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Research article

Minimal translation graphs in semi-Euclidean space

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Abstract: In this paper we study a characterization of minimal translation graphs which are generalization of minimal translation hypersurfaces in semi-Euclidean space.

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1. Introduction

As is wellknown the first non-trivial examples of minimal surfaces in 3-dimensional Euclidean space E^3 are the catenoids, the helicoids and the minimal translation surfaces. A surface is called a translation surface if it is given by an immersion

$$X : U \subset E^2 \to E^3, (x, y) \to (x, y, z),$$

where $z = f(x) + g(y)$. Scherk proved in 1835 that the only minimal translation surfaces (besides the planes) are the surfaces given by

$$z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right|,$$

where a is a non-zero constant [1].

In [2], it has been shown that the minimal translation surfaces are generalized to minimal translation hypersurfaces as follows:

Let M^n $(n \geq 2)$ be a translation hypersurface in E^{n+1} i.e. M^n is the graph of a function

$$F : \mathbb{R}^n \to \mathbb{R} : (x_1, \ldots, x_n) \to F(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n),$$

where f_i is a smooth function of one real variable for $i = 1, \ldots, n$. Then M^n is minimal if and only if either M^n is a hyperplane or a product submanifold $M^n = M^2 \times E^{n-2}$, where M^2 is a minimal translation surface of Scherk in E^3.

In [3], Woestyne parameterized minimal translation surfaces in the 3-dimensional Minkowski space $\mathbb{R}^3_1$ with metric $g = dx_1^2 + dx_2^2 - dx_3^2$, in the following theorems:

Theorem 1. Every minimal, spacelike surface of translation in $\mathbb{R}^3_1$, is congruent to a part of one of the following surfaces:

1. A spacelike plane,
2. The surface of Scherk of the first kind, a parametrization of the surface is $F(x, y) = (x, y, a^{-1} \log(\cosh(ay)/\cosh(ax)))$, with $\tanh^2(ax) + \tanh^2(ay) < 1$ [3].

Theorem 2. Every minimal, timelike surface of translation in $\mathbb{R}^3_1$, is congruent to a part of one of the following surfaces:

1. A timelike plane,
2. The surface of Scherk of the first kind, a parametrization of the surface is $F(x, y) = (x, y, a^{-1} \log(\cosh(ay)/\cosh(-ax)))$ with $\tan h^2(-ax) + \tan h^2(ay) > 1$.
3. The surface of Scherk of the second kind, a parametrization of the surface is $F(x, y) = (x, y, a^{-1} \log(\cosh(ay)/\sinh(-ax)))$.
4. The surface of Scherk of the third kind, a parametrization of the surface is $F(x, y) = (x, y, a^{-1} \log(\sinh(ay)/\sinh(-ax)))$.
5. A flat B-scroll over a null curve, a parametrization of the surface is $F(x, y) = (x, y, \pm x + g(y))$ with $g(y)$ an arbitrary function [3].

Seo, gave a classification of the translation hypersurfaces with constant mean curvature or constant GaussKronecker curvature in Euclidean space or Lorentz Minkowski space in [4]. Also they characterized the minimal translation hypersurfaces in the upper half-space model of hyperbolic space. In particular, they proved the following theorem:

Theorem 3. Let M be a translation hypersurface with constant mean curvature H in $\mathbb{R}^{n+1}$. Then M is congruent to a cylinder $\Sigma \times \mathbb{R}^{n-2}$, where Σ is a constant mean curvature surface in $\mathbb{R}^3$. In particular, if $H = 0$, then M is either a hyperplane or $M = \Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherks minimal translation surface in $\mathbb{R}^3$[4].

And they can obtained a similar result in the LorentzMinkowski space as follows:

Theorem 4. Let M be a spacelike translation hypersurface with constant mean curvature H in L^{n+1}. Then M is congruent to a cylinder $\Sigma \times \mathbb{R}^{n-2}$, where Σ is a constant mean curvature surface in L^3. In particular, if $H = 0$, then M is either a hyperplane or $M = \Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherks maximal spacelike translation surface in L^3 [4].

In [5], Hasanis and Lopez classified and described the construction of all minimal translation surfaces in $\mathbb{R}^3$. In 2019, Aydn and Ogrenmis investigated translation hypersurfaces generated by translating planar curves and classified these translation hypersurfaces with constant Gauss-Kronecker and mean curvature [6]. Recently, many authors have studied the geometry of minimal translational hypersurface [7–12].

In [12], Yang, Zhang and Fu gave a characterization of a class of minimal translation graphs which are generalization of minimal translation hypersurfaces in Euclidean space. In this paper we study a characterization of minimal translation graphs in semi-Euclidean space.

2. Characterization of minimal translation graphs in semi-Euclidean space

By the $(n+1)$-dimensional semi-Euclidean space with index ν, denoted by $\mathbb{R}^{n+1}_{\nu}$, mean $\mathbb{R}^{n+1}$ equipped with the semi-Euclidean metric

$$g = \varepsilon_1 dx_1^2 + \varepsilon_2 dx_2^2 + \cdots + \varepsilon_{n+1} dx_{n+1}^2, \tag{2.1}$$

for which ε_i, $(i = 1, 2, \ldots, n+1)$ is either -1 or 1. We assume $n \geq 2$. The number of minus signs is equal to the index ν and is given by

$$\nu = \frac{1}{2}(n + 1 - \sum_{j=1}^{n+1} \varepsilon_j).$$

Let M^n be a hypersurface of $\mathbb{R}^{n+1}_{\nu}$ for which the induced metric is non-degenerate. Then M^n can locally always be seen as the graph of a function $F : \mathbb{R}^n \to \mathbb{R}$. In what follows, we will assume that f is a function of the coordinates $x_1, \ldots, x_n$. This can easily be achieved possibly by rearranging the coordinates of $\mathbb{R}^{n+1}_{\nu}$. So M^n is locally given by

