Research article

A class of hypersurfaces in $\mathbb{E}^{n+1}_s$ satisfying $\Delta \vec{H} = \lambda \vec{H}$

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Abstract: A nondegenerate hypersurface in a pseudo-Euclidean space $\mathbb{E}^{n+1}_s$ is called to have proper mean curvature vector if its mean curvature $\vec{H}$ satisfies $\Delta \vec{H} = \lambda \vec{H}$ for a constant $\lambda$. In 2013, Arvanitoyeorgos and Kaimakamis conjectured [1]: any hypersurface satisfying $\Delta \vec{H} = \lambda \vec{H}$ in pseudo-Euclidean space $\mathbb{E}^{n+1}_s$ has constant mean curvature. This paper will give further support evidences to this conjecture by proving that a linear Weingarten hypersurface $M^n$ in $\mathbb{E}^{n+1}_s$ satisfying $\Delta \vec{H} = \lambda \vec{H}$ has constant mean curvature if $M^n$ has diagonalizable shape operator with less than seven distinct principal curvatures.

Keywords: proper mean curvature vector; linear Weingarten hypersurfaces; principal curvatures

Mathematics Subject Classification: 53C40, 53C42

1. Introduction

A submanifold in a pseudo-Euclidean space $\mathbb{E}^{n+1}_s$ is called to have proper mean curvature vector if its mean curvature vector $\vec{H}$ satisfies a geometric equation

$$\Delta \vec{H} = \lambda \vec{H}$$

(1.1)

for a constant $\lambda$.

Equation (1.1) was first appeared in Chen’s paper [4] in 1988 where surfaces of $\mathbb{E}^3_s$ satisfying Eq (1.1) were classified. In [6], Chen proved that a submanifold in $\mathbb{E}^{n+1}_s$ satisfying Eq (1.1) is a biharmonic submanifold or 1-type submanifold or null 2-type submanifold. In [10], Defever proved that every hypersurface of $\mathbb{E}^4_s$ satisfying Eq (1.1) has constant mean curvature. And recently, Fu [14] also got the same conclusion for hypersurface of $\mathbb{E}^{n+1}_s$ satisfying Eq (1.1) with three distinct principal curvatures. Noting that when $\lambda = 0$, Eq (1.1) becomes the classical biharmonic submanifolds equation, biharmonic submanifolds have received great attention in recent years, see [9, 12, 13]. Other results about submanifolds satisfying Eq (1.1) have been obtained by Chen [7, 8], or see a survey in book [9].
The study of submanifolds in $\mathbb{E}^{n+1}$ satisfying Eq (1.1) was originated by Ferrández and Lucas [11], and the complete classification of surfaces $M^2_r$ ($r = 0, 1$) in $\mathbb{E}^3_1$ was given. In [2] Arvanitoyeorgos, Defever and Kaimakamis proved that if hypersurface $M^2_r (r = 0, 1, 2, 3)$ of $\mathbb{E}^4_1$ satisfying Eq (1.1) has diagonalizable shape operator, then it has constant mean curvature. Also, Arvanitoyeorgos, Kaimakamis and Magid [3] got the same conclusion for nondegenerate Lorentz hypersurface $M^1_3$ of $\mathbb{E}^4_1$ satisfying Eq (1.1). Based on these results, Arvanitoyeorgos and Kaimakamis [1] in 2013 proposed the following conjecture:

"Any hypersurface of $\mathbb{E}^{n+1}_s$ satisfying $\Delta \vec{H} = \lambda \vec{H}$ has constant mean curvature."

In recent years, Liu and Yang in [18] proved that the conjecture holds for hypersurfaces with at most two distinct principal curvatures in $\mathbb{E}^{n+1}_s$, and for the Lorentz hypersurfaces with at most three distinct principal curvatures in $\mathbb{E}^{n+1}_1$, see [17]. We mention here that, Liu and Yang in [16] gave the following result.

**Theorem 1.1.** Let $M^n_r$ be a nondegenerate linear Weingarten hypersurface in $\mathbb{E}^{n+1}_s$ ($n \geq 4$) satisfying $\Delta \vec{H} = \lambda \vec{H}$. If $M^n_r$ has diagonalizable Weingarten operator with at most three distinct principal curvatures, then the mean curvature of $M^n_r$ is constant.

As the number of different principal curvatures increases, this problem becomes more and more complicated. In this paper, we will deal with hypersurfaces with at most six distinct principal curvatures. Meanwhile, we will be concerned with linear Weingarten hypersurfaces (satisfying $aR + bH = c$, where $R$ is the scalar curvature, $H$ is the mean curvature and $a, b, c$ are constants such that $a^2 + b^2 \neq 0$), which is a natural generalization of hypersurfaces with constant mean curvature and constant scalar curvature.

Precisely, we will consider linear Weingarten hypersurfaces of $\mathbb{E}^{n+1}_s$ satisfying Eq (1.1) and with less than seven distinct principal curvatures. Our result gives further support evidence to Arvanitoyeorgos and Kaimakamis’s conjecture [1].

**Theorem 1.2 (Main theorem).** Let $M^n_r$ be a nondegenerate linear Weingarten hypersurface of $\mathbb{E}^{n+1}_s$ satisfying $\Delta \vec{H} = \lambda \vec{H}$ with less than seven distinct principal curvatures. If $M^n_r$ has diagonalizable shape operator, then $M^n_r$ must have constant mean curvature and constant scalar curvature.

Naturally, we could get a conclusion:

**Corollary 1.3.** Let $M^n_r$ be a nondegenerate linear Weingarten hypersurface in $\mathbb{E}^7$ satisfying $\Delta \vec{H} = \lambda \vec{H}$. If $M^n_r$ has diagonalizable shape operator, $M^n_r$ must have constant scalar curvature and constant mean curvature.

