\} & \text{if } \rho = 0 \\ [\rho q^*, \rho q_*] & \text{if } \rho < 0. \end{cases} \end{aligned}$$

For  $[q_*, q^*], [s_*, s^*] \in \mathcal{K}_C$ , the inclusion “ $\subseteq$ ” is defined by  $[q_*, q^*] \subseteq [s_*, s^*]$ , if and only if  $s_* \leq q_*$ ,  $q^* \leq s^*$ .

**Remark 2.1.** The relation “ $\leq_I$ ” defined on  $\mathcal{K}_C$  by

$$[q_*, q^*] \leq_I [s_*, s^*] \text{ if and only if } q_* \leq s_*, q^* \leq s^*, \quad (4)$$

for all  $[\varrho_*, \varrho^*], [\mathfrak{s}_*, \mathfrak{s}^*] \in \mathcal{K}_C$ , it is an order relation, see [41]. For given  $[\varrho_*, \varrho^*], [\mathfrak{s}_*, \mathfrak{s}^*] \in \mathcal{K}_C$ , we say that  $[\varrho_*, \varrho^*] \leq_I [\mathfrak{s}_*, \mathfrak{s}^*]$  if and only if  $\varrho_* \leq \mathfrak{s}_*, \varrho^* \leq \mathfrak{s}^*$  or  $\varrho_* \leq \mathfrak{s}_*, \varrho^* < \mathfrak{s}^*$ .

Moore [13] initially proposed the concept of Riemann integral for I-V-F, which is defined as follows:

**Theorem 2.2.** [13] If  $\Psi: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C$  is an I-V-F on such that  $\Psi(w) = [\Psi_*(w), \Psi^*(w)]$ . Then  $\Psi$  is Riemann integrable over  $[u, v]$  if and only if,  $\Psi_*$  and  $\Psi^*$  both are Riemann integrable over  $[u, v]$  such that

$$(IR) \int_u^v \Psi(w)dw = [(R) \int_u^v \Psi_*(w)dw, (R) \int_u^v \Psi^*(w)dw]. \quad (5)$$

Let  $\mathbb{R}$  be the set of real numbers. A mapping  $\tilde{\zeta}: \mathbb{R} \rightarrow [0, 1]$  called the membership function distinguishes a fuzzy subset set  $\mathcal{A}$  of  $\mathbb{R}$ . This representation is found to be acceptable in this study.  $\mathbb{F}(\mathbb{R})$  also stand for the collection of all fuzzy subsets of  $\mathbb{R}$ .

A real fuzzy interval  $\tilde{\zeta}$  is a fuzzy set in  $\mathbb{R}$  with the following properties:

- (1)  $\tilde{\zeta}$  is normal i.e. there exists  $w \in \mathbb{R}$  such that  $\tilde{\zeta}(w) = 1$ ;
- (2)  $\tilde{\zeta}$  is upper semi continuous i.e., for given  $w \in \mathbb{R}$ , for every  $w \in \mathbb{R}$  there exist  $\varepsilon > 0$  there exist  $\delta > 0$  such that  $\tilde{\zeta}(w) - \tilde{\zeta}(y) < \varepsilon$  for all  $y \in \mathbb{R}$  with  $|w - y| < \delta$ ;
- (3)  $\tilde{\zeta}$  is fuzzy convex i.e.,  $\tilde{\zeta}((1 - \xi)w + \xi y) \geq \min(\tilde{\zeta}(w), \tilde{\zeta}(y))$ ,  $\forall w, y \in \mathbb{R}$  and  $\xi \in [0, 1]$ ;
- (4)  $\tilde{\zeta}$  is compactly supported i.e.,  $cl\{w \in \mathbb{R} | \tilde{\zeta}(w) > 0\}$  is compact.

The collection of all real fuzzy intervals is denoted by  $\mathbb{F}_0$ .

Let  $\tilde{\zeta} \in \mathbb{F}_0$  be real fuzzy interval, if and only if,  $\theta$ -levels  $[\tilde{\zeta}]^\theta$  is a nonempty compact convex set of  $\mathbb{R}$ . This is represented by

$$[\tilde{\zeta}]^\theta = \{w \in \mathbb{R} | \tilde{\zeta}(w) \geq \theta\},$$

from these definitions, we have

$$[\tilde{\zeta}]^\theta = [\zeta_*(\theta), \zeta^*(\theta)],$$

where

$$\zeta_*(\theta) = \inf\{w \in \mathbb{R} | \tilde{\zeta}(w) \geq \theta\},$$

$$\zeta^*(\theta) = \sup\{w \in \mathbb{R} | \tilde{\zeta}(w) \geq \theta\}.$$

Thus a real fuzzy interval  $\tilde{\zeta}$  can be identified by a parametrized triples

$$\{(\zeta_*(\theta), \zeta^*(\theta), \theta): \theta \in [0, 1]\}.$$

These two end point functions  $\zeta_*(\theta)$  and  $\zeta^*(\theta)$  are used to characterize a real fuzzy interval as a result.

**Proposition 2.3.** [18] Let  $\tilde{\zeta}, \tilde{\theta} \in \mathbb{F}_0$ . Then fuzzy order relation “ $\leq$ ” given on  $\mathbb{F}_0$  by

$$\tilde{\zeta} \preceq \tilde{\theta} \text{ if and only if, } [\tilde{\zeta}]^\theta \leq_I [\tilde{\theta}]^\theta \text{ for all } \theta \in (0, 1],$$

it is partial order relation.

We'll now look at some of the properties of fuzzy intervals using arithmetic operations. Let  $\tilde{\zeta}, \tilde{\theta} \in \mathbb{F}_0$  and  $\rho \in \mathbb{R}$ , then we have

$$[\tilde{\zeta} \tilde{+} \tilde{\theta}]^\theta = [\tilde{\zeta}]^\theta + [\tilde{\theta}]^\theta, \quad (6)$$

$$[\tilde{\zeta} \tilde{\times} \tilde{\theta}]^\theta = [\tilde{\zeta}]^\theta \times [\tilde{\theta}]^\theta, \quad (7)$$

$$[\rho \cdot \tilde{\zeta}]^\theta = \rho \cdot [\tilde{\zeta}]^\theta. \quad (8)$$

For  $\psi \in \mathbb{F}_0$  such that  $\tilde{\zeta} = \tilde{\theta} \tilde{+} \tilde{\psi}$ , we have the existence of the Hukuhara difference of  $\tilde{\zeta}$  and  $\tilde{\theta}$ , which we call the H-difference of  $\tilde{\zeta}$  and  $\tilde{\theta}$ , and denoted by  $\tilde{\zeta} \tilde{-} \tilde{\theta}$ . If H-difference exists, then

$$\begin{aligned} (\psi)_*(\theta) &= (\tilde{\zeta} \tilde{-} \tilde{\theta})_*(\theta) = \zeta_*(\theta) - \theta_*(\theta), \\ (\psi)^*(\theta) &= (\tilde{\zeta} \tilde{-} \tilde{\theta})^*(\theta) = \zeta^*(\theta) - \theta^*(\theta). \end{aligned} \quad (9)$$

**Definition 2.4.** [38] A fuzzy-interval-valued map  $\tilde{\Psi}: [u, v] \subset \mathbb{R} \rightarrow \mathbb{F}_0$  is called F-I-V-F. For each  $\theta \in (0, 1]$ , whose  $\theta$ -levels define the family of I-V-Fs  $\Psi_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C$  are given by  $\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)]$  for all  $w \in [u, v]$ . Here, for each  $\theta \in (0, 1]$ , the end point real functions  $\Psi_*(\cdot, \theta), \Psi^*(\cdot, \theta): [u, v] \rightarrow \mathbb{R}$  are called lower and upper functions of  $\tilde{\Psi}$ .

The following conclusions can be drawn from the preceding literature review [38–40]:

**Definition 2.5.** Let  $\tilde{\Psi}: [u, v] \subset \mathbb{R} \rightarrow \mathbb{F}_0$  be an F-I-V-F. Then fuzzy integral of  $\tilde{\Psi}$  over  $[u, v]$ , denoted by  $(FR) \int_u^v \tilde{\Psi}(w) dw$ , it is given level-wise by

$$[(FR) \int_u^v \tilde{\Psi}(w) dw]^\theta = (IR) \int_u^v \Psi_\theta(w) dw = \left\{ \int_u^v \Psi(w, \theta) dw : \Psi(w, \theta) \in \mathcal{R}_{([u, v], \theta)} \right\}, \quad (10)$$

for all  $\theta \in (0, 1]$ , where  $\mathcal{R}_{([u, v], \theta)}$  denotes the collection of Riemannian integrable functions of I-V-Fs.  $\tilde{\Psi}$  is  $FR$ -integrable over  $[u, v]$  if  $(FR) \int_u^v \tilde{\Psi}(w) dw \in \mathbb{F}_0$ . Note that, if  $\Psi_*(w, \theta), \Psi^*(w, \theta)$  are Lebesgue-integrable, then  $\Psi$  is fuzzy Aumann-integrable function over  $[u, v]$ , see [18,39,40].

