Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 334–348.
DOI: 10.3934/math.2022023
Received: 03 September 2021
Accepted: 08 October 2021
Published: 12 October 2021

Research article

The infinite sums of reciprocals and the partial sums of Chebyshev polynomials

Fan Yang*and Yang Li

School of Economics and Management, Xi'an Shiyou University, Xi'an 710065, Shaanxi, China

* **Correspondence:** Email: 79204721@qq.com; Tel: +86-18602968386.

Abstract: In this paper, the infinite sums of reciprocals and the partial sums derived from Chebyshev polynomials are studied. For the infinite sums of reciprocals, we apply the floor function to the reciprocals of these sums to obtain some new and interesting identities involving the Chebyshev polynomials. Simultaneously, we get several identities about the partial sums of Chebyshev polynomials by the relation of two types of Chebyshev polynomials.

Keywords: Chebyshev polynomials; reciprocal; inequality; floor function
Mathematics Subject Classification: 11B37, 11B39

1. Introduction

The properties of orthogonal polynomials and recursive sequences are popular in number theory. They are important in theoretical research and application. The famous Chebyshev polynomials and Fibonacci polynomials are widely used in the field of function, approximation theory and difference equation. They also promote the development of both the branch of mathematics such as cryptography, combinatorics and application of discipline such as intelligent sensing, satellite positioning. Furthermore, they are close to the Fibonacci numbers and Lucas numbers. Therefore, a large number of scholars have investigated them and get many properties and identities.

In the aspect of sums of reciprocals, Millin [1] originally studied the infinite sums of reciprocal Fibonacci series where the subscript is 2^n. Based on the initial achievement, Good [2] further studied this issue and proved

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}.$$

Afterwards, Ohtsuka and Nakamura [3] deduced the infinite sum of reciprocal Fibonacci series

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \text{ is even and } n \geq 2 \ , \\ F_n - F_{n-1} - 1, & \text{if } n \text{ is odd and } n \geq 1; \end{cases}$$

and the infinite sum of reciprocal square Fibonacci series

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1, & \text{if } n \text{ is even and } n \geq 2 \ , \\ F_n F_{n-1}, & \text{if } n \text{ is odd and } n \geq 1; \end{cases}$$

Similar properties were investigated in several different ways, see reference [4, 5]. Falcón and Plaza [6–8] used Fibonacci polynomials to study Fibonacci numbers and get a lot of identities. For example,

$$\sum_{k=1}^{n-1} F_k(x) F_{n-k}(x) = \frac{(n-1)xF_n(x) + 2nF_{n-1}(x)}{x^2 + 4},$$

$$\sum_{k=1}^{n} F_k(x) = \frac{F_{n+1}(x) + F_n(x) - 1}{x}$$

where n and k are positive integers. This fact allows them to invest some integer sequences in a new and direct way. With these fundamental achievements, Wu and Zhang [9] proceeded generation and deduced the the infinite sum of reciprocal Fibonacci polynomials

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k(x)} \right)^{-1} \right\rfloor = \begin{cases} F_n(x) - F_{n-1}(x), & \text{if } n \text{ is even and } n \geq 2 \ , \\ F_n(x) - F_{n-1}(x) - 1, & \text{if } n \text{ is odd and } n \geq 1; \end{cases}$$

and the the infinite sum of reciprocal square Fibonacci polynomials

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2(x)} \right)^{-1} \right\rfloor = \begin{cases} xF_n(x)F_{n-1}(x) - 1, & \text{if } n \text{ is even and } n \geq 2 \ , \\ xF_n(x)F_{n-1}(x), & \text{if } n \text{ is odd and } n \geq 1; \end{cases}$$

where x is any positive integer. besides, Panda et al. [10] did some research about bounds for reciprocal sums in terms of balancing and Lucas-balancing sequences. Also, Dutta and Ray [11] found some identities about finite reciprocal sums of Fibonacci and Lucas polynomials.

As we know, the first and the second kind of Chebyshev polynomials are usually defined as follows: $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$, $n \geq 0$, with the initial values $T_0(x) = 1$, $T_1(x) = x$; $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$, $n \geq 0$, with the initial values $U_0(x) = 1$, $U_1(x) = 2x$; Then from the second-order linear recurrence sequences we have

$$\begin{aligned} T_n(x) &= \frac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n], \\ U_n(x) &= \frac{1}{2\sqrt{x^2 - 1}}[(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}]. \end{aligned}$$

Based on these sequences,many scholars used these polynomials to study the Fibonacci sequences and the Lucas sequences and have investigated them and got many properties of F_n and L_n.

For example, Zhang [12] used the Chebyshev polynomials and has obtain the general formulas involving F_n and L_n

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} F_{m(a_1+1)} \cdot F_{m(a_2+1)} \cdots F_{m(a_{k+1}+1)} = (-i)^{mn} \frac{F_m^{k+1}}{2^k \cdot k!} U_{n+k}^{(k)} \left(\frac{i^m L_m}{2} \right).$$

$$\sum_{a_1+a_2+\ldots+a_{k+1}=n+k+1} L_{m(a_1+1)} \cdot L_{m(a_2+1)} \cdots L_{m(a_{k+1}+1)}$$

$$= (-i)^{m(n+k+1)} \frac{2}{k!} \sum_{h=0}^{k+1} \left(\frac{i^{m+2} L_m}{2} \right)^h \frac{(k+1)!}{h!(k+1-h)!} U_{n+2k+1-h}^{(k)} \left(\frac{i^m L_m}{2} \right),$$

where k, m are any positive integers, a_1, a_2, $\cdots a_{k+1}$ are nonnegative integers and i is the square root of -1. Wu and Yang [13] also studied Chebyshev polynomials and got a lot of properties. Besides, Dilcher and Stolarsky [14] established several related results involving resultants and discriminants about Chebyshev polynomials. Furthermore, bounds about the discriminant of the Chebyshev polynomials were given by Filipovski [15].