Note that the linear Weingarten submanifolds include all submanifolds with constant scalar curvature (for $b = 0$). Hence we get a conclusion:

**Corollary 1.4.** Let $M^n_r$ be a nondegenerate hypersurface in $\mathbb{E}^{n+1}_s$ satisfying $\Delta \vec{H} = \lambda \vec{H}$ with less than seven distinct principal curvatures. If $M^n_r$ has constant scalar curvature and diagonalizable shape operator, then $M^n_r$ must have constant mean curvature.

This paper is organized as follow. We give some important formulas for linear Weingarten hypersurfaces satisfying Eq (1.1) in Section 2. Under the condition that the mean curvature $H$ is not constant, we give an important proposition concerning the principal curvature of linear Weingarten hypersurfaces in $\mathbb{E}^{n+1}_s$ in Section 3. The detail proof of main Theorem 1.2 is given in Section 4.
2. A necessary result

Let \( x : M^n \rightarrow \mathbb{R}^{n+1} \) be an isometric immersion. \( \nabla \) and \( \bar{\nabla} \) are the Levi-Civita connections of \( M^n \) and \( \mathbb{R}^{n+1} \). Let vector fields \( X \) and \( Y \) be tangent to \( M^n \) and \( \xi \) be a unite normal vector of \( M^n \) satisfying \( \langle \xi, \xi \rangle = \varepsilon = \pm 1 \).

Recall the mean curvature vector \( \bar{H} \) is given by \( \bar{H} = H \xi \), where \( H \) is the mean curvature satisfying

\[
H = \frac{1}{n} \text{tr} A
\]  

(2.1) for the shape operator \( A \). For arbitrary vector fields \( X, Y, Z \) are tangent to \( M^n \), the Gauss equation is given by

\[
R(X, Y)Z = \langle AX, Z \rangle AX - \langle AX, Z \rangle Y + \langle AY, Z \rangle X,
\]

(2.2) where \( R(X, Y)Z \) is defined as

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [X, Y] Z.
\]

(2.3)

The Codazzi equation is given by

\[
(\nabla_X A)Y = (\nabla_Y A)X,
\]

(2.4) where

\[
(\nabla_X A)Y = \nabla_X (AY) - A(\nabla_X Y).
\]

It is well known that \( R, H \) and the squared length of the second fundamental form \( B \) satisfy

\[
R = n^2 H^2 - B.
\]

(2.5)

According to [5], we know \( M^n \) satisfies Eq (1.1) if and only if it satisfies equations

\[
\Delta H + \varepsilon H \text{tr} A^2 = \lambda H,
\]

(2.6)

\[
A(\nabla H) = -\frac{n}{2} \varepsilon H(\nabla H),
\]

(2.7)

where the Laplace operator \( \Delta \) is defined by

\[
\Delta f = -\sum_{i=1}^n \varepsilon_i (e_i e_i - \nabla_{e_i} e_i) f,
\]

(2.8)

for a local orthonormal frame \( \{e_1, e_2, \ldots, e_n\} \) satisfying \( \langle e_i, e_i \rangle = \varepsilon_i = \pm 1 \).

Let \( \nabla_{e_i} e_j = \sum_{k=1}^n \omega^k_{ij} e_k \). According to Eqs (2.2) and (2.3), by a direct calculations about \( < R(e_i, e_j) e_k, e_n > \) and \( < R(e_j, e_k) e_i, e_n > \), we have

\[
\omega^i_{jk}(e_j \omega^j_{i} - \varepsilon_i \omega^j_{kk}) = \omega^j_{ik}(e_i \omega^i_{j} - \varepsilon_k \omega^i_{ij}),
\]

(2.9)
Proposition 2.1. Let $M^n_r$ be a nondegenerate linear Weigartem hypersurface in $\mathbb{E}^{n+1}$ satisfying $\Delta \check{H} = \lambda \check{H}$ with diagonalizable Weingarten operator and less than seven distinct principal curvatures. If $H$ is not constant, then all the principal curvatures of $M^n_r$ are functions with one variable.

Proof. According to the condition that $H$ is not constant, we know that there exists at least one point $p_0$ satisfying $\text{grad} \, H(p_0) \neq 0$. And we will work on an neighborhood of $p_0$. From Eq (2.7), it is to know that $\nabla H$ is an eigenvector of $A$ and $-n\varepsilon H/2$ is the corresponding principal curvature. Therefore, we let $e_n = \frac{\nabla H}{|\nabla H|}$. Under the assumption that the Weingarten operator of $M^n_r$ is diagonalizable, there exits a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ satisfying

$$ A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), $$

where $\lambda_i$ are the principal curvatures and

$$ \lambda_n = -n\varepsilon H/2. $$

It follows from (2.1) that

$$ \sum_{i=1}^{n-1} \lambda_i = -3\lambda_n. $$

We have assumed that the mean curvature of $M^n_r$ is not constant, the Weingarten condition $aR + bH = c$ implies that $a \neq 0$, and from (2.11) we have

$$ R = -\frac{b}{a} H + \frac{c}{a} = \frac{2b\varepsilon}{na} \lambda_n + \frac{c}{a}. $$

Furthermore, from (2.5) and (2.11), we have

$$ R = n^2 H^2 - B = 3\lambda_n^2 - \sum_{i=1}^{n-1} \lambda_i^2. $$

Equations (2.13) and (2.14) force that

$$ \sum_{i=1}^{n-1} \lambda_i^2 = 3\lambda_n^2 + c_1 \lambda_n - c_2, $$

where $c_1 = -2b\varepsilon/na$, $c_2 = c/a$. Since $e_n$ is parallel to $\nabla H$, we have

$$ e_n(H) \neq 0, \quad e_i(H) = 0. $$

According to (2.11), Eq (2.16) is equivalent to

$$ e_n(\lambda_n) \neq 0, \quad e_i(\lambda_n) = 0 $$

for $i = 1, 2, \ldots, n - 1$. We note that $\lambda_n$ is a smooth function depending only on one variable $t$. 