**Theorem 2.6.** Let  $\tilde{\Psi}: [u, v] \subset \mathbb{R} \rightarrow \mathbb{F}_0$  be a F-I-V-F, whose  $\theta$ -levels define the family of I-V-Fs  $\Psi_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C$  are given by  $\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)]$  for all  $w \in [u, v]$  and for all  $\theta \in (0, 1]$ . Then  $\tilde{\Psi}$  is  $FR$ -integrable over  $[u, v]$  if and only if,  $\Psi_*(w, \theta)$  and  $\Psi^*(w, \theta)$  both are  $R$ -integrable over  $[u, v]$ . Moreover, if  $\tilde{\Psi}$  is  $FR$ -integrable over  $[u, v]$ , then

$$[(FR) \int_u^v \tilde{\Psi}(w) dw]^\theta = \left[ (R) \int_u^v \Psi_*(w, \theta) dw, (R) \int_u^v \Psi^*(w, \theta) dw \right] = (IR) \int_u^v \Psi_\theta(w) dw, \quad (11)$$

for all  $\theta \in (0, 1]$ . For all  $\theta \in (0, 1]$ ,  $\mathcal{FR}_{([u, v], \theta)}$  denotes the collection of all  $FR$ -integrable F-I-V-Fs over  $[u, v]$ .

**Definition 2.7.** [42] A set  $K = [u, v] \subset \mathbb{R}^+ = (0, \infty)$  is said to be convex set, if, for all  $w, y \in K, \xi \in [0, 1]$ , we have

$$\frac{wy}{\xi w + (1-\xi)y} \in K. \quad (12)$$

**Definition 2.8.** [42] The  $\Psi: [u, v] \rightarrow \mathbb{R}^+$  is called harmonically convex function on  $[u, v]$  if

$$\Psi\left(\frac{wy}{\xi w + (1-\xi)y}\right) \leq (1-\xi)\Psi(w) + \xi\Psi(y), \quad (13)$$

for all  $w, y \in [u, v], \xi \in [0, 1]$ , where  $\Psi(w) \geq 0$  for all  $w \in [u, v]$ . If (13) is reversed then,  $\Psi$  is called harmonically concave F-I-V-F on  $[u, v]$ .

**Definition 2.11.** [30] The F-I-V-F  $\tilde{\Psi}: [u, v] \rightarrow \mathbb{F}_0$  is called harmonically convex F-I-V-F on  $[u, v]$  if

$$\tilde{\Psi}\left(\frac{wy}{\xi w + (1-\xi)y}\right) \preceq (1-\xi)\tilde{\Psi}(w) \tilde{+} \xi\tilde{\Psi}(y), \quad (14)$$

for all  $w, y \in [u, v], \xi \in [0, 1]$ , where  $\tilde{\Psi}(w) \succeq \tilde{0}$ , for all  $w \in [u, v]$ . If (14) is reversed then,  $\tilde{\Psi}$  is called harmonically concave F-I-V-F on  $[u, v]$ .

**Definition 2.12.** The F-I-V-F  $\tilde{\Psi}: [u, v] \rightarrow \mathbb{F}_0$  is called harmonically convex F-I-V-F on  $[u, v]$  if

$$\tilde{\Psi}\left(\frac{wy}{\xi w + (1-\xi)y}\right) \preceq (1-\xi)\tilde{\Psi}(w) \tilde{+} \xi\tilde{\Psi}(y), \quad (15)$$

for all  $w, y \in [u, v], \xi \in [0, 1]$ , where  $\tilde{\Psi}(w) \succeq \tilde{0}$ , for all  $w \in [u, v]$ . If (15) is reversed then,  $\tilde{\Psi}$  is called harmonically concave F-I-V-F on  $[u, v]$ . The set of all harmonically convex (harmonically concave) F-I-V-F is denoted by

$$\begin{aligned} &HFSX([u, v], \mathbb{F}_0), \\ &(HFSV([u, v], \mathbb{F}_0)). \end{aligned}$$

**Theorem 2.13.** Let  $[u, v]$  be harmonically convex set, and let  $\tilde{\Psi}: [u, v] \rightarrow \mathbb{F}_c(\mathbb{R})$  be a F-I-V-F, whose  $\theta$ -levels define the family of I-V-Fs  $\Psi_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_c^+ \subset \mathcal{K}_c$  are given by

$$\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)], \forall w \in [u, v]. \quad (16)$$

for all  $w \in [u, v], \theta \in [0, 1]$ . Then,  $\tilde{\Psi} \in HFSX([u, v], \mathbb{F}_0)$ , if and only if, for all  $\theta \in [0, 1]$ ,  $\Psi_*(w, \theta), \Psi^*(w, \theta) \in HSX([u, v], \mathbb{R}^+)$ .

*Proof.* The demonstration of proof is similar to proof of Theorem 2.12, see [26].

**Example 2.14.** We consider the F-I-V-Fs  $\tilde{\Psi}: [0, 2] \rightarrow \mathbb{F}_c(\mathbb{R})$  defined by,

$$\tilde{\Psi}(w)(\partial) = \begin{cases} \frac{\partial}{\sqrt{w}} & \partial \in [0, \sqrt{w}] \\ \frac{2-\partial}{2\sqrt{w}} & \partial \in (\sqrt{w}, 2\sqrt{w}] \\ 0 & \text{otherwise.} \end{cases}$$

Then, for each  $\theta \in [0, 1]$ , we have  $\Psi_\theta(w) = [\theta\sqrt{w}, (2-\theta)\sqrt{w}]$ . Since  $\Psi_*(w, \theta), \Psi^*(w, \theta) \in HSX([u, v], \mathbb{R}^+)$ , for each  $\theta \in [0, 1]$ . Hence  $\tilde{\Psi} \in HFSX([u, v], \mathbb{F}_0)$ .

**Remark 2.15.** If  $\mathcal{T}_*(u, \theta) = \mathcal{T}^*(v, \theta)$  with  $\theta = 1$ , then harmonically convex F-I-V-F reduces to the classical harmonically convex function, see [42].

### 3. Fuzzy-interval fractional Hermite-Hadamard inequalities

In this section, we will prove two types of inequalities. First one is  $H.H$  and their variant forms, and the second one is  $H \cdot H$  Fejér inequalities for convex F-I-V-Fs where the integrands are F-I-V-Fs.

**Theorem 3.1.** Let  $\tilde{\Psi} \in HFSX([u, v], \mathbb{F}_0)$ , whose  $\theta$ -levels define the family of I-V-Fs  $\Psi_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$  are given by  $\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)]$  for all  $w \in [u, v]$ ,  $\theta \in [0, 1]$ . If  $\tilde{\Psi} \in \mathcal{FR}_{([u, v], \theta)}$ , then

$$\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \leq \frac{\tilde{\Psi}(u) \tilde{\Psi}(v)}{2}. \quad (17)$$

If  $\tilde{\Psi} \in HFSV([u, v], \mathbb{F}_0)$ , then

$$\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \geq \frac{uv}{v-u} \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \geq \frac{\tilde{\Psi}(u) \tilde{\Psi}(v)}{2}. \quad (18)$$

*Proof.* Let  $\tilde{\Psi} \in HFSX([u, v], \mathbb{F}_0)$ . Then, by hypothesis, we have

$$2\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \leq \tilde{\Psi}\left(\frac{uv}{\xi u + (1-\xi)v}\right) \tilde{\Psi}\left(\frac{uv}{(1-\xi)u + \xi v}\right).$$

Therefore, for each  $\theta \in [0, 1]$ , we have

$$\begin{aligned} 2\Psi_*\left(\frac{2uv}{u+v}, \theta\right) &\leq \Psi_*\left(\frac{uv}{\xi u + (1-\xi)v}, \theta\right) + \Psi_*\left(\frac{uv}{(1-\xi)u + \xi v}, \theta\right), \\ 2\Psi^*\left(\frac{2uv}{u+v}, \theta\right) &\leq \Psi^*\left(\frac{uv}{\xi u + (1-\xi)v}, \theta\right) + \Psi^*\left(\frac{uv}{(1-\xi)u + \xi v}, \theta\right). \end{aligned}$$

