Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 10164–10191.
DOI: 10.3934/math.2021589
Received: 05 March 2021
Accepted: 29 June 2021
Published: 09 July 2021

Research article

Generalized integral inequalities for ABK-fractional integral operators

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Abstract: In this paper, we employ new version of the Atangana-Baleanu integral operator namely ABK-fractional integrals to obtain two general integral identities complying second-order derivatives for a given function. Thus allowing us to derive new generalized Hermite-Hadamard type inequalities via ABK- fractional integrals. Moreover we give several new versions of Mid-point and Trapezoid type inequalities by employing Hölder, Young and Jensen inequalities.

Keywords: differentiable convex functions; Hölder inequality; Young inequality; power mean inequality; Katugampola fractional integral; AB-fractional integral; ABK-fractional integral
Mathematics Subject Classification: 26A33, 26A51, 26D10

1. Introduction

Convexity, with a very old background, has more significance for the theory of inequalities and other areas of mathematics such as probability theory, graph theory, functional analysis and numerical computing. The definition of a classical convex function is:

Definition 1. *(see, e.g., [38]) The mapping $f : [\alpha, \beta] \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if the following condition holds*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

for all $x, y \in [\alpha, \beta]$ and $\lambda \in [0,1]$.

Integral inequalities have a significant role in the expansion of all branches of mathematics. One of the most powerful of these integral inequalities is the Hermite-Hadamard inequality obtained for convex function as follows:

Theorem 1. *(see, e.g., [37]) Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I with $\alpha < \beta$. Then*

$$f\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int\limits_{\alpha}^{\beta} f(x)dx \leq \frac{f(\alpha)+f(\beta)}{2}. \tag{1.2}$$

There are many new results in the literature concerning these type of inequalities. For example, Hermite-Hadamard inequalities for m-convex and (α, m)-convex functions have been proved by Bakula et al. and similar results have been provided by Kırmacı et al. for $s-$convex functions in [1, 2]. In [3], by using the classical definition of convexity, Kavurmacı et al. obtained new Hermite-Hadamard type inequalities. In [4], new Hermite-Hadamard type inequalities involving fractional operators were presented by Set et al. In [5], Bayraktar proved new results related to the inequality for concave and $r-$convex functions. Akdemir et al. [10], introduced new Hermite-Hadamard type inequalities for $GG-$convex functions. Also, in 2002–2005, Guessab and Schmeisser, worked on convexity and Hermite-Hadamard inequality in [31–33].

One of the concepts that have played a significant role in the growth of inequality theory in recent years is fractional analysis. Since fractional calculus was presented toward the end of the nineteenth century, the subject has become a quickly developing area and has discovered numerous applications in different research fields. Fractional integrals are the most commonly used concept in calculus analysis to obtain new generalizations, extensions, and versions of classical integral inequalities. Until today, many definitions of fractional integrals have been presented such as Riemann-Liouville, Weyl, Erdelyi-Kober, Hadamard, Katugampola, conformable fractional integrals et. al. and fractional versions of many inequalities such as Hermite-Hadamard, Simpson, Ostrowski, Grüss, Chebyshev inequalities for these fractional integrals have been obtained. For example, Minkowski and Hermite-Hadamard integral inequalities for Riemann-Liouville fractional integrals were obtained by Z. Dahmani in [12]. In 2016 to 2020, R. Almeida worked on Caputo fractional derivative in [6–8]. Khalil et al. and Abdeljawad worked on conformable fractional calculus in [9, 13], respectively and Akdemir et al. [11] obtained new integral inequalities by motivated from these studies. Also, Chebyshev type inequalities for Katugampola fractional integral operators and fractional integral inequalities for co-ordinated $MT-$convex functions were obtained, respectively, in [15, 16]. Recently, some interesting and new results related to fractional analysis have been presented to the literature by the authors as Sene and Yavuz (see the papers [34–36]). Readers who want to learn more about fractional derivatives and integral operators can look to the book by Samko et al. in [14].

2. Preliminaries

Now let's give the concept of fractional integral, which has been recently defined and attracted many researchers.

Definition 2. $X_c^p(\alpha,\beta)$ $(c \in \mathbb{R})$, $1 \leq p \leq \infty$ *denotes the space of all complex-valued Lebesgue* *measurable functions f for which* $\|f\|_{X_c^p} < \infty$, *where the norm* $\|.\|_{X_c^p}$ *is defined by*

$$\|f\|_{X_c^p} = \left(\int_\alpha^\beta \mid \varkappa^c f(\varkappa) \mid^p \frac{d\varkappa}{\varkappa} \right)^{\frac{1}{p}} \quad (1 \le p < \infty)$$

and for $p = \infty$

$$\|f\|_{X_c^\infty} = ess \sup_{\alpha \le \varkappa \le \beta} \mid \varkappa^c f(\varkappa) \mid .$$

Definition 3. *[17] Let* $[\alpha, \beta] \subset \mathbb{R}$ *be a finite interval. Then, the left and right side Katugampola fractional integrals of order* $\zeta (> 0)$ *of* $f \in X_c^p(\alpha, \beta)$ *are defined by*

$${}^\tau I_{\alpha^+}^\zeta f(x) = \frac{\tau^{1-\zeta}}{\Gamma(\zeta)} \int_\alpha^x \frac{\varkappa^{\tau-1}}{(x^\tau - \varkappa^\tau)^{1-\zeta}} f(\varkappa) d\varkappa, \quad x > \alpha$$

and

$${}^\tau I_{\beta^-}^\zeta f(x) = \frac{\tau^{1-\zeta}}{\Gamma(\zeta)} \int_x^\beta \frac{\varkappa^{\tau-1}}{(\varkappa^\tau - x^\tau)^{1-\zeta}} f(\varkappa) d\varkappa, \quad x < \beta,$$

where $\tau > 0$, *if the integrals exits.*

This fractional integrals are generalized the Riemann-Liouville and Hadamard fractional integrals which are defined as follows:

Definition 4. *Let* $f \in L[\alpha, \beta]$. *The Riemann-Liouville integrals* $J_{\alpha+}^\alpha f$ *and* $J_{\beta-}^\alpha f$ *of order* $\zeta > 0$ *with* $\alpha \ge 0$ *are defined by*

$$J_{\alpha+}^\zeta f(x) = \frac{1}{\Gamma(\zeta)} \int_\alpha^x (x - t)^{\zeta - 1} f(t) dt, \quad x > \alpha$$

and

$$J_{\beta-}^\zeta f(x) = \frac{1}{\Gamma(\zeta)} \int_x^\beta (t - x)^{\zeta - 1} f(t) dt, \quad x < \beta$$

respectively where $\Gamma(\zeta) = \int_0^\infty e^{-t} u^{\alpha - 1} du$. *Here is* $J_{\alpha+}^0 f(x) = J_{\beta-}^0 f(x) = f(x)$.

Recently, the Caputo-Fabrizio (CF) fractional integral has become a term that many researchers use as follows.

Definition 5. *[18] Let* $f \in H^1(0, \beta)$, $\beta > \alpha$, $\zeta \in [0, 1]$ *then, the definition of the new Caputo-Fabrizio fractional derivative is:*

$${}^{CF}D^\zeta f(\varkappa) = \frac{B(\zeta)}{1 - \zeta} \int_\alpha^\varkappa f'(s) exp\left[-\frac{\zeta}{(1 - \zeta)} (\varkappa - s) \right] ds \tag{2.1}$$

where $B(\zeta)$ *is a normalization function such that* $B(0) = B(1) = 1$.

