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Research article

Stability of intuitionistic fuzzy set-valued maps and solutions of integral inclusions

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Abstract: In this paper, new intuitionistic fuzzy fixed point results for sequence of intuitionistic fuzzy set-valued maps in the structure of b-metric spaces are examined. A few nontrivial comparative examples are constructed to keep up the hypotheses and generality of our obtained results. Following the fact that most existing concepts of Ulam-Hyers type stabilities are concerned with crisp mappings, we introduce the notion of stability and well-posedness of functional inclusions involving intuitionistic fuzzy set-valued maps. It is a familiar fact that solution of every functional inclusion is a subset of an appropriate space. In this direction, intuitionistic fuzzy fixed point problem involving (α, β)-level set of an intuitionistic fuzzy set-valued map is initiated. Moreover, novel sufficient criteria for existence of solutions to an integral inclusion are investigated to indicate a possible application of the ideas presented herein.

Keywords: intuitionistic fuzzy set; intuitionistic fuzzy fixed point; b-metric space; Ulam-Hyers stability; integral inclusion
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1. Introduction

Diverse investigations in science and engineering described by nonlinear functional equations can be worked-out by reformulating them to their corresponding fixed point problems. Indeed, an operator equation $\vartheta\jmath = 0$ can be framed as a fixed point (fp) problem $\phi\jmath = \jmath$, where ϕ is a self-mapping with an appropriate domain. The fp theorem (thrm), regularly named as the Banach fp thrm (see [10]), appeared in explicit form in Banach thesis in 1922, where it was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has gained a number of generalizations and modifications by many authors. For some familiar articles in this context, we refer [14, 39] In particular, the idea of family of contraction mappings was initiated by Ciric [14]. Following [14], the study of existence of common fp of infinite family of self-mappings has been examined by several investigators. Not long ago, Allahyari et al. [2] defined some new conditions of contractions for infinite family of self-mappings and established a handful of results on the existence and uniqueness of common fp in the bodywork of complete metric(cms) space, thereby, generalizing the results in [14] and some references therein.

As a natural generalization of the notion of crisp sets, fuzzy set (fs) was introduced originally by Zadeh [47]. Since then, to use this concept, many authors have progressively extended the theory and its applications to other branches of sciences, social sciences and engineering. Heilpern [17] used the idea of fuzzy set to launch a class of fuzzy set-valued maps (fsm) and proved a fp thrm for fuzzy contraction mappings which is a fuzzy analogue of the fp thrm of Nadler [36]. Subsequently, several authors have studied the existence of fp of fsm (see, e.g. [5, 6, 30–35]).

The concept of intuitionistic fuzzy set (IFS) was inaugurated by Atanassov [4] as a further generalization of fs theory. IFS provides a useful mathematical framework to manage inaccuracy and hesitancy due to inadequate information. As a result, it has gained more robust applications in various areas such as medical diagnosis, image processing, drug selection, decision making problems and so on. Meanwhile, researches on IFS has been growing at a rapid rate and different results have been presented in various domains. Not long ago, Azam et al. [5] introduced a new technique for analysing fp results via intuitionistic fsm on a cms. Thereafter, Azam and Tabassum [6] provided some conditions for existence of common coincidence points for three intuitionistic fsm and applied their thrm to investigate existence criteria for solutions of a system of integral equations. Recently, Tabassum et al. [46] established some common fp thrm for a pair of intuitionistic fsm in the setting of $(\mathcal{T}, \mathcal{N}, \alpha)$-cut set of IFS.

Nowadays, stability results for fp problems have become more useful in many branches of applied sciences. Historically, stability theory of functional equations was initiated in 1940 by Ulam [44]. Along with other unsolved mathematical problems, most confusing question posed by Ulam was on stability of group homomorphism. The first affirmative response to this question was provided by Hyers [20], who solved the problem for additive mappings in Banach spaces. Shortly, this type of problem propounded by Ulam was named Ulam-Hyers stability. Later on, Rassias [40] established a notable generalization of the Ulam-Hyers stability of mappings. Thereafter, more than a handful of mathematicians embraced these types of results to carry on their research in various arms of mathematical analysis. For some work on this matter, the interested reader may consult the monograph [23].

Following the above trends, first in this paper, common intuitionistic fuzzy fp thrm for sequence of intuitionistic fsm in the setting of complete b-metric(cbms) spaces are established. Nontrivial examples are constructed to verify the hypotheses of our main results. As far as we know, most available Ulam-Hyers type stability results deal with crisp mappings. With this information, we introduce the idea of stability and well-posedness of functional inclusions involving intuitionistic fsm. It is a known fact that solutions of a functional inclusion is a subset of an appropriate space. Thus, intuitionistic fuzzy fp problem for which the right-hand-side is an (α, β)-level set of an intuitionistic fsm is also considered in this note. From application perspective, sufficient criteria for existence of solutions to an int-incl are examined to point out an additional usability of our obtained results. The ideas established herein are inspired by [2, 14, 17, 21, 44] and generalize, unify as well as complement the results obtained therein.

2. Preliminaries

As a generalization of ms, Czerwik [15] introduced the notion of a bms as follows:

Definition 2.1. [15] Let Λ be a nonempty set (n-set) and $\eta \geq 1$ be a constant. Suppose that the mapping $\varrho : \Lambda \times \Lambda \longrightarrow \mathbb{R}_+$ satisfies the following conditions for all $\jmath, \ell, z \in \Lambda$:

(i) $\varrho(\jmath, \ell) = 0$ if and only if $\jmath = \ell$;
(ii) $\varrho(\jmath, \ell) = \varrho(\ell, \jmath)$;
(iii) $\varrho(\jmath, \ell) \leq \eta\,[\varrho(\jmath, z) + \varrho(z, \ell)]$.

Then, the triple (Λ, ϱ, η) is called a bms.

Example 2.2. [18] Let $\aleph^p(\Delta) = \{\jmath \in \aleph(\Delta) : \|\jmath\|_{\aleph^p} < \infty\}(0 < p < 1)$ be $\aleph^p$ space defined on the unit disk Δ, where $\aleph(\Delta)$ is the set of all holomorphic functions on Δ and

$$\|\jmath\|_{\aleph^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\jmath(re^{\iota\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

Define $\varrho : \aleph^p(\Delta) \times \aleph^p(\Delta) \longrightarrow \mathbb{R}_+$ by

$$\varrho(\jmath, \ell) = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\jmath(re^{\iota\theta}) - \ell(re^{\iota\theta})|^p d\theta \right)^{\frac{1}{p}},$$

for all $\jmath, \ell \in \aleph^p(\Delta)$. Then $(\aleph^p(\Delta), \varrho, \eta)$ is a bms with the parameter $\eta = 2^{\frac{1}{p}-1}$.

Definition 2.3. [12] Let (Λ, ϱ, η) be a bms. A sequence $\{\jmath_n\}_{n\in\mathbb{N}}$ is said to be:

(i) Convergent(cgent) if and only if there exists $\jmath \in \Lambda$ such that $\varrho(\jmath_n, \jmath) \longrightarrow 0$ as $n \longrightarrow \infty$, and we write this as $\lim_{n\longrightarrow\infty} \varrho(\jmath_n, \jmath) = 0$.
(ii) Cauchy(Cchy) if and only if $\varrho(\jmath_n, \jmath_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.
(iii) Complete if every Cchy sequence in Λ is cgent.

Denoted by $\mathcal{K}(\Lambda)$, the class of nonempty compact subsets of Λ. Let (Λ, ϱ, η) be a bms. For $\nabla, \triangle \in \mathcal{K}(\Lambda)$, the function $\aleph : \mathcal{K}(\Lambda) \times \mathcal{K}(\Lambda) \longrightarrow \mathbb{R}_+$, defined by

$$\aleph(\nabla, \triangle) = \begin{cases} \max\left\{\sup_{\jmath\in\nabla} \varrho(\jmath, \triangle), \sup_{\jmath\in\triangle} \varrho(\jmath, \nabla)\right\}, & \text{if it exists} \\ \infty, & \text{otherwise} \end{cases}$$

is called the generalized Hausdorff-Pompeiu bm on $\mathcal{K}(\Lambda)$ induced by the bm ϱ, where $\varrho(\jmath, \nabla) = \inf_{\ell\in\nabla} \varrho(\jmath, \ell)$.

Definition 2.4. Let Λ be a n-set. Then an IFS Ψ in Λ is a set of ordered triples given by

$$\Psi = \{\langle \jmath, \mu_\Psi(\jmath), \nu_\Psi(\jmath)\rangle : \jmath \in \Lambda\},$$

where $\mu_\Psi : \Lambda \longrightarrow [0,1]$ and $\nu_\Psi : \Lambda \longrightarrow [0,1]$ define the degrees of membership and non-membership, respectively of $\jmath$ in Λ and satisfy $0 \leq \mu_\Psi + \nu_\Psi \leq 1$, for each $\jmath \in \Lambda$. Moreover, the degree of non-determinacy (or hesitancy) of $\jmath \in \Lambda$ is defined as

$$h_\Psi(\jmath) = 1 - \mu_\Psi(\jmath) - \nu_\Psi(\jmath).$$

In particular, if $h_\Psi(\jmath) = 0$ for all $\jmath \in \Lambda$, then an IFS reduces to an ordinary fs.