Then

$$\begin{aligned} 2 \int_0^1 \Psi_*\left(\frac{2uv}{u+v}, \theta\right) d\xi &\leq \int_0^1 \Psi_*\left(\frac{uv}{\xi u + (1-\xi)v}, \theta\right) d\xi + \int_0^1 \Psi_*\left(\frac{uv}{(1-\xi)u + \xi v}, \theta\right) d\xi, \\ 2 \int_0^1 \Psi^*\left(\frac{2uv}{u+v}, \theta\right) d\xi &\leq \int_0^1 \Psi^*\left(\frac{uv}{\xi u + (1-\xi)v}, \theta\right) d\xi + \int_0^1 \Psi^*\left(\frac{uv}{(1-\xi)u + \xi v}, \theta\right) d\xi. \end{aligned}$$

It follows that

$$\begin{aligned} \Psi_*\left(\frac{2uv}{u+v}, \theta\right) &\leq \frac{uv}{v-u} \int_u^v \frac{\Psi_*(w, \theta)}{w^2} dw, \\ \Psi^*\left(\frac{2uv}{u+v}, \theta\right) &\leq \frac{uv}{v-u} \int_u^v \frac{\Psi^*(w, \theta)}{w^2} dw. \end{aligned}$$

That is

$$\left[ \Psi_*\left(\frac{2uv}{u+v}, \theta\right), \Psi^*\left(\frac{2uv}{u+v}, \theta\right) \right] \leq_l \frac{uv}{v-u} \left[ \int_u^v \frac{\Psi_*(w, \theta)}{w^2} dw, \int_u^v \frac{\Psi^*(w, \theta)}{w^2} dw \right].$$

Thus,

$$\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw. \quad (19)$$

In a similar way as above, we have

$$\frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \leq \frac{\tilde{\Psi}(u) + \tilde{\Psi}(v)}{2}. \quad (20)$$

Combining (19) and (20), we have

$$\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \leq \frac{\tilde{\Psi}(u) + \tilde{\Psi}(v)}{2}.$$

Hence, the required result.

**Remark 3.2.** If  $\Psi_*(w, \theta) = \Psi^*(w, \theta)$  with  $\theta = 1$ , then Theorem 3.1 reduces to the result for classical harmonically convex function, see [42]:

$$\Psi\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} (R) \int_u^v \frac{\Psi(w)}{w^2} dw \leq \frac{\Psi(u) + \Psi(v)}{2}.$$

**Example 3.3.** We consider the FIVFs  $\tilde{\Psi}: [0, 2] \rightarrow \mathbb{F}_C(\mathbb{R})$ , as in Example 2.14. Then, for each  $\theta \in [0, 1]$ , we have  $\Psi_\theta(w) = [\theta\sqrt{w}, (2-\theta)\sqrt{w}]$  is harmonically convex FIVF. Since,  $\Psi_*(w, \theta) = \theta\sqrt{w}$ ,  $\Psi^*(w, \theta) = (2-\theta)\sqrt{w}$ . We now compute the following:

$$\Psi_*\left(\frac{2uv}{u+v}, \theta\right) \leq \frac{uv}{v-u} \int_u^v \frac{\Psi_*(w, \theta)}{w^2} dw \leq \frac{\Psi_*(u, \theta) + \Psi_*(v, \theta)}{2},$$

$$\Psi_*\left(\frac{2uv}{u+v}, \theta\right) = \Psi_*(0, \theta) = 0,$$

$$\frac{uv}{v-u} \int_u^v \frac{\Psi_*(w, \theta)}{w^2} dw = \frac{0}{2} \int_0^2 \frac{\theta\sqrt{w}}{w^2} dw = 0,$$

$$\frac{\Psi_*(u, \theta) + \Psi_*(v, \theta)}{2} = \frac{\theta}{\sqrt{2}},$$

for all  $\theta \in [0, 1]$ . That means

$$0 \leq 0 \leq \frac{\theta}{\sqrt{2}}.$$

Similarly, it can be easily show that

$$\Psi^*\left(\frac{2uv}{u+v}, \theta\right) \leq \frac{uv}{v-u} \int_u^v \frac{\Psi^*(w, \theta)}{w^2} dw \leq \frac{\Psi^*(u, \theta) + \Psi^*(v, \theta)}{2}.$$

for all  $\theta \in [0, 1]$ , such that

$$\Psi^*\left(\frac{2uv}{u+v}, \theta\right) = \Psi^*(0, \theta) = 0,$$



$$\frac{uv}{v-u} \int_u^v \frac{\Psi^*(w, \theta)}{w^2} dw = \frac{0}{2} \int_0^2 \frac{(2-\theta)\sqrt{w}}{w^2} dw = 0,$$

$$\frac{\Psi^*(u, \theta) + \Psi^*(v, \theta)}{2} = \frac{(2-\theta)}{\sqrt{2}}.$$

From which, we have

$$0 \leq 0 \leq \frac{(2-\theta)}{\sqrt{2}},$$

that is

$$[0, 0] \leq_l [0, 0] \leq_l \frac{1}{\sqrt{2}} [\theta, (2-\theta)], \text{ for all } \theta \in [0, 1].$$

Hence,

$$\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \leq \frac{\tilde{\Psi}(u) \tilde{\Psi}(v)}{2}.$$

**Theorem 3.4.** Let  $\tilde{\Psi} \in HFSX([u, v], \mathbb{F}_0)$ , whose  $\theta$ -levels define the family of I-V-Fs  $\Psi_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$  are given by  $\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)]$  for all  $w \in [u, v]$ ,  $\theta \in [0, 1]$ . If  $\tilde{\Psi} \in \mathcal{FR}_{([u, v], \theta)}$ , then

$$\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \leq_{\triangleright_2} \frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \leq_{\triangleright_1} \frac{\tilde{\Psi}(u) \tilde{\Psi}(v)}{2}, \quad (21)$$

where

$$\triangleright_1 = \frac{1}{2} \left[ \frac{\tilde{\Psi}(u) \tilde{\Psi}(v)}{2} \tilde{\Psi}\left(\frac{2uv}{u+v}\right) \right],$$

$$\triangleright_2 = \frac{1}{2} \left[ \tilde{\Psi}\left(\frac{4uv}{u+3v}\right) \tilde{\Psi}\left(\frac{4uv}{3u+v}\right) \right],$$

and  $\triangleright_1 = [\triangleright_{1*}, \triangleright_{1*}^*]$ ,  $\triangleright_2 = [\triangleright_{2*}, \triangleright_{2*}^*]$ . If  $\tilde{\Psi} \in HFSV([u, v], \mathbb{F}_0)$ , then inequality (21) is reversed.

*Proof.* Take  $\left[u, \frac{2uv}{u+v}\right]$ , we have

$$2\tilde{\Psi}\left(\frac{u \frac{4uv}{u+v}}{\xi u + (1-\xi) \frac{2uv}{u+v}} + \frac{u \frac{4uv}{u+v}}{(1-\xi)u + \xi \frac{2uv}{u+v}}\right)$$

$$\leq \tilde{\Psi}\left(\frac{u \frac{2uv}{u+v}}{\xi u + (1-\xi) \frac{2uv}{u+v}}\right) \tilde{\Psi}\left(\frac{u \frac{2uv}{u+v}}{(1-\xi)u + \xi \frac{2uv}{u+v}}\right).$$

Therefore, for every  $\theta \in [0, 1]$ , we have

$$\begin{aligned}
& 2\Psi_* \left( \frac{u \frac{4uv}{u+v}}{\xi u + (1-\xi) \frac{2uv}{u+v}} + \frac{u \frac{4uv}{u+v}}{(1-\xi)u + \xi \frac{2uv}{u+v}}, \theta \right) \\
& \leq \Psi_* \left( \frac{u \frac{2uv}{u+v}}{\xi u + (1-\xi) \frac{2uv}{u+v}}, \theta \right) + \Psi_* \left( \frac{u \frac{2uv}{u+v}}{(1-\xi)u + \xi \frac{2uv}{u+v}}, \theta \right), \\
& 2\Psi^* \left( \frac{u \frac{4uv}{u+v}}{\xi u + (1-\xi) \frac{2uv}{u+v}} + \frac{u \frac{4uv}{u+v}}{(1-\xi)u + \xi \frac{2uv}{u+v}}, \theta \right) \\
& \leq \Psi^* \left( \frac{u \frac{2uv}{u+v}}{\xi u + (1-\xi) \frac{2uv}{u+v}}, \theta \right) + \Psi^* \left( \frac{u \frac{2uv}{u+v}}{(1-\xi)u + \xi \frac{2uv}{u+v}}, \theta \right).
\end{aligned}$$