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Substituting $H = -2\varepsilon_\lambda \lambda_n / n$ into Eq (2.6), from (2.8), (2.15) and (2.17), we get

$$
\varepsilon_n e_i e_n(\lambda_n) = e_n(\lambda_n) \left( \sum_{i=1}^{n-1} \varepsilon_i \omega_i^n \right) + \varepsilon \lambda_n \left( 4\lambda_n^2 + c_1 \lambda_n - c_2 \right) = -\lambda \lambda_n.
$$

(2.18)

Since $\lambda_n = \lambda_n(t)$ and $e_n(\lambda_n) \neq 0$, from (2.18) we have $\sum_{i=1}^{n-1} \varepsilon_i \omega_i^n$ is a function with the variable $t$, we denote it by $f_i(t)$, that is

$$
\sum_{i=1}^{n-1} \varepsilon_i \omega_i^n = e_n(\lambda_n) \left( 4\lambda_n^2 + c_1 \lambda_n - c_2 \right) + \lambda \lambda_n = f_i(t).
$$

(2.19)

It follows from (2.17) that $[e_i, e_j](H) = 0$, which forces to

$$
\omega_i^j = \omega_j^i,
$$

(2.20)

for $i, j = 1, 2, \ldots, n - 1$ and $i \neq j$. By simple calculations about $\nabla_{e_i} \langle e_j, e_i \rangle = 0$ and $\nabla_{e_i} \langle e_i, e_j \rangle = 0$, we have

$$
\omega_{ij} = 0, \quad \varepsilon_i \omega_{ij} + \varepsilon_j \omega_{ji} = 0,
$$

(2.21)

for $i, j, k = 1, 2, 3, \ldots, n$ and $i \neq j \neq k$. Putting $X = e_i$ and $Y = e_j$, Codazzi equation (2.4) gives

$$
e_i(\lambda_j) e_j + \sum_{k=1}^{n} \omega_{ij}^k (\lambda_j - \lambda_k) e_k = e_j(\lambda_i) e_i + \sum_{k=1}^{n} \omega_{ij}^k (\lambda_i - \lambda_k) e_k
$$

for $i \neq j$. So we get

$$
e_i(\lambda_j) = (\lambda_i - \lambda_j) \omega_{ij}^j,
$$

(2.22)

$$
(\lambda_i - \lambda_j) \omega_{ij}^j = (\lambda_k - \lambda_j) \omega_{jk}^j,
$$

(2.23)

for $i, j, k = 1, 2, \ldots, n$ and $i \neq j \neq k$.

In the case $\lambda_j = \lambda_n$ for $j = 1, \cdots, n - 1$, putting $i = n$ into (2.22) we get $e_n(\lambda_n) = e_n(\lambda_j) = 0$, which is contradict to (2.17), so we get $\lambda_j \neq \lambda_n, j = 1, \cdots, n - 1$.

For $j = n, i \neq n$ in (2.22) we have

$$
e_i(\lambda_n) = (\lambda_i - \lambda_n) \omega_{ni}^n.
$$

From (2.17) and $\lambda_i \neq \lambda_n$ for $i = 1, \cdots, n - 1$, we have $\omega_{ni}^n = 0$, which together with the second equation of (2.21) forces that $\omega_{ij}^n = 0, i = 1, \cdots, n$.

For $j = n, k, i \neq n$ in (2.23) we have

$$
(\lambda_i - \lambda_n) \omega_{ki}^n = (\lambda_k - \lambda_n) \omega_{kj}^n,
$$

which together with (2.20) yields $\omega_{ij}^n = 0$ for $i, j = 1, 2, \cdots, n - 1$ and $i \neq j$.
From Eq (2.3), we have

\[
< R(e_n, e_i)e_n, e_i > = e_1[e_n(\omega^i_m) + \sum_{k=1}^{m} \omega^k_m \omega^i_k - e_n(\omega^i_m) - \sum_{k=1}^{m} \omega^k_m \omega^i_k - \sum_{k=1}^{m} \omega^k_m \omega^i_k + \sum_{k=1}^{m} \omega^k_m \omega^i_k] = e_i e_n(\omega^i_m) + e_i(\omega^i_m)^2. 
\] (2.24)

From Eq (2.2), we have

\[
< R(e_n, e_i)e_n, e_i > = -e_n e_i \lambda_n \lambda_i. 
\] (2.25)

Hence Eqs (2.24) and (2.25) force to

\[
e_n(e_n^i \omega^i_m) = e_n(e_n^i \omega^i_m)^2 + \lambda_n \lambda_i, 
\] (2.26)

for \( i = 1, 2, \cdots, n - 1 \). Summing both sides of Eq (2.26) and from (2.12) and (2.19), we have

\[
\sum_{i=1}^{n-1} (e_n^i \omega^i_m)^2 = e_n e_n(f_1) + 3e_n^2 \lambda_n \lambda_i \approx f_2(t). 
\] (2.27)

Putting \( i = n \) into (2.22) and according to (2.21), we have

\[
e_n(\lambda_i) = e_n e_i(\lambda_i - \lambda_n) \omega^i_m. 
\] (2.28)

Similarly, taking sum of Eq (2.28) and from (2.12) and (2.19), we have

\[
\sum_{i=1}^{n-1} \lambda_i^2 \omega^i_m^2 = \lambda_n f_1 - 3e_n^2 \lambda_n \lambda_i \approx g_1(t). 
\] (2.29)

Multiplying on both sides of Eq (2.26) by \( e_i^2 \omega^i_m^2 \) and Summing for \( i = 1, 2, \cdots, n - 1 \), then from (2.27) and (2.29) we obtain

\[
\sum_{i=1}^{n-1} (e_n^i \omega^i_m)^3 = \frac{1}{2} e_n e_n(f_2) - e_n^2 \lambda_n g_1 \approx f_3(t). 
\] (2.30)