We denote the collection of all intuitionistic fs in Λ by $(IFS)^\Lambda$.

Definition 2.5. [4] Let $L = \{(\alpha,\beta) : \alpha + \beta \leq 1,\ (\alpha,\beta) \in (0,1] \times [0,1)\}$ and Ψ is an intuitionistic fs in Λ. Then the (α,β)-level set of Ψ is defined as

$$[\Psi]_{(\alpha,\beta)} = \{\jmath \in \Lambda : \mu_\Psi(\jmath) \geq \alpha \text{ and } \nu_\Psi(\jmath) \leq \beta\}.$$

Definition 2.6. [43] Let Λ be a n-set. A mapping $F = \langle \mu_F, \nu_F \rangle : \Lambda \longrightarrow (IFS)^\Lambda$ is called an intuitionistic fsm. A point $u \in \Lambda$ is called an intuitionistic fuzzy fp of F if there exist $(\alpha,\beta) \in (0,1] \times [0,1)$ such that $u \in [Fu]_{(\alpha,\beta)}$.

Definition 2.7. [6] An IFS Ψ in a ms Λ is said to be an approximate quantity if and only if $[\Psi]_{(\alpha,\beta)}$ is compact and convex in Ψ for each $(\alpha,\beta) \in (0,1] \times [0,1)$ with

$$\sup_{\jmath\in\Lambda} \mu_\Psi(\jmath) = 1 \text{ and } \inf_{\jmath\in\Lambda} \nu_\Psi(\jmath) = 0.$$

We denote the collection of all approximate quantities in Λ by $W(\Lambda)$.

Remark 1. Any crisp set M can be represented as an intuitionistic fs by its intuitionistic characteristic function $\mathfrak{M} = \langle F_M, \urcorner_M \rangle$ defined as:

$$F_M(\jmath) = \begin{cases} 1, & \text{if } \jmath \in M \\ 0, & \text{if } \jmath \notin M, \end{cases} \qquad \urcorner_M(\jmath) = \begin{cases} 0, & \text{if } \jmath \in M \\ 1, & \text{if } \jmath \notin M. \end{cases}$$

Consistent with Azam and Tabassum [6], for each $(\alpha,\beta) \in (0,1]\times[0,1)$ such that $[\Psi_1]_{(\alpha,\beta)}, [\Psi_2]_{(\alpha,\beta)} \in \mathcal{K}(\Lambda)$ and $p(\Psi_1,\Psi_2), \varrho(\infty,\infty) : \mathcal{K}(\Lambda) \times \mathcal{K}(\Lambda) \longrightarrow \mathbb{R}$, we define the following distance functions:

$$p_{(\alpha,\beta)}(\Psi_1,\Psi_2) = \varrho\left([\Psi_1]_{(\alpha,\beta)}, [\Psi_2]_{(\alpha,\beta)}\right).$$

$$p(\Psi_1,\Psi_2) = \sup_{(\alpha,\beta)} p_{(\alpha,\beta)}(\Psi_1,\Psi_2).$$

$$D_{(\alpha,\beta)}(\Psi_1,\Psi_2) = H([\Psi_1]_{(\alpha,\beta)}, [\Psi_2]_{(\alpha,\beta)}).$$

$$\varrho_{(\infty,\infty)}(\Psi_1,\Psi_2) = \sup_{(\alpha,\beta)} D_{(\alpha,\beta)}(\Psi_1,\Psi_2).$$

From [17], we note that $p_{(\alpha,\beta)}$ is nondecreasing for each $(\alpha,\beta) \in (0,1] \times [0,1)$.

Samet et al. [42] introduced the idea of θ-admissible mappings in the following manner.

Definition 2.8. [42] Let Λ be a n-set, $G : \Lambda \longrightarrow \Lambda$ and $\theta : \Lambda \times \Lambda \longrightarrow \mathbb{R}_+$ be mappings. Then G is said to be θ-admissible if for all $\jmath, \ell \in \Lambda$,

$$\theta(\jmath,\ell) \geq 1 \text{ implies } \theta(G\jmath, Gy) \geq 1.$$

Definition 2.9. Let Λ be a n-set and $\{G_n\}_{n\geq 1}$ be a family of single-valued mappings on Λ. A point $u \in \Lambda$ is called a fp of this family if $G_n(u) = u$ for each $n \in \mathbb{N}$.

The following well-known result for a family of generalized contractions was established by Ciric [14].

Theorem 2.10. *Let (Λ, ϱ) be a cms and $\{G_i\}_{i\in\Delta}$ be a family single-valued mappings on Λ. If there exists $j \in \Delta$ and $\eta \in (0,1)$ such that for all $\jmath, \ell \in \Lambda$,*

$$\varrho(G_i x, G_j \ell) \leq \ \eta \max\left\{\varrho(\jmath, \ell), \varrho(\jmath, G_i x), \varrho(\ell, G_j y), \frac{1}{2}\left[\varrho(\jmath, G_j y) + \varrho(\ell, G_i \jmath)\right]\right\},$$

then G_i have a unique common fp in Λ for each $i \in \Delta$.

For some recent extensions of Thrm 2.10, see [1, 2, 21, 27] and references therein.

3. Main results

Definition 3.1. Let Λ be an arbitrary n-set and $F_n = \langle \mu_{F_n}, \nu_{F_n} \rangle : \Lambda \longrightarrow (IFS)^{\Lambda} (n \in \mathbb{N})$ be a sequence of intuitionistic fsm. Then $\{F_n\}_{n\geq 1}$ is said to be (α, β)-admissible, if there exist $(\alpha, \beta) \in (0, 1] \times [0, 1)$ and a function $\rho : \Lambda \times \Lambda \longrightarrow \mathbb{R}_+$ such that for each $\jmath \in \Lambda$ and $\ell \in [F_n \jmath]_{(\alpha,\beta)}$ with $\rho(\jmath, \ell) \geq 1$, we have $\rho(\ell, z) \geq 1$ for all $z \in [F_{n+1}\ell]_{(\alpha,\beta)}$.

Theorem 3.2. *Let (Λ, ϱ, η) be a cbms, $F_n : \Lambda \longrightarrow (IFS)^{\Lambda}$ be a sequence of intuitionistic fsm and $0 < k_{i,j}$ $(i, j \in \mathbb{N})$ with $k_{i,i+1} \neq 1$ for all $i \in \mathbb{N}$. Assume:*

(i) $\limsup_{i\longrightarrow\infty} k_{i,j} < 1$*, for each $j \in \mathbb{N}$;*
(ii) $\sum_{n=1}^{\infty} \Omega_n < \infty$*, where* $\Omega_n = \prod_{i=1}^{n} \frac{k_{i,i+1}}{1-k_{i,i+1}}$*;*
(iii) there exist $(\alpha, \beta) \in (0, 1] \times [0, 1)$ and a function $\rho : \Lambda \times \Lambda \longrightarrow \mathbb{R}_+$ such that for each $\jmath \in \Lambda$, $[F_n \jmath]_{(\alpha,\beta)}$ is a nonempty compact subset of Λ, and

$$\rho(\jmath, \ell)\aleph\left([F_i \jmath]_{(\alpha,\beta)}, [F_j \ell]_{(\alpha,\beta)}\right) \leq \frac{k_{i,j}}{\eta}\left[\varrho(\jmath, [F_j y]_{(\alpha,\beta)}) + \varrho(\ell, [F_i x]_{(\alpha,\beta)})\right] \tag{3.1}$$

for all $\jmath, \ell \in \Lambda$, $i, j \in \mathbb{N}$ with $\jmath \neq \ell$ and $i \neq j$;
(iv) there exists $\jmath_0 \in \Lambda$, $\jmath_1 \in [F_1 \jmath_0]_{(\alpha,\beta)}$ with $\jmath_0 \neq \jmath_1$ and $\rho(\jmath_0, \jmath_1) \geq 1$;
(v) $\{F_n\}_{n\geq 1}$ is (α, β)-admissible;
(vi) for each sequence $\{\jmath_n\}_{n\geq 1}$ in Λ with $\rho(\jmath_n, \jmath_{n+1}) \geq 1$ and $\jmath_n \longrightarrow u$, we have $\rho(\jmath_n, u) \geq 1$, for all $n \in \mathbb{N}$.
(vii) there exists $\eta \geq 1$ such that the series $\sum_{n=1}^{\infty} \Omega_n \eta^n$ is cgent.

Then there exists $u \in \Lambda$ such that $u \in \bigcap_{n=1}^{\infty} [F_n u]_{(\alpha,\beta)}$.

