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Research article

On the character sums analogous to high dimensional Kloosterman sums

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Abstract: The main purpose of this paper is using the properties of the classical Gauss sums and the analytic methods to study the computational problem of one kind of character sums analogous to high dimensional Kloosterman sums, and give some interesting identities for it.

Keywords: character sums; high dimensional Kloosterman sums; analytic method; identity
Mathematics Subject Classification: 11L03, 11L05, 11L40

1. Introduction

Let $q > 1$ be an integer, h, c_i and b are integers with $h > 1$ and $i = 1,\ 2,\ \cdots,h$. The h-dimensional Kloosterman sums $K(c_1, c_2, \cdots, c_h, b; q)$ is defined as:

$$K(c_1, c_2, \cdots, c_h, b; q) = \sideset{}{'}\sum_{a_1=1}^{q} \sideset{}{'}\sum_{a_2=1}^{q} \cdots \sideset{}{'}\sum_{a_h=1}^{q} e\left(\frac{c_1a_1 + c_2a_2 + \cdots + c_ha_h + b\overline{a_1a_2\cdots a_h}}{q}\right),$$

where as usual, $e(y) = e^{2\pi iy}$, $\sideset{}{'}\sum_{a=1}^{q}$ denotes the summation over all $1 \le a \le q$ such that $(a, q) = 1$, and $c \cdot \overline{c} \equiv 1 \bmod q$.

This sum occupies a very important position in the research of number theory, and many classical problems in analytic number theory are closely related to it. For these reasons, many scholars have studied the properties of $K(c_1, c_2, \cdots, c_h, b; q)$, and obtained a series of important results. For example, R. A. Smith [1] studied the properties of the n-dimensional Kloosterman sums, and obtained a sharp upper bound estimate for

$$S(m, n; q) = \sideset{}{'}\sum_{a_1=1}^{q} \cdots \sideset{}{'}\sum_{a_n=1}^{q} e\left(\frac{a_1 + a_2 + \cdots + a_n + m\overline{a_1} \cdot \overline{a_2} \cdots \overline{a_n}}{q}\right).$$

W. P. Zhang and D. Han [2] studied the fourth power mean of the 2-dimensional Kloostermann sums, and proved the identity

$$\begin{aligned}&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}e\left(\frac{ma+b+\overline{ab}}{p}\right)\right|^4\\ =&\begin{cases}7p^5-18p^4-\left(b_p+6\right)p^3-6p^2-3p & \text{if } p\equiv 1 \bmod 6;\\ 7p^5-22p^4-\left(b_p-14\right)p^3-6p^2-3p & \text{if } p\equiv 5 \bmod 6,\end{cases}\end{aligned}$$

where b_p is an integer satisfying $|b_p|<2p^{\frac{3}{2}}$.

W. P. Zhang and X. X. Li [3] obtained the identity

$$\begin{aligned}&\sum_{\chi \bmod p}\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(ab)e\left(\frac{a+b+m\overline{ab}}{p}\right)\right|^4\\ =&\ (p-1)(2p^5-7p^4+2p^3+8p^2+4p+1).\end{aligned}$$

In this paper, we considered the character sums analogous to high dimensional Kloosterman sums as follows:

$$S(m,h,\chi;p)=\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+\cdots+a_h+m\overline{a_1\cdots a_h}\right), \tag{1.1}$$

where p is an odd prime, χ is any non-principal Dirichlet character mod p, h is any fixed positive integer, and m is any integer.

About summation (1.1), X. X. Lv and W. P. Zhang [4] studied the hybrid power mean involving $S(m,2,\chi;p)$ and two-term exponential sums, and proved the following results.