Differentiating (2.29) along \( e_n \), from (2.28) and (2.26), we have

\[
e_n(g_1) = 2 \sum_{i=1}^{n-1} e_n \lambda_i (e_n \omega^i_m)^2 - \sum_{i=1}^{n-1} e_n \lambda_i (e_n \omega^i_m)^2 + \sum_{i=1}^{n-1} \lambda_i^2. 
\]

Furthermore, from (2.15) and (2.27) we have

\[
\sum_{i=1}^{n-1} \lambda_i (e_n \omega^i_m)^2 = \frac{1}{2} \{e_n e_n(g_1) + \lambda_n f_2 - e_n \lambda_n (3\lambda_n^2 + c_1 \lambda_n - c_2) \} \approx g_2(t). 
\] (2.31)

Multiplying on the Eq (2.26) by \( (e_n \omega^i_m)^2 \) and summing for \( i \), we get

\[
\frac{1}{3} \sum_{i=1}^{n-1} e_n ((e_n \omega^i_m)^3) = \sum_{i=1}^{n-1} e_n^2 (e_n \omega^i_m)^4 + \sum_{i=1}^{n-1} \lambda_n \lambda_i (e_n \omega^i_m)^2. 
\]
According to (2.30) and (2.31), we obtain

$$\sum_{i=1}^{n-1} (e_i \omega_{ii}^n)^4 = \frac{1}{3} e_n e_n (f_3) - e_n \lambda_n g_2 \doteq f_4(t).$$  \hspace{1cm} (2.32)$$

In the same way, we can get

$$\sum_{i=1}^{n-1} (e_i \omega_{ii}^n)^5 = \frac{1}{4} e_n e_n (f_4) - \lambda_n g_3 \doteq f_5(t).$$  \hspace{1cm} (2.33)$$

Based on the discussions above, from (2.19), (2.27), (2.30), (2.32) and (2.33), we conclude that

$$\sum_{i=1}^{n-1} (e_i \omega_{ii}^n)^k = f_k(t), \quad \text{for} \quad k = 1, \ldots, 5.$$  \hspace{1cm} (2.34)$$

Next, taking into account Eq (2.23) and the second Eq of (2.21) for distinct principal curvatures \(\lambda_i, \lambda_j, \lambda_k\) \((i, j, k = 1, 2, \ldots, n - 1)\), we have

$$e_i \omega_{ij}^k (\lambda_k - \lambda_i) = e_j \omega_{ij}^k (\lambda_i - \lambda_j) = e_k \omega_{ij}^k (\lambda_j - \lambda_k),$$  \hspace{1cm} (2.35)$$

$$\omega_{ij}^k \omega_{ij}^l + \omega_{ik}^j \omega_{kj}^l + \omega_{jk}^i \omega_{ki}^l = 0.$$  \hspace{1cm} (2.36)$$

It follows from (2.9) and (2.21) that

$$\omega_{ij}^k (e_i \varepsilon_k \omega_{kk}^n - \omega_{ii}^n) = \omega_{ij}^k (e_j \varepsilon_k \omega_{mj}^n - \omega_{jj}^n) = \omega_{ij}^k (e_j \varepsilon_k \omega_{jj}^n - \omega_{mk}^n).$$  \hspace{1cm} (2.37)$$

Since \(\lambda_i \neq \lambda_j\), it follows from (2.21), (2.22) that

$$\omega_{ji}^j = \omega_{jj}^j = 0.$$  \hspace{1cm} (2.38)$$

Computing \(\langle R(e_i, e_j) e_i, e_j \rangle\) by (2.3) and (2.2), we can get

$$e_i (\omega_{ji}^j) + \sum_{k=1}^{n} \omega_{ji}^k \omega_{kj}^l - e_j (\omega_{ii}^n) - \sum_{k=1}^{n} \omega_{ki}^j \omega_{ij}^k - \sum_{k=1}^{n} (\omega_{ij}^k - \omega_{kj}^k) \omega_{ki}^l = -e_i \varepsilon_j \lambda_i \lambda_j,$$  \hspace{1cm} (2.39)$$

for \(\lambda_i \neq \lambda_j\), \(i, j = 1, 2, \ldots, n - 1\).

From (2.21), (2.36) and (2.38), Eq (2.39) becomes

$$e_n \omega_{ii}^n \omega_{jj}^n - 2 e_k \sum_{k=1,k\neq l(i,j)}^{n-1} \omega_{ji}^k \omega_{ji}^l = -e_i \varepsilon_j \lambda_i \lambda_j.$$  \hspace{1cm} (2.40)$$

for \(\lambda_i \neq \lambda_j\), where \(l(i, j)\) stands for the indexes satisfying \(\lambda_{l(i,j)} = \lambda_i\) or \(\lambda_j\).

Since \(\lambda_j \neq \lambda_n\), \(j = 1, 2, \ldots, n - 1\), there are at most five distinct principal curvatures except \(\lambda_n\). And we assume the principal curvatures are \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\) in sequence and the corresponding multiplicities are \(n_1, n_2, n_3, n_4, n_5\) \((\sum_{i=1}^{5} n_i = n - 1)\). Equations (2.35) and (2.37) imply that \(\lambda_i \neq \lambda_j\) is equivalent to \(\omega_{ii}^n \neq \omega_{ij}^n\) \((i, j = 1, 2, \ldots, n - 1)\).