A variety of sums about Chebyshev polynomials are hot issues in the number theory all the time. For example, Cesarano [16] gained several conclusions about the generating function of Chebyshev polynomials

$$\sum_{n=0}^{\infty} \xi^n T_{n+l}(x) = \frac{(1-\xi x)T_l(x) - \xi(1-x^2)U_{l-1}}{1-2\xi x+\xi^2}$$

and the identical equation

$$\sum_{n=0}^{\infty} \xi^n U_{n-1+l}(x) = \frac{\xi T_l(x) - (1-\xi x)U_{l-1}}{1-2\xi x+\xi^2}$$

In this, ξ is a real number and $-1 < \xi < 1$. Furthermore, Knopfmacher et al. [17] did some research and got the result as follows:

$$\frac{1}{U_m(x)} = \frac{1}{m+1} \sum_{j=1}^{m} \frac{(-1)^{j+1} \sin^2 \theta_j}{x - \cos \theta_j}$$

and the identical equation

$$\frac{1+U_{m-1}(x)}{U_m(x)} = \frac{1}{m+1} \sum_{j=1}^{m} \frac{[1+(-1)^{j+1}] \sin^2 \theta_j}{x - \cos \theta_j},$$

where $\theta_j = \frac{j\pi}{m+1}$, m is a positive integer.

In this paper, we combine Ohtsuka and Falcón's ideas. Then we consider the subseries of infinite sums derived from the reciprocals of the Chebyshev polynomials and prove the following:

Theorem 1. For any positive integer n, m and x, we have the following formula

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{T_{mk}(x)} \right)^{-1} \right\rfloor = T_{mn}(x) - T_{mn-m}(x)$$

Theorem 2. For any positive integer n, and x, we have the following formula

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k(x)} \right)^{-1} \right\rfloor = U_n(x) - U_{n-1}(x) - 1.$$

Theorem 3. For any positive integer n, and x, we have the following formula

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k^2(x)} \right)^{-1} \right\rfloor = U_n^2(x) - U_{n-1}^2(x) - 1.$$

With Falcón's enlightening, we can apply similar method into deduction of partial sums of Chebyshev polynomials. For convenient expression, we firstly set

$$G_n(x) = U_{n-1}(x)U_n(x) + U_{n-1}(x)U_{n+1}(x)$$

$$M_n(x) = U_{n-2}(x)U_n(x) + U_{n-1}(x)U_n(x)$$

and obtain:
Theorem 4. For any positive integer n,

$$\begin{aligned} \sum_{k=0}^{2n+1} k^2 U_k(x) &= \frac{1}{2}(2n+1)U'_{2n+1}(x) - G'_n(x) + 1 \\ &\quad + (n+1)U'_{2n+2}(x) + M_n(x). \end{aligned}$$

Theorem 5. For any positive integer n,

$$\begin{aligned} \sum_{k=0}^{2n} k^2 T_k(x) &= \frac{1}{2}(2n+1)^2 U_{2n-1}(x) - U'_{2n}(x) \\ &\quad + 2(n+1)^2 U_{2n+2}(x) - U'_{2n+1}(x). \end{aligned}$$

Theorem 6. For any positive integer n,

$$\begin{aligned} \sum_{k=0}^{2n} k^3 T_k(x) &= 4(n+1)^3 U_{2n+2}(x) + \frac{1}{2}(2n+1)^3 U_{2n+1}(x) - G_n(x) \\ &\quad - (3n + \frac{3}{2})U'_{2n-1}(x) - 3nU'_{2n}(x) + 3M'_n(x) - 1. \end{aligned}$$

2. Some lemmas

In order to prove the results of the infinite sums of reciprocal Chebyshev polynomials, several lemmas are needed.

Let $\alpha = x + \sqrt{x^2 - 1}$ and $\beta = x - \sqrt{x^2 - 1}$, then we have the following lemmas.
Lemma 1. For any positive integer n, we have

$$U_n^2(x) = 1 + U_{n-1}(x)U_{n+1}(x),$$

$$U_n^2(x) = 4x^2 + U_{n-2}(x)U_{n+2}(x).$$

Proof. From the definition of Chebyshev polynomials, we have

$$\begin{aligned}
U_n^2(x) - U_{n-1}(x)U_{n+1}(x) &= \frac{(\alpha^{n+1} - \beta^{n+1})^2 - (\alpha^n - \beta^n)(\alpha^{n+2} - \beta^{n+2})}{(\alpha - \beta)^2} \\
&= \frac{\alpha^{2n+2} + \beta^{2n+2} + \alpha^2 + \beta^2 - 2 - \alpha^{2n+2} - \beta^{2n+2}}{(\alpha - \beta)^2} = 1. \\
U_n^2(x) - U_{n-2}(x)U_{n+2}(x) &= \frac{(\alpha^{n+1} - \beta^{n+1})^2 - (\alpha^{n-1} - \beta^{n-1})(\alpha^{n+3} - \beta^{n+3})}{(\alpha - \beta)^2} \\
&= \frac{\alpha^{2n+2} + \beta^{2n+2} + \alpha^4 + \beta^4 - 2 - \alpha^{2n+2} - \beta^{2n+2}}{(\alpha - \beta)^2} \\
&= (\alpha + \beta)^2 = 4x^2.
\end{aligned}$$