Proof. From Conditions (*iv*) and (3.1), we have

$$\begin{aligned}\varrho(\jmath_1,[F_2\jmath_1]_{(\alpha,\beta)}) \leq&\rho(\jmath_0,\jmath_1)\aleph([F_1\jmath_0]_{(\alpha,\beta)},[F_2\jmath_1]_{(\alpha,\beta)})\\ \leq&\frac{k_{1,2}}{\eta}\Big[\varrho(\jmath_0,[F_2\jmath_1]_{(\alpha,\beta)})+\varrho(\jmath_1,[F_1\jmath_0]_{(\alpha,\beta)})\Big]\\ =&\frac{k_{1,2}}{\eta}\Big[\varrho(\jmath_0,[F_2\jmath_1]_{(\alpha,\beta)})\Big]\\ \leq&\frac{k_{1,2}}{\eta}\Big[\eta\Big(\varrho(\jmath_0,\jmath_1)+\varrho(\jmath_1,[F_2\jmath_1]_{(\alpha,\beta)})\Big)\Big]\\ =&k_{1,2}\Big(\varrho(\jmath_0,\jmath_1)+\varrho(\jmath_1,[F_2\jmath_1]_{(\alpha,\beta)})\Big).\end{aligned} \tag{3.2}$$

(3.2) yields

$$\varrho(\jmath_1,[F_2\jmath_1]_{(\alpha,\beta)}) \leq\frac{k_{1,2}}{1-k_{1,2}}\varrho(\jmath_0,\jmath_1). \tag{3.3}$$

Since $[F_2\jmath_1]_{(\alpha,\beta)} \in \mathcal{K}(\Lambda)$, there exists $\jmath_2 \in [F_2\jmath_1]_{(\alpha,\beta)}$ such that from (3.3), $\varrho(\jmath_1,\jmath_2) \leq \frac{k_{1,2}}{1-k_{1,2}}\varrho(\jmath_0,\jmath_1)$. Using (α,β)-admissibility of $\{F_n\}_{n\geq 1}$, we can find $(\alpha,\beta) \in (0,1]\times[0,1)$ satisfying $\rho(\jmath_1,\jmath_2) \geq 1$. Analogously, we have

$$\begin{aligned}\varrho(\jmath_2,[F_3\jmath_2]_{(\alpha,\beta)}) \leq&\frac{k_{2,3}}{1-k_{2,3}}\varrho(\jmath_1,\jmath_2).\\ \leq&\frac{k_{2,3}}{1-k_{2,3}}\frac{k_{1,2}}{1-k_{1,2}}\varrho(\jmath_0,\jmath_1).\end{aligned} \tag{3.4}$$

And by hypotheses, there exists $\jmath_3 \in [F_3\jmath_2]_{(\alpha,\beta)}$ such that $\varrho(\jmath_2,\jmath_3) \leq \frac{k_{2,3}}{1-k_{2,3}}\frac{k_{1,2}}{1-k_{1,2}}\varrho(\jmath_0,\jmath_1)$. Recursively, we see $\{\jmath_n\}_{n\geq 1}$ such that $\jmath_{n+1} \in [F_{n+1}\jmath_n]_{(\alpha,\beta)}$, with $\rho(\jmath_n,\jmath_{n+1}) \geq 1$, and

$$\varrho(\jmath_n,\jmath_{n+1}) \leq \Omega_n\varrho(\jmath_0,\jmath_1),\ n\in\mathbb{N}. \tag{3.5}$$

From (3.5), using triangular inequality in (Λ,ϱ,η), for all $p \geq 1$, we obtain

$$\begin{aligned}\varrho(\jmath_{n+p},\jmath_n) \leq&\ \eta(\varrho(\jmath_{n+p},\jmath_{n+1})+\varrho(\jmath_{n+1},\jmath_n))\\ \leq&\ \frac{1}{\eta^{n-1}}\sum_{l=n}^{n+p-1}\eta^l\varrho(\jmath_l,\jmath_{l+1})\\ \leq&\ \frac{1}{\eta^{n-1}}\sum_{l=n}^{n+p-1}\Omega_l\eta^l\varrho(\jmath_0,\jmath_1)\\ \leq&\ \frac{1}{\eta^{n-1}}\sum_{l=n}^{\infty}\Omega_l\eta^l\varrho(\jmath_0,\jmath_1).\end{aligned} \tag{3.6}$$

Letting $n \longrightarrow \infty$ in (3.6) and applying Condition (*vii*), we find that $\lim_{n\longrightarrow\infty}\varrho(\jmath_{n+p},\jmath_n) = 0$. This shows that $\{\jmath_n\}_{n\geq 1}$ is a Cchy sequence in Λ. The completeness of Λ implies that there exists $u \in \Lambda$ such that $\jmath_n \longrightarrow u$ as $n \longrightarrow \infty$. By Condition (*iv*), $\rho(\jmath_n,u) \geq 1$, for all $n \in \mathbb{N}$. Now, we shall show that u is a common intuitionistic fuzzy fp of F_n. Assume contrary; that is, $u \notin [F_n u]_{(\alpha,\beta)}$ for some $n \in \mathbb{N}$ so that $\varrho(u, [F_n u]_{(\alpha,\beta)}) > 0$. Let $p > 0$ be arbitrary. Then,

$$\begin{aligned}
\varrho(u, [F_p u]_{(\alpha,\beta)}) &\leq \eta(\varrho(u, \jmath_n) + \varrho(\jmath_n, [F_p u]_{(\alpha,\beta)})) \\
&\leq \eta\varrho(u, \jmath_n) + \eta\rho(\jmath_{n-1}, u)\aleph([F_n \jmath_{n-1}]_{(\alpha,\beta)}, [F_p u]_{(\alpha,\beta)}) \\
&\leq \eta\varrho(u, \jmath_n) + \eta\left(\frac{k_n, p}{\eta}\right)\left[\varrho(\jmath_{n-1}, [F_p u]_{(\alpha,\beta)}) + \varrho(u, [F_n x_{n-1}]_{(\alpha,\beta)})\right] \\
&\leq \eta\varrho(u, \jmath_n) + k_{n,p}\left[\varrho(\jmath_{n-1}, [F_p u]_{(\alpha,\beta)}) + \varrho(u, \jmath_n)\right].
\end{aligned} \tag{3.7}$$

From (3.7), we have

$$\begin{aligned}
\varrho(u, [F_p u]_{(\alpha,\beta)}) &\leq \left[(\limsup_{n\longrightarrow\infty} k_{n,p})\varrho(u, [F_p u]_{(\alpha,\beta)})\right] \\
&< \varrho(u, [F_p u]_{(\alpha,\beta)}),
\end{aligned}$$

a contradiction. Consequently, $u \in \bigcap_{n=1}^{\infty}[F_n u]_{(\alpha,\beta)}$. □

Following Heilpern [17], we apply Thrm 3.2 to deduce the next result in association with $\varrho_{(\infty,\infty)}$-metric for intuitionistic fs. It is pertinent to note that the study of fp of intuitionistic fsm in connection with $\varrho_{(\infty,\infty)}$-metric is very significant in computing Hausdorff dimensions. These dimensions help us to comprehend the notions of ε^{∞}-space which is of great importance in higher energy physics (see, e.g. [16]). Note that $\varrho_{(\infty,\infty)}$ is a metric on Λ (induced by the Hausdorff metric $\aleph$) and the completeness of (Λ, ϱ, η) implies the completeness of the corresponding bms $(\Lambda, \aleph, \eta)$ and $(\mathcal{K}(\Lambda), \varrho_{(\infty,\infty)}, \eta)$. Furthermore,

$$(\Lambda, \varrho, \eta) \longmapsto (\mathcal{K}(\Lambda), \aleph, \eta) \longmapsto (\mathcal{K}(\Lambda), \varrho_{(\infty,\infty)}, \eta),$$

are isometric embeddings by means of $\jmath \longrightarrow \{\jmath\}$(non-fs) and $M \longrightarrow \chi_M$, respectively, where χ_M is the characteristic function of the crisp set M. For similar observations, see [7].

Theorem 3.3. *Let (Λ, ϱ, η) be a cbms and $F_n : \Lambda \longrightarrow W(\Lambda)$ be a sequence of intuitionistic fsm. Assume that conditions $(i), (ii), (iv), (v)$, (vi) and (vii) of Thrm 3.2 hold. If Condition (iii) of the thrm is replaced with:*

$$\rho(\jmath, \ell)\varrho_{(\infty,\infty)}(F_i(\jmath), F_j(\ell)) \leq \frac{k_{i,j}}{\eta}\left[p(\jmath, F_j(\ell)) + p(\ell, F_i(\jmath))\right]$$

for all $\jmath, \ell \in \Lambda$ with $\jmath \neq \ell$ and $i \neq j$. Then there exists $u \in \Lambda$ such that $\{u\} \subset F_i(u)$ for each $i \in \mathbb{N}$.