In the following, let \(g\) be the number of distinct principal curvatures.
Firstly, $g = 6$. In this case, $e_a^a\omega_{aa}^n, e_b^b\omega_{bb}^n, \cdots, e_c^c\omega_{cc}^n$ are different from each other. Five equations in (2.34) could transfer into

$$n_1(e_a^a\omega_{aa}^n)^k + n_2(e_b^b\omega_{bb}^n)^k + n_3(e_c^c\omega_{cc}^n)^k + n_4(e_d^d\omega_{dd}^n)^k + n_5(e_e^e\omega_{ee}^n)^k = f_k$$  \hspace{1cm} (2.41)

for $k = 1, 2, \cdots, 5$. Differentiating (2.41) with respect to $e_i$ for any $i = 1, 2, \cdots, n - 1$, and from $e_i(f_k) = 0$ for $k = 1, 2, \cdots, 5$, we can get a homogeneous equation system with five equations and five unknowns $e_i(e_j\omega_{jj}^n)$ for $j = a, b, c, d, e$ and $k = 1, 2, \cdots, 5$

$$n_1(e_a^a\omega_{aa}^n)^{k-1}e_i(e_a^a\omega_{aa}^n) + n_2(e_b^b\omega_{bb}^n)^{k-1}e_i(e_b^b\omega_{bb}^n) + n_3(e_c^c\omega_{cc}^n)^{k-1}e_i(e_c^c\omega_{cc}^n)$$

$$+ n_4(e_d^d\omega_{dd}^n)^{k-1}e_i(e_d^d\omega_{dd}^n) + n_5(e_e^e\omega_{ee}^n)^{k-1}e_i(e_e^e\omega_{ee}^n) = 0.$$  \hspace{1cm} (2.42)

Since $e_j\omega_{jj}^n$ are mutually different, it follows that the determinant

$$\begin{vmatrix}
  n_1 & n_2 & n_3 & n_4 & n_5 \\
  n_1e_a^a\omega_{aa}^n & n_2e_b^b\omega_{bb}^n & n_3e_c^c\omega_{cc}^n & n_4e_d^d\omega_{dd}^n & n_5e_e^e\omega_{ee}^n \\
  n_1(e_a^a\omega_{aa}^n)^2 & n_2(e_b^b\omega_{bb}^n)^2 & n_3(e_c^c\omega_{cc}^n)^2 & n_4(e_d^d\omega_{dd}^n)^2 & n_5(e_e^e\omega_{ee}^n)^2 \\
  n_1(e_a^a\omega_{aa}^n)^3 & n_2(e_b^b\omega_{bb}^n)^3 & n_3(e_c^c\omega_{cc}^n)^3 & n_4(e_d^d\omega_{dd}^n)^3 & n_5(e_e^e\omega_{ee}^n)^3 \\
  n_1(e_a^a\omega_{aa}^n)^4 & n_2(e_b^b\omega_{bb}^n)^4 & n_3(e_c^c\omega_{cc}^n)^4 & n_4(e_d^d\omega_{dd}^n)^4 & n_5(e_e^e\omega_{ee}^n)^4
\end{vmatrix} \neq 0,$$

which together with (2.42) forces that

$$e_i(e_j\omega_{jj}^n) = 0,$$

for $i, j = 1, 2, \cdots, n - 1$. Moreover, from

$$e_i e_n(e_j\omega_{jj}^n) - e_n e_i(e_j\omega_{jj}^n) = [e_i, e_n](e_j\omega_{jj}^n) = \sum_{k=1}^{n-1} (\omega_{ji}^k - \omega_{ji}^k) e_k(e_j\omega_{jj}^n),$$

we get

$$e_i e_n(e_j\omega_{jj}^n) = 0,$$

for $i, j = 1, 2, \cdots, n - 1$. From (2.26) we have

$$\lambda_i e_i(\lambda_j) = e_i e_n(e_j\omega_{jj}^n) - 2e_n e_i(e_j\omega_{jj}^n) e_i(e_j\omega_{jj}^n) = 0,$$  \hspace{1cm} (2.43)

hence

$$e_i(\lambda_j) = 0,$$

for $i, j = 1, 2, \cdots, n - 1$, which means that $\lambda_i$ is a function with only one variable $t$ for any $i$.

Secondly, we consider the case $g \leq 5$. In this case, except $\lambda_n$, there are up to four distinct principal curvatures. It is very similar discussion to above, the number of different $e_i\omega_{ii}^n$ for $i = 1, \cdots, n$ is up to four. If four ones of $e_i\omega_{ii}^n$ are different, we only consider Eq (2.41) for $k = 1, \cdots, 4$. A similar discussion to $g = 6$ yields the conclusion of Proposition 2.1. If less than four ones of $e_i\omega_{ii}^n$ are different, the discussion is quite similar. We prove Proposition 2.1. \hfill $\square$
3. The proof of Theorem 1.2

In this part, we begin with the assumption that the mean curvature of \( M^r_n \) is not constant, then we will deduce a contradiction. Our proof of Theorem 1.2 are divided into the following four cases according to \( g \).

Firstly, \( g = 6 \). In this case, we assume that the principal curvatures are \( \lambda_a, \lambda_b, \lambda_c, \lambda_d, \lambda_e, \lambda_n \) and the corresponding multiplicities are \( n_1, n_2, n_3, n_4, n_5, 1 \) with \( \sum_{i=1}^5 n_i = n - 1 \). According to Eqs (2.35)–(2.37), we divide into the following three cases:

(i) Precisely three \( \omega_{ij}^p \) are not zero for different \( i, j, k \). We assume \( \omega_{ab}^c \neq 0, \omega_{ab}^d \neq 0 \) and \( \omega_{ab}^f \neq 0 \). From Eqs (2.35) and (2.37), we have

\[
\frac{e_a \omega_{ab}^p - e_b \omega_{ab}^p}{\lambda_a - \lambda_b} = \frac{e_a \omega_{ab}^p - e_j \omega_{ab}^{p'} - e_j \omega_{ab}^{p'}}{\lambda_a - \lambda_j} = \frac{e_b \omega_{ab}^p - e_j \omega_{ab}^{p'}}{\lambda_b - \lambda_j},
\]

for \( j = c, d, e \). It follows from Proposition 2.1 that \( \lambda_i \) and \( \omega_{ij}^p \) depend on a parameter \( t \), so there are two smooth functions \( \alpha(t) \) and \( \beta(t) \) satisfying

\[
e_i \omega_{ij}^p = \alpha \lambda_i + \beta, \tag{3.1}
\]

for \( i = 1, 2, \cdots, n - 1 \). Differentiating Eq (3.1) along \( e_n \) and using (2.26) and (2.28), we have

\[
e_n(\alpha) = e_n[(\lambda_n(\alpha^2 + 1) + \alpha \beta), \tag{3.2}
\]

\[
e_n(\beta) = e_n(\lambda_n \alpha + \beta). \tag{3.3}
\]