The related CF fractional derivative integral formula can be given as follows.

Definition 6. *[19] Suppose that* $f \in H^1(0, \beta)$, $\beta > \alpha$, $\zeta \in [0, 1]$ *then, the definition of the left and right side of* CF *fractional integral is:*

$$\left({}_\alpha^{CF} I^\zeta \right) f(\varkappa) = \frac{1 - \zeta}{B(\zeta)} f(\varkappa) + \frac{\zeta}{B(\zeta)} \int_\alpha^\varkappa f(y) dy,$$

and

$$\left({}^{CF}I_{\beta}^{\zeta}\right) f(\varkappa) = \frac{1-\zeta}{B(\zeta)} f(\varkappa) + \frac{\zeta}{B(\zeta)} \int_{\varkappa}^{\beta} f(y)dy$$

where $B(\zeta)$ is normalization function.

For recent results concerning these operators, we refer to [20, 21].

Fractional calculus is a branch of mathematical analysis that studies the various unique possibilities of describing real or complex numbers powers of the differentiation and integration operators. The fractional calculus has a significant part in different scientific fields because of its few applications in dynamic problems including hydrodynamics, signals, dynamics, fluid, viscoelastic theory, control theory, biology, computer networking, image processing, and lots of others. Fractional calculus is an important technique with significant applications in science and engineering. Atangana and Baleanu have recently introduced AB-fractional calculus and have drawn a large number of scientists in various scientific fields to investigate various topics. Atangana and Baleanu recommended a greatly improved variant of a derivative with no singular kernel. Their derivative depends on the notable generalized Mittag-Leffler function. It very well may be reviewed that, the Mittag-Leffler function has been acquainted giving a reaction to a traditional inquiry of complex analysis.

The Atangana-Baleanu fractional integral operator is defined as follows.

Definition 7. *[22] The AB-fractional integral of a function $f \in H^1(\alpha, \beta)$ is given by*

$${}^{AB}_{\alpha}I_{\varkappa}^{\zeta}\{f(\varkappa)\} = \frac{1-\zeta}{B(\zeta)} f(\varkappa) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_{\alpha}^{\varkappa} f(y)(\varkappa - y)^{\zeta-1}dy$$

where $\beta > \alpha, \zeta \in [0, 1]$.

Similarly, [23], the authors gave opposite side of the AB-fractional integral operator is given by

$${}^{AB}I_{\beta}^{\zeta}\{f(\varkappa)\} = \frac{1-\zeta}{B(\zeta)} f(\varkappa) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_{\varkappa}^{\beta} f(y)(y - \varkappa)^{\zeta-1}dy.$$

For recent results related to AB-fractional operators see [24–27, 40, 41].

In [30], a connection between the Atangana-Baleanu and the Riemann-Liouville fractional integrals of a function with respect to a monotone function with nonsingular kernel is given as follows:

Lemma 1. *For any L^1 function $f(x)$ on an interval $[\alpha, \beta]$ with $x \in [\alpha, \beta]$, the ζth Atangana-Baleanu (AB) fractional integral of a function $f(x)$ with respect to another function can be represented as follows:*

$${}^{AB}_{\psi(x)}I_{\alpha}^{\zeta}\{f(\varkappa)\} = \frac{\zeta}{B(\zeta)}\, {}_{\psi(x)}J_{\alpha}^{\zeta}\{f(\varkappa)\} + \frac{1-\zeta}{B(\zeta)} f(\varkappa), \qquad 0 < \zeta < 1 \tag{2.2}$$

where $B(\zeta)$ is as before and $\psi(x)$ is an increasing positive monotone function on the interval $[\alpha, \beta]$ with $\psi'(x) \in L^1(\alpha, \beta)$.

The left and right side of ABK-fractional integrals, the special case of (2.2), was given by Kashuri as:

Definition 8. *[28] Let $[\alpha,\beta] \subset \mathbb{R}$ be a finite interval. Then, the left and right side ABK-fractional integrals of order $\zeta \in (0,1)$ of $f \in X_c^p(\alpha,\beta)$ are defined by*

$$ {}^{ABK\tau}_{\alpha^+}I^{\zeta}_{\varkappa}\{f(\varkappa)\} = \frac{1-\zeta}{B(\zeta)}f(\varkappa) + \frac{\tau^{1-\zeta}\zeta}{B(\zeta)\Gamma(\zeta)}\int_{\alpha}^{\varkappa}\frac{y^{\tau-1}}{(\varkappa^{\tau}-y^{\tau})^{1-\zeta}}f(y)dy, \ \ \varkappa > \alpha \geq 0 \tag{2.3}$$

and

$$ {}^{ABK\tau}_{\beta^-}I^{\zeta}_{\varkappa}\{f(\varkappa)\} = \frac{1-\zeta}{B(\zeta)}f(\varkappa) + \frac{\tau^{1-\zeta}\zeta}{B(\zeta)\Gamma(\zeta)}\int_{\varkappa}^{\beta}\frac{y^{\tau-1}}{(y^{\tau}-\varkappa^{\tau})^{1-\zeta}}f(y)dy, \ \ \varkappa < \beta, \tag{2.4}$$

where $\tau > 0$ and $B(\zeta) > 0$ satisfies the property $B(0) = B(1) = 1$.

Since the normalization function $B(\zeta) > 0$ is positive, it directly implies that the fractional *ABK*-integral of a positive function is positive. Note that, if $\tau \to 1$ then we recover the *AB*-fractional integral. The Atangana-Baleanu fractional integral operator differs positively from other fractional integral operators with its time memory effect feature in the solutions of the real world problems and pysical phenomenon with the superior features of its kernel structure. In addition to preserving the features of this unique operator, *ABK*- fractional integral operators have started to be used widely in various disciplines in many application areas in the literature as they form a general form.

Here, we aim to establish certain new generalizations of Hermite-Hadamard type integral inequalities for twice differentiable convex functions via the fractional integral operators (2.3) and (2.4). The inequalities presented here, being very general, are pointed out to be specialized to yield many new and known inequalities associated with some known fractional integrals.

3. Main results

We start with the following lemma for *ABK*-fractional integral operators, which is the main motivation of this paper:

Lemma 2. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^{\tau},\beta^{\tau}] \to \mathbb{R}$ be a twice differentiable mapping on $(\alpha^{\tau},\beta^{\tau})$ with $0 \leq \alpha < \beta$. Then the following equality is valid for ABK-fractional integral operators*

$$\begin{aligned} & {}^{ABK\tau}_{\alpha^+}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^{\tau}-\alpha^{\tau})^{\zeta}f(\alpha^{\tau}) + (\beta^{\tau}-\varkappa^{\tau})^{\zeta}f(\beta^{\tau})}{\tau^{\zeta}B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^{\tau})}{B(\zeta)} \\ = & \frac{(\varkappa^{\tau}-\alpha^{\tau})^{\zeta+1}}{\tau^{\zeta}(\zeta+1)B(\zeta)\Gamma(\zeta)}f'(\alpha^{\tau}) + \frac{(\varkappa^{\tau}-\alpha^{\tau})^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^{\tau})^{\zeta+1}k^{\tau-1}f''\left(k^{\tau}\varkappa^{\tau}+(1-k^{\tau})\alpha^{\tau}\right)dk \\ & -\frac{(\beta^{\tau}-\varkappa^{\tau})^{\zeta+1}}{\tau^{\zeta}(\zeta+1)B(\zeta)\Gamma(\zeta)}f'(\beta^{\tau}) + \frac{(\beta^{\tau}-\varkappa^{\tau})^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(k^{\tau})^{\zeta+1}k^{\tau-1}f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\varkappa^{\tau}\right)dk \end{aligned}$$

where $g(u) = u^{\tau}$, $\varkappa^{\tau} \in [\alpha^{\tau},\beta^{\tau}]$.