Let p be an odd prime with $3\nmid(p-1)$. Then for any non-principal character χ modulo p, one has the identities

$$\begin{aligned}&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{a}\overline{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb+\overline{b}}{p}\right)\right|^2\\ =&\ \sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{a}\overline{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2=p^2\cdot\left(p^2-p-1\right).\end{aligned}$$

If p is an odd prime with $3\mid(p-1)$. Then for any three-th character χ mod p (i.e., there exists a character χ_1 mod p such that $\chi=\chi_1^3$), then one has the asymptotic formulae

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{a}\overline{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb+\overline{b}}{p}\right)\right|^2=3p^4+N(p),$$

and

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{a}\overline{b}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2=3p^4+N_1(p),$$

where $N(p)$ and $N_1(p)$ satisfy $|N(p)| \leq 9 \cdot p^{\frac{7}{2}}$ and $|N_1(p)| \leq 15 \cdot p^3$ respectively.

X. L. Xu, J. F. Zhang and W. P. Zhang [5] generalized the results in [4] to the high dimensional cases, and obtained the following conclusions:

Theorem A. Let p be an odd prime, $h \geq 1$ is an integer with $(h+1, p-1) = 1$. Then for any non-principal character χ mod p, one has the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a_1=1}^{p-1} \cdots \sum_{a_h=1}^{p-1} \chi\left(a_1 + \cdots + a_h + m\overline{a_1 \cdots a_h}\right) \right|^2 \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^{h+1}+a}{p}\right) \right|^2$$

$$= p^h \cdot \left(p^2 - p - 1\right).$$

Theorem B. Let p be an odd prime, h is an integer with $(h+1) \mid (p-1)$, χ is any non-principal character mod p. If χ is a $(h+1)$-th character mod p (That is, there exists a character χ_1 mod p such that $\chi = \chi_1^{h+1}$), then one has

$$\sum_{m=1}^{p-1} \left| \sum_{a_1=1}^{p-1} \cdots \sum_{a_h=1}^{p-1} \chi\left(a_1 + \cdots + a_h + m\overline{a_1 \cdots a_h}\right) \right|^2 \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^{h+1}+a}{p}\right) \right|^2$$

$$= (h+1) \cdot p^{h+2} + O\left(h^2 \cdot p^{h+1}\right).$$

In addition, some other results related to the Kloosterman sums can also be found in the references [6–12], we no longer list here.

The main purpose of this paper is using the analytic methods and the properties of Gauss sums to study the computational problem of the fourth power mean of (1.1) with $h = 2$, and give some interesting results. That is, we will prove the following:

Theorem 1.1. Let p be an odd prime with $p \equiv 1 \bmod 3$, χ be any 3-th non-principal character modulo p. Then we have the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi\left(a + b + m\overline{ab}\right) \right|^4$$

$$= 15p^4(p-1) + \frac{2(p-1)\overline{\tau}^3(\chi)\chi^3(3)}{p^2} \cdot E_9(\chi) + \frac{2(p-1)\tau^3(\chi)\overline{\chi}^3(3)}{p^2} \cdot \overline{E}_9(\chi),$$

where $E_n(\chi) = \tau^n(\chi_1) + \tau^n(\chi_1\lambda) + \tau^n(\chi_1\overline{\lambda})$, $\overline{E}_n(\chi)$ denotes the complex conjugate of $E_n(\chi)$, and $E_n(\chi)$ satisfies the third-order linear recursion formula

$$E_{n+3}(\chi) = E_1(\chi) \cdot E_{n+2}(\chi) - \tau(\chi) \cdot \overline{\chi}(3) \cdot \overline{E}_1(\chi) \cdot E_{n+1}(\chi) + p \cdot \tau(\chi) \cdot \overline{\chi}(3) \cdot E_n(\chi),$$

the first three terms of $E_n(\chi)$ are

$E_0(\chi) = 3$; $E_1(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^3}{p}\right)$; $E_2(\chi) = E_1^2(\chi) - 2\tau(\chi)\overline{\chi}(3) \cdot \overline{E}_1(\chi)$, and

$\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums, $|\tau(\chi)| = \sqrt{p}$.