Differentiating Eq (2.12) along \( e_n \) and using (2.28), we have

\[
3e_n(\lambda_n) = \sum_{i=1}^{n-1} e_i e_j (\lambda_n - \lambda_i) \omega_{ij}^p. \tag{3.4}
\]

Substituting (3.1) into (3.4) and from (2.12), (2.15) we have

\[
3e_n(\lambda_n) = e_n[\alpha(-6\lambda_n^2 - c_1 \lambda_n + c_2) + (n + 2)\beta \lambda_n]. \tag{3.5}
\]

Substituting (3.1) into (2.18) and from (2.12) we get

\[
e_n e_n(\lambda_n) = e_n[e_n(\lambda_n)[-3\lambda_n \alpha + (n - 1)\beta] + e_{3\lambda_n} (4\lambda_n^2 + c_1 \lambda_n - c_2) - \lambda \lambda_n]. \tag{3.6}
\]

Differentiating (3.5) along \( e_n \) and using (3.2), (3.3), then comparing to (3.6), we eliminate \( e_n e_n(\lambda_n) \) as follows

\[
2(n - 4)\beta e_n(\lambda_n) = -6(e_n + 2\epsilon) \lambda_n^3 + (-c_1 \lambda_n + c_2)(e_n + 3\epsilon) \lambda_n + 3\lambda \lambda_n. \tag{3.7}
\]

Furthermore, eliminating \( e_n(\lambda_n) \) between (3.5) and (3.7), we get

\[
2(n - 4)\beta[e_n(-6\lambda_n^2 - c_1 \lambda_n + c_2) + (n + 2)\beta \lambda_n] =
3e_n[-6(e_n + 2\epsilon) \lambda_n^3 + (-c_1 \lambda_n + c_2)(e_n + 3\epsilon) \lambda_n + 3\lambda \lambda_n]. \tag{3.8}
\]
Similarly, differentiating Eq (3.7) along \( e_n \) and from (3.5), (3.6), (3.7), we have
\[
\{72(e_n + 2\epsilon)\lambda_n^3 - 18(e_n + 2\epsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1\alpha - (a_2\lambda_n^3 + a_3\lambda_n)\beta = 0, \tag{3.9}
\]
where \( a_i = a_i(e_n, \epsilon, \lambda, n) \) are given by
\[
a_1 = (-c_1\lambda_n + c_2)\{(c_1\lambda_n + c_2)(e_n + 3\epsilon) + 3\lambda\}, \\
a_2 = 36(e_n + 2\epsilon) + 24(n - 4)\epsilon, \\
a_3 = \{2(n - 1)e_n + 18\epsilon\}(-c_1\lambda_n + c_2) + 18\lambda.
\tag{3.10}
\]
Differentiating Eq (3.9) along \( e_n \) and from (3.2), (3.3), (3.5), (3.7), we get
\[
\{288(e_n + 2\epsilon)\lambda_n^3 - 36(-c_1\lambda_n + c_2)(e_n + 2\epsilon)\lambda_n\} \cdot (n - 4)\alpha\epsilon \lambda_n \\
+ \{72(e_n + 2\epsilon)\lambda_n^4 - 18(e_n + 2\epsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1\epsilon\{(e_n(\alpha^2 + 1) + \alpha\beta\}
- 3\{3a_2\lambda_n^3 + a_3\}[-6(e_n + 2\epsilon)\lambda_n^3 + (-c_1\lambda_n + c_2)(e_n + 3\epsilon)\lambda_n + 3\lambda\lambda_n] \\
- (a_2\lambda_n^3 + a_3\lambda_n)\epsilon\beta(\lambda_n\alpha + \beta) = 0.
\tag{3.11}
\]
For convenience, Eq (3.11) could be rewritten as
\[
q_1(\lambda_n)\alpha^2 + q_2(\lambda_n)\alpha\beta + q_3(\lambda_n)\beta^2 + q_4(\lambda_n) = 0, \tag{3.12}
\]
where
\[
q_1(\lambda_n) = \{288(e_n + 2\epsilon)\lambda_n^3 - 36(-c_1\lambda_n + c_2)(e_n + 2\epsilon)\lambda_n\} \cdot (n - 4)\alpha\epsilon \lambda_n \\
+ \{72(e_n + 2\epsilon)\lambda_n^4 - 18(e_n + 2\epsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1\epsilon\{(e_n(\alpha^2 + 1) + \alpha\beta\}
- 3\{3a_2\lambda_n^3 + a_3\}[-6(e_n + 2\epsilon)\lambda_n^3 + (-c_1\lambda_n + c_2)(e_n + 3\epsilon)\lambda_n + 3\lambda\lambda_n].
\tag{3.13}
\]
In the same way, (3.8) and (3.9) could be rewritten as
\[
p_1(\lambda_n)\beta^2 + p_2(\lambda_n)\alpha\beta + p_3(\lambda_n) = 0, \tag{3.14}
\]
\[
h_1(\lambda_n)\alpha + h_2(\lambda_n)\beta = 0, \tag{3.15}
\]
where
\[
p_1(\lambda_n) = 2(n - 4)(n + 2)\lambda_n, \\
p_2(\lambda_n) = 2(n - 4)(-6\lambda_n^2 - c_1\lambda_n + c_2), \\
p_3(\lambda_n) = -3\epsilon\{6(e_n + 2\epsilon)\lambda_n^3 + (-c_1\lambda_n + c_2)(e_n + 3\epsilon)\lambda_n + 3\lambda\lambda_n],
\tag{3.16}
\]
\[
h_1(\lambda_n) = 72(e_n + 2\epsilon)\lambda_n^4 - 18(e_n + 2\epsilon)(-c_1\lambda_n + c_2)\lambda_n^2 + a_1, \\
h_2(\lambda_n) = -(a_2\lambda_n^3 + a_3\lambda_n).
Combining (3.12), (3.14) with (3.15) to eliminate $\alpha$, we have

$$P_1\beta^2 + P_2 = 0,$$

(3.17)

$$Q_1\beta^2 + Q_2 = 0.$$  

(3.18)