$$x_{n+1} = F(x_1, \ldots x_n). \tag{2.2}$$

Assume that M^n is minimal. This means that the mean curvature vector vanishes at every point. The graph M^n in the semi-Euclidean space $\mathbb{R}^n_{\nu}$ is minimal if and only if

$$\sum_{j=1}^{n} \varepsilon_j \left[\frac{\partial^2 F}{\partial x_j^2} \left(\sum_{i=1}^{n} \varepsilon_i \left(\frac{\partial F}{\partial x_i} \right)^2 + \varepsilon_{n+1} \right) - \frac{\partial F}{\partial x_j} \sum_{i=1}^{n} \varepsilon_i \frac{\partial F}{\partial x_i} \frac{\partial^2 F}{\partial x_i \partial x_j} \right] = 0. \tag{2.3}$$

One easily calculates that minimality condition above [13].

A hypersurface M^n in the semi-Euclidean space $\mathbb{R}^{n+1}_{\nu}$ is called translation graph if it is the graph of the function given by

$$F(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_{n-1}(x_{n-1}) + f_n(u),$$

where $u = \sum\limits_{i=1}^{n} c_i x_i$, c_i are constants, $c_n \neq 0$ and each f_i is a smooth function of one real variable for $i = 1, 2, \ldots, n$. Additionally in this paper, we assume that the following condition are provided:

$$\sum_{i=1}^{n} \varepsilon_i c_i^2 \neq 0 \;\; and \;\; \sum_{\substack{j=1 \\ j \neq i}}^{n} \varepsilon_j c_j^2 \neq 0, \;\; for\ all \;\; i = 1, \ldots, n-1. \tag{2.4}$$

f_i vanishes nowhere for $i = 1, 2, \ldots, n$, otherwise M^n is a non-degenerate hyperplane.

The minimality condition (2.3) can be rewritten as

$$\varepsilon_{n+1} \sum_{j=1}^{n} \varepsilon_j F_{jj} + \sum_{i \neq j}^{n} \varepsilon_i \varepsilon_j \left(F_i^2 F_{jj} - F_i F_j F_{ij} \right) = 0. \tag{2.5}$$

Then we calculate the partial derivatives in the Eq (2.5) for the translation graph,

$$F_i = f_i' + c_i f_n', \quad F_n = c_n f_n', \tag{2.6}$$

$$F_{ii} = f_{ii}' + c_i^2 f_n'', \quad F_{nn} = c_n^2 f_n'', \tag{2.7}$$

$$F_{ij} = c_i c_j f_n'', \quad F_{in} = c_i c_n f_n'', \tag{2.8}$$

for $1 \le i \le n-1$. Since M^n is minimal, we substitute (2.6)–(2.8) into (2.5) and we obtain

$$\begin{aligned} &\varepsilon_{n+1}\sum_{i=1}^{n-1}\varepsilon_i f_i'' + \left(\varepsilon_{n+1}\sum_{i=1}^{n}\varepsilon_i c_i^2 + \varepsilon_n c_n^2 \sum_{i=1}^{n-1}\varepsilon_i f_i'^2\right) f_n'' + \varepsilon_n c_n^2 \left(\sum_{i=1}^{n-1}\varepsilon_i f_i''\right) f_n'^2 \\ &+ \sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j (f_i' + c_i f_n')^2 f_j'' + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j (c_i f_j' - c_j f_i')^2 f_n'' = 0. \end{aligned} \tag{2.9}$$

Since $c_n \neq 0$, we take the derivative of the Eq (2.9) with respect to x_n, we have

$$\begin{aligned} &\left[\varepsilon_{n+1}\sum_{i=1}^{n}\varepsilon_i c_i^2 + \varepsilon_n c_n^2 \sum_{i=1}^{n-1}\varepsilon_i f_i'^2 + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j (c_i f_j' - c_j f_i')^2\right] f_n''' \\ &+ 2\left[\sum_{i,j=1}^{n-1}\varepsilon_i (\sum_{\substack{j=1\\ j\neq i}}^{n}\varepsilon_j c_j^2) f_i''\right] f_n' f_n'' + 2\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j c_i f_i' f_j'' f_n'' = 0. \end{aligned} \tag{2.10}$$

According to the Eq (2.10), we get following cases:

Case 1. $f_n''' = 0$.

With proper translation, $f_n = mu^2$ for a constant $m \neq 0$ such that $f_n = mu^2$. If $m = 0$, then M^n would not be a translation graph. According to this, we rewrite (2.10)

$$2m^2 u \sum_{i=1}^{n-1}\varepsilon_i (\sum_{\substack{j=1\\ j\neq i}}^{n}\varepsilon_j c_j^2) f_i'' + m\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j c_i f_i' f_j'' = 0. \tag{2.11}$$

Since $m \neq 0$, we get

$$\sum_{i=1}^{n-1}\varepsilon_i (\sum_{\substack{j=1\\ j\neq i}}^{n}\varepsilon_j c_j^2) f_i'' = 0, \quad \sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j c_i f_i' f_j'' = 0. \tag{2.12}$$

In the first equation of (2.12), each f_i'' depends on a different variable, then f_i'' has to be a constant for $i = 1, \ldots, n-1$. Also, let be $f_i(x_i) = a_i x_i^2$, where a_i is constant. Then from (2.12) we obtain following equations

$$\sum_{i=1}^{n-1}\varepsilon_i (\sum_{\substack{j=1\\ j\neq i}}^{n}\varepsilon_j c_j^2) a_i = 0, \quad a_i c_i \sum_{\substack{j=1\\ j\neq i}}^{n-1}\varepsilon_j a_j = 0, \quad \textit{where} \;\; i = 1, \ldots, n-1. \tag{2.13}$$