Lemma 2. For any positive integer n, we have

$$T_n^2(x) = T_{n-1}(x)T_{n+1}(x) + 1 - x^2,$$

$$T_n^2(x) = T_{n-2}(x)T_{n+2}(x) + 4x^2(1 - x^2).$$

Proof. From the definition of Chebyshev polynomials, we have

$$\begin{aligned}
T_n^2(x) - T_{n-1}(x)T_{n+1}(x) &= \frac{1}{4}[(\alpha^n + \beta^n)^2 - (\alpha^{n-1} + \beta^{n-1})(\alpha^{n+1} + \beta^{n+1})] \\
&= -\frac{1}{4}[\alpha^2 + \beta^2 - 2] = -\frac{1}{4}(\alpha - \beta)^2 = 1 - x^2. \\
T_n^2(x) - T_{n-2}(x)T_{n+2}(x) &= \frac{1}{4}[(\alpha^n + \beta^n)^2 - (\alpha^{n-2} + \beta^{n-2})(\alpha^{n+2} + \beta^{n+2})] \\
&= \frac{1}{4}[\alpha^{2n} + \beta^{2n} - \alpha^4 - \beta^4 + 2 - \alpha^{2n} - \beta^{2n}] \\
&= -\frac{1}{4}(\alpha + \beta)^2(\alpha - \beta)^2 = 4x^2(1 - x^2).
\end{aligned}$$

Lemma 3. For any positive integer n and m, we have

$$T_n(T_m(x)) = T_{nm}(x),$$

$$U_n(T_m(x)) = \frac{U_{m(n+1)-1}(x)}{U_{m-1}(x)}.$$

Proof. See Reference [12].

Lemma 4. For any positive integer n and x, we have

$$\frac{1}{T_n(x)} + \frac{1}{T_{n+1}(x)} \quad < \quad \frac{1}{T_n(x) - T_{n-1}(x)} - \frac{1}{T_{n+2}(x) - T_{n+1}(x)},$$

$$\frac{1}{T_n(x)}+\frac{1}{T_{n+1}(x)} \quad > \quad \frac{1}{T_n(x)-T_{n-1}(x)+1}-\frac{1}{T_{n+2}(x)-T_{n+1}(x)+1}.$$

Proof. The first inequality equivalent to

$$\frac{T_n(x)+T_{n+1}(x)}{T_n(x)T_{n+1}(x)}<\frac{T_{n+2}(x)-T_{n+1}(x)-T_n(x)+T_{n-1}(x)}{(T_n(x)-T_{n-1}(x))(T_{n+2}(x)-T_{n+1}(x))}, \tag{2.1}$$

or

$$\begin{aligned}&[T_n(x)+T_{n+1}(x)](T_n(x)-T_{n-1}(x)+1)(T_{n+2}(x)-T_{n+1}(x)+1)<\\&T_n(x)T_{n+1}(x)[T_{n+2}(x)-T_{n+1}(x)-T_n(x)+T_{n-1}(x)],\end{aligned}$$

Then we have

$$\begin{aligned}&T_n^2(x)T_{n+2}(x)+T_{n+1}^2(x)T_{n-1}(x)<\\&T_{n-1}(x)T_{n+2}(x)T_n(x)+T_{n-1}(x)T_{n+2}(x)T_{n+1}(x),\end{aligned}$$

applying Lemma 2, inequality (2.1) is equivalent to

$$(1-x^2)[T_{n-1}(x)+T_{n+2}(x)]<0. \tag{2.2}$$

For any positive x and $n\geq 1$, $1-x^2<0$ and $T_{n-1}(x)+T_{n+2}(x)>0$. Thus it is very easy to check inequality (2.2) is true. Similarly, we can consider the second inequality of Lemma 4. The second inequality is equivalent to

$$\frac{T_n(x)+T_{n+1}(x)}{T_n(x)T_{n+1}(x)}>\frac{T_{n+2}(x)-T_{n+1}(x)-T_n(x)+T_{n-1}(x)}{(T_n(x)-T_{n-1}(x)+1)(T_{n+2}(x)-T_{n+1}(x)+1)}, \tag{2.3}$$

or

$$\begin{aligned}&T_n^2(x)T_{n+2}(x)-T_{n-1}(x)T_n(x)T_{n+2}(x)-T_{n-1}(x)T_{n+1}(x)T_{n+2}(x)+T_n(x)T_{n+2}(x)\\&+T_{n+1}(x)T_{n+2}(x)+T_{n+1}^2(x)T_{n-1}(x)-T_{n+1}^2(x)+T_n^2(x)-T_n(x)T_{n-1}(x)\\&-T_{n+1}(x)T_{n-1}(x)+T_n(x)+T_{n+1}(x)>0,\end{aligned}$$

applying Lemma 2, inequality (2.3) is equivalent to

$$(T_{n+1}(x)-(x^2-1))T_{n+2}(x)-(T_n(x)+(x^2-1))T_{n-1}(x)+T_n(x)+T_{n+1}(x)>0. \tag{2.4}$$