Proof. Employing integration by parts gives

$$\frac{(\varkappa^{\tau}-\alpha^{\tau})^{\zeta+1}}{\tau^{\zeta-1}B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^{\tau})^{\zeta}k^{\tau-1}f'\left(k^{\tau}\varkappa^{\tau}+(1-k^{\tau})\alpha^{\tau}\right)dk \tag{3.1}$$

$$= \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}}{\tau^{\zeta-1}B(\zeta)\Gamma(\zeta)}\Bigg[-\frac{(1-k^\tau)^{\zeta+1}}{\tau(\zeta+1)}f'\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\Bigg|_0^1$$

$$+(\varkappa^\tau-\alpha^\tau)\int_0^1\frac{(1-k^\tau)^{\zeta+1}}{\zeta+1}k^{\tau-1}f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)dk\Bigg]$$

$$= \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}}{\tau^{\zeta-1}B(\zeta)\Gamma(\zeta)}\Bigg[\frac{1}{\tau(\zeta+1)}f'(\alpha^\tau)+\frac{(\varkappa^\tau-\alpha^\tau)}{\zeta+1}\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)dk\Bigg].$$

Similarly

$$\begin{aligned}
&-\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+1}}{\tau^{\zeta-1}B(\zeta)\Gamma(\zeta)}\int_0^1(k^\tau)^{\zeta}k^{\tau-1}f'\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)dk \\
=\;&-\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+1}}{\tau^{\zeta-1}B(\zeta)\Gamma(\zeta)}\Bigg[\frac{(k^\tau)^{\zeta+1}}{\tau(\zeta+1)}f'\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\Bigg|_0^1 \\
&-\frac{(\beta^\tau-\varkappa^\tau)}{\zeta+1}\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)dk\Bigg] \\
=\;&-\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+1}}{\tau^{\zeta-1}B(\zeta)\Gamma(\zeta)}\Bigg[\frac{1}{\tau(\zeta+1)}f'(\beta^\tau)-\frac{(\beta^\tau-\varkappa^\tau)}{(\zeta+1)}\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)dk\Bigg].
\end{aligned} \tag{3.2}$$

By adding (3.1) and (3.2), we obtain

$$\begin{aligned}
&{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)} \\
=\;&\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}f'(\alpha^\tau)+\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)dk \\
&-\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+1}}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}f'(\beta^\tau)+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)dk.
\end{aligned}$$

So, the proof is completed. □

Remark 1. *For $\tau\to 1$ in Lemma 2 , we will get Lemma 2.1 in [29].*

Now, by using this important equality, we will establish the generalizations of the Hermite-Hadamard type inequalities via ABK-fractional integral operators.

Theorem 2. *Let $\zeta\in(0,1)$ and $\tau>0$ and $f:[\alpha^\tau,\beta^\tau]\to\mathbb{R}$ be a twice differentiable mapping on (α^τ,β^τ) with $0\le\alpha<\beta$ and $f''\in X^\tau_c(\alpha^\tau,\beta^\tau)$. If $|f''|$ is a convex on $[\alpha^\tau,\beta^\tau]$, then following inequality for ABK-fractional integral operators*

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}$$

$$-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|$$

$$\leq\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\frac{|f''(\varkappa^\tau)|}{\tau(\zeta+2)(\zeta+3)}+\frac{|f''(\alpha^\tau)|}{\tau(\zeta+3)}\right]$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\frac{|f''(\beta^\tau)|}{\tau(\zeta+3)}+\frac{|f''(\varkappa^\tau)|}{\tau(\zeta+2)(\zeta+3)}\right]$$

where $g(u)=u^\tau$, $\varkappa^\tau\in[\alpha^\tau,\beta^\tau]$, $\zeta\in(0,1)$, $B(\zeta)$ is normalization function.

Proof. Taking into account the identity obtained in Lemma 2, we may write

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}$$

$$-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|$$

$$=\Bigg|\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)dk$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)dk\Bigg|$$

$$\leq\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|dk.$$

By using convexity of $|f''|$, it yields

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}$$

$$-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|$$

$$\leq\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|dk$$

$$\leq\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}[k^\tau|f''(\varkappa^\tau)|+(1-k^\tau)|f''(\alpha^\tau)|]dk$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}[k^\tau|f''(\beta^\tau)|+(1-k^\tau)|f''(\varkappa^\tau)|]dk$$

$$= \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\frac{|f''(\varkappa^\tau)|}{\tau(\zeta+2)(\zeta+3)}+\frac{|f''(\alpha^\tau)|}{\tau(\zeta+3)}\right]$$
$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\frac{|f''(\beta^\tau)|}{\tau(\zeta+3)}+\frac{|f''(\varkappa^\tau)|}{\tau(\zeta+2)(\zeta+3)}\right]$$

and the proof is completed. □

Remark 2. *For $\tau \to 1$ in Theorem 2 , we will get Theorem 2.1 in [29].*

Corollary 1. *Set $\varkappa^\tau = \frac{\alpha^\tau+\beta^\tau}{2}$ in Theorem 2, we obtain*

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$
$$-\frac{(\beta^\tau-\alpha^\tau)^{\zeta+1}}{2^{\zeta+1}\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}\big[f'(\alpha^\tau)-f'(\beta^\tau)\big]-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$
$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[2\frac{\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|}{\tau(\zeta+2)(\zeta+3)}+\frac{|f''(\alpha^\tau)|}{\tau(\zeta+3)}+\frac{|f''(\beta^\tau)|}{\tau(\zeta+3)}\right].$$

Remark 3. *For $\tau \to 1$ in Corollary 1 , we will get Corollary 2.1 in [29].*

Theorem 3. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^\tau,\beta^\tau] \to \mathbb{R}$ be a twice differentiable mapping on (α^τ,β^τ) with $0 \leq \alpha < \beta$ and $f'' \in X^\tau_c(\alpha^\tau,\beta^\tau)$. If $|f''|^s$ is a convex on $[\alpha^\tau,\beta^\tau]$, then the following inequality for ABK-fractional integral operators*

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}$$
$$-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|$$
$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left(\frac{|f''(\varkappa^\tau)|^s+|f''(\alpha^\tau)|^s}{2\tau}\right)^{\frac{1}{s}}$$
$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left(\frac{|f''(\beta^\tau)|^s+|f''(\varkappa^\tau)|^s}{2\tau}\right)^{\frac{1}{s}}$$

where $g(u) = u^\tau$, $r^{-1}+s^{-1} = 1$, $\varkappa^\tau \in [\alpha^\tau,\beta^\tau]$, $\zeta \in (0,1)$, $s > 1$, $B(\zeta)$ is normalization function.