Theorem 1.2. Let p be a prime with $p \equiv 1 \bmod 3$, then we have

$$\frac{1}{p-1}\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+m\overline{ab}}{p}\right)\right|^4 = 19p^4 + 4p\left(d^2-2p\right)\left(d^4-4pd^2+p^2\right),$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol modulo p, d is uniquely determined by $4p = d^2 + 27c^2$ and $d \equiv 1 \bmod 3$.

From our theorems we can also deduce the following:

Corollary 1.1. Let p be an odd prime with $p \equiv 1 \bmod 3$, g be any primitive root modulo p. Then we have the identity

$$\frac{1}{3}\sum_{i=0}^{2}\left(\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+g^i\cdot\overline{ab}}{p}\right)\right)^2 = 3p^2.$$

Corollary 1.2. Let p be a prime, χ be any non-principal character modulo p. Then we have the estimate

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right)\right|^4 \le 27\cdot p^4\cdot(p-1).$$

Some notes: Let p be an odd prime, $h \ge 1$ is an integer with $(h+1) \mid (p-1)$. If χ is not a $(h+1)$-th character modulo p, then it is easy to prove that

$$\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_h+m\overline{a_1\cdots a_h}\right)\right| = 0.$$

Let p be an odd prime, $k \ge 1$ be an integer with $(k+1, p-1) = 1$. Then for any integer m with $(m, p) = 1$ and any non-principal character χ mod p, we have

$$\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right| = p^{\frac{k}{2}}.$$

So in these cases, our conclusions are trivial.

2. Some lemmas

In some places of this paper, we need to use the definition and properties of the classical Gauss sums $\tau(\chi)$ and character sums, these contents can be found in some analytic number theory books, such as [13], here we will not repeat the related contents. First we have the following:

Lemma 2.1. Let p be an odd prime, $k \ge 1$ is an integer with $(k+1) \mid (p-1)$, χ is any non-principal character mod p, and m is an integer with $(m, p) = 1$. If χ is not a $(k+1)$-th character mod p, then we have the identity

$$\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right) = 0.$$

If χ is a $(k+1)$-th character modulo p, then we have the identity

$$\begin{aligned}&\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\\ =\ & \chi_1(m)\left[\frac{\tau^{k+1}(\overline{\chi}_1)}{\tau(\overline{\chi})}+\overline{\lambda}(m)\frac{\tau^{k+1}(\overline{\chi}_1\lambda)}{\tau(\overline{\chi})}+\cdots+\overline{\lambda}^k(m)\frac{\tau^{k+1}(\overline{\chi}_1\lambda^k)}{\tau(\overline{\chi})}\right],\end{aligned}$$

where λ denotes any $(k+1)$-order character modulo p, and $\chi=\chi_1^{k+1}$.

Proof. If $(k+1)\mid(p-1)$, and χ is not a $(k+1)$-th character mod p, then there exists an integer $1<r<p-1$ such that $r^{k+1}\equiv\overline{r}^{k+1}\equiv 1 \bmod p$ and $\chi(r)\neq 1$. So from the properties of the reduced residue system modulo p we have

$$\begin{aligned}&\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\\ =\ &\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(ra_1+ra_2+\cdots+ra_k+m\overline{r}^k\overline{a_1\cdots a_k}\right)\\ =\ &\chi(r)\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{r}^{k+1}\overline{a_1\cdots a_k}\right)\\ =\ &\chi(r)\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right).\end{aligned}\tag{2.1}$$

Since $\chi(r)\neq 1$, so from (2.1) we have the identity

$$\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)=0.\tag{2.2}$$

If $(k+1)\mid(p-1)$, and χ is a $(k+1)$-th character mod p, let $\chi=\chi_1^{k+1}$, and λ is a $k+1$-order character modulo p. Then for any integer m with $(m,p)=1$, note that the identity

$$1+\lambda(b)+\lambda^2(b)+\cdots+\lambda^k(b)=\begin{cases}k+1, & \text{if } (b,p)=1 \text{ and } b\equiv c^{k+1} \bmod p;\\ 0, & \text{otherwise.}\end{cases}$$