For any positive x and $n\geq 1$,

$$(T_{n+1}(x)-(x^2-1))T_{n+2}(x)-(T_n(x)+(x^2-1))T_{n-1}(x)>0$$

Thus it is very easy to check inequality (2.4) is true.
Lemma 5. For any positive integer n and x,

$$\frac{1}{U_n(x)}+\frac{1}{U_{n+1}(x)} \quad > \quad \frac{1}{U_n(x)-U_{n-1}(x)}-\frac{1}{U_{n+2}(x)-U_{n+1}(x)},$$

$$\frac{1}{U_n(x)}+\frac{1}{U_{n+1}(x)} \quad < \quad \frac{1}{U_n(x)-U_{n-1}(x)-1}-\frac{1}{U_{n+2}(x)-U_{n+1}(x)-1}.$$

Prove. The first inequality is equivalent to

$$\frac{U_{n+1}(x)+U_n(x)}{U_{n+1}(x)U_n(x)} > \frac{U_{n+2}(x)-U_{n+1}(x)-U_n(x)+U_{n-1}(x)}{(U_n(x)-U_{n-1}(x))(U_{n+2}(x)-U_{n+1}(x))}, \tag{2.5}$$

or

$$\begin{aligned}&[U_{n+1}(x)+U_n(x)](U_n(x)-U_{n-1}(x))(U_{n+2}(x)-U_{n+1}(x)) > \\ &U_{n+1}(x)U_n(x)[U_{n+2}(x)-U_{n+1}(x)-U_n(x)+U_{n-1}(x)],\end{aligned}$$

Then we have

$$\begin{aligned}&U_n^2(x)U_{n+2}(x)+U_{n+1}^2(x)U_{n-1}(x) > \\ &U_n(x)U_{n+2}(x)U_{n-1}(x)+U_{n-1}(x)U_{n+1}(x)U_{n+2}(x),\end{aligned}$$

applying Lemma 1, inequality (2.5) is equivalent to

$$U_{n+2}(x)+U_{n-1}(x) > 0. \tag{2.6}$$

For any positive x and $n \geq 1$, it is very easy to check inequality (2.6) is true. Similarly, we can consider the second inequality of Lemma 5.

$$\frac{U_{n+1}(x)+U_n(x)}{U_{n+1}(x)U_n(x)} < \frac{U_{n+2}(x)-U_{n+1}(x)-U_n(x)+U_{n-1}(x)}{(U_n(x)-U_{n-1}(x)-1)(U_{n+2}(x)-U_{n+1}(x)-1)}, \tag{2.7}$$

or

$$\begin{aligned}&U_n^2(x)U_{n+2}(x)-U_{n-1}(x)U_n(x)U_{n+2}(x)-U_{n-1}(x)U_{n+1}(x)U_{n+2}(x)-U_n(x)U_{n+2}(x)\\ &-U_{n+1}(x)U_{n+2}(x)+U_{n+1}^2(x)U_{n-1}(x)+U_{n+1}^2(x)-U_n^2(x)+U_n(x)U_{n-1}(x)\\ &+U_{n+1}(x)U_{n-1}(x)+U_n(x)+U_{n+1}(x) < 0,\end{aligned}$$

applying Lemma 1, inequality (2.7) equivalent to

$$U_{n+2}(x)+U_{n-1}(x)+U_n(x)U_{n-1}(x)+U_n(x)+U_{n+1}(x) < U_{n+1}(x)U_{n+2}(x). \tag{2.8}$$

For any positive x and $n \geq 1$, it is very easy to check inequality (2.8) is true.
Lemma 6. For any positive integers n and x, we have

$$\begin{aligned}\frac{1}{U_n^2(x)}+\frac{1}{U_{n+1}^2(x)} \quad &> \quad \frac{1}{U_n^2(x)-U_{n-1}^2(x)}-\frac{1}{U_{n+2}^2(x)-U_{n+1}^2(x)},\\ \frac{1}{U_n^2(x)}+\frac{1}{U_{n+1}^2(x)} \quad &< \quad \frac{1}{U_n^2(x)-U_{n-1}^2(x)-1}-\frac{1}{U_{n+2}^2(x)-U_{n+1}^2(x)-1}.\end{aligned}$$

Proof. The first inequality is equivalent to

$$\frac{U_n^2(x)+U_{n+1}^2(x)}{U_n^2(x)U_{n+1}^2(x)} > \frac{U_{n+2}^2(x)-U_{n+1}^2(x)-U_n^2(x)+U_{n-1}^2(x)}{(U_{n+2}^2(x)-U_{n+1}^2(x))(U_n^2(x)-U_{n-1}^2(x))}, \tag{2.9}$$

or

$$[U_n^2(x)+U_{n+1}^2(x)](U_{n+2}^2(x)-U_{n+1}^2(x))(U_n^2(x)-U_{n-1}^2(x)) >$$
$$U_n^2(x)U_{n+1}^2(x)[U_{n+2}^2(x)-U_{n+1}^2(x)-U_n^2(x)+U_{n-1}^2(x)],$$