In consequence, we obtain

$$\begin{aligned}
\frac{1}{2} \Psi_* \left( \frac{4uv}{u+3v}, \theta \right) & \leq \frac{uv}{v-u} \int_u^{\frac{2uv}{u+v}} \frac{\Psi_*(w, \theta)}{w^2} dw, \\
\frac{1}{2} \Psi^* \left( \frac{4uv}{u+3v}, \theta \right) & \leq \frac{uv}{v-u} \int_u^{\frac{2uv}{u+v}} \frac{\Psi^*(w, \theta)}{w^2} dw.
\end{aligned}$$

That is

$$\frac{1}{2} \left[ \Psi_* \left( \frac{4uv}{u+3v}, \theta \right), \Psi^* \left( \frac{4uv}{u+3v}, \theta \right) \right] \leq \frac{uv}{v-u} \left[ \int_u^{\frac{2uv}{u+v}} \frac{\Psi_*(w, \theta)}{w^2} dw, \int_u^{\frac{2uv}{u+v}} \frac{\Psi^*(w, \theta)}{w^2} dw \right].$$

It follows that

$$\frac{1}{2} \tilde{\Psi} \left( \frac{4uv}{u+3v} \right) \leq \frac{uv}{v-u} \int_u^{\frac{2uv}{u+v}} \frac{\tilde{\Psi}(w)}{w^2} dw. \quad (22)$$

In a similar way as above, we have

$$\frac{1}{2} \tilde{\Psi} \left( \frac{4uv}{3u+v} \right) \leq \frac{uv}{v-u} \int_{\frac{2uv}{u+v}}^v \frac{\tilde{\Psi}(w)}{w^2} dw. \quad (23)$$

Combining (22) and (23), we have

$$\frac{1}{2} \left[ \tilde{\Psi} \left( \frac{4uv}{u+3v} \right) \mp \tilde{\Psi} \left( \frac{4uv}{3u+v} \right) \right] \leq \frac{uv}{v-u} \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw. \quad (24)$$

Therefore, for every  $\theta \in [0, 1]$ , by using Theorem 3.1, we have

$$\begin{aligned}
\Psi_* \left( \frac{2uv}{u+v}, \theta \right) & \leq \frac{1}{2} \left[ \Psi_* \left( \frac{4uv}{u+3v}, \theta \right) + \Psi_* \left( \frac{4uv}{3u+v}, \theta \right) \right], \\
\Psi^* \left( \frac{2uv}{u+v}, \theta \right) & \leq \frac{1}{2} \left[ \Psi^* \left( \frac{4uv}{u+3v}, \theta \right) + \Psi^* \left( \frac{4uv}{3u+v}, \theta \right) \right],
\end{aligned}$$

$$\begin{aligned}
&= \triangleright_{2*}, \\
&= \triangleright_2^*, \\
&\leq \frac{uv}{v-u} \int_u^v \frac{\Psi_*(w, \theta)}{w^2} dw, \\
&\leq \frac{uv}{v-u} \int_u^v \frac{\Psi^*(w, \theta)}{w^2} dw, \\
&\leq \frac{1}{2} \left[ \frac{\Psi_*(u, \theta) + \Psi_*(v, \theta)}{2} + \Psi_*\left(\frac{2uv}{u+v}, \theta\right) \right], \\
&\leq \frac{1}{2} \left[ \frac{\Psi^*(u, \theta) + \Psi^*(v, \theta)}{2} + \Psi^*\left(\frac{2uv}{u+v}, \theta\right) \right], \\
&= \triangleright_{1*}, \\
&= \triangleright_1^*, \\
&\leq \frac{1}{2} \left[ \frac{\Psi_*(u, \theta) + \Psi_*(v, \theta)}{2} + \frac{1}{2} (\Psi_*(u, \theta) + \Psi_*(v, \theta)) \right], \\
&\leq \frac{1}{2} \left[ \frac{\Psi^*(u, \theta) + \Psi^*(v, \theta)}{2} + \frac{1}{2} (\Psi^*(u, \theta) + \Psi^*(v, \theta)) \right], \\
&= \frac{1}{2} [\Psi_*(u, \theta) + \Psi_*(v, \theta)], \\
&= \frac{1}{2} [\Psi^*(u, \theta) + \Psi^*(v, \theta)],
\end{aligned}$$

that is

$$\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \leq \triangleright_2 \leq \frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \leq \triangleright_1 \leq \frac{1}{2} [\tilde{\Psi}(u) \tilde{\Psi}(v)].$$

**Theorem 3.5.** Let  $\tilde{\Psi} \in HFSX([u, v], \mathbb{F}_0)$  and  $\tilde{\mathcal{P}} \in HFSX([u, v], \mathbb{F}_0)$ , whose  $\theta$ -levels  $\Psi_\theta, \mathcal{P}_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$  are defined by  $\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)]$  and  $\mathcal{P}_\theta(w) = [\mathcal{P}_*(w, \theta), \mathcal{P}^*(w, \theta)]$  for all  $w \in [u, v], \theta \in [0, 1]$ , respectively. If  $\tilde{\Psi} \tilde{\times} \tilde{\mathcal{P}} \in \mathcal{FR}_{([u, v], \theta)}$ , then

$$\frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w) \tilde{\times} \tilde{\mathcal{P}}(w)}{w^2} dw \leq \frac{\tilde{\mathcal{M}}(u, v)}{3} \tilde{\times} \frac{\tilde{\mathcal{N}}(u, v)}{6},$$

where  $\tilde{\mathcal{M}}(u, v) = \tilde{\Psi}(u) \tilde{\times} \tilde{\mathcal{P}}(u) \tilde{\times} \tilde{\Psi}(v) \tilde{\times} \tilde{\mathcal{P}}(v)$ ,  $\tilde{\mathcal{N}}(u, v) = \tilde{\Psi}(u) \tilde{\times} \tilde{\mathcal{P}}(v) \tilde{\times} \tilde{\Psi}(v) \tilde{\times} \tilde{\mathcal{P}}(u)$ , and  $\mathcal{M}_\theta(u, v) = [\mathcal{M}_*((u, v), \theta), \mathcal{M}^*((u, v), \theta)]$  and  $\mathcal{N}_\theta(u, v) = [\mathcal{N}_*((u, v), \theta), \mathcal{N}^*((u, v), \theta)]$ .

*Proof.* Since  $\tilde{\Psi}, \tilde{\mathcal{P}}$  are harmonically convex F-I-V-Fs then, for each  $\theta \in [0, 1]$  we have

$$\begin{aligned}
\Psi_*\left(\frac{uv}{(1-\xi)u + \xi v}, \theta\right) &\leq \xi \Psi_*(u, \theta) + (1-\xi) \Psi_*(v, \theta), \\
\Psi^*\left(\frac{uv}{(1-\xi)u + \xi v}, \theta\right) &\leq \xi \Psi^*(u, \theta) + (1-\xi) \Psi^*(v, \theta).
\end{aligned}$$

And

$$\mathcal{P}_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \leq \xi \mathcal{P}_*(u, \theta) + (1-\xi) \mathcal{P}_*(v, \theta),$$

$$\mathcal{P}^* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \leq \xi \mathcal{P}^*(u, \theta) + (1-\xi) \mathcal{P}^*(v, \theta).$$