From the properties of the classical Gauss sums we have

$$\begin{aligned}&\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\\ =\ &\frac{1}{\tau(\overline{\chi})}\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(b)e\left(\frac{b(a_1+a_2+\cdots+a_k+m\overline{a_1a_1\cdots a_k})}{p}\right)\\ =\ &\frac{1}{\tau(\overline{\chi})}\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}e\left(\frac{a_1+a_2+\cdots+a_k}{p}\right)\sum_{b=1}^{p-1}\overline{\chi}(b)e\left(\frac{mb^{k+1}\overline{a_1a_2\cdots a_k}}{p}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\tau(\overline{\chi})}\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} e\left(\frac{a_1+a_2+\cdots+a_k}{p}\right)\\
&\quad\times\sum_{b=1}^{p-1}\left(1+\lambda(b)+\lambda^2(b)+\cdots+\lambda^k(b)\right)\overline{\chi}_1(b)e\left(\frac{mb\overline{a_1a_2\cdots a_k}}{p}\right)\\
&= \chi_1(m)\left[\frac{\tau^{k+1}(\overline{\chi}_1)}{\tau(\overline{\chi})}+\overline{\lambda}(m)\frac{\tau^{k+1}(\overline{\chi}_1\lambda)}{\tau(\overline{\chi})}+\cdots+\overline{\lambda}^k(m)\frac{\tau^{k+1}(\overline{\chi}_1\lambda^k)}{\tau(\overline{\chi})}\right]. \qquad (2.3)
\end{aligned}
$$

Now Lemma 2.1 follows from formulae (2.2) and (2.3).

Lemma 2.2. Let p be an odd prime, $k > 1$ is an integer with $(k+1, p-1) = 1$. Then for any non-principal character χ mod p and integer m with $(m, p) = 1$, we have the identity

$$\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\right| = p^{\frac{k}{2}}.$$

Proof. Note that $(k+1, p-1) = 1$, so if a passes through a reduced residue system modulo p, then a^{k+1} also pass through a reduced residue system modulo p. Let $h(k+1) \equiv 1 \bmod (p-1)$, then from the method of proving (2.3) we have

$$
\begin{aligned}
&\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi\left(a_1+a_2+\cdots+a_k+m\overline{a_1\cdots a_k}\right)\\
&= \frac{1}{\tau(\overline{\chi})}\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} e\left(\frac{a_1+a_2+\cdots+a_k}{p}\right)\sum_{b=1}^{p-1}\overline{\chi}(b)e\left(\frac{mb^{k+1}\overline{a_1a_2\cdots a_k}}{p}\right)\\
&= \frac{1}{\tau(\overline{\chi})}\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} e\left(\frac{a_1+a_2+\cdots+a_k}{p}\right)\sum_{b=1}^{p-1}\overline{\chi}(b^h)e\left(\frac{mb^{h(k+1)}\overline{a_1a_2\cdots a_k}}{p}\right)\\
&= \frac{1}{\tau(\overline{\chi})}\cdot\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} e\left(\frac{a_1+a_2+\cdots+a_k}{p}\right)\sum_{b=1}^{p-1}\overline{\chi}^h(b)e\left(\frac{mb\overline{a_1a_2\cdots a_k}}{p}\right)\\
&= \frac{\chi^h(m)\cdot\tau\left(\overline{\chi}^h\right)}{\tau(\overline{\chi})}\cdot\left(\sum_{a=1}^{p-1}\overline{\chi}^h(a)e\left(\frac{a}{p}\right)\right)^k = \chi^h(m)\cdot\frac{\tau^{k+1}\left(\overline{\chi}^h\right)}{\tau(\overline{\chi})}. \qquad (2.4)
\end{aligned}
$$

Note that $|\tau\left(\overline{\chi}^h\right)| = |\tau(\overline{\chi})| = \sqrt{p}$, from (2.4) we can deduce Lemma 2.2.