Then we have

$$U_n^4(x)U_{n+2}^2(x)-U_n^2(x)U_{n+2}^2(x)U_{n-1}^2(x)-U_{n+1}^2(x)U_{n+2}^2(x)U_{n-1}^2(x)$$
$$+U_{n+1}^4(x)U_{n-1}^2(x)>0,$$

applying Lemma 1, inequality (2.9) is equivalent to

$$U_{n+2}^2(x)+2U_{n-1}(x)U_{n+1}(x)U_{n+2}^2(x)+U_{n-1}^2(x)+2U_n(x)U_{n+2}(x)U_{n-1}^2(x)>0. \tag{2.10}$$

For any positive x and $n \geq 1$, it is very easy to check inequality (2.10) is true. Similarly, we can consider the second inequality of Lemma 6. The second inequality is equivalent to

$$\frac{U_n^2(x)+U_{n+1}^2(x)}{U_n^2(x)U_{n+1}^2(x)} < \frac{U_{n+2}^2(x)-U_{n+1}^2(x)-U_n^2(x)+U_{n-1}^2(x)}{(U_{n+2}^2(x)-U_{n+1}^2(x)-1)(U_n^2(x)-U_{n-1}^2(x)-1)}, \tag{2.11}$$

or

$$U_n^4(x)U_{n+2}^2(x)-U_n^4(x)-U_n^2(x)U_{n+2}^2(x)U_{n-1}^2(x)-U_{n+1}^2(x)U_{n+2}^2(x)U_{n-1}^2(x)$$
$$+U_{n+1}^4(x)U_{n-1}^2(x)+U_n^2(x)U_{n-1}^2(x)+U_{n+1}^2(x)U_{n-1}^2(x)-U_n^2(x)U_{n+2}^2(x)$$
$$-U_{n+1}^2(x)U_{n+2}^2(x)+U_{n+1}^4(x)+U_n^2(x)+U_{n+1}^2(x)<0,$$

applying Lemma 1, inequality (2.11) is equivalent to

$$U_n^2(x)U_{n-1}^2(x)+U_n^2(x)+U_{n+1}^2(x)+U_{n+2}^2(x)+2U_{n-1}(x)U_{n+1}(x)U_{n+2}^2(x)+U_{n-1}^2(x)$$
$$2U_n(x)U_{n+2}(x)U_{n-1}^2(x)+2U_n(x)U_{n+2}(x)<U_{n+1}^2(x)U_{n+2}^2(x)+2U_{n-1}(x)U_{n+1}(x). \tag{2.12}$$

For any positive x and $n \geq 1$, it is very easy to check inequality (2.12) is true.

Aiming to prove the results of the partial sums of Chebyshev polynomials, the lemmas below are necessary.

Lemma 7. For any positive integer $n \geq 2$

$$T_n(x) = \frac{1}{2}U_n(x)-\frac{1}{2}U_{n-2}(x)$$
$$\sum_{k=1}^{n} T_k(x) = \frac{1}{2}U_n(x)+\frac{1}{2}U_{n-1}(x)-\frac{1}{2}$$

Prove. The general term formula of Chebyshev polynomials is as follows

$$T_n(x) = \frac{1}{2}[(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n]$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}]$$

For convenient proving, we set $\alpha = x + \sqrt{x^2-1}$, $\beta = x - \sqrt{x^2-1}$, and easily verify $\alpha + \beta = 2x$, $\alpha\beta = 1$. Thus, according to the definition we get

$$\begin{aligned}
\frac{1}{2}U_n(x) - \frac{1}{2}U_{n-2}(x) &= \frac{1}{2}(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - \frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}) \\
&= \frac{1}{2(\alpha-\beta)}[\alpha^{n-1}(\alpha^2-1) - \beta^{n-1}(\beta^2-1)] \\
&= \frac{1}{2(\alpha-\beta)}[\alpha^{n-1}(\alpha^2-\alpha\beta) - \beta^{n-1}(\beta^2-\alpha\beta)] \\
&= \frac{1}{2(\alpha-\beta)}[\alpha^{n}(\alpha-\beta) + \beta^{n}(\alpha-\beta)] \\
&= \frac{1}{2}(\alpha^n + \beta^n).
\end{aligned}$$

This proves the first equation. And next we prove the second equation

$$\begin{aligned}
\sum_{k=1}^{n} T_k(x) &= \frac{1}{2}\sum_{k=2}^{n} U_k(x) - \frac{1}{2}\sum_{k=2}^{n} U_{k-2}(x) + T_1(x) \\
&= \frac{1}{2}\sum_{k=2}^{n} U_k(x) - \frac{1}{2}\sum_{k=0}^{n-2} U_k(x) + T_1(x) \\
&= \frac{1}{2}U_n(x) + \frac{1}{2}U_{n-1}(x) - T_1(x) - \frac{1}{2} + T_1(x) \\
&= \frac{1}{2}U_n(x) + \frac{1}{2}U_{n-1}(x) - \frac{1}{2}.
\end{aligned}$$

This proves Lemma 7.