Proof. By using Lemma 2, we get

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}$$
$$-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|$$
$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|dk.$$

By applying Hölder inequality, we get

$$\begin{aligned}
&\left|{}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}\right.\\
&\left.-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\right|\\
\leq\ &\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(\int_0^1(1-k^\tau)^{(\zeta+1)r}k^{\tau-1}dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|^s dk\right)^{\frac{1}{s}}\right]\\
&+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(\int_0^1(k^\tau)^{(\zeta+1)r}k^{\tau-1}dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|^s dk\right)^{\frac{1}{s}}\right].
\end{aligned}$$

Using convexity of $|f''|^s$, we get

$$\begin{aligned}
\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|^s dk &\leq \int_0^1 k^{\tau-1}\Big[k^\tau\left|f''(\varkappa^\tau)\right|^s+(1-k^\tau)\left|f''(\alpha^\tau)\right|^s\Big]dk,\\
\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|^s dk &\leq \int_0^1 k^{\tau-1}\Big[k^\tau\left|f''(\beta^\tau)\right|^s+(1-k^\tau)\left|f''(\varkappa^\tau)\right|^s\Big]dk.
\end{aligned}$$

After simplification, we get desired result. □

Remark 4. *For $\tau\to 1$ in Theorem 3 , we will get Theorem 2.2 in [29].*

Corollary 2. *Set $\varkappa^\tau=\frac{\alpha^\tau+\beta^\tau}{2}$ in Theorem 3, we obtain*

$$\begin{aligned}
&\left|{}^{ABK\tau}_{\alpha^+}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\Big[f(\alpha^\tau)+f(\beta^\tau)\Big]\right.\\
&\left.-\frac{(\beta^\tau-\alpha^\tau)^{\zeta+1}}{2^{\zeta+1}\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}\Big[f'(\alpha^\tau)-f'(\beta^\tau)\Big]-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\right|\\
\leq\ &\frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left[\left(\frac{\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s+\left|f''(\alpha^\tau)\right|^s}{2\tau}\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\frac{\left|f''(\beta^\tau)\right|^s+\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s}{2\tau}\right)^{\frac{1}{s}}\right].
\end{aligned}$$

Remark 5. *For $\tau\to 1$ in Corollary 2 , we will get Corollary 2.2 in [29].*

Theorem 4. *Let $\zeta\in(0,1)$ and $\tau>0$ and $f:[\alpha^\tau,\beta^\tau]\to\mathbb{R}$ be a twice differentiable mapping on (α^τ,β^τ) with $0\leq\alpha<\beta$ and $f''\in X^\tau_c(\alpha^\tau,\beta^\tau)$. If $|f''|^s$ is a convex on $[\alpha^\tau,\beta^\tau]$, then the following inequality for ABK-fractional integral operators*

$$\left|{}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}\right.$$

$$-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau r(\zeta r+r+1)}+\frac{|f''(\varkappa^\tau)|^s+|f''(\alpha^\tau)|^s}{2\tau s}\right)$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau r(\zeta r+r+1)}+\frac{|f''(\beta^\tau)|^s+|f''(\varkappa^\tau)|^s}{2\tau s}\right)$$

where $g(u)=u^\tau$, $r^{-1}+s^{-1}=1$, $\varkappa^\tau\in[\alpha^\tau,\beta^\tau]$, $\zeta\in(0,1)$, $s>1$, $B(\zeta)$ *is normalization function.*

Proof. By using Lemma 2, we have

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}$$

$$-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|dk.$$

By using the Young inequality as $xy\leq\frac{1}{r}x^r+\frac{1}{s}y^s$

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}$$

$$-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\frac{1}{r}\int_0^1(1-k^\tau)^{(\zeta+1)r}k^{\tau-1}dk+\frac{1}{s}\int_0^1k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|^s dk\right]$$

$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\frac{1}{r}\int_0^1(k^\tau)^{(\zeta+1)r}k^{\tau-1}dk+\frac{1}{s}\int_0^1k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|^s dk\right].$$

Now by employing convexity of $|f''|^s$ along with simple computations give the required result. □

Remark 6. *For* $\tau\to1$ *in Theorem 4 , we will get Theorem 2.3 in [29].*

Corollary 3. *Set* $\varkappa^\tau=\frac{\alpha^\tau+\beta^\tau}{2}$ *in Theorem 4, we obtain*

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\left[f(\alpha^\tau)+f(\beta^\tau)\right]$$

$$-\frac{(\beta^\tau-\alpha^\tau)^{\zeta+1}}{2^{\zeta+1}\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[f'(\alpha^\tau)-f'(\beta^\tau)\right]-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{2}{\tau r(\zeta r+r+1)}+\frac{2\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s+|f''(\alpha^\tau)|^s+|f''(\beta^\tau)|^s}{2\tau s}\right).$$

Remark 7. *For $\tau \to 1$ in Corollary 3 , we will get Corollary 2.3 in [29].*

Theorem 5. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^\tau, \beta^\tau] \to \mathbb{R}$ be a twice differentiable mapping on $(\alpha^\tau, \beta^\tau)$ with $0 \le \alpha < \beta$ and $f'' \in X_c^\tau(\alpha^\tau, \beta^\tau)$. If $|f''|^s$ is a convex on $[\alpha^\tau, \beta^\tau]$, then the following inequality for ABK-fractional integral operators*

$$\begin{aligned}
&\Bigg| {}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)} \\
&-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1} f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1} f'(\beta^\tau)}{\tau^\zeta(\zeta+1) B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg| \\
\le\ & \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(\frac{1}{\tau(\zeta+2)}\right)^{1-\frac{1}{s}}\left(\frac{|f''(\varkappa^\tau)|^s}{\tau(\zeta+2)(\zeta+3)}+\frac{|f''(\alpha^\tau)|^s}{\tau(\zeta+3)}\right)^{\frac{1}{s}}\right] \\
&+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(\frac{1}{\tau(\zeta+2)}\right)^{1-\frac{1}{s}}\left(\frac{|f''(\beta^\tau)|^s}{\tau(\zeta+3)}+\frac{|f''(\varkappa^\tau)|^s}{\tau(\zeta+2)(\zeta+3)}\right)^{\frac{1}{s}}\right]
\end{aligned}$$

where $g(u) = u^\tau$, $\varkappa^\tau \in [\alpha^\tau, \beta^\tau]$, $\zeta \in (0,1)$, $s \ge 1$, $B(\zeta)$ is normalization function.