Lemma 2.3. Let p be an odd prime with $p \equiv 1 \bmod 3$. Then for any 3-th character $\chi = \chi_1^3$ modulo p, we have the identity

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right)\right|^4\\
&= 15p^4(p-1)+\frac{2(p-1)\overline{\tau}^3(\chi)\chi^3(3)}{p^2}\cdot\left(\tau^9(\chi_1)+\tau^9(\chi_1\lambda)+\tau^9(\chi_1\overline{\lambda})\right)\\
&\quad+\frac{2(p-1)\tau^3(\chi)\overline{\chi}^3(3)}{p^2}\cdot\left(\overline{\tau}^9(\chi_1)+\overline{\tau}^9(\chi_1\lambda)+\overline{\tau}^9(\chi_1\overline{\lambda})\right),
\end{aligned}
$$

where λ denotes any cubic-order character modulo p.

Proof. If $(3, p-1) = 3$, then for any non-principal character χ mod p with $\chi = \chi_1^3$, taking $k = 2$ in Lemma 2.1 we have

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right) = \frac{\chi_1(m)}{\tau(\overline{\chi})}\left(\tau^3(\overline{\chi}_1) + \overline{\lambda}(m)\tau^3(\overline{\chi}_1\lambda) + \lambda(m)\tau^3(\overline{\chi}_1\overline{\lambda})\right). \tag{2.5}$$

Note that the identity

$$\sum_{m=1}^{p-1}\lambda(m) = \sum_{m=1}^{p-1}\overline{\lambda}(m) = 0,$$

from (2.5) we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right)\right|^4 \\
= \ & \frac{1}{p^2}\sum_{m=1}^{p-1}\left|\tau^3(\overline{\chi}_1) + \overline{\lambda}(m)\tau^3(\overline{\chi}_1\lambda) + \lambda(m)\tau^3(\overline{\chi}_1\overline{\lambda})\right|^4 \\
= \ & p^4\cdot\sum_{m=1}^{p-1}\left(3 + \lambda(m)\frac{\overline{\tau}^3(\overline{\chi}_1\lambda)}{\overline{\tau}^3(\overline{\chi}_1)} + \overline{\lambda}(m)\frac{\overline{\tau}^3(\overline{\chi}_1\overline{\lambda})}{\overline{\tau}^3(\overline{\chi}_1)} + \overline{\lambda}(m)\frac{\tau^3(\overline{\chi}_1\lambda)}{\tau^3(\overline{\chi}_1)}\right. \\
& \left. +\lambda(m)\frac{\tau^3(\overline{\chi}_1\overline{\lambda})}{\tau^3(\overline{\chi}_1)} + \frac{\overline{\lambda}(m)}{p^3}\overline{\tau}^3(\overline{\chi}_1\lambda)\tau^3(\overline{\chi}_1\overline{\lambda}) + \frac{\lambda(m)}{p^3}\tau^3(\overline{\chi}_1\lambda)\overline{\tau}^3(\overline{\chi}_1\overline{\lambda})\right)^2 \\
= \ & p^4(p-1)\left(15 + 2\frac{\overline{\tau}^3(\overline{\chi}_1\lambda)\overline{\tau}^3(\overline{\chi}_1\overline{\lambda})}{\overline{\tau}^6(\overline{\chi}_1)} + 2\frac{\overline{\tau}^6(\overline{\chi}_1\lambda)\tau^3(\overline{\chi}_1\overline{\lambda})}{p^3\cdot\overline{\tau}^3(\overline{\chi}_1)} + 2\frac{\tau^3(\overline{\chi}_1\lambda)\tau^3(\overline{\chi}_1\overline{\lambda})}{\tau^6(\overline{\chi}_1)}\right. \\
& \left. +2\frac{\overline{\tau}^3(\overline{\chi}_1\lambda)\tau^6(\overline{\chi}_1\overline{\lambda})}{p^3\cdot\tau^3(\overline{\chi}_1)} + 2\frac{\tau^6(\overline{\chi}_1\lambda)\overline{\tau}^3(\overline{\chi}_1\overline{\lambda})}{p^3\cdot\tau^3(\overline{\chi}_1)} + 2\frac{\tau^3(\overline{\chi}_1\lambda)\overline{\tau}^6(\overline{\chi}_1\overline{\lambda})}{p^3\cdot\overline{\tau}^3(\overline{\chi}_1)}\right).
\end{aligned} \tag{2.6}$$