Proof. Let $\jmath \in \Lambda$ and $(\alpha, \beta) = (1, 0)$, then by hypotheses, $[F_n \jmath]_{(1,0)}$ is a nonempty compact subset of Λ for each $n \in \mathbb{N}$. Now, for all $\jmath, \ell \in \Lambda$,

$$\begin{aligned}
\rho(\jmath, \ell)D_{(1,0)}(F_i(\jmath), F_j(\ell)) &= \rho(\jmath, \ell)\aleph([F_i \jmath]_{(1,0)}, [F_j \ell]_{(1,0)}) \\
&\leq \rho(\jmath, \ell) \sup_{(\alpha,\beta)\in(0,1]\times[0,1)} \aleph([F_i \jmath]_{(\alpha,\beta)}, [F_j \ell]_{(\alpha,\beta)}) \\
&= \rho(\jmath, \ell)\varrho_{(\infty,\infty)}(F_i(\jmath), F_j(\ell)).
\end{aligned}$$

Since $[F_n \jmath]_{(1,0)} \subseteq [F_n \jmath]_{(\alpha,\beta)} \in \mathcal{K}(\Lambda)$ for each $(\alpha,\beta) \in (0,1] \times [0,1)$ and $n \in \mathbb{N}$, therefore, $\varrho(\jmath, [F_n \jmath]_{(\alpha,\beta)}) \leq \varrho(\jmath, [F_n \jmath]_{(1,0)})$ for each $(\alpha,\beta) \in (0,1] \times [0,1)$; this implies that $p(\jmath, F_n(\jmath)) \leq \varrho(\jmath, [F_n \jmath]_{(1,0)})$. Hence, for all $\jmath, \ell \in \Lambda$ and $i, j \in \mathbb{N}$,

$$\rho(\jmath,\ell)\aleph([F_i \jmath]_{(1,0)}, [F_j \ell]_{(1,0)}) \leq \frac{k_{i,j}}{\eta}\left[\varrho(\jmath, [F_j y]_{(1,0)}) + \varrho(\ell, [F_i \jmath]_{(1,0)})\right].$$

Consequently, Thrm 3.2 can be applied to find $u \in \Lambda$ such that $u \in \bigcap_{i=1}^{\infty} [F_i u]_{(1,0)}$. □

Theorem 3.4. *Let (Λ, ϱ, η) be a cbms, $F_n : \Lambda \longrightarrow (IFS)^{\Lambda}$ be a sequence of intuitionistic fsm and $0 < k_{i,j}$ $(i, j \in \mathbb{N})$ with $k_{i,i+1} \neq 1$ for all $i \in \mathbb{N}$. Assume:*

(i) $\limsup_{i \longrightarrow \infty} k_{i,j} < 1$*, for each* $j \in \mathbb{N}$*;*
(ii) $\sum_{n=1}^{\infty} \Omega_n < \infty$*, where* $\Omega_n = \prod_{i=1}^{n} \frac{k_{i,i+1}}{1-k_{i,i+1}}$*;*
(iii) there exist $(\alpha,\beta) \in (0,1] \times [0,1)$ and a function $\rho : \Lambda \times \Lambda \longrightarrow \mathbb{R}_+$ such that for each $\jmath \in \Lambda$, $[F_n \jmath]_{(\alpha,\beta)}$ is a nonempty compact subset of Λ, and

$$\begin{aligned}\rho(\jmath,\ell)\aleph\left([F_i \jmath]_{(\alpha,\beta)}, [F_j \ell]_{(\alpha,\beta)}\right) \leq & \frac{k_{i,j}}{\eta} \max\{\varrho(\jmath,\ell), \varrho(\jmath, [F_i \jmath]_{(\alpha,\beta)}),\\ & \varrho(\ell, [F_j \ell]_{(\alpha,\beta)}), \varrho(\jmath, [F_j y]_{(\alpha,\beta)}), \varrho(\ell, [F_i \jmath]_{(\alpha,\beta)})\}\end{aligned} \tag{3.8}$$

for all $\jmath, \ell \in \Lambda$, $i, j \in \mathbb{N}$ with $\jmath \neq \ell$ and $i \neq j$;
(iv) there exists $\jmath_0 \in \Lambda$, $\jmath_1 \in [F_1 \jmath_0]_{(\alpha,\beta)}$ with $\jmath_0 \neq \jmath_1$ and $\rho(\jmath_0, \jmath_1) \geq 1$;
(v) $\{F_n\}_{n \geq 1}$ is (α,β)-admissible;
(v i) for each sequence $\{\jmath_n\}_{n \geq 1}$ in Λ with $\rho(\jmath_n, \jmath_{n+1}) \geq 1$ and $\jmath_n \longrightarrow u$, we have $\rho(\jmath_n, u) \geq 1$, for all $n \in \mathbb{N}$.
(vii) there exists $\eta \geq 1$ such that the series $\sum_{n=1}^{\infty} \Omega_n \eta^n$ is cgent.

Then there exists $u \in \Lambda$ such that $u \in \bigcap_{n=1}^{\infty} [F_n u]_{(\alpha,\beta)}$.

Proof. Using condition (*iv*) and (3.8), we have

$$\begin{aligned}\varrho(\jmath_1, [F_2 \jmath_1]_{(\alpha,\beta)}) \leq & \rho(\jmath_0, \jmath_1)\aleph([F_1 \jmath_0]_{(\alpha,\beta)}, [F_2 \jmath_1]_{(\alpha,\beta)})\\ \leq & \frac{k_{1,2}}{\eta} \max\{\varrho(\jmath_0, \jmath_1), \varrho(\jmath_0, [F_1 \jmath_0]_{(\alpha,\beta)}),\\ & \varrho(\jmath_1, [F_2 \jmath_1]_{(\alpha,\beta)}), \varrho(\jmath_0, [F_2 \jmath_1]_{(\alpha,\beta)}), \varrho(\jmath_1, [F_1 \jmath_0]_{(\alpha,\beta)})\}\\ \leq & \frac{k_{1,2}}{\eta} \max\{\varrho(\jmath_0, \jmath_1), \varrho(\jmath_1, [F_2 \jmath_1]_{(\alpha,\beta)}), \varrho(\jmath_0, [F_2 \jmath_1]_{(\alpha,\beta)})\}\\ \leq & \frac{k_{1,2}}{\eta} \max\{\varrho(\jmath_0, \jmath_1), \varrho(\jmath_1, [F_2 \jmath_1]_{(\alpha,\beta)}),\\ & \eta(\varrho(\jmath_0, \jmath_1) + \varrho(\jmath_1, [F_2 \jmath_1]_{(\alpha,\beta)}))\}\\ \leq & \frac{k_{1,2}}{\eta} [\eta(\varrho(\jmath_0, \jmath_1) + \varrho(\jmath_1, [F_2 \jmath_1]_{(\alpha,\beta)}))]\\ = & k_{1,2}[\varrho(\jmath_0, \jmath_1) + \varrho(\jmath_1, [F_2 \jmath_1]_{(\alpha,\beta)})].\end{aligned} \tag{3.9}$$

From (3.9), we have

$$\varrho(\jmath_1, [F_2 \jmath_1]_{(\alpha,\beta)}) \leq \frac{k_{1,2}}{1-k_{1,2}} \varrho(\jmath_0, \jmath_1). \tag{3.10}$$

Since $[F_2 \jmath_1]_{(\alpha,\beta)} \in \mathcal{K}(\Lambda)$, there exists $\jmath_2 \in [F_2 \jmath_1]_{(\alpha,\beta)}$ such that (3.10) gives $\varrho(\jmath_1, \jmath_2) \leq \frac{k_{1,2}}{1-k_{1,2}}\varrho(\jmath_0, \jmath_1)$. By (α,β)-admissibility of $\{F_n\}_{n\geq 1}$, we can find $(\alpha,\beta) \in (0,1] \times [0,1)$ such that $\rho(\jmath_1, \jmath_2) \geq 1$. On similar steps, we have

$$\begin{aligned}\varrho(\jmath_2, [F_3 \jmath_2]_{(\alpha,\beta)}) &\leq \frac{k_{2,3}}{1-k_{2,3}}\varrho(\jmath_1, \jmath_2)\\ &\leq \frac{k_{2,3}}{1-k_{2,3}}\frac{k_{1,2}}{1-k_{1,2}}\varrho(\jmath_0, \jmath_1).\end{aligned}$$

Following the proof of Thrm 3.2, we generate a Cchy sequence $\{\jmath_n\}_{n\geq 1}$ with $\rho(\jmath_n, \jmath_{n+1}) \geq 1$ such that $\jmath_n \longrightarrow u$ as $n \longrightarrow \infty$, for some $u \in \Lambda$. By Condition (vi), $\rho(\jmath_n, u) \geq 1$ $n \in \mathbb{N}$. Now, we show that u is a common intuitionistic fuzzy fp of F_n, $n \in \mathbb{N}$. Assume otherwise that $u \notin [F_n u]_{(\alpha,\beta)}$ so that $\varrho(u, [F_n u]_{(\alpha,\beta)}) > 0$. Let $p > 0$ be arbitrary. Then, consider

$$\begin{aligned}\varrho(u, [F_p u]_{(\alpha,\beta)}) \leq& \eta(\varrho(u, \jmath_n) + \varrho(\jmath_n, [F_p u]_{(\alpha,\beta)}))\\ \leq& \eta\varrho(u, \jmath_n) + \eta\rho(\jmath_{n-1}, u)\aleph([F_n \jmath_{n-1}]_{(\alpha,\beta)}, [F_p u]_{(\alpha,\beta)})\\ \leq& \eta\varrho(u, \jmath_n) + \eta\left(\frac{k_{n,p}}{\eta}\right)\max\{\varrho(\jmath_{n-1}, u), \varrho(\jmath_{n-1}, [F_p x_{n-1}]_{(\alpha,\beta)}),\\ &\varrho(u, [F_p u]_{(\alpha,\beta)}), \varrho(\jmath_{n-1}, [F_p u]_{(\alpha,\beta)}), \varrho(u, [F_n x_{n-1}]_{(\alpha,\beta)})\}\\ \leq& \eta\varrho(u, \jmath_n) + k_{n,p}\max\{\varrho(\jmath_{n-1}, u), \varrho(\jmath_{n-1}, \jmath_n),\\ &\varrho(u, [F_p u]_{(\alpha,\beta)}), \varrho(\jmath_{n-1}, [F_p u]_{(\alpha,\beta)}), \varrho(u, \jmath_n)\}.\end{aligned} \tag{3.11}$$