From the definition of harmonically convexity of F-I-V-Fs it follows that  $\tilde{\Psi}(w) \geq \tilde{0}$  and  $\tilde{\mathcal{P}}(w) \geq \tilde{0}$ , so

$$\begin{aligned} & \Psi_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \times \mathcal{P}_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \\ & \leq (\xi \Psi_*(u, \theta) + (1-\xi) \Psi_*(v, \theta)) (\xi \mathcal{P}_*(u, \theta) + (1-\xi) \mathcal{P}_*(v, \theta)) \\ & = \Psi_*(u, \theta) \times \mathcal{P}_*(u, \theta) [(\xi)(\xi)] + \Psi_*(v, \theta) \times \mathcal{P}_*(v, \theta) [(1-\xi)(1-\xi)] \\ & \quad + \Psi_*(u, \theta) \mathcal{P}_*(v, \theta) \xi(1-\xi) + \Psi_*(v, \theta) \times \mathcal{P}_*(u, \theta) \xi(1-\xi), \\ & \Psi^* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \times \mathcal{P}^* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \\ & \leq (\xi \Psi^*(u, \theta) + (1-\xi) \Psi^*(v, \theta)) (\xi \mathcal{P}^*(u, \theta) + (1-\xi) \mathcal{P}^*(v, \theta)) \\ & = \Psi^*(u, \theta) \times \mathcal{P}^*(u, \theta) [(\xi)(\xi)] + \Psi^*(v, \theta) \times \mathcal{P}^*(v, \theta) [(1-\xi)(1-\xi)] \\ & \quad + \Psi^*(u, \theta) \times \mathcal{P}^*(v, \theta) \xi(1-\xi) + \Psi^*(v, \theta) \times \mathcal{P}^*(u, \theta) \xi(1-\xi). \end{aligned}$$

Integrating both sides of above inequality over  $[0, 1]$  we get

$$\begin{aligned} \int_0^1 \Psi_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \times \mathcal{P}_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) &= \frac{uv}{v-u} \int_u^v \frac{\Psi_*(w, \theta) \times \mathcal{P}_*(w, \theta)}{w^2} dw \\ &\leq (\Psi_*(u, \theta) \times \mathcal{P}_*(u, \theta) + \Psi_*(v, \theta) \times \mathcal{P}_*(v, \theta)) \int_0^1 (\xi)(\xi) d\xi \\ &\quad + (\Psi_*(u, \theta) \times \mathcal{P}_*(v, \theta) + \Psi_*(v, \theta) \times \mathcal{P}_*(u, \theta)) \int_0^1 \xi(1-\xi) d\xi, \\ \int_0^1 \Psi^* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \times \mathcal{P}^* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) &= \frac{uv}{v-u} \int_u^v \frac{\Psi^*(w, \theta) \times \mathcal{P}^*(w, \theta)}{w^2} dw \\ &\leq (\Psi^*(u, \theta) \times \mathcal{P}^*(u, \theta) + \Psi^*(v, \theta) \times \mathcal{P}^*(v, \theta)) \int_0^1 (\xi)(\xi) d\xi \\ &\quad + (\Psi^*(u, \theta) \times \mathcal{P}^*(v, \theta) + \Psi^*(v, \theta) \times \mathcal{P}^*(u, \theta)) \int_0^1 \xi(1-\xi) d\xi. \end{aligned}$$

It follows that,

$$\frac{uv}{v-u} \int_u^v \Psi_*(w, \theta) \times \mathcal{P}_*(w, \theta) dw \leq \frac{\mathcal{M}_*((u, v), \theta)}{3} + \frac{\mathcal{N}_*((u, v), \theta)}{6}$$

$$\frac{uv}{v-u} \int_u^v \Psi^*(w, \theta) \times \mathcal{P}^*(w, \theta) dw \leq \frac{\mathcal{M}^*((u, v), \theta)}{3} + \frac{\mathcal{N}^*((u, v), \theta)}{6},$$

that is

$$\begin{aligned} & \frac{uv}{v-u} \left[ \int_u^v \Psi_*(w, \theta) \times \mathcal{P}_*(w, \theta) dw, \int_u^v \Psi^*(w, \theta) \times \mathcal{P}^*(w, \theta) dw \right] \\ & \leq_I \frac{1}{3} [\mathcal{M}_*((u, v), \theta), \mathcal{M}^*((u, v), \theta)] + \frac{1}{6} [\mathcal{N}_*((u, v), \theta), \mathcal{N}^*((u, v), \theta)]. \end{aligned}$$

Thus,

$$\frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w) \tilde{\times} \tilde{\mathcal{P}}(w)}{w^2} dw \leq \frac{\tilde{\mathcal{M}}(u, v)}{3} \tilde{\mp} \frac{\tilde{\mathcal{N}}(u, v)}{6}.$$

**Theorem 3.6.** Let  $\tilde{\Psi}, \tilde{\mathcal{P}} \in HFSX([u, v], \mathbb{F}_0)$ , whose  $\theta$ -levels  $\Psi_\theta, \mathcal{P}_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$  are defined by  $\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)]$  and  $\mathcal{P}_\theta(w) = [\mathcal{P}_*(w, \theta), \mathcal{P}^*(w, \theta)]$  for all  $w \in [u, v]$ ,  $\theta \in [0, 1]$ , respectively. If  $\tilde{\Psi} \tilde{\times} \tilde{\mathcal{P}} \in \mathcal{FR}_{([u, v], \theta)}$ , then

$$2\tilde{\Psi} \left( \frac{2uv}{u+v} \right) \tilde{\times} \tilde{\mathcal{P}} \left( \frac{2uv}{u+v} \right) \leq \frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w) \tilde{\times} \tilde{\mathcal{P}}(w)}{w^2} dw + \frac{\tilde{\mathcal{M}}(u, v)}{6} \tilde{\mp} \frac{\tilde{\mathcal{N}}(u, v)}{3},$$

where  $\tilde{\mathcal{M}}(u, v) = \tilde{\Psi}(u) \tilde{\times} \tilde{\mathcal{P}}(u) \tilde{\mp} \tilde{\Psi}(v) \tilde{\times} \tilde{\mathcal{P}}(v)$ ,  $\tilde{\mathcal{N}}(u, v) = \tilde{\Psi}(u) \tilde{\times} \tilde{\mathcal{P}}(v) \tilde{\mp} \tilde{\Psi}(v) \tilde{\times} \tilde{\mathcal{P}}(u)$ , and  $\mathcal{M}_\theta(u, v) = [\mathcal{M}_*((u, v), \theta), \mathcal{M}^*((u, v), \theta)]$  and  $\mathcal{N}_\theta(u, v) = [\mathcal{N}_*((u, v), \theta), \mathcal{N}^*((u, v), \theta)]$ .

*Proof.* By hypothesis, for each  $\theta \in [0, 1]$ , we have

$$\begin{aligned} & \Psi_* \left( \frac{2uv}{u+v}, \theta \right) \times \mathcal{J}_* \left( \frac{2uv}{u+v}, \theta \right) \\ & \Psi^* \left( \frac{2uv}{u+v}, \theta \right) \times \mathcal{J}^* \left( \frac{2uv}{u+v}, \theta \right) \\ & \leq \frac{1}{4} \left[ \begin{aligned} & \Psi_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times \mathcal{J}_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \\ & + \Psi_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times \mathcal{J}_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \end{aligned} \right] \\ & + \frac{1}{4} \left[ \begin{aligned} & \Psi_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \times \mathcal{J}_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \\ & + \Psi_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times \mathcal{J}_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \end{aligned} \right], \\ & \leq \frac{1}{4} \left[ \begin{aligned} & \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times \mathcal{J}^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \\ & + \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times \mathcal{J}^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \end{aligned} \right] \\ & + \frac{1}{4} \left[ \begin{aligned} & \Psi^* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \times \mathcal{J}^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \\ & + \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times \mathcal{J}^* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \end{aligned} \right], \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4} \left[ \Psi_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times J_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \right. \\
&\quad \left. + \Psi_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \times J_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \right] \\
&\quad + \frac{1}{4} \left[ \begin{aligned} &(\xi \Psi_*(u, \theta) + (1-\xi) \Psi_*(v, \theta)) \\ &\times ((1-\xi) J_*(u, \theta) + \xi J_*(v, \theta)) \\ &+ ((1-\xi) \Psi_*(u, \theta) + \xi \Psi_*(v, \theta)) \\ &\times (\xi J_*(u, \theta) + (1-\xi) J_*(v, \theta)) \end{aligned} \right], \\
&\leq \frac{1}{4} \left[ \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times J^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \right. \\
&\quad \left. + \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times J^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \right] \\
&\quad + \frac{1}{4} \left[ \begin{aligned} &(\xi \Psi^*(u, \theta) + (1-\xi) \Psi^*(v, \theta)) \\ &\times ((1-\xi) J^*(u, \theta) + \xi J^*(v, \theta)) \\ &+ ((1-\xi) \Psi^*(u, \theta) + \xi \Psi^*(v, \theta)) \\ &\times (\xi J^*(u, \theta) + (1-\xi) J^*(v, \theta)) \end{aligned} \right] \\
&= \frac{1}{4} \left[ \Psi_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times J_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \right. \\
&\quad \left. + \Psi_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times J_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \right] \\
&\quad + \frac{1}{4} \left[ \begin{aligned} &\{(\xi)(\xi) + (1-\xi)(1-\xi)\} \mathcal{N}_*((u, v), \theta) \\ &+ \{\xi(1-\xi) + \xi(1-\xi)\} \mathcal{M}_*((u, v), \theta) \end{aligned} \right], \\
&= \frac{1}{4} \left[ \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times J^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \right. \\
&\quad \left. + \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \times J^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \right] \\
&\quad + \frac{1}{4} \left[ \begin{aligned} &\{(\xi)(\xi) + (1-\xi)(1-\xi)\} \mathcal{N}^*((u, v), \theta) \\ &+ \{\xi(1-\xi) + \xi(1-\xi)\} \mathcal{M}^*((u, v), \theta) \end{aligned} \right].
\end{aligned}$$