Now we substitute $f_n = mu^2$ and $f_i(x_i) = a_i x_i^2$ for $i = 1, \ldots, n-1$ in the Eq (2.9), we find

$$\begin{aligned}
&4m^2 \left[\sum_{i=1}^{n-1} \varepsilon_i (\sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_j c_j^2) a_i \right] u^2 + 8m \left(\sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \varepsilon_i \varepsilon_j a_i c_i a_j x_i \right) u - 4m \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \varepsilon_i \varepsilon_j a_i a_j c_i c_j x_i x_j \\
&+ 4 \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \left(m \sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_j c_j^2 + \sum_{\substack{j=1\\ j\neq i}}^{n-1} \varepsilon_j a_j \right) x_i^2 + \varepsilon_{n+1} \sum_{i=1}^{n-1} \varepsilon_i a_i + \varepsilon_{n+1} m \sum_{i=1}^{n} \varepsilon_i c_i^2 = 0.
\end{aligned} \tag{2.14}$$

According to (2.13) and (2.14), we obtain

$$4m \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \varepsilon_i \varepsilon_j a_i a_j c_i c_j x_i x_j = 4 \sum_{i=1}^{n-1} \varepsilon_i a_i^2 \left(m \sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_j c_j^2 + \sum_{\substack{j=1\\ j\neq i}}^{n-1} \varepsilon_j a_j \right) x_i^2 + \varepsilon_{n+1} \sum_{i=1}^{n-1} \varepsilon_i a_i + \varepsilon_{n+1} m \sum_{i=1}^{n} \varepsilon_i c_i^2 \tag{2.15}$$

Since the above equation is a quadratic polynomial with $x_1, \ldots, x_{n-1}$, by the arbitrariness of x_i, we get

$$a_i^2 \left(m \sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_j c_j^2 + \sum_{\substack{j=1\\ j\neq i}}^{n-1} \varepsilon_j a_j \right) = 0, \quad for \;\; i = 1, \ldots, n-1, \tag{2.16}$$

$$\sum_{i=1}^{n-1} \varepsilon_i a_i + m \sum_{i=1}^{n} \varepsilon_i c_i^2 = 0 \tag{2.17}$$

and

$$a_i a_j c_i c_j = 0, \quad for \;\; i, j = 1, \ldots, n-1, \quad i \neq j. \tag{2.18}$$

According to (2.16) and (2.17), we find

$$a_i^3 = -m\,(c_i a_i)^2, \quad for \;\; i = 1, \ldots, n-1. \tag{2.19}$$

From (2.18), we can see that at most one $a_k c_k \neq 0$. Without loss of generality, we assume $a_{k_0} c_{k_0} \neq 0$ and every $a_k c_k = 0$ for $k \neq k_0$. From (2.19), we get $a_{k_0} \neq 0$ and $a_k = 0$ for $k \neq k_0$. According to this, with the first equation of (2.13) and the assumption (2.4), we have a contradiction. Therefore, we obtain that every $a_k = 0$ for $k = 1, \ldots, n-1$ and $f_i''(x_i) = 0$. By substituting this equalities in (2.9) we obtain $m \sum_{i=1}^{n} \varepsilon_i c_i^2 = 0$, which is a contradiction with the assumption (2.4).

Case 2. $f_n''' \neq 0$.

If we divide by f_n''' on both sides of the Eq (2.10), we obtain

$$\begin{aligned}
&\left[\varepsilon_{n+1} \sum_{i=1}^{n} \varepsilon_i c_i^2 + \varepsilon_n c_n^2 \sum_{i=1}^{n-1} \varepsilon_i f_i'^2 + \frac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \varepsilon_i \varepsilon_j (c_i f_j' - c_j f_i')^2 \right] \\
&+ 2 \left[\sum_{i=1}^{n-1} \varepsilon_i (\sum_{\substack{j=1\\ j\neq i}}^{n} \varepsilon_j c_j^2) f_i'' \right] \frac{f_n' f_n''}{f_n'''} + 2 \left[\sum_{\substack{i,j=1\\ i\neq j}}^{n-1} \varepsilon_i \varepsilon_j c_i f_i' f_j'' \right] \frac{f_n''}{f_n'''} = 0
\end{aligned} \tag{2.20}$$

Differentiating (2.20) with respect to u, we get

$$\left[\sum_{i=1}^{n-1}\varepsilon_i(\sum_{\substack{j=1\\ j\neq i}}^{n}\varepsilon_j c_j^2)f_i''\right]\left(\frac{f_n'f_n''}{f_n'''}\right)_u+\left[\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j c_i f_i' f_j''\right]\left(\frac{f_n''}{f_n'''}\right)_u=0 \tag{2.21}$$

We have 3 possibilities.

Case 2a. $\left(\frac{f_n''}{f_n'''}\right)_u \neq 0$.