From (3.11), we have

$$\varrho(u, [F_p u]_{(\alpha,\beta)}) \leq \left(\limsup_{n\longrightarrow\infty} k_{n,p}\right)\varrho(u, [F_p u]_{(\alpha,\beta)}) \tag{3.12}$$
$$< \varrho(u, [F_p u]_{(\alpha,\beta)}) \tag{3.13}$$

a contradiction. It follows that $u \in \bigcap_{n=1}^{\infty}[F_n u]_{(\alpha,\beta)}$. □

Theorem 3.5. *Let (Λ, ϱ, η) be a cbms, $F_n : \Lambda \longrightarrow (IFS)^{\Lambda}$ be a sequence of intuitionistic fsm and $0 < l_{i,j}, k_{i,j}$ $(i, j \in \mathbb{N})$ with $l_{i,i+1} \neq 1$ for all $i \in \mathbb{N}$. Assume:*

(i) $\limsup_{i\longrightarrow\infty} l_{i,j} < 1$, *and* $\limsup_{i\longrightarrow\infty} k_{i,j} < 1$ *for each $j \in \mathbb{N}$;*
(ii) $\sum_{n=1}^{\infty}\Omega_n^* < \infty$, *where* $\Omega_n^* = \prod_{i=1}^{n}\frac{k_{i,i+1}}{1-\frac{l_{i,i+1}}{\eta}}$*;*
(iii) there exist $(\alpha,\beta) \in (0,1] \times [0,1)$ and a function $\rho : \Lambda \times \Lambda \longrightarrow \mathbb{R}_+$ such that for each $\jmath \in \Lambda$, $[F_n \jmath]_{(\alpha,\beta)}$ is a nonempty compact subset of Λ, and

$$\begin{aligned}&\rho(\jmath, \ell)\aleph\left([F_i \jmath]_{(\alpha,\beta)}, [F_j \ell]_{(\alpha,\beta)}\right)\\ &\leq \frac{l_{i,j}}{\eta}\varrho(\ell, [F_j y]_{(\alpha,\beta)})\varphi[\varrho(\jmath, [F_i \jmath]_{(\alpha,\beta)}), \varrho(\jmath, \ell)] + k_{i,j}\varrho(\jmath, \ell)\end{aligned} \tag{3.14}$$

for all $\jmath, \ell \in \Lambda$, $i, j \in \mathbb{N}$ with $\jmath \neq \ell$ and $i \neq j$; where $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function with the property that $\varphi(t,t) = 1$ for all $t \in \mathbb{R}_+$ and for any $t_1, s_1, t_2, s_2 \in \mathbb{R}_+$, we have $t_1 \leq t_2$, $s_1 = s_2$ implies $\varphi(t_1, s_1) \leq \varphi(t_2, s_2)$;

(iv) *there exists* $\jmath_0 \in \Lambda$, $\jmath_1 \in [F_1\jmath_0]_{(\alpha,\beta)}$ *with* $\jmath_0 \neq \jmath_1$ *and* $\rho(\jmath_0, \jmath_1) \geq 1$;

(v) $\{F_n\}_{n\geq 1}$ *is* (α,β)*-admissible;*

(vi) *for each sequence* $\{\jmath_n\}_{n\geq 1}$ *in* Λ *with* $\rho(\jmath_n, \jmath_{n+1}) \geq 1$ *and* $\jmath_n \longrightarrow u$, *we have* $\rho(\jmath_n, u) \geq 1$, *for all* $n \in \mathbb{N}$.

(vii) *there exists* $\eta \geq 1$ *such that the series* $\sum_{n=1}^{\infty} \Omega_n^* \eta^n$ *is cgent.*

Then there exists $u \in \Lambda$ *such that* $u \in \bigcap_{n=1}^{\infty} [F_n u]_{(\alpha,\beta)}$.

Proof. From (iv) and (3.14), we have

$$\begin{aligned}\varrho(\jmath_1, [F_2\jmath_1]_{(\alpha,\beta)}) \leq& \rho(\jmath_0, \jmath_1)\aleph([F_1\jmath_0]_{(\alpha,\beta)}, [F_2\jmath_1]_{(\alpha,\beta)})\\ \leq& \frac{l_{1,2}}{\eta}\varrho(\jmath_1, [F_2\jmath_1]_{(\alpha,\beta)})\varphi(\varrho(\jmath_0, [F_1\jmath_0]_{(\alpha,\beta)}), \varrho(\jmath_0, \jmath_1))\\ &+ k_{1,2}\varrho(\jmath_0, \jmath_1)\\ \leq& \frac{l_{1,2}}{\eta}\varrho(\jmath_1, [F_2\jmath_1]_{(\alpha,\beta)})\varphi(\varrho(\jmath_0, \jmath_1), \varrho(\jmath_0, \jmath_1)) + k_{1,2}\varrho(\jmath_0, \jmath_1)\\ \leq& \frac{l_{1,2}}{\eta}\varrho(\jmath_1, [F_2\jmath_1]_{(\alpha,\beta)}) + k_{1,2}\varrho(\jmath_0, \jmath_1).\end{aligned} \tag{3.15}$$

From (3.15), we have

$$\varrho(\jmath_1, [F_2\jmath_1]_{(\alpha,\beta)}) \leq \frac{k_{1,2}}{1 - \frac{l_{1,2}}{\eta}}\varrho(\jmath_0, \jmath_1). \tag{3.16}$$

Since $[F_2\jmath_1]_{(\alpha,\beta)} \in \mathcal{K}(\Lambda)$, we can find $\jmath_2 \in [F_2\jmath_1]_{(\alpha,\beta)}$ such that (3.16) becomes $\varrho(\jmath_1, \jmath_2) \leq \frac{k_{1,2}}{1-\frac{l_{1,2}}{\eta}}\varrho(\jmath_0, \jmath_1)$.

On similar steps, we get

$$\begin{aligned}\varrho(\jmath_2, [F_3\jmath_2]_{(\alpha,\beta)}) \leq& \frac{k_{2,3}}{1 - \frac{l_{2,3}}{\eta}}\varrho(\jmath_1, \jmath_2)\\ \leq& \left(\frac{k_{2,3}}{1 - \frac{l_{2,3}}{\eta}}\right)\left(\frac{k_{1,2}}{1 - \frac{l_{1,2}}{\eta}}\right)\varrho(\jmath_0, \jmath_1).\end{aligned}$$

By hypotheses, there exists $\jmath_3 \in [F_3\jmath_2]_{(\alpha,\beta)}$ such that

$$\varrho(\jmath_2, \jmath_3) \leq \left(\frac{k_{2,3}}{1 - \frac{l_{2,3}}{\eta}}\right)\left(\frac{k_{1,2}}{1 - \frac{l_{1,2}}{\eta}}\right)\varrho(\jmath_0, \jmath_1).$$

Recursively, we generate a sequence $\{\jmath_n\}_{n\geq 1}$ such that $\jmath_{n+1} \in [F_{n+1}\jmath_n]_{(\alpha,\beta)}$ with $\rho(\jmath_n, \jmath_{n+1}) \geq 1$, and

$$\varrho(\jmath_n, \jmath_{n+1}) \leq \Omega_n^* d(\jmath_0, \jmath_1),\ n \in \mathbb{N}. \tag{3.17}$$

From here, following the proof of Thrm 3.2, we deduce that $\{\jmath_n\}_{n\geq 1}$ is a Cchy sequence in Λ, and the completeness of Λ implies that there exists $u \in \Lambda$ such that $\jmath_n \longrightarrow u$ as $n \longrightarrow \infty$. By Condition (vi), $\rho(\jmath_n, u) \geq 1$, $n \in \mathbb{N}$. Next, we shall show that u is a common intuitionistic fuzzy fp of F_n. For this, assume contrary, that is, $u \notin [F_n u]_{(\alpha,\beta)}$ for all $(\alpha,\beta) \in (0,1] \times [0,1)$. Let $p > 0$ be arbitrary. Then, we have

$$\begin{aligned}
\varrho(u,[F_p u]_{(\alpha,\beta)}) \leq & \eta(\varrho(u,\jmath_n) + \varrho(\jmath_n,[F_p u]_{(\alpha,\beta)})) \\
\leq & \eta\varrho(u,\jmath_n) + \eta\rho(\jmath_{n-1},u)\aleph([F_n x_{n-1}]_{(\alpha,\beta)},[F_p u]_{(\alpha,\beta)}) \\
\leq & \eta\varrho(u,\jmath_n) + \eta\left(\frac{l_{n,p}}{\eta}\right)\varrho(u,[F_p u]_{(\alpha,\beta)})\varphi(\varrho(\jmath_{n-1},[F_n x_{n-1}]_{(\alpha,\beta)}),\varrho(\jmath_{n-1},u)) \\
& + k_{n,p}\varrho(\jmath_{n-1},u) \\
\leq & \eta\varrho(u,\jmath_n) + l_{n,p}\varrho(u,[F_p u]_{(\alpha,\beta)})\varphi(\varrho(\jmath_{n-1},\jmath_n),\varrho(\jmath_{n-1},u)) \\
& + k_{n,p}\varrho(\jmath_{n-1},u).
\end{aligned} \tag{3.18}$$

From (3.18), by using the continuity of the function φ, we obtain

$$\begin{aligned}
\varrho(u,[F_p u]_{(\alpha,\beta)}) \leq & \left(\limsup_{n\longrightarrow\infty} l_{n,p}\right)\varrho(u,[F_p u]_{(\alpha,\beta)}) \\
< & \varrho(u,[F_p u]_{(\alpha,\beta)})
\end{aligned}$$

a contradiction. This proves that $u \in \bigcap_{n=1}^{\infty}[F_n u]_{(\alpha,\beta)}$. □

We construct Example 3.6 to verify the assumptions of Thrm 3.2.