Integrating over  $[0, 1]$ , we have

$$\begin{aligned}
&2\Psi_* \left( \frac{2uv}{u+v}, \theta \right) \times J_* \left( \frac{2uv}{u+v}, \theta \right) \\
&\leq \frac{1}{v-u} (R) \int_u^v \Psi_*(w, \theta) \times J_*(w, \theta) dw + \mathcal{M}_*((u, v), \theta) \int_0^1 \xi(1-\xi) d\xi \\
&\quad + \mathcal{N}_*((u, v), \theta) \int_0^1 (\xi)(\xi) d\xi,
\end{aligned}$$

$$\begin{aligned}
& 2\Psi^*\left(\frac{2uv}{u+v}, \theta\right) \times J^*\left(\frac{2uv}{u+v}, \theta\right) \\
& \leq \frac{1}{v-u} (R) \int_u^v \Psi^*(w, \theta) \times J^*(w, \theta) dw + \mathcal{M}^*((u, v), \theta) \int_0^1 \xi(1-\xi) d\xi \\
& + \mathcal{N}^*((u, v), \theta) \int_0^1 (\xi)(\xi) d\xi,
\end{aligned}$$

that is

$$2\tilde{\Psi}\left(\frac{2uv}{u+v}\right) \tilde{\mathcal{P}}\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w) \tilde{\mathcal{P}}(w)}{w^2} dw + \frac{\tilde{\mathcal{M}}(u, v)}{6} \tilde{\mathcal{N}}(u, v).$$

The theorem has been proved.

First, we will propose the following inequality linked with the right part of the classical  $H - H$  Fejér inequality for harmonically convex F-I-V-Fs through fuzzy order relation, which is said to be 2nd fuzzy  $H - H$  Fejér inequality.

**Theorem 3.7.** (Second fuzzy  $H - H$  Fejér inequality) Let  $\tilde{\Psi} \in HFSX([u, v], \mathbb{F}_0)$ , whose  $\theta$ -levels define the family of I-V-Fs  $\Psi_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_C^+$  are given by  $\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)]$  for all  $w \in [u, v]$ ,  $\theta \in [0, 1]$ . If  $\tilde{\Psi} \in \mathcal{FR}_{([u, v], \theta)}$  and  $\nabla: [u, v] \rightarrow \mathbb{R}$ ,  $\nabla\left(\frac{1}{\frac{1}{u} + \frac{1}{v} - \frac{1}{w}}\right) = \nabla(w) \geq 0$ , then

$$(FR) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} \nabla(w) dw \leq \frac{\tilde{\Psi}(u) \tilde{\mathcal{P}}(v)}{2} \int_0^1 \frac{\nabla(w)}{w^2} dw. \quad (25)$$

If  $\tilde{\Psi} \in HFSV([u, v], \mathbb{F}_0)$ , then inequality (25) is reversed.

*Proof.* Let  $\Psi$  be a harmonically convex F-I-V-F. Then, for each  $\theta \in [0, 1]$ , we have

$$\begin{aligned}
& \Psi_*\left(\frac{uv}{(1-\xi)u + \xi v}, \theta\right) \nabla\left(\frac{uv}{(1-\xi)u + \xi v}\right) \\
& \leq (\xi\Psi_*(u, \theta) + (1-\xi)\Psi_*(v, \theta)) \nabla\left(\frac{uv}{(1-\xi)u + \xi v}\right), \\
& \Psi^*\left(\frac{uv}{(1-\xi)u + \xi v}, \theta\right) \nabla\left(\frac{uv}{(1-\xi)u + \xi v}\right) \\
& \leq (\xi\Psi^*(u, \theta) + (1-\xi)\Psi^*(v, \theta)) \nabla\left(\frac{uv}{(1-\xi)u + \xi v}\right).
\end{aligned} \quad (26)$$

And

$$\begin{aligned}
& \Psi_*\left(\frac{uv}{\xi u + (1-\xi)v}, \theta\right) \nabla\left(\frac{uv}{\xi u + (1-\xi)v}\right) \\
& \leq ((1-\xi)\Psi_*(u, \theta) + \xi\Psi_*(v, \theta)) \nabla\left(\frac{uv}{\xi u + (1-\xi)v}\right),
\end{aligned}$$

$$\begin{aligned} & \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) \\ & \leq ((1-\xi)\Psi^*(u, \theta) + \xi\Psi^*(v, \theta)) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right). \end{aligned} \quad (27)$$

After adding (26) and (27), and integrating over  $[0, 1]$ , we get

$$\begin{aligned} & \int_0^1 \Psi_* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) d\xi \\ & + \int_0^1 \Psi_* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi \\ & \leq \int_0^1 \left[ \Psi_*(u, \theta) \left\{ \xi \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) + (1-\xi) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) \right\} \right. \\ & \quad \left. + \Psi_*(v, \theta) \left\{ (1-\xi) \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) + \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) \right\} \right] d\xi, \\ & \int_0^1 \Psi^* \left( \frac{uv}{(1-\xi)u + \xi v}, \theta \right) \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) d\xi \\ & + \int_0^1 \Psi^* \left( \frac{uv}{\xi u + (1-\xi)v}, \theta \right) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi \\ & \leq \int_0^1 \left[ \Psi^*(u, \theta) \left\{ \xi \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) + (1-\xi) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) \right\} \right. \\ & \quad \left. + \Psi^*(v, \theta) \left\{ (1-\xi) \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) + \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) \right\} \right] d\xi. \\ & = 2\Psi_*(u, \theta) \int_0^1 \xi \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) d\xi + 2\Psi_*(v, \theta) \int_0^1 \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi, \\ & = 2\Psi^*(u, \theta) \int_0^1 \xi \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) d\xi + 2\Psi^*(v, \theta) \int_0^1 \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi. \end{aligned}$$

Since  $\nabla$  is symmetric, then

$$\begin{aligned} & = 2[\Psi_*(u, \theta) + \Psi_*(v, \theta)] \int_0^1 \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi, \\ & = 2[\Psi^*(u, \theta) + \Psi^*(v, \theta)] \int_0^1 \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi. \end{aligned} \quad (28)$$

Since

$$\begin{aligned} & \int_0^1 \Psi_*(\xi u + (1-\xi)v, \theta) \nabla \left( \frac{uv}{(1-\xi)u + \xi v} \right) d\xi \\ & = \int_0^1 \Psi_*((1-\xi)u + \xi v, \theta) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi \\ & = \frac{uv}{v-u} \int_u^v \Psi_*(w, \theta) \nabla(w) dw \end{aligned}$$



$$\begin{aligned}
& \int_0^1 \Psi^*((1-\xi)u + \xi v, \theta) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi \\
&= \int_0^1 \Psi^*(\xi u + (1-\xi)v, \theta) \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi \\
&= \frac{uv}{v-u} \int_u^v \Psi^*(w, \theta) \nabla(w) dw.
\end{aligned} \tag{29}$$

From (28) and (29), we have

$$\begin{aligned}
\frac{uv}{v-u} \int_u^v \Psi_*(w, \theta) \nabla(w) dw &\leq [\Psi_*(u, \theta) + \Psi_*(v, \theta)] \int_0^1 \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi, \\
\frac{uv}{v-u} \int_u^v \Psi^*(w, \theta) \nabla(w) dw &\leq [\Psi^*(u, \theta) + \Psi^*(v, \theta)] \int_0^1 \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi,
\end{aligned}$$

that is

$$\begin{aligned}
& \left[ \frac{uv}{v-u} \int_u^v \Psi_*(w, \theta) \nabla(w) dw, \frac{uv}{v-u} \int_u^v \Psi^*(w, \theta) \nabla(w) dw \right] \\
&\leq_I [\Psi_*(u, \theta) + \Psi_*(v, \theta), \Psi^*(u, \theta) + \Psi^*(v, \theta)] \int_0^1 \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi,
\end{aligned}$$

hence

$$\frac{uv}{v-u} (FR) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} \nabla(w) dw \leq [\tilde{\Psi}(u) \tilde{\Psi}(v)] \int_0^1 \xi \nabla \left( \frac{uv}{\xi u + (1-\xi)v} \right) d\xi,$$

this concludes the proof.