In this case,

$$f_n = af_n' + bu, \tag{2.22}$$

where a, b are constants. Since $f_n''' \neq 0$, then $a \neq 0$. By solving this equation we obtain

$$f_n(u) = ke^{\frac{u}{a}} + bu + ab,$$

where k is a nonzero constant. According to this equation, we get

$$\left(\frac{f_n'f_n''}{f_n'''}\right)_u = \frac{k}{a}e^{\frac{u}{a}} \neq 0.$$

Thus, according to (2.21), we obtain

$$\sum_{i=1}^{n-1}\varepsilon_i(\sum_{\substack{j=1\\ j\neq i}}^{n}\varepsilon_j c_j^2)f_i'' = 0. \tag{2.23}$$

Since each f_i'' depends on a different variable, then f_i'' has to be a constant for $i = 1, \ldots, n-1$. Let be $f_i(x_i) = a_i x_i^2$, where a_i is constant. From (2.23) we obtain

$$\sum_{i=1}^{n-1}\varepsilon_i(\sum_{\substack{j=1\\ j\neq i}}^{n}\varepsilon_j c_j^2)a_i = 0.$$

Hence, from (2.20) we have

$$\varepsilon_{n+1}\sum_{i=1}^{n}\varepsilon_i c_i^2 + \varepsilon_n c_n^2\sum_{i=1}^{n-1}\varepsilon_i(2a_ix_i)^2 + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j(2c_ia_jx_j - 2c_ja_ix_i)^2 + 8a\sum_{\substack{i,j=1\\ i\neq j}}^{n-1}\varepsilon_i\varepsilon_j c_ia_ia_jx_i = 0.$$

This is a contradiction.

Case 2b. $\sum_{i=1}^{n-1}\varepsilon_i(\sum_{\substack{j=1\\ j\neq i}}^{n}\varepsilon_j c_j^2)f_i'' \neq 0$.

Let be $\sum_{i=1}^{n-1} \varepsilon_i (\sum_{\substack{j=1 \\ j\neq i}}^{n} \varepsilon_j c_j^2) f_i'' = 0$. Then each f_i'' is constant for $i = 1, \ldots, n-1$. Also we can write $f_i(x_i) = a_i x_i^2$, where a_i is constant. From the assumption, we have

$$\sum_{i=1}^{n-1} \varepsilon_i (\sum_{\substack{j=1 \\ j\neq i}}^{n} \varepsilon_j c_j^2) a_i = 0.$$

According to (2.21), we get

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{n-1} \varepsilon_i \varepsilon_j c_i f_i' f_j'' = 0.$$

From (2.20), we obtain

$$\varepsilon_{n+1} \sum_{i=1}^{n} \varepsilon_i c_i^2 + \varepsilon_n c_n^2 \sum_{i=1}^{n-1} \varepsilon_i (2a_i x_i)^2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n-1} \varepsilon_i \varepsilon_j (2c_i a_j x_j - 2c_j a_i x_i)^2 = 0,$$

which is a contradiction. Also it must be $\sum_{i=1}^{n-1} \varepsilon_i (\sum_{\substack{j=1 \\ j\neq i}}^{n} \varepsilon_j c_j^2) f_i'' \neq 0$. According to Cases 2a and 2b, we can rewrite (2.21)

$$\frac{\sum_{\substack{i,j=1 \\ i\neq j}}^{n-1} \varepsilon_i \varepsilon_j c_i f_i' f_j''}{\sum_{i=1}^{n-1} \varepsilon_i (\sum_{\substack{j=1 \\ j\neq i}}^{n} \varepsilon_j c_j^2) f_i''} = -\frac{\left(\frac{f_n' f_n''}{f_n'''}\right)_u}{\left(\frac{f_n''}{f_n'''}\right)_u} = m, \tag{2.24}$$

where m is constant. Thus we have

$$\left(\frac{f_n' f_n''}{f_n'''}\right)_u = -m \left(\frac{f_n''}{f_n'''}\right)_u.$$

By integration of this equation, we get

$$\frac{f_n' f_n''}{f_n'''} = -m \frac{f_n''}{f_n'''} + c \tag{2.25}$$

where c is a constant. Thus we have $f_n' f_n'' = -m f_n'' + c f_n'''$. By integration of this equation, we obtain

$$f_n'^2 + 2m f_n' = 2c f_n'' + c_0, \tag{2.26}$$

where c_0 is a constant. By solving this ODE, after a translation, we find

$$f_n = \begin{cases} -mu - 2c \ln \cos\left(\dfrac{\sqrt{-(m^2+c_0)}}{2c}u\right), & if\, m^2 + c_0 < 0 \\ -mu - 2c \ln \cosh\left(\dfrac{\sqrt{m^2+c_0}}{2c}u\right), & if\, m^2 + c_0 > 0 \ and \ \left|\dfrac{f_n' + m}{\sqrt{m^2+c_0}}\right| < 1 \\ -mu - 2c \ln \sinh\left(\dfrac{\sqrt{m^2+c_0}}{2c}u\right), & if\, m^2 + c_0 > 0 \ and \ \left|\dfrac{f_n' + m}{\sqrt{m^2+c_0}}\right| > 1 \\ -mu - 2c \ln |u|, & if\, m^2 + c_0 = 0. \end{cases}$$

Moreover, from (2.24), we get

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} \varepsilon_i \varepsilon_j c_i f_i' f_j'' = m \sum_{i=1}^{n-1} \varepsilon_i (\sum_{\substack{j=1 \\ j \neq i}}^{n} \varepsilon_j c_j^2) f_i''. \tag{2.27}$$

Since

$$\sum_{i=1}^{n-1} \varepsilon_i (\sum_{\substack{j=1 \\ j \neq i}}^{n} \varepsilon_j c_j^2) f_i'' \neq 0,$$

all f_i'' functions dont vanish for $i = 1, \ldots, n-1$. Let be $f_{i_0}'' \neq 0$. By differentiating the Eq (2.27) with respect to x_{i_0}, we obtain

$$\left(m \sum_{\substack{i=1 \\ i \neq i_0}}^{n} \varepsilon_i c_i^2 - \sum_{\substack{i=1 \\ i \neq i_0}}^{n-1} \varepsilon_i c_i f_i' \right) \frac{f_{i_0}'''}{f_{i_0}''} = c_{i_0} \sum_{\substack{i=1 \\ i \neq i_0}}^{n-1} \varepsilon_i f_i''. \tag{2.28}$$

Thus we get the following case.