Example 3.6. Let $\Lambda = [0,1]$ be equipped with the metric $\varrho(\jmath,\ell) = |\jmath - \ell|^2$ for all $\jmath,\ell \in \Lambda$. Then $(\Lambda,\varrho,\eta = 2)$ is a cbms. But (Λ,ϱ,η) is not a metric, since for $\jmath = 0$ and $\ell = 2$ and $z = \frac{1}{3}$, we have

$$\varrho(\jmath,\ell) = 4 > \frac{26}{9} = \varrho(\jmath,z) + \varrho(z,\ell).$$

Now, define a sequence of intuitionistic fsm $F_n = \langle\mu_{F_n},\nu_{F_n}\rangle : \Lambda \longrightarrow (IFS)^{\Lambda}$ by

$$\mu_{F_n(\jmath)}(t) = \begin{cases} \frac{1}{2+n^2}, & \text{if } 0 \leq t \leq \frac{1}{6} + \frac{\jmath}{n+8} \\ \frac{1}{5+n^2}, & \text{if } \frac{1}{6} + \frac{\jmath}{n+8} < t \leq \frac{1}{3} + \frac{\jmath}{n+7} \\ \frac{1}{9+n^2}, & \text{if } \frac{1}{3} + \frac{\jmath}{n+7} < t \leq 1, \end{cases}$$

$$\nu_{F_n(\jmath)}(t) = \begin{cases} \frac{1}{50+n^2}, & \text{if } 0 \leq t \leq \frac{1}{6} + \frac{\jmath}{n+8} \\ \frac{1}{10+n^2}, & \text{if } \frac{1}{6} + \frac{\jmath}{n+8} < t \leq \frac{1}{3} + \frac{\jmath}{n+7} \\ \frac{1}{27+n^2}, & \text{if } \frac{1}{3} + \frac{\jmath}{n+7} < t \leq 1. \end{cases}$$

If $(\alpha,\beta) = \left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)$, for all $n \in \mathbb{N}$. Then, for each $\jmath \in \Lambda$, we have

$$[F_n(\jmath)]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)} = \left[0,\frac{1}{6} + \frac{\jmath}{n+8}\right].$$

Take $k_{i,j} = \frac{1}{3} + \frac{1}{(i-j)^2+11}$. Then, $\limsup_i k_{i,j} = \frac{1}{3} < 1$, for all $j \in \mathbb{N}$; and $\Omega_n = \prod_{i=1}^{n}\frac{k_{i,i+1}}{1-k_{i,i+1}} = \left(\frac{5}{12}\right)^n$. Therefore, $\sum_{n=1}^{\infty}\Omega_n = \sum_{n=1}^{\infty}\left(\frac{5}{12}\right)^n < \infty$. Clearly, since $\eta = 2$, then $\sum_{n=1}^{\infty}\Omega_n\eta^n = \sum_{n=1}^{\infty}\left(\frac{10}{12}\right)^n$. Hence, by D'Alembert's ration test, the series $\sum\Omega_n\eta^n$ is cgent. Moreover, define $\rho : \Lambda \times \Lambda \longrightarrow \mathbb{R}_+$ by

$$\rho(\jmath,\ell) = \begin{cases} 1, & \text{if } \jmath,\ell \in \left[0,\frac{1}{4}\right) \\ 0, & \text{otherwise.} \end{cases}$$

Now, we prove that

$$\begin{aligned}&\rho(\jmath,\ell)\aleph\left([F_i\jmath]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)},[F_j\ell]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)}\right)\\ \leq&\frac{k_{i,j}}{\eta}\left[\varrho(\jmath,[F_j\ell]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)})+\varrho(\ell,[F_i\jmath]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)})\right]\end{aligned}$$

Notice that for $\jmath=\ell=0$,

$$\begin{aligned}&\rho(0,0)\aleph\left([F_i0]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)},[F_j0]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)}\right)\\ =0\leq&\frac{k_{i,j}}{\eta}\left[\varrho(0,[F_j0]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)})+\varrho(0,[F_i0]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)})\right]\end{aligned}$$

for all $i,j\in\mathbb{N}$ and $\eta\geq1$. Next, for $\jmath,\ell\in(0,1]$ with $\jmath\neq\ell$, and for all $i,j\in\mathbb{N}$ with $i\neq j$, we have

$$\begin{aligned}&\rho(\jmath,\ell)\aleph\left([F_i\jmath]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)},[F_j\ell]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)}\right)\\ =&\aleph\left(\left[0,\frac{1}{6}+\frac{\jmath}{i+8}\right],\left[0,\frac{1}{6}+\frac{\ell}{j+8}\right]\right)=\left|\frac{\jmath}{i+8}-\frac{\ell}{j+8}\right|^2\leq\left|\frac{\jmath}{i+8}\right|^2\leq\frac{1}{|i+8|^2}\leq\frac{1}{8^2}\\ \leq&\left(\frac{1}{6}+\frac{1}{2(i-j)^2+22}\right)\left[|\jmath|^2+|\ell|^2\right]\\ \leq&\frac{1}{2}\left(\frac{1}{3}+\frac{1}{(i-j)^2+11}\right)\left[\varrho(\jmath,[F_jy]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)})+\varrho(\ell,[F_i\jmath]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)})\right]\\ =&\frac{k_{i,j}}{\eta}\left[\varrho(\jmath,[F_jy]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)})+\varrho(\ell,[F_i\jmath]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)})\right].\end{aligned}$$

Moreover, for $\jmath_0=0$ and $\jmath_1=\frac{1}{10}$, we get $\jmath_1\in[F_1\jmath_0]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)}=\left[0,\frac{1}{6}\right]$ and $\rho(\jmath_0,\jmath_1)\geq1$. It is easy to see that $\{F_n\}_{n\geq1}$ is (α,β)-admissible. Therefore, all the assumptions of Thrm 3.2 are satisfied. Hence, we can see that there exists $u=0\in\Lambda$ such that $0\in\bigcap_{n=1}^{\infty}[F_n0]_{\left(\frac{1}{2+n^2},\frac{1}{50+n^2}\right)}$.

Remark 2. Theorems 3.2, 3.4 and 3.5 are improvements of the main results in [1,2,21], even when Λ is a metric space; while Thrm 3.3 complements the result of Heilpern [17] even with $\eta=1$.

4. Stability and well-posedeness of intuitionistic fuzzy fixed point inclusions

Stability is a notion associated with the limiting behaviours of a system. It has been examined in the setting of both discrete and continuous dynamical systems (see, e.g. [41]) The study of the relationship between the cgence of a sequence of mappings and their fp has also been deeply investigated in different framework (see, e.g. [8]). Since set-valued mappings often have more fp than their corresponding single-valued mappings, their set of fp become more interesting for the study of stability.

In this section, we introduce the idea of Ulam-Hyers stability for fp problems in the frame of intuitionistic fsm. To this end, a few auxiliary definitions are inaugurated as follows.

Definition 4.1. Let (Λ, ϱ, η) be a bms and $F_n : \Lambda \longrightarrow (IFS)^{\Lambda}$ be a sequence of intuitionistic fuzzy set-valued maps. Then, the intuitionistic fuzzy fixed point inclusion:

$$\jmath \in [F_n \jmath]_{(\alpha,\beta)}, \ \jmath \in \Lambda, \ n \in \mathbb{N} \tag{4.1}$$

is said to be

(i) Ulam-Hyers stable, if there exist a real number $\varsigma > 0$ and $(\alpha, \beta) \in (0, 1] \times [0, 1)$ such that for every $\epsilon > 0$ and any solution ℓ^* of the inequation:

$$\varrho(\ell, [F_n \ell]_{(\alpha,\beta)}) \leq \epsilon \tag{4.2}$$

there exists a solution u of the inclusion (4.1) such that

$$\varrho(\ell^*, u) \leq \varsigma\epsilon. \tag{4.3}$$

(ii) Well-posed, if $F_n : \Lambda \longrightarrow (IFS)^{\Lambda}$ has an intuitionistic fuzzy fp u in Λ, for each $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} \varrho(\jmath_n, u) = 0$ for every sequence $\{\jmath_n\}_{n\geq 1}$ in Λ such that $\lim_{n\longrightarrow\infty} \varrho(\jmath_n, [F_n \jmath_n]_{(\alpha,\beta)}) = 0$, for some $(\alpha, \beta) \in (0, 1] \times [0, 1)$.