Moreover, combining (3.17) with (3.18) to eliminate $\beta^2$, we obtain

$$P_2Q_1 - P_1Q_2 = 0,$$

(3.19)

where $P_i, Q_i (i = 1, 2)$ are given by

$$P_1 = h_1^2q_1 - h_1h_2q_2 + h_2^2q_3 = l_{11}A_n^{11} + \cdots,$$

$$P_2 = h_1^2q_4 = l_{21}A_n^{13} + \cdots,$$

$$Q_1 = h_1^2p_1 - h_1h_2p_2 = l_{31}A_n^9 + \cdots,$$

$$Q_2 = h_1^2p_3 = l_{41}A_n^{11} + \cdots,$$

where $l_{ij} = l_i(\varepsilon, \varepsilon_n, n) (i = 1, 2, 3, 4)$. Substituting $P_i$ and $Q_i$ into Eq (3.19), we could get

$$\sum_{i=0}^{22} L_iA_n^i = 0,$$

(3.20)

where $L_i = L_i(\varepsilon, \varepsilon_n, n)$ are constant, and the coefficient $L_{22}$ is given by

$$L_{22} = l_{21}l_{31} - l_{11}l_{41}$$

$$= 72^2(\varepsilon_n + 2\varepsilon)^2a_2^2(2(n - 4)(n + 2)(\varepsilon_n + 2\varepsilon)(72 + 54a_2\varepsilon_n)$$

$$+ 18\varepsilon_n(\varepsilon_n + 2\varepsilon)^2[3456(n - 4) - 72])$$

$$- 72^2(\varepsilon_n + 2\varepsilon)^2 \times 72a_2(\varepsilon_n + 2\varepsilon) \times 18\varepsilon_n(\varepsilon_n + 2\varepsilon) \times$$

$$\{576(\varepsilon_n + 2\varepsilon)(n - 4)(n + 2) + 72(\varepsilon_n + 2\varepsilon) - a_2\}$$

$$- 72^2(\varepsilon_n + 2\varepsilon)^2 \times 72a_2(\varepsilon_n + 2\varepsilon)$$

$$\times 12(n - 4)(\varepsilon_n + 2\varepsilon)(72 + 54\varepsilon_n a_2)$$

$$+ 72^4(\varepsilon_n + 2\varepsilon)^4 \times 18a_2\varepsilon_n(\varepsilon_n + 2\varepsilon).$$

(3.21)

Since $\varepsilon = \pm 1$, $\varepsilon_n = \pm 1$, we consider $L_{22}$ in the following

- $\varepsilon = 1$, $\varepsilon_n = 1$, in this case

$$a_2 = 108 + 24(n - 4),$$

and

$$L_{22} = 46656 \times (4478976(n + 2)(n - 4)^4 + 60715008(n + 2)(n - 4)^3$$

$$+ 80621568(n - 4)^3 - 209392128(n + 2)(n - 4)^2 + 712157184(n - 4)^2$$

$$- 1763596800(n + 2)(n - 4) + 1531809792(n - 4) + 90699264)$$

$$= 46656 \times [4478976(n - 4)^2 - 209392128(n + 2)(n - 4)^2$$

$$+ 46656 \times (60715008(n - 4)^2 - 1763596800)(n + 2)(n - 4)$$

$$+ 46656 \times (80621568(n - 4)^3 + 712157184(n - 4)^2$$

$$+ 1531809792(n - 4) + 90699264).$$

(3.22)
So $L_{22} > 0$ if $n \geq 100$, and $L_{22} \neq 0$ if $1 \leq n \leq 100$ with the tool of MATLAB.

- $\varepsilon = 1, \varepsilon_n = -1$, in this case
  \[ a_2 = -36 - 24(n - 4), \]

and
  \[ L_{22} = 5184 \times (746496(n + 2)(n - 4)^4 + 6635520(n + 2)(n - 4)^3 \]
  \[ + 12317184(n - 4)^3 - 8087040(n + 2)(n - 4)^2 + 34805376(n - 4)^2 \]
  \[ - 22021632(n + 2)(n - 4) + 22254912(n - 4) + 3359232]. \]

In the same way with the first case, we also get $L_{22} \neq 0$.

- $\varepsilon = -1, \varepsilon_n = 1$, in this case
  \[ a_2 = 36 + 24(n - 4), \]

and
  \[ L_{22} = 5184 \times (-1492992(n + 2)(n - 4)^4 - 6801408(n + 2)(n - 4)^3 \]
  \[ + 62705664(n - 4)^3 + 75893377(n + 2)(n - 4)^2 + 186625000(n - 4)^2 \]
  \[ + 21648384(n + 2)(n - 4) + 138848256(n - 4)]. \]

In the same way with the first case, we also get $L_{22} \neq 0$.

- $\varepsilon = -1, \varepsilon_n = -1$, in this case
  \[ a_2 = -108 - 24(n - 4), \]

and
  \[ L_{22} = 46659 \times (-4478976(n + 2)(n - 4)^4 - 60715008(n + 2)(n - 4)^3 \]
  \[ - 80621568(n - 4)^3 + 209392128(n + 2)(n - 4)^2 - 739031040(n - 4)^2 \]
  \[ - 195507303(n + 2)(n - 4) - 1638645216(n - 4) - 272097792]. \]

In the same way with the first case, we also get $L_{22} \neq 0$.