Case 2c. $m \sum\limits_{\substack{i=1 \\ i \neq i_0}}^{n} \varepsilon_i c_i^2 - \sum\limits_{\substack{i=1 \\ i \neq i_0}}^{n-1} \varepsilon_i c_i f_i' = 0$.

Let be $m \sum\limits_{\substack{i=1 \\ i \neq i_0}}^{n} \varepsilon_i c_i^2 - \sum\limits_{\substack{i=1 \\ i \neq i_0}}^{n-1} \varepsilon_i c_i f_i' \neq 0$. From (2.28), we get

$$\frac{f_{i_0}'''}{c_{i_0} f_{i_0}''} = \frac{\sum\limits_{\substack{i=1 \\ i \neq i_0}}^{n-1} \varepsilon_i f_i''}{m \sum\limits_{\substack{i=1 \\ i \neq i_0}}^{n} \varepsilon_i c_i^2 - \sum\limits_{\substack{i=1 \\ i \neq i_0}}^{n-1} \varepsilon_i c_i f_i'} = a, \tag{2.29}$$

where a is a constant. From (2.29), we obtain

$$f_{i_0} = b_{i_0} e^{a c_{i_0} x_{i_0}} - \frac{d_{i_0}}{a c_{i_0}} x_{i_0}, \tag{2.30}$$

where b_{i_0} and d_{i_0} are constants. According to (2.29), we get

$$\varepsilon_i \left(f_i'' + ac_i f_i'\right) = ma \sum_{\substack{i=1\\ i\neq i_0}}^{n} \varepsilon_i c_i^2,$$

for each $i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n - 1$. By solving the equation, we obtain

$$f_i = b_i e^{-ac_i x_i} + \frac{m \sum\limits_{\substack{i=1\\ i\neq i_0}}^{n} \varepsilon_i c_i^2}{\varepsilon_i c_i} x_i, \tag{2.31}$$

where b_i are constants for $i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n - 1$. By differentiating (2.27) with respect to x_k for $k = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n - 1$, we get

$$f_k''' \left(\sum_{\substack{i=1\\ i\neq k}}^{n-1} \varepsilon_i c_i f_i' - m \sum_{\substack{i=1\\ i\neq k}}^{n} \varepsilon_i c_i^2 \right) + c_k f_k'' \sum_{\substack{i=1\\ i\neq k}}^{n-1} \varepsilon_i f_i'' = 0. \tag{2.32}$$

We substitute (2.30) and (2.31) in (2.32) and we obtain

$$2ac_k^3 b_k \sum_{\substack{i=1\\ i\neq k, i\neq i_0}}^{n} \varepsilon_i c_i^2 b_i e^{-a(c_i x_i + c_k x_k)} = c_k^3 b_k B_k e^{-ac_k x_k}, \tag{2.33}$$

where

$$B_k = (n-3)m \sum_{\substack{i=1\\ i\neq i_0}}^{n} \varepsilon_i c_i^2 - m \sum_{\substack{i=1\\ i\neq k}}^{n} \varepsilon_i c_i^2 - \varepsilon_{i_0} \frac{d_{i_0}}{a}$$

for $k = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n - 1$. Since x_k is arbitrary and from (2.31), we get $c_k^3 b_k c_i^2 b_i = 0$. Hence $c_k b_k c_i b_i = 0$ for i, $k = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n - 1$ and $i \neq k$. Thus, there is at most one $c_k b_k \neq 0$. Let be all $c_k b_k = 0$ for $k = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n - 1$ and $k \neq j_0$. It follows

$$f_{j_0}' = -ab_{j_0} c_{j_0} e^{-ac_{j_0} x_{j_0}} + \frac{m \sum\limits_{\substack{i=1\\ i\neq i_0}}^{n} \varepsilon_i c_i^2}{\varepsilon_{j_0} c_{j_0}} \tag{2.34}$$

and

$$f_k' = \frac{m \sum\limits_{\substack{k=1\\ k\neq i_0}}^{n} \varepsilon_k c_k^2}{\varepsilon_k c_k} \tag{2.35}$$

for $k = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n - 1$ and $k \neq j_0$. By substituting (2.30), (2.34) and (2.35) in (2.27), we obtain

$$\varepsilon_{i_0}(n-3)m \left(\sum_{\substack{i=1\\ i\neq i_0}}^{n} \varepsilon_i c_i^2 \right) a^2 b_{i_0} c_{i_0}^2 e^{ac_{i_0} x_{i_0}} + \varepsilon_{j_0} a^2 b_{j_0} c_{j_0}^2 C e^{-ac_{j_0} x_{j_0}} = 0,$$

where

$$C = (n-3)m\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2 - m\sum_{\substack{i=1\\i\neq j_0}}^{n}\varepsilon_i c_i^2 - \varepsilon_{i_0}\frac{d_{i_0}}{a}.$$

We rewrite the Eq (2.20) for $n = 3$

$$\begin{aligned}&\varepsilon_4\sum_{i=1}^{3}\varepsilon_i c_i^2 + \varepsilon_3 c_3^2\sum_{i=1}^{2}\varepsilon_i {f_i'}^2 + \varepsilon_1\varepsilon_2(c_1 f_2' - c_2 f_1')^2\\ &\quad + 2\left[\sum_{i=1}^{2}\varepsilon_i(\sum_{\substack{j=1\\j\neq i}}^{3}\varepsilon_j c_j^2)f_i''\right]\frac{f_3' f_3''}{f_3'''} + 2\left[\sum_{\substack{i,j=1\\i\neq j}}^{2}\varepsilon_i\varepsilon_j c_i f_i' f_j''\right]\frac{f_3''}{f_3'''} = 0.\end{aligned} \tag{2.36}$$