Theorem 4.2. *Let (Λ, ϱ, η) be a cbms with parameter $\eta \geq 1$ and $F_n : \Lambda \longrightarrow (IFS)^{\Lambda}$ be a sequence of intuitionistic fsm. In addition to the hypotheses of Thrm 3.2, suppose that the following conditions hold:*

(S1) For every $\eta \geq 1$, there exists $\epsilon > \eta$ such that $\limsup_{i\longrightarrow\infty} k_{i,j} \leq \frac{1}{1+\epsilon}$ for all $i, j \in \mathbb{N}$. ($k_{i,j}$ is as defined in Theorem (3.2)).

(S2) $\rho(\jmath, \ell) \geq 1$ for any $\jmath, \ell \in \Lambda$ satisfying (4.1).

(S3) $\rho(\jmath_n, u) \geq 1$ for any sequence $\{\jmath_n\}_{n\geq 1}$ in Λ such that $\lim_{n\longrightarrow\infty} \varrho(\jmath_n, [F_n \jmath_n]_{(\alpha,\beta)}) = 0$ and $u \in \bigcap_{n=1}^{\infty} [F_n u]_{(\alpha,\beta)}$.

Then, the intuitionistic fuzzy fp inclusion (4.1) is Ulam-Hyers stable if conditions $(S1)-(S2)$ hold, and well-posed if $(S3)$ is satisfied.

Proof. First, we will show that the intuitionistic fuzzy fp inclusion (4.1) is Ulam-Hyers stable. For each parameter $\eta \geq 1$, let $\epsilon > 0$ be chosen such that $\epsilon > \eta$. Let ℓ^* be a solution of (4.1), which means that $\varrho(\ell^*, [F_n \ell^*]_{(\alpha,\beta)}) \leq \epsilon$. Taking Thrm 3.2 into consideration, we know that there exists $u \in \Lambda$ such that $u \in [F_n u]_{(\alpha,\beta)}$, for each $n \in \mathbb{N}$ so that $\varrho(u, [F_n u]_{(\alpha,\beta)}) = 0 \leq \epsilon$, which means u verifies (4.1) so that by hypothesis, $\rho(\ell^*, u) \geq 1$. Now, using the triangle inequality in (Λ, ϱ, η) and the contraction condition 3.1, for all $p \geq 1$, we have

$$\begin{aligned}
\varrho(\ell^*, u) &= \varrho(\ell^*, [F_p u]_{(\alpha,\beta)}) \\
&\leq \eta[\varrho(\ell^*, [F_n \ell^*]_{(\alpha,\beta)}) + \aleph([F_n \ell^*]_{(\alpha,\beta)}, [F_p u]_{(\alpha,\beta)})] \\
&\leq \eta[\varrho(\ell^*, [F_n \ell^*]_{(\alpha,\beta)}) + \eta\rho(\ell^*, u)\aleph([F_n \ell^*]_{(\alpha,\beta)}, [F_p u]_{(\alpha,\beta)})] \\
&\leq \eta[\varrho(\ell^*, [F_n \ell^*]_{(\alpha,\beta)}) + \eta\left(\frac{k_{n,p}}{\eta}\right)[\varrho(\ell^*, [F_p u]_{(\alpha,\beta)}) + \varrho(u, [F_n \ell^*]_{(\alpha,\beta)})] \\
&\leq \eta\epsilon + k_{n,p}[\varrho(\ell^*, u) + \varrho(u, [F_n \ell^*]_{(\alpha,\beta)})] \\
&\leq \eta\epsilon + k_{n,p}[\varrho(\ell^*, u) + \eta\varrho(u, \ell^*) + \eta\varrho(\ell^*, [F_n \ell^*]_{(\alpha,\beta)})] \\
&= \eta\epsilon + k_{n,p}[\varrho(\ell^*, u)(1 + \eta) + \eta\epsilon].
\end{aligned} \tag{4.4}$$

By condition $(S1)$, the inequality (4.4) becomes

$$\begin{aligned}\varrho(\ell^*, u) \leq\ & \eta\epsilon + \frac{\varrho(\ell^*, u)(1+\eta)}{1+\epsilon} + \frac{\eta\epsilon}{1+\epsilon} \\ =\ & \left(\frac{2+\epsilon}{\epsilon-\eta}\right)\eta\epsilon = \varsigma\epsilon,\end{aligned}$$

where $\varsigma = \left(\frac{2+\epsilon}{\epsilon-\eta}\right)\eta > 0$; that is, $\varrho(\ell^*, u) \leq \varsigma\epsilon$.

Next, in view of condition $(S3)$, we will show that the intuitinoistic fuzzy fp inclusion (4.1) is well-posed. Since $u \in [F_n u]_{(\alpha,\beta)}$, for each $n \in \mathbb{N}$, then for all $p \geq 1$, we have

$$\begin{aligned}\varrho(\jmath_n, u) \leq\ & \eta[\varrho(\jmath_n, [F_n \jmath_n]_{(\alpha,\beta)}) + \aleph([F_n \jmath_n]_{(\alpha,\beta)}, [F_p u]_{(\alpha,\beta)})] \\ \leq\ & \eta\varrho(\jmath_n, [F_n \jmath_n]_{(\alpha,\beta)}) + \eta\rho(\jmath_n, u)\aleph([F_n \jmath_n]_{(\alpha,\beta)}, [F_p u]_{(\alpha,\beta)}) \\ \leq\ & \eta\varrho(\jmath_n, [F_n \jmath_n]_{(\alpha,\beta)}) + \eta\left(\frac{k_{n,p}}{\eta}\right)[\varrho(\jmath_n, [F_p u]_{(\alpha,\beta)}) + \varrho(u, [F_n \jmath_n]_{(\alpha,\beta)})] \\ \leq\ & \eta\varrho(\jmath_n, [F_n \jmath_n]_{(\alpha,\beta)}) + \eta k_{n,p}[2\varrho(\jmath_n, u) + \varrho(\jmath_n, [F_n \jmath_n]_{(\alpha,\beta)})] \\ \leq\ & \left(\frac{1+k_{n,p}}{1-2\eta k_{n,p}}\right)\varrho(\jmath_n, [F_n x_n]_{(\alpha,\beta)}).\end{aligned} \tag{4.5}$$

Letting $n \longrightarrow \infty$ in (4.5) and noting that $\lim_{n\longrightarrow\infty} \varrho(\jmath_n, [F_n \jmath_n]_{(\alpha,\beta)}) = 0$, we obtain $\lim_{n\longrightarrow\infty} \varrho(\jmath_n, u) = 0$, proving the well-posedness of (4.1). □

5. Applications to Fredholm integral inclusions

In this section, we apply Thrm 3.4 to investigate sufficient conditions for existence of solutions to a system of Fredholm int-incl. For basic concepts of int-incl, we refer the interested reader to [3, 25] and references therein.

Consider the following system of int-incl of Fredholm type:

$$\jmath(t) \in \left[f(t) + \int_a^b L_n(t, s, \jmath(s))\varrho s,\ t \in [a, b] = J,\right] \tag{5.1}$$

where $\jmath \in C(J, \mathbb{R})$ is an unknown function, $f \in C(J, \mathbb{R})$ is a given real-valued function and $L_n : J \times J \times \mathbb{R} \longrightarrow F_{cv}(\mathbb{R})$ is a given multivalued map, for each $n \in \mathbb{N}$, where we denote the family of nonempty compact and convex subsets of $\mathbb{R}$ by $F_{cv}(\mathbb{R})$, and the set of all real-valued continuous functions on J is represented by $C(J, \mathbb{R})$. By the function space $L^1(J)$, we mean the Lebesgue space $L^p(J)$ for $p = 1$ over J.

Let $\Lambda = C(J, \mathbb{R})$ and define a metric $\varrho : \Lambda \times \Lambda \longrightarrow \mathbb{R}_+$ as $\varrho(\jmath, \ell) = |\jmath - \ell|^2$, for all $\jmath, \ell \in \Lambda$. Then $(\Lambda, \varrho, \eta = 2)$ is a cbms. Now, we study solvability conditions of (5.1) under the following assumptions:

Theorem 5.1. *Let $F_n : \Lambda \longrightarrow (IFS)^\Lambda$ be a sequence of intuitionistic fsm whose (α,β)-level set is given as*

$$[F_n \jmath]_{(\alpha,\beta)} = \left\{\jmath \in \Lambda : \ell(t) \in f(t) + \int_a^b L_n(t, s, \jmath(s))\varrho s,\ t \in [a, b] = J\right\}.$$

Also, let $\varrho : \Lambda \times \Lambda \longrightarrow \mathbb{R}$ be a given function. Suppose that the following hypotheses are satisfied:

(H1) *the set-valued map* $L_n : J \times J \times \mathbb{R} \longrightarrow F_{cv}(\mathbb{R})$ *is such that for every* $\jmath \in \Lambda$*, the map* $L_{n_\jmath}(t,s) := L_n(t,s,\jmath(s))$ *is lower semicontinuous for each* $n \in \mathbb{N}$*;*
Let $0 < k_{i,j}(i, j \in \mathbb{N})$ *with* $k_{i,i+1} \neq 1$ *for each* $i \in \mathbb{N}$ *satisfies:*

(H2) $\lim_{i\longrightarrow\infty} \sup k_{i,j} < 1$*, for each* $j \in \mathbb{N}$*;*