Next, we construct first  $H \cdot H$  Fejér inequality for harmonically convex F-I-V-F, which generalizes first  $H - H$  Fejér inequality for harmonically convex function.

**Theorem 3.8.** (First fuzzy fractional  $H - H$  Fejér inequality) Let  $\tilde{\Psi} \in HFSX([u, v], \mathbb{F}_0)$ , whose  $\theta$ -levels define the family of I-V-Fs  $\Psi_\theta: [u, v] \subset \mathbb{R} \rightarrow \mathcal{K}_c^+$  are given by  $\Psi_\theta(w) = [\Psi_*(w, \theta), \Psi^*(w, \theta)]$  for all  $w \in [u, v]$ ,  $\theta \in [0, 1]$ . If  $\tilde{\Psi} \in \mathcal{FR}_{([u, v], \theta)}$  and  $\nabla: [u, v] \rightarrow \mathbb{R}$ ,

$\nabla \left( \frac{1}{\frac{1}{u} + \frac{1}{v} - \frac{1}{w}} \right) = \nabla(w) \geq 0$ , then

$$\tilde{\Psi} \left( \frac{2uv}{u+v} \right) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \leq (FR) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} \nabla(w) dw. \tag{30}$$

If  $\tilde{\Psi} \in HFSV([u, v], \mathbb{F}_0)$ , then inequality (30) is reversed.

*Proof.* Since  $\Psi$  is a harmonically convex, then for  $\theta \in [0, 1]$ , we have

$$\begin{aligned}\Psi_*\left(\frac{2uv}{u+v}, \theta\right) &\leq \frac{1}{2}\left(\Psi_*\left(\frac{uv}{(1-\xi)u+\xi v}, \theta\right) + \Psi_*\left(\frac{uv}{\xi u+(1-\xi)v}, \theta\right)\right) \\ \Psi^*\left(\frac{2uv}{u+v}, \theta\right) &\leq \frac{1}{2}\left(\Psi^*\left(\frac{uv}{(1-\xi)u+\xi v}, \theta\right) + \Psi^*\left(\frac{uv}{\xi u+(1-\xi)v}, \theta\right)\right).\end{aligned}\quad (31)$$

By multiplying (31) by  $\nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right) = \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right)$  and integrate it by  $\xi$  over  $[0, 1]$ , we obtain

$$\begin{aligned}&\Psi_*\left(\frac{2uv}{u+v}, \theta\right) \int_0^1 \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right) d\xi \\ &\leq \frac{1}{2} \left( \int_0^1 \Psi_*\left(\frac{uv}{(1-\xi)u+\xi v}, \theta\right) \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right) d\xi \right. \\ &\quad \left. + \int_0^1 \Psi_*\left(\frac{uv}{\xi u+(1-\xi)v}, \theta\right) \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right) d\xi \right) \\ &\Psi^*\left(\frac{2uv}{u+v}, \theta\right) \int_0^1 \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right) d\xi \\ &\leq \frac{1}{2} \left( \int_0^1 \Psi^*\left(\frac{uv}{(1-\xi)u+\xi v}, \theta\right) \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right) d\xi \right. \\ &\quad \left. + \int_0^1 \Psi^*\left(\frac{uv}{\xi u+(1-\xi)v}, \theta\right) \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right) d\xi \right).\end{aligned}\quad (32)$$

Since

$$\begin{aligned}&\int_0^1 \Psi_*\left(\frac{uv}{(1-\xi)u+\xi v}, \theta\right) \nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right) d\xi \\ &= \int_0^1 \Psi_*\left(\frac{uv}{\xi u+(1-\xi)v}, \theta\right) \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right) d\xi \\ &= \frac{uv}{v-u} \int_u^v \Psi_*(w, \theta) \nabla(w) dw, \\ &\int_0^1 \Psi^*\left(\frac{uv}{\xi u+(1-\xi)v}, \theta\right) \nabla\left(\frac{uv}{\xi u+(1-\xi)v}\right) d\xi \\ &= \int_0^1 \Psi^*\left(\frac{uv}{(1-\xi)u+\xi v}, \theta\right) \nabla\left(\frac{uv}{(1-\xi)u+\xi v}\right) d\xi \\ &= \frac{uv}{v-u} \int_u^v \Psi^*(w, \theta) \nabla(w) dw.\end{aligned}\quad (33)$$

From (32) and (33), we have

$$\begin{aligned}\Psi_*\left(\frac{2uv}{u+v}, \theta\right) &\leq \frac{1}{\int_u^v \nabla(w) dw} \int_u^v \Psi_*(w, \theta) \nabla(w) dw, \\ \Psi^*\left(\frac{2uv}{u+v}, \theta\right) &\leq \frac{1}{\int_u^v \nabla(w) dw} \int_u^v \Psi^*(w, \theta) \nabla(w) dw.\end{aligned}$$

From which, we have

$$\leq_I \frac{1}{\int_u^v \nabla(w) dw} \left[ \int_u^v \Psi_*(w, \theta) \nabla(w) dw, \int_u^v \Psi^*(w, \theta) \nabla(w) dw \right],$$

that is

$$\tilde{\Psi} \left( \frac{2uv}{u+v} \right) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} dw \leq (FR) \int_u^v \frac{\tilde{\Psi}(w)}{w^2} \nabla(w) dw.$$

Then we complete the proof.

**Remark 3.9.** If  $\nabla(w) = 1$ , then from Theorems 3.7 and 3.8, we obtain inequality (17). If  $\Psi_*(w, \theta) = \Psi^*(w, \theta)$  with  $\theta = 1$ , then Theorems 3.7 and 3.8 reduce to classical first and second classical  $H - H$  Fejér inequality for classical harmonically convex function.

#### 4. Conclusions and future plan

Several novel conclusions in convex analysis and associated optimization theory can be obtained using this new class of functions known as harmonically convex F-I-V. The main findings include some new bounds with error estimations via fuzzy Riemann integrals. All of these papers aim to provide new estimations and optimal approaches. But, the main motivation of this paper is that we obtained new method by using fuzzy integrals for harmonically convex F-I-V-Fs calculus. The authors anticipate that this study may inspire more research in a variety of pure and applied sciences fields.

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#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

#### Conflict of interest

The authors declare that they have no competing interests.