According to (2.25) and (2.27), (2.36) becomes

$$\varepsilon_4(\varepsilon_1 c_1^2+\varepsilon_2 c_2^2+\varepsilon_3 c_3^2)+\varepsilon_3 c_3^2(\varepsilon_1 {f_1'}^2+\varepsilon_2 {f_2'}^2)+\varepsilon_1\varepsilon_2(c_1 f_2' - c_2 f_1')^2+2c(\varepsilon_1(\varepsilon_2 c_2^2+\varepsilon_3 c_3^2)f_1''+\varepsilon_2(\varepsilon_1 c_1^2+\varepsilon_3 c_3^2)f_2'') = 0. \tag{2.37}$$

By differentiating the Eq (2.37) with respect to x_1, we obtain

$$\varepsilon_3 c_3^2 f_1' f_1'' - \varepsilon_2 c_2(c_1 f_2' - c_2 f_1')f_1'' + c(\varepsilon_2 c_2^2 + \varepsilon_3 c_3^2)f_1''' = 0.$$

If we arrange the equation above, then we get

$$(\varepsilon_2 c_2^2 + \varepsilon_3 c_3^2)\left(f_1' + c\frac{f_1'''}{f_1''}\right) = \varepsilon_2 c_1 c_2 f_2'. \tag{2.38}$$

From this equation, f_2'' is constant and $f_2'' = 0$. This is a contradiction. Also it must be

$$m\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2 - \sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i c_i f_i' = 0.$$

We showed that $m\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2 - \sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i c_i f_i' = 0$ and f_i'' are constants for $1 \leq i \leq n-1,\ i \neq i_0$. Then $f_i'(x_i) = a_i$, where a_i are constants for $i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n-1$ and

$$m\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2 = \sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i c_i a_i \tag{2.39}$$

We rewrite (2.20),

$$\begin{aligned}&\varepsilon_{n+1}\sum_{i=1}^{n}\varepsilon_i c_i^2+\varepsilon_n c_n^2\sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i a_i^2+\varepsilon_{i_0}{f'_{i_0}}^2\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2+\sum_{\substack{j=1\\j\neq i_0}}^{n-1}\left(\varepsilon_j a_j^2\sum_{\substack{i=1\\i\neq j}}^{n-1}\varepsilon_i c_i^2\right)\\&-2\varepsilon_{i_0}c_{i_0}f'_{i_0}\sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i c_i a_i-\sum_{\substack{j=1\\j\neq i_0}}^{n-1}\left(\varepsilon_j c_j a_j\sum_{\substack{i=1\\i\neq j,i\neq i_0}}^{n-1}\varepsilon_i c_i a_i\right)\\&+2\varepsilon_{i_0}f''_{i_0}\left(\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2\right)\frac{f'_n f''_n}{f'''_n}+2\varepsilon_{i_0}f''_{i_0}\left(\sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i c_i a_i\right)\frac{f''_n}{f'''_n}=0\end{aligned}\tag{2.40}$$

According to (2.25) and (2.39), we rewrite (2.40)

$$\begin{aligned}&\varepsilon_{n+1}\sum_{i=1}^{n}\varepsilon_i c_i^2+\varepsilon_n c_n^2\sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i a_i^2+\varepsilon_{i_0}{f'_{i_0}}^2\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2+\sum_{\substack{j=1\\j\neq i_0}}^{n-1}\left(\varepsilon_j a_j^2\sum_{\substack{i=1\\i\neq j}}^{n-1}\varepsilon_i c_i^2\right)\\&-2\varepsilon_{i_0}c_{i_0}f'_{i_0}\sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i c_i a_i-\sum_{\substack{j=1\\j\neq i_0}}^{n-1}\left(\varepsilon_j c_j a_j\sum_{\substack{i=1\\i\neq j,i\neq i_0}}^{n-1}\varepsilon_i c_i a_i\right)+2\varepsilon_{i_0}cf''_{i_0}\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2=0.\end{aligned}\tag{2.41}$$

We arrange this equation

$$\varepsilon_{i_0}\left(\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2\right){f'_{i_0}}^2-2\varepsilon_{i_0}c_{i_0}\left(\sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i c_i a_i\right)f'_{i_0}+2\varepsilon_{i_0}c\left(\sum_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2\right)f''_{i_0}+B=0,\tag{2.42}$$

with

$$\begin{aligned}B&=\varepsilon_{n+1}\sum_{i=1}^{n}\varepsilon_i c_i^2+\varepsilon_n c_n^2\sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i a_i^2+\sum_{\substack{j=1\\j\neq i_0}}^{n-1}\left(\varepsilon_j a_j^2\sum_{\substack{i=1\\i\neq j}}^{n-1}\varepsilon_i c_i^2\right)-\sum_{\substack{j=1\\j\neq i_0}}^{n-1}\left(\varepsilon_j c_j a_j\sum_{\substack{i=1\\i\neq j,i\neq i_0}}^{n-1}\varepsilon_i c_i a_i\right)\\&=\varepsilon_{n+1}\sum_{i=1}^{n}\varepsilon_i c_i^2+\left(\varepsilon_n c_n^2+\varepsilon_{i_0}c_{i_0}^2\right)\sum_{\substack{i=1\\i\neq i_0}}^{n-1}\varepsilon_i a_i^2+\frac{1}{2}\sum_{\substack{i,j=1\\i\neq j,\ i,j\neq i_0}}^{n-1}\varepsilon_i\varepsilon_j(a_i c_j-a_j c_i)^2.\end{aligned}$$

From (2.39), we rewrite (2.42)

$${f'_{i_0}}^2-2mc_{i_0}f'_{i_0}+2cf''_{i_0}+\frac{B}{\varepsilon_{i_0}\left(\sum\limits_{\substack{i=1\\i\neq i_0}}^{n}\varepsilon_i c_i^2\right)}=0.$$