Lemma 8. For any positive integer n

$$\sum_{k=1}^{2n} U_k(x) = U_{n-1}(x)U_n(x) + U_{n-1}(x)U_{n+1}(x), \tag{2.13}$$

$$\sum_{k=1}^{2n-1} U_k(x) = U_{n-2}(x)U_n(x) + U_{n-1}(x)U_n(x). \tag{2.14}$$

Prove. In accordance of the general term formula of Chebyshev polynomials, it is not hard to get

$$\begin{aligned}
U_{2n+1}(x) &= U_n(x)U_{n+1}(x) - U_n(x)U_{n-1}(x), \\
U_{2n+2}(x) &= U_{n+1}^2(x) - U_{n+1}(x)U_{n-1}(x) - 1, \\
U_{n+1}^2(x) &= U_{n+2}(x)U_n(x) + 1.
\end{aligned}$$

Easily test that when $n = 1$, identical Eq (2.13) is right. Supposing that $n = m$, Eq (2.13) is right. Then when $n = m + 1$,

$$\begin{aligned}\sum_{k=1}^{2m+2} U_k(x) &= U_{m-1}(x)U_{m+1}(x) + U_{m-1}(x)U_m(x) + U_{2m+1}(x) + U_{2m+2}(x)\\ &= U_{m+1}(x)U_m(x) + U_{m+1}^2 - 1\\ &= U_m(x)U_{m+2}(x) + U_{m+1}(x)U_m(x).\end{aligned}$$

Applying mathematical induction, it is not hard to prove identical Eq (2.14). This proves Lemma 8.
Lemma 9. For any positive integers n,

$$\begin{aligned}\sum_{k=0}^{2n} kT_k(x) &= \frac{1}{2}(2n+1)U_{2n-1}(x) + (n+1)U_{2n}(x) - G_n(x)\\ \sum_{k=0}^{2n} kU_k(x) &= \frac{1}{2}U'_{n+1}(x) + \frac{1}{2}U'_n(x) - G_n(x)\end{aligned}$$

Prove. According to Lemma 7, we have

$$\sum_{k=0}^{n+1} T_k(x) = \frac{U_{n+1}(x) + U_n(x) + 1}{2}.$$

Through derivation on the left and right sides,we get

$$\sum_{k=0}^{n} (k+1)U_k(x) = \frac{U'_{n+1}(x) + U'_n(x)}{2}.$$

Applying Lemma 7 and Lemma 8, we obtain

$$\begin{aligned}\sum_{k=1}^{2n} kU_k(x) &= \frac{1}{2}U'_{2n+1}(x) + \frac{1}{2}U'_{2n}(x) - \sum_{k=1}^{2n} U_k(x)\\ &= \frac{1}{2}U'_{2n+1}(x) + \frac{1}{2}U'_{2n}(x) - G_n(x)\\ \sum_{k=1}^{2n} kT_k(x) &= x + \frac{1}{2}\sum_{k=2}^{2n} kU_k(x) - \frac{1}{2}\sum_{k=2}^{2n} kU_{k-2}(x)\\ &= x + \frac{1}{2}\sum_{k=2}^{2n} kU_k(x) - \frac{1}{2}\sum_{k=0}^{2n-2} (k+2)U_k(x)\\ &= (n + \frac{1}{2})U_{2n-1}(x) + (n+1)U_{2n}(x) - \sum_{k=1}^{2n} U_k(x)\\ &= (n + \frac{1}{2})U_{2n-1}(x) + (n+1)U_{2n}(x) - G_n(x).\end{aligned}$$

This proves Lemma 9.

3. Proof of the theorems

In this section, we will prove our theorems. For the infinite sums of reciprocal Chebyshev polynomials, firstly we prove Theorem 1. For any positive integer n and x, using Lemma 4, we have

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{T_k(x)} &= \sum_{k=s}^{\infty}\left(\frac{1}{T_{2k-1}(x)}+\frac{1}{T_{2k}(x)}\right)\\ &< \sum_{k=s}^{\infty}\left(\frac{1}{T_{2k-1}(x)-T_{2k-2}(x)}-\frac{1}{T_{2k+1}(x)-T_{2k}(x)}\right)\\ &= \frac{1}{T_n(x)-T_{n-1}(x)}\end{aligned}$$

In the similar way, we have

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{T_k(x)} &= \sum_{k=s}^{\infty}\left(\frac{1}{T_{2k-1}(x)}+\frac{1}{T_{2k}(x)}\right)\\ &> \sum_{k=s}^{\infty}\left(\frac{1}{T_{2k-1}(x)-T_{2k-2}(x)+1}-\frac{1}{T_{2k+1}(x)-T_{2k}(x)+1}\right)\\ &= \frac{1}{T_n(x)-T_{n-1}(x)+1}.\end{aligned}$$

And then we have

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{T_k(x)}\right)^{-1}\right\rfloor = T_n(x)-T_{n-1}(x)$$

and then let $x = T_m(x)$, according to Lemma 3, we can get

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{T_{mk}(x)}\right)^{-1}\right\rfloor = T_{mn}(x)-T_{mn-m}(x)$$

This proved Theorem 1.