## References

1. S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA monographs, Victoria University, 2004.
2. J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Elsevier, 1992.
3. F. X. Chen, A note on Hermite-Hadamard inequalities for products of convex functions, *J. Appl. Math.*, **2013** (2013), 935020. doi: 10.1155/2013/935020.
4. S. S. Dragomir, Inequalities of Hermite-Hadamard type for  $h$ -convex functions on linear spaces, *Proyecciones*, **34** (2015), 323–341. doi: 10.4067/S0716-09172015000400002.
5. S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.*, **167** (1992), 49–56. doi: 10.1016/0022-247X(92)90233-4.
6. S. Dragomir, J. Pecaric, L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.*, **21** (1995), 335–341.
7. B. Pachpatte, On some inequalities for convex functions, *RGMIA Res. Rep. Collect.*, **6** (2003), 1–9.
8. J. R. Wang, X. Z. Li, C. Zhu, Refinements of Hermite-Hadamard type inequalities involving fractional integrals, *Bull. Belg. Math. Soc. Simon Stevin*, **20** (2013), 655–666. doi: 10.36045/bbms/1382448186.
9. M. Z. Sarikaya, F. Ertugral, On the generalized Hermite-Hadamard inequalities, *Ann. Univ. Craiova Math. Comput. Sci. Ser.*, **47** (2020), 193–213.
10. M. Z. Sarikaya, H. Yildirim, On generalization of the Riesz potential, *Indian J. Math. Math. Sci.*, **3** (2007), 231–235.
11. F. Ertugral, M. Z. Sarikaya, Simpson type integral inequalities for generalized fractional integral, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.*, **113** (2019), 3115–3124. doi: 10.1007/s13398-019-00680-x.
12. K. L. Tseng, S. R. Hwang, New Hermite-Hadamard-type inequalities and their applications, *Filomat*, **30** (2016), 3667–3680. doi: 10.2298/FIL1614667T.
13. R. E. Moore, *Interval analysis*, Prentice Hall, Englewood Cliffs, 1966.
14. Y. Chalco-Cano, A. Flores-Franulić, H. Román-Flores, Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative, *Comput. Appl. Math.*, **31** (2012), 457–472. doi: 10.1590/S1807-03022012000300002.
15. Y. Chalco-Cano, W. A. Lodwick, W. Condori-Equice, Ostrowski type inequalities and applications in numerical integration for interval-valued functions, *Soft Comput.*, **19** (2015), 3293–3300. doi: 10.1007/s00500-014-1483-6.
16. H. Román-Flores, Y. Chalco-Cano, W. A. Lodwick, Some integral inequalities for interval-valued functions, *Comput. Appl. Math.*, **37** (2018), 1306–1318. doi: 10.1007/s40314-016-0396-7.
17. T. M. Costa, Jensen's inequality type integral for fuzzy-interval-valued functions, *Fuzzy Sets Syst.*, **327** (2017), 31–47. doi: 10.1016/j.fss.2017.02.001.
18. T. M. Costa, H. Román-Flores, Some integral inequalities for fuzzy-interval-valued functions, *Inf. Sci.*, **420** (2017), 110–125. doi: 10.1016/j.ins.2017.08.055.

19. A. Flores-Franulič, Y. Chalco-Cano, H. Román-Flores, An Ostrowski type inequality for interval-valued functions, In: *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, (2013), 1459–1462. doi: 10.1109/IFSA-NAFIPS.2013.6608617.
20. H. Román-Flores, Y. Chalco-Cano, G. N. Silva, A note on Gronwall type inequality for interval-valued functions, In: *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, (2013), 1455–1458. doi: 10.1109/IFSA-NAFIPS.2013.6608616.
21. E. Sadowska, Hadamard inequality and a refinement of Jensen inequality for set valued functions, *Results Math.*, **32** (1997), 332–337. doi: 10.1007/BF03322144.
22. F. C. Mitroi, K. Nikodem, S. Wasowicz, Hermite-Hadamard inequalities for convex set-valued functions, *Demonstr. Math.*, **46** (2013), 655–662. doi: 10.1515/dema-2013-0483.
23. K. Nikodem, J. L. Sánchez, L. Sánchez, Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps, *Math. Aeterna*, **4** (2014), 979–987.
24. M. B. Khan, M. A. Noor, K. I. Noor, Y. M. Chu, New Hermite-Hadamard type inequalities for  $(h_1, h_2)$ -convex fuzzy-interval-valued functions, *Adv. Differ. Equations*, **2021** (2021), 6–20. doi: 10.1186/s13662-021-03245-8.
25. M. B. Khan, P. O. Mohammed, M. A. Noor, Y. S. Hamed, New Hermite-Hadamard inequalities in fuzzy-interval fractional calculus and related inequalities, *Symmetry*, **13** (2021), 673. doi: 10.3390/sym13040673.
26. M. B. Khan, P. O. Mohammed, M. A. Noor, A. M. Alsharif, K. I. Noor, New fuzzy-interval inequalities in fuzzy-interval fractional calculus by means of fuzzy order relation, *AIMS Math.*, **6** (2021), 10964–10988. doi: 10.3934/math.2021637.
27. M. B. Khan, M. A. Noor, L. Abdullah, Y. M. Chu, Some new classes of preinvex fuzzy-interval-valued functions and inequalities, *Int. J. Comput. Intell. Syst.*, **14** (2021), 1403–1418. doi: 10.2991/ijcis.d.210409.001.
28. M. B. Khan, L. Abdullah, M. A. Noor, K. I. Noor, New Hermite-Hadamard and Jensen inequalities for log- $h$ -convex fuzzy-interval-valued functions, *Int. J. Comput. Intell. Syst.*, **14** (2021), 155. doi: 10.1007/s44196-021-00004-1.
29. P. Liu, M. B. Khan, M. A. Noor, K. I. Noor, K. I. Noor, New Hermite-Hadamard and Jensen inequalities for log- $s$ -convex fuzzy-interval-valued functions in the second sense, *Complex Intell. Syst.*, **2021** (2021), 1–15. doi: 10.1007/s40747-021-00379-w.
30. G. Sana, M. B. Khan, M. A. Noor, P. O. Mohammed, Y. M. Chu, Harmonically convex fuzzy-interval-valued functions and fuzzy-interval Riemann-Liouville fractional integral inequalities, *Int. J. Comput. Intell. Syst.*, **2021** (2021), 1809–1822. doi: 10.2991/ijcis.d.210620.001.
31. M. B. Khan, P. O. Mohammed, M. A. Noor, K. M. Abualnaja, Fuzzy integral inequalities on coordinates of convex fuzzy interval-valued functions, *Math. Biosci. Eng.*, **18** (2021), 6552–6580. doi: 10.3934/mbe.2021325.
32. M. B. Khan, M. A. Noor, H. M. Al-Bayatti, K. I. Noor, Some new inequalities for LR-log- $h$ -convex interval-valued functions by means of pseudo order relation, *Appl. Math. Inf. Sci.*, **15** (2021), 459–470.
33. M. B. Khan, P. O. Mohammed, M. A. Noor, D. Baleanu, J. L. G. Guirao, Some new fractional estimates of inequalities for LR- $p$ -convex interval-valued functions by means of pseudo order relation, *Axioms*, **10** (2021), 175. doi: 10.3390/axioms10030175.

34. P. Liu, M. B. Khan, M. A. Noor, K. I. Noor, On strongly generalized preinvex fuzzy mappings, *J. Math.*, **2021** (2021), 6657602. doi: 10.1155/2021/6657602.
35. M. B. Khan, M. A. Noor, K. I. Noor, A. T. Ab Ghani, L. Abdullah, Extended perturbed mixed variational-like inequalities for fuzzy mappings, *J. Math.*, **2021** (2021), 6652930. doi: 10.1155/2021/6652930.
36. M. B. Khan, M. A. Noor, K. I. Noor, H. Almusawa, K. S. Nisar, Exponentially preinvex fuzzy mappings and fuzzy exponentially mixed variational-like inequalities, *Int. J. Anal. Appl.*, **19** (2021), 518–541. doi: 10.28924/2291-8639-19-2021-518.
37. M. B. Khan, M. A. Noor, K. I. Noor, Y. M. Chu, Higher-order strongly preinvex fuzzy mappings and fuzzy mixed variational-like inequalities, *Int. J. Comput. Intell. Syst.*, **14** (2021), 1856–1870. doi: 10.2991/ijcis.d.210616.001.
38. B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets Syst.*, **151** (2005), 581–599. doi: 10.1016/j.fss.2004.08.001.
39. R. Goetschel Jr., W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets Syst.*, **18** (1986), 31–43. doi: 10.1016/0165-0114(86)90026-6.
40. O. Kaleva, Fuzzy differential equations, *Fuzzy Sets Syst.*, **24** (1987), 301–317. doi: 10.1016/0165-0114(87)90029-7.
41. U. W. Kulish, W. L. Miranker, *Computer arithmetic in theory and practice*, New York: Academic Press, 1981.
42. İ. İşcan, Hermite-Hadamard type inequalities for harmonically convex functions, *Hacet. J. Math. Stat.*, **43** (2014), 935–942.
43. M. A. Noor, K. I. Noor, M. U. Awan, S. Costache, Some integral inequalities for harmonically  $h$ -convex functions, *U.P.B. Sci. Bull., Seri. A*, **77** (2015), 5–16.
44. M. B. Khan, M. A. Noor, P. O. Mohammed, J. L. G. Guirao, K. I. Noor, Some integral inequalities for generalized convex fuzzy-interval-valued functions via fuzzy Riemann integrals, *Int. J. Comput. Intell. Syst.*, **14** (2021), 158. doi: 10.1007/s44196-021-00009-w.
45. M. B. Khan, H. M. Srivastava, P. O. Mohammed, J. L. Guirao, Fuzzy mixed variational-like and integral inequalities for strongly preinvex fuzzy mappings, *Symmetry*, **13** (2021), 1816. doi: 10.3390/sym13101816.



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