Proof. By using Lemma 2, we get

$$\begin{aligned}
&\Bigg| {}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)} \\
&-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1} f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1} f'(\beta^\tau)}{\tau^\zeta(\zeta+1) B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg| \\
\le\ & \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|dk \\
&+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|dk.
\end{aligned}$$

By applying power mean inequality, we get

$$\begin{aligned}
&\Bigg| {}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)} \\
&-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1} f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1} f'(\beta^\tau)}{\tau^\zeta(\zeta+1) B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg| \\
\le\ & \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}dk\right)^{1-\frac{1}{s}}\left(\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|^s dk\right)^{\frac{1}{s}}\right] \\
&+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}dk\right)^{1-\frac{1}{s}}\left(\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|^s dk\right)^{\frac{1}{s}}\right].
\end{aligned}$$

By using convexity of $|f''|^s$, we obtain

$$\Bigg| {}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}$$

$$
-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|
$$

$$
\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}dk\right)^{1-\frac{1}{s}}
$$

$$
\left(\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}\Big[k^\tau|f''(\varkappa^\tau)|^s+(1-k^\tau)|f''(\alpha^\tau)|^s\Big]dk\right)^{\frac{1}{s}}\Bigg]
$$

$$
+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}dk\right)^{1-\frac{1}{s}}
$$

$$
\left(\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}\Big[k^\tau|f''(\beta^\tau)|^s+(1-k^\tau)|f''(\varkappa^\tau)|^s\Big]dk\right)^{\frac{1}{s}}\Bigg]
$$

$$
= \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(\frac{1}{\tau(\zeta+2)}\right)^{1-\frac{1}{s}}\left(\frac{|f''(\varkappa^\tau)|^s}{\tau(\zeta+2)(\zeta+3)}+\frac{|f''(\alpha^\tau)|^s}{\tau(\zeta+3)}\right)^{\frac{1}{s}}\right]
$$

$$
+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(\frac{1}{\tau(\zeta+2)}\right)^{1-\frac{1}{s}}\left(\frac{|f''(\beta^\tau)|^s}{\tau(\zeta+3)}+\frac{|f''(\varkappa^\tau)|^s}{\tau(\zeta+2)(\zeta+3)}\right)^{\frac{1}{s}}\right].
$$

So, the proof is completed. □

Remark 8. *For $\tau \to 1$ in Theorem 5 , we will get Theorem 2.4 in [29].*

Corollary 4. *Set $\varkappa^\tau = \frac{\alpha^\tau+\beta^\tau}{2}$ in Theorem 5, we obtain*

$$
\Bigg|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\Big[f(\alpha^\tau)+f(\beta^\tau)\Big]
$$

$$
-\frac{(\beta^\tau-\alpha^\tau)^{\zeta+1}}{2^{\zeta+1}\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}\Big[f'(\alpha^\tau)-f'(\beta^\tau)\Big]-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|
$$

$$
\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta+2)}\right)^{1-\frac{1}{s}}\Bigg[\left(\frac{\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s}{\tau(\zeta+2)(\zeta+3)}+\frac{|f''(\alpha^\tau)|^s}{\tau(\zeta+3)}\right)^{\frac{1}{s}}
$$

$$
+\left(\frac{|f''(\beta^\tau)|^s}{\tau(\zeta+3)}+\frac{\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s}{\tau(\zeta+2)(\zeta+3)}\right)^{\frac{1}{s}}\Bigg].
$$

Remark 9. *For $\tau \to 1$ in Corollary 4 , we will get Corollary 2.4 in [29].*

Theorem 6. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^\tau,\beta^\tau] \to \mathbb{R}$ be twice differentiable mapping on (α^τ,β^τ)*

with $0 \leq \alpha < \beta$ *and* $f'' \in X_c^\tau(\alpha^\tau, \beta^\tau)$. *If* $|f''|^s$ *is a concave for* $s > 1$, *then we have*

$$\left| {}^{ABK\tau}_{\alpha^+}I_{\varkappa}^{\zeta}\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I_{\varkappa}^{\zeta}\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)} \right.$$
$$\left. -\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)} \right|$$
$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta+2)}\right)\left|f''\left(\frac{1}{\zeta+3}\varkappa^\tau+\frac{\zeta+2}{\zeta+3}\alpha^\tau\right)\right|$$
$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta+2)}\right)\left|f''\left(\frac{\zeta+2}{\zeta+3}\beta^\tau+\frac{1}{\zeta+3}\varkappa^\tau\right)\right|$$

where $g(u) = u^\tau$, $\varkappa^\tau \in [\alpha^\tau, \beta^\tau]$, $\zeta \in (0, 1)$.

Proof. Using Jensen integral inequality and Lemma 2, we have

$$\left| {}^{ABK\tau}_{\alpha^+}I_{\varkappa}^{\zeta}\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I_{\varkappa}^{\zeta}\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)} \right.$$
$$\left. -\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)} \right|$$
$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|dk$$
$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|dk$$
$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}dk\right)\left|f''\left(\frac{\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)dk}{\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}dk}\right)\right|$$
$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}dk\right)\left|f''\left(\frac{\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)dk}{\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}dk}\right)\right|.$$

By computing the above integrals we have

$$\left| {}^{ABK\tau}_{\alpha^+}I_{\varkappa}^{\zeta}\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I_{\varkappa}^{\zeta}\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)} \right.$$
$$\left. -\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)} \right|$$
$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta+2)}\right)\left|f''\left(\frac{1}{\zeta+3}\varkappa^\tau+\frac{\zeta+2}{\zeta+3}\alpha^\tau\right)\right|$$
$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta+2)}\right)\left|f''\left(\frac{\zeta+2}{\zeta+3}\beta^\tau+\frac{1}{\zeta+3}\varkappa^\tau\right)\right|.$$

□

Remark 10. *For $\tau \to 1$ in Theorem 6 , we will get Theorem 2.5 in [29].*

Corollary 5. *Set $\varkappa^\tau = \frac{\alpha^\tau+\beta^\tau}{2}$ in Theorem 6, we obtain*

$$\left| {}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) + {}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) - \frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\left[f(\alpha^\tau)+f(\beta^\tau)\right]\right.$$
$$\left. -\frac{(\beta^\tau-\alpha^\tau)^{\zeta+1}}{2^{\zeta+1}\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[f'(\alpha^\tau)-f'(\beta^\tau)\right] - \frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\right|$$
$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta+2)}\right)\left[\left|f''\left(\frac{\alpha^\tau+\beta^\tau}{2(\zeta+3)}+\frac{\zeta+2}{\zeta+3}\alpha^\tau\right)\right|\right.$$
$$\left. +\left|f''\left(\frac{\zeta+2}{\zeta+3}\beta^\tau+\frac{\alpha^\tau+\beta^\tau}{2(\zeta+3)}\right)\right|\right].$$

Remark 11. *For $\tau \to 1$ in Corollary 5 , we will get Corollary 2.5 in [29].*

Theorem 7. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^\tau, \beta^\tau] \to \mathbb{R}$ be a twice differentiable mapping on $(\alpha^\tau, \beta^\tau)$ with $0 \leq \alpha < \beta$ and $f'' \in X^\tau_c(\alpha^\tau, \beta^\tau)$. If $|f''|^s$ is a concave mapping, we have*

$$\left| {}^{ABK\tau}_{\alpha^+}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}\right.$$
$$\left. -\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\right|$$
$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left|f''\left(\frac{\alpha^\tau+\varkappa^\tau}{2\tau}\right)\right|$$
$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left|f''\left(\frac{\beta^\tau+\varkappa^\tau}{2\tau}\right)\right|$$

where $g(u) = u^\tau$, $r^{-1} + s^{-1} = 1$, $\varkappa^\tau \in [\alpha^\tau, \beta^\tau]$, $\zeta \in (0,1)$, $s > 1$.