From the triplication formula for Gauss sums (see [14]) we have

$$\tau(\chi) = \tau\left(\chi_1^3\right) = \frac{1}{p}\cdot\chi(3)\cdot\tau(\chi_1)\cdot\tau(\chi_1\lambda)\cdot\tau\left(\chi_1\overline{\lambda}\right). \tag{2.7}$$

Note that $\tau(\chi)\cdot\overline{\tau}(\chi) = p$, combining (2.6) and (2.7) we have the identity

$$\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right)\right|^4 \\
= \ & 15p^4(p-1) + \frac{2(p-1)\overline{\tau}^3(\chi)\chi^3(3)}{p^2}\cdot\left(\tau^9(\chi_1) + \tau^9(\chi_1\lambda) + \tau^9(\chi_1\overline{\lambda})\right) \\
& +\frac{2(p-1)\tau^3(\chi)\overline{\chi}^3(3)}{p^2}\cdot\left(\overline{\tau}^9(\chi_1) + \overline{\tau}^9(\chi_1\lambda) + \overline{\tau}^9(\chi_1\overline{\lambda})\right).
\end{aligned}$$

This proves Lemma 2.3.

Lemma 2.4. Let p be a prime with $p \equiv 1 \bmod 6$, χ be any sixth-order character mod p. Then about the classical Gauss sums $\tau(\chi)$, the following holds.

$$\tau^3(\chi) + \tau^3(\overline{\chi}) = \begin{cases} p^{\frac{1}{2}}\left(d^2 - 2p\right) & \text{if } p = 12h+1, \\ -i \cdot p^{\frac{1}{2}}\left(d^2 - 2p\right) & \text{if } p = 12h+7, \end{cases}$$

where $i^2 = -1$, d is uniquely determined by $4p = d^2 + 27c^2$ and $d \equiv 1 \bmod 3$.
Proof. This result is Lemma 2.3 in S. Chowla, J. Cowles and M. Cowles [15], so the proof details are omitted.

3. Proof of the theorems

Now we use the several basic lemmas of the previous section to complete the proofs of our theorems. First we prove Theorem 1.1. Let $\chi = \chi_1^3$. Then from Lemma 2.3 we have

$$\begin{aligned} &\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right)\right|^4 = 15p^4(p-1) \\ &+\frac{2(p-1)\overline{\tau}^3(\chi)\chi^3(3)}{p^2}\cdot E_9(\chi) + \frac{2(p-1)\tau^3(\chi)\overline{\chi}^3(3)}{p^2}\cdot \overline{E}_9(\chi), \end{aligned} \tag{3.1}$$

where $\overline{E}_n(\chi)$ denotes the complex conjugate of $E_n(\chi)$.

Let $r_1 = \tau(\chi_1)$, $r_2 = \tau(\chi_1\lambda)$ and $r_3 = \tau(\chi_1\overline{\lambda})$. It is clear that

$$E_1(\chi) = \tau(\chi_1) + \tau(\chi_1\lambda) + \tau(\chi_1\overline{\lambda}) = \sum_{a=1}^{p-1}\chi(a)e\left(\frac{a^3}{p}\right). \tag{3.2}$$

From (2.7) we have

$$r_1r_2r_3 = \tau(\chi_1)\cdot\tau(\chi_1\lambda)\cdot\tau(\chi_1\overline{\lambda}) = p\cdot\tau(\chi)\cdot\overline{\chi}(3). \tag{3.3}$$