By solving the equation, we find

$$f_{i_0}=\begin{cases}2c\ln\cos\dfrac{\sqrt{-A}}{2c}x_{i_0}+mc_{i_0}x_{i_0}, & if\, A<0\\ 2c\ln\cosh\dfrac{\sqrt{A}}{2c}x_{i_0}+mc_{i_0}x_{i_0}, & if\, A>0\ and\ \left|\dfrac{f'_{i_0}-mc_{i_0}}{\sqrt{A}}\right|<1\\ 2c\ln\sinh\dfrac{\sqrt{A}}{2c}x_{i_0}+mc_{i_0}x_{i_0}, & if\, A>0\ and\ \left|\dfrac{f'_{i_0}-mc_{i_0}}{\sqrt{A}}\right|>1\\ 2c\ln\left|x_{i_0}\right|+mc_{i_0}x_{i_0}, & if\, A=0\end{cases} \tag{2.43}$$

with

$$A=\frac{B}{\varepsilon_{i_0}\sum\limits_{\substack{i=1\\ i\neq i_0}}^{n}\varepsilon_i c_i^2}-m^2c_{i_0}^2.$$

Moreover $f_i'(x_i)=a_i$, where a_i are constants for $i=1,\ldots,i_0-1,i_0+1,\ldots,n-1$ and we rewrite (2.9) again

$$\begin{aligned}&\Bigg[\varepsilon_{n+1}\sum_{i=1}^{n}\varepsilon_i c_i^2+\varepsilon_n c_n^2\sum_{\substack{i=1\\ i\neq i_0}}^{n-1}\varepsilon_i a_i^2+\varepsilon_{i_0}f_{i_0}'^2\sum_{\substack{i=1\\ i\neq i_0}}^{n}\varepsilon_i c_i^2+\sum_{\substack{j=1\\ j\neq i_0}}^{n-1}\left(\varepsilon_j a_j^2\sum_{\substack{i=1\\ i\neq j}}^{n-1}\varepsilon_i c_i^2\right)\\ &-2\varepsilon_{i_0}c_{i_0}f'_{i_0}\sum_{\substack{i=1\\ i\neq i_0}}^{n-1}\varepsilon_i c_i a_i-\sum_{\substack{j=1\\ j\neq i_0}}^{n-1}\left(\varepsilon_j c_j a_j\sum_{\substack{i=1\\ i\neq j,i\neq i_0}}^{n-1}\varepsilon_i c_i a_i\right)\Bigg]f_n''\\ &+\left[\varepsilon_{n+1}\varepsilon_{i_0}+\varepsilon_{i_0}\sum_{\substack{i=1\\ i\neq i_0}}^{n-1}\varepsilon_i a_i^2+\left(\varepsilon_{i_0}\sum_{\substack{i=1\\ i\neq i_0}}^{n}\varepsilon_i c_i^2\right)f_n'^2+2\left(\varepsilon_{i_0}\sum_{\substack{i=1\\ i\neq i_0}}^{n-1}\varepsilon_i a_i c_i\right)f_n'\right]f_{i_0}''=0.\end{aligned} \tag{2.44}$$

According to (2.41) and (2.44), we get

$$\left[\varepsilon_{n+1}-2c\left(\sum_{\substack{i=1\\ i\neq i_0}}^{n}\varepsilon_i c_i^2\right)f_n''+\sum_{\substack{i=1\\ i\neq i_0}}^{n-1}\varepsilon_i a_i^2+\left(\sum_{\substack{i=1\\ i\neq i_0}}^{n}\varepsilon_i c_i^2\right)f_n'^2+2\left(\sum_{\substack{i=1\\ i\neq i_0}}^{n-1}\varepsilon_i a_i c_i\right)f_n'\right]f_{i_0}''=0. \tag{2.45}$$

Since $f_{i_0}''\neq 0$, by substituting (2.39) into (2.45) and we obtain

$$f_n'^2+2mf_n'=2cf_n''-\frac{\varepsilon_{n+1}+\sum\limits_{\substack{i=1\\ i\neq i_0}}^{n-1}\varepsilon_i a_i^2}{\sum\limits_{\substack{i=1\\ i\neq i_0}}^{n}\varepsilon_i c_i^2}.$$

When considered this equation with (2.26), then

$$c_0 = -\frac{\varepsilon_{n+1} + \sum\limits_{\substack{i=1\\i\neq i_0}}^{n-1} \varepsilon_i a_i^2}{\sum\limits_{\substack{i=1\\i\neq i_0}}^{n} \varepsilon_i c_i^2}.$$

According to (2.39), we get

$$m^2 + c_0 = \frac{-\varepsilon_{n+1} \sum\limits_{\substack{i=1\\i\neq i_0}}^{n} \varepsilon_i c_i^2 - \varepsilon_n c_n^2 \sum\limits_{\substack{i=1\\i\neq i_0}}^{n-1} \varepsilon_i a_i^2 - \left[\sum\limits_{\substack{i=1\\i\neq i_0}}^{n-1} \varepsilon_i a_i^2 \sum\limits_{\substack{i=1\\i\neq i_0}}^{n-1} \varepsilon_i c_i^2 - \left(\sum\limits_{\substack{i=1\\i\neq i_0}}^{n-1} \varepsilon_i c_i a_i \right)^2 \right]}{\left(\sum\limits_{\substack{i=1\\i\neq i_0}}^{n} \varepsilon_i c_i^2 \right)^2}$$