Proof. Taking into account Lemma 2 along with Hölder's inequality yields

$$\left| {}^{ABK\tau}_{\alpha^+}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\} + {}^{ABK\tau}_{\beta^-}I^{\zeta}_{\varkappa}\{(f\circ g)(\varkappa)\} - \frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}\right.$$
$$\left. -\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)} - \frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\right|$$
$$\leq \frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\int_0^1(1-k^\tau)^{(\zeta+1)r}k^{\tau-1}dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|^s dk\right)^{\frac{1}{s}}$$
$$+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\int_0^1(k^\tau)^{(\zeta+1)r}k^{\tau-1}dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|^s dk\right)^{\frac{1}{s}}.$$

Now by employing concavity of $|f''|^s$ under the essence of Jensen's integral inequality gives

$$\begin{aligned}\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|^s dk &= \int_0^1 k^o k^{\tau-1}\left|f''\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)\right|^s dk\\ &\leq \left(\int_0^1 k^0 dk\right)\left|f''\left(\frac{\int_0^1 k^{\tau-1}\left(k^\tau\varkappa^\tau+(1-k^\tau)\alpha^\tau\right)dk}{\int_0^1 k^0 dk}\right)\right|^s\\ &= \left|f''\left(\frac{\alpha^\tau+\varkappa^\tau}{2\tau}\right)\right|^s\end{aligned}$$

Similarly

$$\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|^s dk \leq \left|f''\left(\frac{\beta^\tau+\varkappa^\tau}{2\tau}\right)\right|^s.$$

So, we obtain

$$\begin{aligned}&\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}+{}^{ABK\tau}_{\beta^-}I^\zeta_\varkappa\{(f\circ g)(\varkappa)\}-\frac{(\varkappa^\tau-\alpha^\tau)^\zeta f(\alpha^\tau)+(\beta^\tau-\varkappa^\tau)^\zeta f(\beta^\tau)}{\tau^\zeta B(\zeta)\Gamma(\zeta)}\\ &-\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+1}f'(\alpha^\tau)-(\beta^\tau-\varkappa^\tau)^{\zeta+1}f'(\beta^\tau)}{\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}-\frac{2(1-\zeta)f(\varkappa^\tau)}{B(\zeta)}\Bigg|\\ \leq\ &\frac{(\varkappa^\tau-\alpha^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left|f''\left(\frac{\alpha^\tau+\varkappa^\tau}{2\tau}\right)\right|\\ &+\frac{(\beta^\tau-\varkappa^\tau)^{\zeta+2}}{\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left|f''\left(\frac{\beta^\tau+\varkappa^\tau}{2\tau}\right)\right|.\end{aligned}$$

□

Remark 12. *For $\tau\to 1$ in Theorem 7 , we will get Theorem 2.6 in [29].*

Corollary 6. *Set $\varkappa^\tau=\frac{\alpha^\tau+\beta^\tau}{2}$ in Theorem 7, we obtain*

$$\begin{aligned}&\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\Big[f(\alpha^\tau)+f(\beta^\tau)\Big]\\ &-\frac{(\beta^\tau-\alpha^\tau)^{\zeta+1}}{2^{\zeta+1}\tau^\zeta(\zeta+1)B(\zeta)\Gamma(\zeta)}\Big[f'(\alpha^\tau)-f'(\beta^\tau)\Big]-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|\\ \leq\ &\frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left[\left|f''\left(\frac{3\alpha^\tau+\beta^\tau}{4\tau}\right)\right|+\left|f''\left(\frac{3\beta^\tau+\alpha^\tau}{4\tau}\right)\right|\right].\end{aligned}$$

Remark 13. *For $\tau\to 1$ in Corollary 6 , we will get Corollary 2.6 in [29].*

4. Some related results via ABK-Fractional integral operators for convex function

Now, we will utilize another change of Lemma 2 by considering an alternate technique to eliminate the first derivative of the function. Substituting $\varkappa^{\tau} = \frac{\alpha^{\tau}+\beta^{\tau}}{2}$ in Lemma 2 lead us to new results.

Lemma 3. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^{\tau},\beta^{\tau}] \to \mathbb{R}$ be a twice differentiable mapping on $(\alpha^{\tau},\beta^{\tau})$ with $0 \leq \alpha < \beta$. Then the following equality for ABK-fractional integral operators*

$$
\begin{aligned}
& {}^{ABK\tau}_{\alpha^{+}}I^{\zeta}_{(\frac{\alpha^{\tau}+\beta^{\tau}}{2})^{1/\tau}} f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right) + {}^{ABK\tau}_{\beta^{-}}I^{\zeta}_{(\frac{\alpha^{\tau}+\beta^{\tau}}{2})^{1/\tau}} f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right) - \frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta}}{2^{\zeta}\tau^{\zeta}B(\zeta)\Gamma(\zeta)}\Big[f(\alpha^{\tau})+f(\beta^{\tau})\Big] \\
& -\frac{2(1-\zeta)f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)}{B(\zeta)} \\
= {} & \frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1 \Big((1-k^{\tau})^{\zeta+1}-1\Big)k^{\tau-1}f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)dk \\
& +\int_0^1 \Big((k^{\tau})^{\zeta+1}-1\Big)k^{\tau-1}f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)dk\Bigg].
\end{aligned}
$$

where $\zeta \in (0,1), \varkappa^{\tau} \in [\alpha^{\tau},\beta^{\tau}]$, and $\Gamma(.)$ is Gamma function.

Proof. By setting $\varkappa^{\tau} = \frac{\alpha^{\tau}+\beta^{\tau}}{2}$ in Lemma 2, we get

$$
\begin{aligned}
& {}^{ABK\tau}_{\alpha^{+}}I^{\zeta}_{(\frac{\alpha^{\tau}+\beta^{\tau}}{2})^{1/\tau}} f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right) + {}^{ABK\tau}_{\beta^{-}}I^{\zeta}_{(\frac{\alpha^{\tau}+\beta^{\tau}}{2})^{1/\tau}} f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right) - \frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta}}{2^{\zeta}\tau^{\zeta}B(\zeta)\Gamma(\zeta)}\Big[f(\alpha^{\tau})+f(\beta^{\tau})\Big] \\
& -\frac{2(1-\zeta)f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)}{B(\zeta)} \\
= {} & \frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta+1}}{2^{\zeta+1}\tau^{\zeta}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Big[f'(\alpha^{\tau})-f'(\beta^{\tau})\Big] + \frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)} \\
& \Bigg[\int_0^1 (1-k^{\tau})^{\zeta+1}k^{\tau-1}f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)dk + \int_0^1 (k^{\tau})^{\zeta+1}k^{\tau-1} \\
& f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)dk\Bigg].
\end{aligned} \tag{4.1}
$$

Then, we can write

$$
\begin{aligned}
& \Big[f'(\alpha^{\tau})-f'(\beta^{\tau})\Big] = -\int_{\alpha^{\tau}}^{\beta^{\tau}} f''(x)dx = -\Bigg[\frac{\beta^{\tau}-\alpha^{\tau}}{2}\int_0^1 \tau k^{\tau-1}f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)dk \\
& +\frac{\beta^{\tau}-\alpha^{\tau}}{2}\int_0^1 \tau k^{\tau-1}f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)dk\Bigg].
\end{aligned} \tag{4.2}
$$