Note that $r_i \cdot \overline{r}_i = p$, $i = 1,\ 2,\ 3$. So we have

$$\begin{aligned} &E_2(\chi) = r_1^2 + r_2^2 + r_3^2 = (r_1+r_2+r_3)^2 - 2r_1r_2 - 2r_1r_3 - 2r_2r_3 \\ = \;& E_1^2(\chi) - \frac{2r_1r_2r_3}{p}\cdot(\overline{r}_1 + \overline{r_2} + \overline{r}_3) = E_1^2(\chi) - 2\tau(\chi)\overline{\chi}(3)\cdot\overline{E}_1(\chi). \end{aligned} \tag{3.4}$$

For any integer $n \geq 0$, it is easy to get the identity

$$\begin{aligned} &E_{n+3}(\chi) \\ = \;& (r_1+r_2+r_3)\cdot E_{n+2}(\chi) - (r_1r_2 + r_1r_3 + r_2r_3)\cdot E_{n+1}(\chi) + r_1r_2r_3\cdot E_n(\chi) \\ = \;& E_1(\chi)\cdot E_{n+2}(\chi) - \tau(\chi)\cdot\overline{\chi}(3)\cdot\overline{E}_1(\chi)\cdot E_{n+1}(\chi) + p\cdot\tau(\chi)\cdot\overline{\chi}(3)\cdot E_n(\chi). \end{aligned}$$

Now note that $E_0(\chi) = 3$, combining (3.2) and (3.4) and the above identity we may immediately deduce Theorem 1.1.

Now we prove Theorem 1.2. Taking $\chi = \chi_2 = \left(\frac{*}{p}\right)$ as the Legendre symbol modulo p in (3.1). It is clear that $\chi = \chi_1^3 = \chi_2$. We discuss in two different conditions $p = 12k+1$ and $p = 12k+7$ respectively.

If $p = 12k + 1$, then $\chi_2(3) = 1$, $\tau(\chi_2) = \sqrt{p}$ (see [13], chapter 9.10), from (3.1) we have

$$\begin{aligned} &\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+m\overline{ab}}{p}\right)\right|^4 \\ = \;& 15p^4(p-1) + \frac{4(p-1)}{\sqrt{p}}\cdot\left[\tau^9(\chi_2) + \tau^9(\chi_2\lambda) + \tau^9(\chi_2\overline{\lambda})\right]. \end{aligned} \tag{3.5}$$

Note that $\chi_2\lambda$ is a sixth-order character modulo p and $\tau(\chi_2\lambda)\cdot\tau(\chi_2\overline{\lambda}) = p$, from Lemma 2.4 we have

$$\begin{aligned} &\tau^9(\chi_2\lambda) + \tau^9(\chi_2\overline{\lambda}) = \left(\tau^3(\chi_2\lambda) + \tau^3(\chi_2\overline{\lambda})\right)\cdot\left(\left(\tau^3(\chi_2\lambda) + \tau^3(\chi_2\overline{\lambda})\right)^2 - 3p^3\right) \\ = \;& p^{\frac{3}{2}}\cdot\left(d^2 - 2p\right)^3 - 3p^{\frac{7}{2}}\cdot\left(d^2 - 2p\right). \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) we have the identity

$$\begin{aligned} &\frac{1}{p-1}\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+m\overline{ab}}{p}\right)\right|^4 \\ = \;& 19p^4 + 4p\left(d^2 - 2p\right)\cdot\left(d^4 - 4pd^2 + p^2\right). \end{aligned} \tag{3.7}$$

If $p = 12k + 7$, then $\chi_2(3) = -1$, $\tau(\chi_2) = i\cdot\sqrt{p}$ and $\tau(\chi_2\lambda)\cdot\tau(\chi_2\overline{\lambda}) = -p$, from Lemma 2.4 and the method of proving (3.6) we have

$$\tau^9(\chi_2\lambda) + \tau^9(\chi_2\overline{\lambda}) = i\cdot p^{\frac{3}{2}}\cdot\left(d^2 - 2p\right)^3 - 3\cdot i\cdot p^{\frac{7}{2}}\cdot\left(d^2 - 2p\right). \tag{3.8}$$