(H3) $\sum_{n=1}^{\infty} \Omega_n < \infty$*, where* $\Omega_n = \prod_{i=1}^{n} \frac{k_{i,i+1}}{1-k_{i,i+1}}$*;*

(H4) *there exist* $\omega(t,.) \in L^1(J)$ *such that for all* $\eta \geq 1$ *and* $s, t \in J$*,* $\sup_{t\in J}\left(\int_a^b \omega(t,s)\varrho s\right)^2 \leq \frac{k_{i,j}}{\eta}$*; where* $k_{i,j}$ *satisfies* $(H2)-(H3)$*, and for all* $i, j \in \mathbb{N}$*,*

$$\aleph(L_{i_\jmath}(t,s), L_{j_\ell}(t,s)) \leq \omega(t,s)(|\jmath(s) - \ell(s)|^2);$$

(H5) *there exist* $\jmath_0 \in \Lambda$*,* $\jmath_1 \in [F_1\jmath_0]_{(\alpha,\beta)}$ *with* $\jmath_0 \neq \jmath_1$ *such that* $\varrho(\jmath_0, \jmath_1) \geq 0$*;*

(H6) *for each* $\jmath \in \Lambda$ *and* $\ell \in [F_n\jmath]_{(\alpha,\beta)}$ *with* $\varrho(\jmath,\ell) \geq 0$*, we have* $\varrho(\ell, z) \geq 0$ *for all* $z \in [F_{n+1}\ell]_{(\alpha,\beta)}$*;*

(H7) *if* $\{\jmath_n\}_{n\geq 1}$ *is a sequence in* Λ *such that* $\jmath_n \longrightarrow u \in \Lambda$ *and* $\varrho(\jmath_n, \jmath_{n+1}) \geq 0$*, then* $\varrho(\jmath_n, u) \geq 0$*, for all* $n \in \mathbb{N}$*;*

(H8) *there exists* $\eta \geq 1$ *such that the series* $\sum_{n=1}^{\infty} \Omega_n \eta^n$ *is cgent.*

Then the system of integral inclusions 5.1 has at least one solution in Λ*.*

Proof. Consider a function $\rho : \Lambda \times \Lambda \longrightarrow \mathbb{R}$ defined by

$$\rho(\jmath, \ell) = \begin{cases} 1, & \text{if } \varrho(\jmath,\ell) \geq 0 \text{ for all } \jmath \neq \ell \\ 0, & \text{otherwise.} \end{cases}$$

Observe that the set of solutions of (5.1) coincides with the intuitionistic fuzzy fp of sequence of intuitionistic fsm $F_n : \Lambda \longrightarrow (IFS)^{\Lambda}$. Also, note that the hypotheses $(H2), (H3), (H5), (H6)$ and $(H7)$ imply that conditions $(i), (ii), (iv), (v)$ and (vi), respectively of Thrm 3.4 hold. Next, we will verify that condition (iii) and the assumptions on the (α,β)-level set $[F_n\jmath]_{(\alpha,\beta)}$ hold for each $\jmath \in \Lambda$. To this end, let $\jmath \in \Lambda$ be arbitrary. Since the multivalued map $L_{n_\jmath} : J \times J \longrightarrow F_{cv}(\mathbb{R})$ is lower semicontinuous, it follows from Michael's selection thrm ([28, Thrm 1]) that there exists a continuous map $\rho_\jmath : J \times J \longrightarrow \mathbb{R}$ such that $\rho_\jmath(t,s) \in L_{n_\jmath}(t,s)$, for each $(t,s) \in J \times J$. Therefore, $f(t) + \int_a^b \rho_\jmath(t,s)\varrho s \in [F_n\jmath]_{(\alpha,\beta)}$. So, $[F_n\jmath]_{(\alpha,\beta)}$ is nonempty for each $n \in \mathbb{N}$. One can easily see that $[F_n\jmath]_{(\alpha,\beta)}$ is a closed subset of Λ. Further, given that $f \in C(J)$ and $L_{n_\jmath}(t,s)$ is continuous on $J \times J$, their range sets are compact. Hence, $[F_n\jmath]_{(\alpha,\beta)}$ is also compact. Take $\jmath_1, \jmath_2 \in \Lambda$, then there exists $(\alpha,\beta) \in (0,1] \times [0,1)$ such that $[F_n\jmath_1]_{(\alpha,\beta)}$ and $[F_n\jmath_2]_{(\alpha,\beta)}$ are nonempty compact subsets of Λ for each $n \in \mathbb{N}$. Let $\ell_1 \in [F_n\jmath_1]_{(\alpha,\beta)}$ be arbitrary such that

$$\ell_1(t) \in f(t) + \int_a^b L_{n_\jmath}(t,s,\jmath_1(s))\varrho s,\ t \in J.$$

This means for each $(t,s) \in J \times J$, there exists $\rho_{\jmath_1} \in L_{n_{\jmath_1}}(t,s)$ such that

$$\ell_1(t) = f(t) + \int_a^b \rho_{\jmath_1}(t,s)\varrho s,\ t \in J.$$

Since from $(H4)$,

$$\aleph\left(L_i(t,s,\jmath_1(s)), L_j(t,s,\jmath_2(s))\right) \leq \omega(t,s)\left(|\jmath_1(s) - \jmath_2(s)|^2\right)$$

for each $t, s \in J$ and $i, j \in \mathbb{N}$, so there exists $\rho_{\jmath_2} \in L_{n_{\jmath_2}}(t, s)$ for each $n \in \mathbb{N}$ such that

$$|\rho_{\jmath_1}(t, s) - \rho_{\jmath_2}(t, s)|^2 \leq \omega(t, s)\left(|\jmath_1(s) - \jmath_2(s)|^2\right)$$

for all $(t, s) \in J \times J$. Now, consider the set-valued map $\mathfrak{M}$ defined by

$$\mathfrak{M}(t, s) = L_{n_{\jmath_2}}(t, s) \cap \left\{\vartheta \in \mathbb{R} : |\rho_{\jmath_1}(t, s) - \vartheta| \leq \omega(t, s)\left(|\jmath_1(s) - \jmath_2(s)|\right)\right\}.$$

Taking into account the fact that from $(H1)$, $\mathfrak{M}$ is lower semicontinuous, therefore, by Michael's selection thrm, there exists a continuous map $\rho_{\jmath_2} : J \times J \longrightarrow \mathbb{R}$ such that $\rho_{\jmath_2}(t, s) \in \mathfrak{M}(t, s)$, for all $(t, s) \in J \times J$. Then,

$$\ell_2(t) \ = \ f(t) + \int_a^b \rho_{\jmath_1}(t, s)\varrho s \in f(t) + \int_a^b L_n(t, s, \jmath_2(s))\varrho s, \ t \in J.$$

Thus, $\ell_2 \in [F_n \jmath 2]_{(\alpha,\beta)}$, and

$$\begin{aligned}
|\ell_1(t) - \ell_2(t)|^2 &\leq \left(\int_a^b |\rho_{\jmath_1}(t, s) - \rho_{\jmath_2}(t, s)|\varrho s\right)^2 \\
&\leq \sup\left(\int_a^b \omega(t, s)\varrho s\right)^2 |\jmath_1(s) - \jmath_2(s)|^2 \\
&\leq \frac{k_{i,j}}{\eta}(|\jmath_1(s) - \jmath_2(s)|^2) \\
&\leq \frac{k_{i,j}}{\eta} \max\{|\jmath_1(s) - \jmath_2(s)|^2, |\jmath_1(s) - L_i(t, s, \jmath_1(s))|^2, |\jmath_2(s) - L_j(t, s, \jmath_2(s))|^2, \\
&\qquad |\jmath_1(s) - L_j(t, s, \jmath_2(s))|^2, |\jmath_2(s) - L_i(t, s, \jmath_1(s))|^2\}.
\end{aligned}$$

The above inequality yields

$$\begin{aligned}
\aleph([F_i \jmath_1]_{(\alpha,\beta)}, [F_j \jmath_2]_{(\alpha,\beta)}) \leq \ & \frac{k_{i,j}}{\eta} \max\{\varrho(\jmath_1, \jmath_2), \varrho(\jmath_1, [F_i \jmath_1]_{(\alpha,\beta)}), \\
& \varrho(\jmath_2, [F_j \jmath_2]_{(\alpha,\beta)}), \varrho(\jmath_1, [F_j \jmath_2]_{(\alpha,\beta)}), \varrho(\jmath_2, [F_i \jmath_1]_{(\alpha,\beta)})\}.
\end{aligned} \tag{5.2}$$

Taking $\jmath = \jmath_1$ and $\jmath_2 = \ell$ in (5.2) gives condition (iii) of Theorem 3.4. Thus, all the hypotheses of Theorem 3.4 are satisfied. So, we conclude that there exists a solution of the system of integral inclusions (5.1). □

6. Conclusions

In the setting of bms, new common intuitionistic fuzzy fp thrm for sequence of intuitionistic fsm are proved. The established ideas improve some important existing fp thrm for single-valued and crisp set-valued maps in the corresponding literature. Moreover, the ideas of Ulam-Hyers stability and well-posedness of intuitionistic fuzzy fp problems are initiated to complement their crisp set-valued counterparts. Sufficient conditions for existence of solutions of int-incl are examined to indicate an application of the established results herein.

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Conflict of interest

The authors declare that they have no competing interests.

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