Putting equality (4.2) in (4.1), we get

$$
\begin{aligned}
&{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) + {}^{ABK\tau}_{\beta^-} I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) - \frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\left[f(\alpha^\tau)+f(\beta^\tau)\right] \\
&-\frac{2(1-\zeta) f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)} \\
= \; & \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\int_0^1\left((1-k^\tau)^{\zeta+1}-1\right)k^{\tau-1}f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)dk\right. \\
&\left.+\int_0^1\left((k^\tau)^{\zeta+1}-1\right)k^{\tau-1}f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)dk\right].
\end{aligned}
$$

□

Remark 14. *For $\tau \to 1$ in Lemma 3 , we will get Lemma 3.1 in [29].*

Theorem 8. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^\tau, \beta^\tau] \to \mathbb{R}$ be a twice differentiable mapping on $(\alpha^\tau, \beta^\tau)$ with $0 \le \alpha < \beta$ and $f'' \in X_c^\tau(\alpha^\tau, \beta^\tau)$. If $|f''|$ is a convex on $[\alpha^\tau, \beta^\tau]$, then the following inequality for ABK-fractional integral operators*

$$
\begin{aligned}
&\left| {}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) + {}^{ABK\tau}_{\beta^-} I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) - \frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\left[f(\alpha^\tau)+f(\beta^\tau)\right]\right. \\
&\left.-\frac{2(1-\zeta) f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\right| \\
\le \; & \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\left(|f''(\alpha^\tau)|+|f''(\beta^\tau)|\right)\left(\frac{1}{\tau(\zeta+3)}-\frac{1}{2\tau}\right)\right. \\
&\left.+2\left|f''\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|\left(\frac{1}{\tau(\zeta+2)(\zeta+3)}-\frac{1}{2\tau}\right)\right]
\end{aligned}
$$

where $\varkappa^\tau \in [\alpha^\tau, \beta^\tau]$, $\zeta \in (0,1)$.

Proof. Employing Lemma 3 gives

$$
\begin{aligned}
&\left| {}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) + {}^{ABK\tau}_{\beta^-} I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) - \frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\left[f(\alpha^\tau)+f(\beta^\tau)\right]\right. \\
&\left.-\frac{2(1-\zeta) f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\right| \\
= \; & \left|\frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\int_0^1\left((1-k^\tau)^{\zeta+1}-1\right)k^{\tau-1}f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)dk\right.\right. \\
&\left.\left.+\int_0^1\left((k^\tau)^{\zeta+1}-1\right)k^{\tau-1}f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)dk\right]\right| \\
\le \; & \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\int_0^1\left((1-k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk\right.
\end{aligned}
$$

$$+\int_0^1\left((k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk\Bigg].$$

By using convexity of $|f''|$, we get

$$\Bigg|\,{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$

$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1\left((1-k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left(k^\tau\left|f''\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|+(1-k^\tau)\left|f''(\alpha^\tau)\right|\right)dk$$

$$+\int_0^1\left((k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left(k^\tau\left|f''(\beta^\tau)\right|+(1-k^\tau)\left|f''\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|\right)dk\Bigg]$$

$$= \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(|f''(\alpha^\tau)|+|f''(\beta^\tau)|\right)\left(\frac{1}{\tau(\zeta+3)}-\frac{1}{2\tau}\right)$$

$$+2\left|f''\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|\left(\frac{1}{\tau(\zeta+2)(\zeta+3)}-\frac{1}{2\tau}\right)\Bigg]$$

and the proof is completed. □

Remark 15. *For $\tau\to 1$ in Theorem 8 , we will get Theorem 3.1 in [29].*

Theorem 9. *Let $\zeta\in(0,1)$ and $\tau>0$ and $f:[\alpha^\tau,\beta^\tau]\to\mathbb{R}$ be a twice differentiable mapping on (α^τ,β^τ) with $0\leq\alpha<\beta$ and $f''\in X_c^\tau(\alpha^\tau,\beta^\tau)$. If $|f''|^s$ is a convex on $[\alpha^\tau,\beta^\tau]$, then the following inequality for ABK-fractional integral operators*

$$\Bigg|\,{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$

$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left(\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}-1\right)$$

$$\times\Bigg[\left(\frac{\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s+|f''(\alpha^\tau)|^s}{2\tau}\right)^{\frac{1}{s}}+\left(\frac{|f''(\beta^\tau)|^s+\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s}{2\tau}\right)^{\frac{1}{s}}\Bigg]$$

where $r^{-1}+s^{-1}=1$, $\varkappa^\tau\in[\alpha^\tau,\beta^\tau]$, $\zeta\in(0,1)$, $s>1$.

Proof. Utilizing Lemma 3, we have

$$\Bigg|\,{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$

$$-\frac{2(1-\zeta)f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1\left((1-k^{\tau})^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)\right|dk$$

$$+\int_0^1\left((k^{\tau})^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)\right|dk\Bigg]$$

$$= \frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1(1-k^{\tau})^{\zeta+1}k^{\tau-1}\left|f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)\right|dk$$

$$-\int_0^1 k^{\tau-1}\left|f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)\right|dk$$

$$+\int_0^1(k^{\tau})^{\zeta+1}k^{\tau-1}\left|f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)\right|dk-\int_0^1 k^{\tau-1}\left|f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)\right|dk\Bigg].$$

By applying Hölder inequality, we get

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^{\tau}+\beta^{\tau}}{2})^{1/\tau}}f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^{\tau}+\beta^{\tau}}{2})^{1/\tau}}f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)-\frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta}}{2^{\zeta}\tau^{\zeta}B(\zeta)\Gamma(\zeta)}\left[f(\alpha^{\tau})+f(\beta^{\tau})\right]$$

$$-\frac{2(1-\zeta)f\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\beta^{\tau}-\alpha^{\tau})^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\int_0^1(1-k^{\tau})^{(\zeta+1)r}k^{\tau-1}dk\right)^{\frac{1}{r}}$$

$$\left(\int_0^1 k^{\tau-1}\left|f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)\right|^s dk\right)^{\frac{1}{s}}$$

$$-\left(\int_0^1(k^0)^r dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)\right|^s dk\right)^{\frac{1}{s}}$$

$$+\left(\int_0^1(k^{\tau})^{(\zeta+1)r}k^{\tau-1}dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)\right|^s dk\right)^{\frac{1}{s}}$$

$$-\left(\int_0^1(k^0)^r dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)\right|^s dk\right)^{\frac{1}{s}}\Bigg].$$

By using convexity of $|f''|^s$, we obtain

$$\int_0^1 k^{\tau-1}\left|f''\left(k^{\tau}\frac{\alpha^{\tau}+\beta^{\tau}}{2}+(1-k^{\tau})\alpha^{\tau}\right)\right|^s dk \leq \int_0^1 k^{\tau-1}\left[k^{\tau}\left|f''\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)\right|^s+(1-k^{\tau})\,|f''(\alpha^{\tau})|^s\right]dk,$$

$$\int_0^1 k^{\tau-1}\left|f''\left(k^{\tau}\beta^{\tau}+(1-k^{\tau})\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)\right|^s dk \leq \int_0^1 k^{\tau-1}\left[k^{\tau}\,|f''(\beta^{\tau})|^q+(1-k^{\tau})\left|f''\left(\frac{\alpha^{\tau}+\beta^{\tau}}{2}\right)\right|^s\right]dk.$$