Depending on the epsilones (-1 or +1) in the above equation, $m^2 + c_0$ can be positive, negative and zero. With suitable translation, we get $f_i = 0$ for $i = 1, \ldots, i_0 - 1, i_0 + 1, \ldots, n - 1$ and following the equations

$$f_{i_0} = \begin{cases} 2c \ln \cos \frac{\sqrt{-M}}{2c} x_{i_0}, \ if\ M < 0 \\ 2c \ln \cosh \frac{\sqrt{M}}{2c} x_{i_0}, \ if\ M > 0 \ and \ \left| \frac{f'_{i_0}}{\sqrt{M}} \right| < 1 \\ 2c \ln \sinh \frac{\sqrt{M}}{2c} x_{i_0}, \ if\ M > 0 \ and \ \left| \frac{f'_{i_0}}{\sqrt{M}} \right| > 1 \\ 2c \ln \left| x_{i_0} \right|, \qquad if\ M = 0 \end{cases} \tag{2.46}$$

and

$$f_n = \begin{cases} -2c \ln \cos \left(\frac{\sqrt{-N}}{2c} (c_1 x_1 + \cdots + c_n x_n) \right), \ if\ N < 0 \\ -2c \ln \cosh \left(\frac{\sqrt{N}}{2c} (c_1 x_1 + \cdots + c_n x_n) \right), \ if\ N > 0 \ and \ \left| \frac{f'_n}{\sqrt{N}} \right| < 1 \\ -2c \ln \sinh \left(\frac{\sqrt{N}}{2c} (c_1 x_1 + \cdots + c_n x_n) \right), \ if\ N > 0 \ and \ \left| \frac{f'_n}{\sqrt{N}} \right| > 1 \\ -2c \ln \left| c_1 x_1 + \cdots + c_n x_n \right|, \qquad if\ N = 0 \end{cases} \tag{2.47}$$

where

$$M = \varepsilon_{i_0} \varepsilon_{n+1} \frac{\sum\limits_{i=1}^{n} \varepsilon_i c_i^2}{\sum\limits_{\substack{i=1\\i\neq i_0}}^{n} \varepsilon_i c_i^2}, \quad N = -\frac{\varepsilon_{n+1}}{\sum\limits_{\substack{i=1\\i\neq i_0}}^{n} \varepsilon_i c_i^2}.$$

Therefore we complete the proof of the following main theorem.

Main theorem. M^n is a non-degenerate minimal translation graph in semi-Euclidean space $\mathbb{R}^{n+1}_\nu$, if it is congruent to a part of one of the following surfaces:

1. A non-degenerate hyperplane,
2. A hypersurface parameterized by

$$\phi(x_1, \dots, x_n) = (x_1, \dots, x_n, F(x_1, \dots, x_n)), \quad F(x_1, \dots, x_n) = f_{i_0}(x_{i_0}) + f_n(u)$$

   where $u = \sum_{i=1}^{n} c_i x_i$, c_i are constants, $c_n \neq 0$, with the conditions in the Eq (2.4), for a unique i_0, $1 \leq i_0 \leq n-1$, such that f_{i_0} and f_n one of the previous forms in (2.46) and (2.47), respectively. In additionally, $f_k(x_k) = 0$ for $k \neq i_0$ and $1 \leq k \leq n-1$.

3. Conclusions

Semi-Euclidean spaces are important in applications of general relativity which is the explanation of gravity in modern physics. In this study, we have a characterization of minimal translation graphs which are generalization of minimal translation hypersurfaces in semi-Euclidean space. Also, we obtain the main theorem by which we classify all non-degenerate minimal translation graphs.

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Conflict of interest

The author declares no conflict of interest.

References

1. L. Verstraellen, J. Walrave, S. Yaprak, The minimal translation surfaces in Euclidean space, *Soochow J. Math.*, **20** (1994), 77–82.
2. F. Dillen, L. Verstraellen, G. Zafindratafa, *A generalization of the translation surfaces of Scherk differential geometry in honor of Radu Rosca*, Meeting on Pure and Applied Differential Geometry, Leuven, KU Leuven, Department Wiskunde, (1991), 107–109.
3. I. Van de Woestijne, Minimal surfaces of the 3-dimensional Minkowski space, In: M. Boyom, J. M. Morvan, L. Verstraelen, P. Brard, B. Laget, C. M. Marle, *Geometry and topology of submanifolds, II*, Avignon: World Scientific, (1990), 344–369.
4. K. Seo, Translation hypersurfaces with constant curvature in space forms, *Osaka J. Math.*, **50** (2013), 631–641.
5. T. Hasanis, R. López, Classification and construction of minimal translation surfaces in Euclidean space, *Results Math.,* **75** (2020), 2–22.
6. M. E. Aydn, A. O. Ogrenmis, Translation hypersurfaces with constant curvature in 4-dimensional isotropic space, 2017. Available from: `https://arxiv.org/abs/1711.09051`.

7. J. Inoguchi, R. López, M. I. Munteanu, Minimal translation surfaces in the Heisenberg group Nil_3, *Geom. Dedicata*, **161** (2012), 221–231.
8. R. López, M. I. Munteanu, Minimal translation surfaces in Sol_3, *J. Math. Soc. Jpn.*, **64** (2012), 985–1003.
9. M. Moruz, M. I. Munteanu, Minimal translation hypersurfaces in E^4, *J. Math. Anal. Appl.*, **439** (2016), 798–812.
10. M. I. Munteanu, O. Palmas, G. Ruiz-Hernandez, Minimal translation surfaces in Euclidean spaces, *Mediter. J. Math.,* **13** (2016), 2659–2676.
11. R. López, Ó. Perdemo, Minimal translation surfaces in Euclidean space, *J. Geom. Anal.*, **27** (2017), 2926–2937.
12. D. Yang, J. J. Zhang, Y. Fu, A note on minimal translation graphs in Euclidean space, *Mathematics*, **7** (2019), 889.
13. I. Van de Woestyne, Minimal homothetical hypersurfaces of a semi-Euclidean, *Results Math.*, **27** (1995), 333–342.

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