Combining (3.1) and (3.8) we also have the identity

$$\begin{aligned} &\frac{1}{p-1}\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+m\overline{ab}}{p}\right)\right|^4 \\ = \;& 19p^4 + 4p\left(d^2 - 2p\right)\cdot\left(d^4 - 4pd^2 + p^2\right). \end{aligned} \tag{3.9}$$

It is clear that Theorem 1.2 follows from (3.7) and (3.9).

Now we prove Corollary 1.1. From (2.5) and the method of proving (2.6) we have

$$\begin{aligned} &\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+m\overline{ab}}{p}\right)\right|^2 \\ = \;& \frac{1}{p}\cdot\sum_{m=1}^{p-1}\left|\tau^3(\chi_2) + \overline{\lambda}(m)\tau^3(\chi_2\lambda) + \lambda(m)\tau^3(\chi_2\overline{\lambda})\right|^2 \\ = \;& p^2\cdot\sum_{m=1}^{p-1} 3 = 3p^2(p-1). \end{aligned} \tag{3.10}$$

Let g be any primitive root modulo p, note that for any integer r,

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+g^{3r}\cdot m\cdot \overline{ab}}{p}\right)\right| = \left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{g^r a+g^r b+g^r\cdot m\cdot \overline{ab}}{p}\right)\right|$$
$$= \left|\left(\frac{g^r}{p}\right)\cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+m\overline{ab}}{p}\right)\right| = \left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+m\overline{ab}}{p}\right)\right|. \tag{3.11}$$

So from (3.10) and (3.11) and the properties of primitive roots and the reduced residue system modulo p we have

$$\begin{aligned} 3p^2(p-1) &= \sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+m\overline{ab}}{p}\right)\right|^2 \\ &= \sum_{i=0}^{2}\sum_{k=0}^{\frac{p-1}{3}-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+g^{3k+i}\cdot \overline{ab}}{p}\right)\right|^2 \\ &= \sum_{i=0}^{2}\sum_{k=0}^{\frac{p-1}{3}-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+g^{i}\cdot \overline{ab}}{p}\right)\right|^2 \\ &= \frac{p-1}{3}\cdot\sum_{i=0}^{2}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+g^{i}\cdot \overline{ab}}{p}\right)\right|^2 \end{aligned}$$

which implies the identity

$$\sum_{i=0}^{2}\left(\frac{1}{3}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{a+b+g^{i}\cdot \overline{ab}}{p}\right)\right)^2 = p^2.$$

This proves Corollary 1.1.

It is easy to deduce Corollary 1.2 from our theorems. In fact, if $(3, p-1) = 1$, then from Lemma 2.1 and Lemma 2.2 we have the estimate

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right)\right|^4 \le p^4(p-1). \tag{3.12}$$

If $3 \mid (p-1)$ and χ is not a 3-th character modulo p, then from Lemma 2.1 we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right)\right|^4 = 0. \tag{3.13}$$

If $3 \mid (p-1)$ and χ is a 3-th character modulo p, then note that the estimate

$$|E_9(\chi)| \le 3\cdot p^{\frac{9}{2}}, \text{ and } |\tau(\chi)| = \sqrt{p},$$

from (3.1) we have the estimate

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi\left(a+b+m\overline{ab}\right)\right|^4 \le 15p^4(p-1)+12p^4(p-1) = 27p^4(p-1). \tag{3.14}$$

Combining (3.12)–(3.14) we may immediately deduce Corollary 1.2.

This completes the proofs of our all results.

4. Conclusions

The main result of this paper is using the elementary and analytic methods to give an exact calculating formula for the fourth power mean of the character sums analogous to 2-dimensional Kloosterman sums. The result is a new contribution to the relevant fields.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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