Next, Theorem 2 will be proved. For any positive integer n and x, using Lemma 5, we have

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{U_k(x)} &= \sum_{k=s}^{\infty}\left(\frac{1}{U_{2k-1}(x)}+\frac{1}{U_{2k}(x)}\right)\\ &< \sum_{k=s}^{\infty}\left(\frac{1}{U_{2k-1}(x)-U_{2k-2}(x)-1}-\frac{1}{U_{2k+1}(x)-U_{2k}(x)-1}\right)\\ &= \frac{1}{U_n(x)-U_{n-1}(x)-1}.\end{aligned}$$

In the similar way, we have

$$\sum_{k=n}^{\infty}\frac{1}{U_k(x)} = \sum_{k=s}^{\infty}\left(\frac{1}{U_{2k-1}(x)}+\frac{1}{U_{2k}(x)}\right)$$

$$
\begin{aligned}
&> \sum_{k=s}^{\infty}\left(\frac{1}{U_{2k-1}(x)-U_{2k-2}(x)}-\frac{1}{U_{2k+1}(x)-U_{2k}(x)}\right)\\
&= \frac{1}{U_n(x)-U_{n-1}(x)}.
\end{aligned}
$$

And then we have

$$
U_n(x)-U_{n-1}(x)-1<\left(\sum_{k=n}^{\infty}\frac{1}{U_k(x)}\right)^{-1}<U_n(x)-U_{n-1}(x).
$$

that is

$$
\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{U_k(x)}\right)^{-1}\right\rfloor=U_n(x)-U_{n-1}(x)-1.
$$

This proved Theorem 2.

Then we shall prove Theorem 3. Using Lemma 6, we can get

$$
\begin{aligned}
\sum_{k=n}^{\infty}\frac{1}{U_k^2(x)} &= \sum_{k=s}^{\infty}\left(\frac{1}{U_{2k-1}^2(x)}+\frac{1}{U_{2k}^2(x)}\right)\\
&< \sum_{k=s}^{\infty}\left(\frac{1}{U_{2k-1}^2(x)-U_{2k-2}^2(x)-1}-\frac{1}{U_{2k+1}^2(x)-U_{2k}^2(x)-1}\right)\\
&= \frac{1}{U_n^2(x)-U_{n-1}^2(x)-1}.
\end{aligned}
$$

In the similar way, we have

$$
\begin{aligned}
\sum_{k=n}^{\infty}\frac{1}{U_k^2(x)} &= \sum_{k=s}^{\infty}\left(\frac{1}{U_{2k-1}^2(x)}+\frac{1}{U_{2k}^2(x)}\right)\\
&> \sum_{k=s}^{\infty}\left(\frac{1}{U_{2k-1}^2(x)-U_{2k-2}^2(x)}-\frac{1}{U_{2k+1}^2(x)-U_{2k}^2(x)}\right)\\
&= \frac{1}{U_n^2(x)-U_{n-1}^2(x)}
\end{aligned}
$$

and then we can get

$$
\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{U_k^2(x)}\right)^{-1}\right\rfloor=U_n^2(x)-U_{n-1}^2(x)-1.
$$

This proved Theorem 3.

For the partial sums of Chebyshev polynomials, firstly we shall prove Theorem 4. According to Lemma 9, we have

$$
\sum_{k=0}^{2n+2}kT_k(x)=\frac{1}{2}(2n+3)U_{2n+1}(x)+(n+2)U_{2n+2}(x)-G_{n+1}(x)
$$

Through simultaneous derivation on the left and right sides, we deduce

$$\sum_{k=1}^{2n+1} k^2 U_{k-1}(x) = \frac{1}{2}(2n+3)U'_{2n+1}(x) + (n+2)U'_{2n+2}(x) - G'_n(x).$$

According to Lemma 8 and Lemma 9 we get

$$\begin{aligned}\sum_{k=0}^{2n+1} k^2 U_k(x) &= \sum_{k=1}^{2n+2} k^2 U_{k-1}(x) - 2\sum_{k=1}^{2n+1} kU_k(x) - \sum_{k=0}^{2n+1} U_k(x) \\ &= \sum_{k=1}^{2n+2} k^2 U_{k-1}(x) - 2\sum_{k=1}^{2n+1} (k+1)U_k(x) + \sum_{k=0}^{2n+1} U_k(x) \\ &= (n+\frac{1}{2})U'_{2n+1}(x) + (n+1)U'_{2n+2}(x) - G'_n(x) + M_n(x) + 1.\end{aligned}$$

Applying Lemma 7 and Lemma 8, we get

$$\begin{aligned}\sum_{k=0}^{2n} k^2 T_k(x) &= x + \frac{1}{2}\sum_{k=2}^{2n} k^2 U_k(x) - \frac{1}{2}\sum_{k=2}^{2n} k^2 U_{k-2}(x) \\ &= x + \frac{1}{2}\sum_{k=2}^{2n} k^2 U_k(x) - \frac{1}{2}\sum_{k=0}^{2n-2} (k+2)^2 U_k(x).\end{aligned}$$

Simplify the above, we have

$$\sum_{k=0}^{2n} k^2 T_k(x) = \frac{1}{2}(2n+1)^2 U_{2n-1}(x) + 2(n+1)^2 U_{2n+2}(x) - U'_{2n+1}(x) - U'_{2n}(x)$$

This proved Theorem 4 and Theorem 5.