Therefore $\lambda_n$ is constant and so $H = -2\varepsilon \lambda_n / n$ is a constant, this contradicts the assumption that the mean curvature is not constant.

(ii) Precisely two ones of $\omega_{ij}^k$ are not zero for different $i, j, k$. And we assume $\omega_{ab}^c \neq 0, \omega_{ab}^d \neq 0$ and $\omega_{ij}^k = 0$. In this case, it follows from (2.40) that
  \[ \varepsilon \varepsilon_n \omega_{ii}^n \omega_{ee}^n = -\varepsilon \lambda_i \lambda_e, \quad (3.23) \]

for $i = 1, \cdots, n_1 + n_2 + n_3 + n_4$. Similarly, from Eqs (2.37) and (2.35), we have
  \[ \varepsilon \omega_{ii}^n = \alpha \lambda_i + \beta, \quad (3.24) \]

where $\alpha$ and $\beta$ are smooth functions satisfying (3.2) and (3.3). From (3.23) and (3.24), we get
  \[ \alpha \varepsilon \varepsilon_n \omega_{ee}^n = -\varepsilon \lambda_e. \quad (3.25) \]

According to (3.25), (3.24) and (3.23), we get
  \[ \beta \lambda_e = 0. \quad (3.26) \]
This forces that $\omega_{ee}^c$ and $\lambda_e$ are entirely determined by $\alpha$ and $\beta$. Substituting (3.24)–(3.26) into (3.4), and differentiating it along $e_n$, from (2.26), (2.28), (3.2), (3.3) we know $\lambda_n$ is a constant by similar discussion as case (i), which is also a contradiction.

(iii) Precisely one of $\omega_{ij}^k$ ($i, j, k$ are different) is not zero. We assume $\omega_{ab}^c \neq 0$ and $\omega_{ij}^k = 0$ for all other different triplets $i, j, k$. In this case, according to (2.40), we have

$$e_i e_a \omega_{ab}^c \omega_{ij}^n = -e_j \lambda_i \lambda_j,$$

(3.27)

$i = 1, \cdots, n_1 + n_2 + n_3$ and $j = n_3 + 1, \cdots, n - 1$. It is very similar to Case (ii), we know that $\lambda_n$ is constant, which gives also a contradiction.

Secondly, considering $g = 5$, in this case we assume the principal curvatures are $\lambda_a, \lambda_b, \lambda_c, \lambda_d = \lambda_c, \lambda_n$ and the corresponding multiplicities are $n_1, n_2, n_3, n_4, n_5, 1 (\Sigma_{i=1}^5 n_i = n - 1)$. According to Eqs (2.35)–(2.37), we only consider the case of existing one, and two ones of $\omega_{ij}^k$ are not zero. A similar discussion to $g = 6$ gives the conclusion.

Thirdly, $g = 4$, in this case, we assume the principal curvatures are $\lambda_a, \lambda_b, \lambda_c = \lambda_d = \lambda_c, \lambda_n$ and the corresponding multiplicities are $n_1, n_2, n_3 + n_4 + n_5, 1 (\Sigma_{i=1}^5 n_i = n - 1)$. According to Eqs (2.35)–(2.37), we only consider the case of existing one $\omega_{ij}^k$ are not zero. A similar discussion can gives the conclusion.

At last, $g < 4$. We could refer to the proof of Liu and Yang [16] with adding additional condition $aR + bH = c$.

In conclusion, we have proved that mean curvature $H$ is constant, and so the scalar curvature is constant as well because of the linear Weingarten condition $aR + bH = c$. We complete the proof of Theorem 1.2. □

Interestingly, hypersurfaces satisfying $\Delta \hat{H} = \lambda \hat{H}$ have constant mean curvature and constant scalar curvature under some geometric assumptions. At the end of this paper, we supply two examples of such hypersurfaces satisfying $\Delta \hat{H} = \lambda \hat{H}$.

Example 3.1. A hypersurface $M^n = \mathbb{R}^r \times \mathbb{H}^{n-r}_{-c}(c)$ in $\mathbb{H}^{n+1}_c$ for $c < 0$. It is easy to check that the hypersurface $M^n$ has two distinct principal curvature

$$\lambda_1 = \cdots = \lambda_r = 0, \quad \lambda_{r+1} = \cdots = \lambda_n = \sqrt{-c}.$$

The mean curvature $H$ and the scalar curvature $R$ of this hypersurface $M^n$ are constants.

Example 3.2. ([15]) A hypersurface $M^n$ in $\mathbb{H}^{n+1}_c$ is given by

$$f(t, s, x_1, u_2, \ldots, u_{n-2})$$

$$= \left( \frac{s}{2} (r^2 + \sum_{j=1}^{n-2} u_j^2) + \phi + a_1 u_1^2 + a_2 u_2^2 + a_3 u_3^2 + a_4 \sum_{i=4}^{n-2} u_i^2, \right.$$

$$u_1(s + 2a_1), \quad st, \quad u_2(s + 2a_2), \quad u_3(s + 2a_3),$$

$$u_4(s + 2a_4), \quad u_5(s + 2a_5), \quad \ldots, \quad u_{n-2}(s + 2a_{n-2}),$$

$$a_1 u_1^2 + a_2 u_2^2 + a_3 u_3^2 + a_4 \sum_{i=4}^{n-2} u_i^2 + \frac{s}{2} (r^2 + \sum_{j=1}^{n-2} u_j^2) + \phi - s \right),$$

where $\phi = \phi(s)$ is a smooth function such that $2\phi' - 1 > 0$, $s, t, u_1, u_2, \ldots, u_{n-2} \in \mathbb{R}$, $2a_1 + s > 0$ and $a_i$ are all distinct constants for $i = 1, 2, 3, 4$. In Remark 3 of [15], the author showed that $M^n$ has constant mean curvature, and thus has constant scalar curvature by the linear Weingarten assumption.
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Conflict of interest

The authors declare no conflict of interest.

References


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