Theorem 6 shall be proved below. According to Lemma 7 and Lemma 8, we have

$$\begin{aligned}\sum_{k=0}^{2n} k^3 T_k(x) &= x + \frac{1}{2}\sum_{k=2}^{2n} k^3 U_k(x) - \frac{1}{2}\sum_{k=2}^{2n} k^3 U_{k-2}(x) \\ &= x + \frac{1}{2}\sum_{k=2}^{2n} k^3 U_k(x) - \frac{1}{2}\sum_{k=0}^{2n-2} (k+2)^3 U_k(x) \\ &= -\sum_{k=2}^{2n-2} (3k^2 + 6k + 4)U_k(x) + \frac{1}{2}(2n+1)^3 U_n(x) \\ &\quad +4(n+1)^3 U_{2n+2}(x) - 26x - 4 \\ &= 4(n+1)^3 U_{2n+2}(x) + \frac{1}{2}(2n+1)^3 U_{2n+1}(x) - G_n(x) \\ &\quad -(3n+\frac{3}{2})U'_{2n-1}(x) - 3nU'_{2n}(x) + 3M'_n(x) - 1.\end{aligned}$$

This proved Theorem 6.

4. Conclusions

In this paper, the infinite sums of reciprocals and the partial sums derived from Chebyshev polynomials are studied. For the infinite sums of reciprocals, we apply the floor function to the reciprocals of these sums to obtain Theorem 1, Theorem 2 and Theorem 3 involving the Chebyshev polynomials. Simultaneously, we get Theorem 4, Theorem 5 and Theorem 6 about the partial sums of Chebyshev polynomials by the relation of two types of Chebyshev polynomials. Our results can enrich the related research domain with respect to orthogonal polynomials and recursive sequences. Besides, the results are hoped to be applied into other branches of mathematics or other disciplines out of mathematics.

Acknowledgments

The authors would like to thank Xi'an Shiyou University for the support of this research.

Conflict of interest

The authors declare there is no conflicts of interest in this paper.

References

1. D. A. Millin, Problem H-237, *Fibonacci Quart.,* **12** (1974), 309.
2. I. J. Good, A reciprocal series of Fibonacci numbers, *Fibonacci Quart.,* **12** (1974), 346.
3. H. Ohtsuka, S. Nakamura, On the sum of the reciprocal of Fibonacci numbers, *Fibonacci Quart.,* **46** (2009), 153–159.
4. W. Zhang, T. Wang, The infinite sum of the reciprocal of Pell numbers, *Appl. Math. Comput.,* **218** (2012), 6164–6167. doi: 10.1016/j.amc.2011.11.090.
5. G. Zhang, The infinite sum of reciprocal of the Fibonacci numbers, *J. Math. Res. Expo.,* **31** (2011), 1030–1034. doi: 10.3770/j.issn:1000-341X.2011.06.010.
6. S. Falcón, Á. Plaza, On *k*-Fibonacci sequences and polynomials and their derivatives, *Chaos Solution. Fract.,* **39** (2009), 1005–1019. doi: 10.1016/j.chaos.2007.03.007.
7. S. Falcón, Á. Plaza, The *k*-Fibonacci sequences and the Pascal 2-triangle, *Chaos Solution. Fract.,* **38** (2007), 38–49. doi: 10.1016/j.chaos.2006.10.022.
8. S. Falcón, Á. Plaza, The *k*-Fibonacci hyperbolic functions, *Chaos Solution. Fract.,* **38** (2008), 409–420. doi: 10.1016/j.chaos.2006.11.019.
9. Z. Wu, W. Zhang, The sums of the reciprocal of Fibonacci polynomials and Lucas polynomials, *J. Inequal. Appl.,* **134** (2012). doi: 10.1186/1029-242X-2012-134.
10. G. K. Panda, T. Komatsu, R. K. Davala, Reciprocal sums of sequences involving balancing and lucas-balancing numbers, *Math. Rep.,* **20** (2018), 201–214.
11. U. K. Dutta, P. K. Ray, On the finite reciprocal sums of fibonacci and lucas polynomials, *AIMS Math.,* **4** (2019), 1569–1581. doi: 10.3934/math.2019.6.1569.

12. W. Zhang, Some identities involving the Fibonacci numbers and Lucas numbers, *Fibonacci Quart.*, **42** (2004), 149–154.

13. X. Wu, G. Yang, The general formula of Chebyshev polynomials, *J. Wuhan Trans. U.*, **24** (2000), 573–576.

14. K. Dilcher, K. Stolarsky, Resultants and discriminants of chebyshev and related polynomials, *T. Am. Math. Soc.*, **357** (2005), 965–981. doi: 10.1090/S0002-9947-04-03687-6.

15. S. Filipovski, A lower bound for the discriminant of polynomials related to Chebyshev polynomials, *Discrete Math.*, **344** (2021). doi: 10.1016/j.disc.2020.112235.

16. C. Cesarano, Identities and generating functions on Chebyshev polynomials, *Georgian Math. J.*, **19** (2012), 427–440. doi; 10.1515/gmj-2012-0031.

17. A. Knopfmacher, T. Mansour, A. O. Munagi, Staircase words and Chebyshev polynomials, *Appl. Anal. discr. Math.*, **4** (2010), 81–95.