By computation the integrals that is in the above inequalities, we get desired result. □

Remark 16. *For $\tau \to 1$ in Theorem 9 , we will get Theorem 3.2 in [29].*

Theorem 10. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^\tau, \beta^\tau] \to \mathbb{R}$ be a twice differentiable mapping on $(\alpha^\tau, \beta^\tau)$ with $0 \le \alpha < \beta$ and $f'' \in X_c^\tau(\alpha^\tau, \beta^\tau)$. If $|f''|^s$ is a convex on $[\alpha^\tau, \beta^\tau]$, then the following inequality for ABK-fractional integral operators*

$$\begin{aligned}
&\Bigg| {}^{ABK\tau}_{\alpha^+} I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) + {}^{ABK\tau}_{\beta^-} I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) - \frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big] \\
&-\frac{2(1-\zeta) f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg| \\
\le\ & \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\left[\frac{2}{\tau r(\zeta r + r + 1)} - \frac{2}{\tau r}\right]
\end{aligned}$$

where $r^{-1} + s^{-1} = 1$, $\varkappa^\tau \in [\alpha^\tau, \beta^\tau]$, $\zeta \in (0,1)$, $s > 1$, $B(\zeta)$ is normalization function.

Proof. By using Lemma 3, we get

$$\begin{aligned}
&\Bigg| {}^{ABK\tau}_{\alpha^+} I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) + {}^{ABK\tau}_{\beta^-} I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) - \frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big] \\
&-\frac{2(1-\zeta) f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg| \\
\le\ & \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1 \left((1-k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk \\
&+\int_0^1 \left((k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk\Bigg] \\
=\ & \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk \\
&-\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk \\
&+\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk - \int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk\Bigg].
\end{aligned}$$

By using the Young inequality as $xy \le \frac{1}{r}x^r + \frac{1}{s}y^s$

$$\begin{aligned}
&\Bigg| {}^{ABK\tau}_{\alpha^+} I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) + {}^{ABK\tau}_{\beta^-} I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}} f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right) - \frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big] \\
&-\frac{2(1-\zeta) f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg| \\
\le\ & \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\frac{1}{r}\int_0^1 (1-k^\tau)^{(\zeta+1)r}k^{\tau-1}dk
\end{aligned}$$

$$+\frac{1}{s}\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|^s dk$$
$$-\frac{1}{r}\int_0^1 (k^0)^r dk-\frac{1}{s}\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|^s dk$$
$$+\frac{1}{r}\int_0^1 (k^\tau)^{(\zeta+1)r}k^{\tau-1}dk+\frac{1}{s}\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|^s dk$$
$$-\frac{1}{r}\int_0^1 (k^0)^r dk-\frac{1}{s}\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|^s dk\Bigg].$$

By using convexity of $|f''|^s$ and by a simple computation, we have the desired result. □

Remark 17. *For $\tau\to 1$ in Theorem 10 , we will get Theorem 3.3 in [29].*

Theorem 11. *Let $\zeta\in(0,1)$ and $\tau>0$ and $f:[\alpha^\tau,\beta^\tau]\to\mathbb{R}$ be a twice differentiable mapping on (α^τ,β^τ) with $0\le\alpha<\beta$ and $f''\in X_c^\tau(\alpha^\tau,\beta^\tau)$. If $|f''|^s$ is a convex on $[\alpha^\tau,\beta^\tau]$, then the following inequality for ABK-fractional integral operators*

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$
$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$
$$\le\frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\frac{1}{\tau(\zeta+2)}\right)^{1-\frac{1}{s}}\left(\frac{\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s}{\tau(\zeta+2)(\zeta+3)}+\frac{|f''(\alpha^\tau)|^s}{\tau(\zeta+3)}\right)^{\frac{1}{s}}$$
$$+\left(\frac{1}{\tau(\zeta+2)}\right)^{1-\frac{1}{s}}\left(\frac{|f''(\beta^\tau)|^s}{\tau(\zeta+3)}+\frac{\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s}{\tau(\zeta+2)(\zeta+3)}\right)^{\frac{1}{s}}-\left(\frac{\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s+|f''(\alpha^\tau)|^s}{2\tau}\right)^{\frac{1}{s}}$$
$$-\left(\frac{|f''(\beta^\tau)|^s+\left|f''(\frac{\alpha^\tau+\beta^\tau}{2})\right|^s}{2\tau}\right)^{\frac{1}{s}}\Bigg]$$

where $\varkappa^\tau\in[\alpha^\tau,\beta^\tau]$, $\zeta\in(0,1)$, $s\ge 1$, $B(\zeta)$ is normalization function.

Proof. By using Lemma 3, we get

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$
$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$
$$\le\frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1\left((1-k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$+\int_0^1 \left((k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk\Big]$$

$$= \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Big[\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$-\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$+\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk-\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk\Big].$$

By applying power mean inequality, we get

$$\Big|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\left[f(\alpha^\tau)+f(\beta^\tau)\right]$$

$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Big|$$

$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}$$

$$\Big[\left(\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}dk\right)^{1-\frac{1}{s}}\left(\int_0^1 (1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|^s dk\right)^{\frac{1}{s}}$$

$$-\left(\int_0^1 k^0 dk\right)^{1-\frac{1}{s}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|^s dk\right)^{\frac{1}{s}}$$

$$+\left(\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}dk\right)^{1-\frac{1}{s}}\left(\int_0^1 (k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|^s dk\right)^{\frac{1}{s}}$$

$$-\left(\int_0^1 k^0 dk\right)^{1-\frac{1}{s}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|^s dk\right)^{\frac{1}{s}}\Big].$$

By using convexity of $|f''|^s$ and after further simplifications, we have the desired result. □

Remark 18. *For $\tau \to 1$ in Theorem 11 , we will get Theorem 3.4 in [29].*

Theorem 12. *Let $\zeta \in (0,1)$ and $\tau > 0$ and $f : [\alpha^\tau,\beta^\tau] \to \mathbb{R}$ be a twice differentiable mapping on (α^τ,β^τ) with $0 \leq \alpha < \beta$ and $f'' \in X^\tau_c(\alpha^\tau,\beta^\tau)$. If $|f''|^s$ is a concave for $s > 1$, then we have*

$$\Big|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\left[f(\alpha^\tau)+f(\beta^\tau)\right]$$

$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Big|$$

$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\frac{1}{\tau(\zeta+2)}\right)\left|f''\left(\frac{\alpha^\tau+\beta^\tau}{2(\zeta+3)}+\frac{\zeta+2}{\zeta+3}\alpha^\tau\right)\right|$$

$$+\left(\frac{1}{\tau(\zeta+2)}\right)\left|f''\left(\frac{\zeta+2}{\zeta+3}\beta^\tau+\frac{\alpha^\tau+\beta^\tau}{2(\zeta+3)}\right)\right|-\left|f''\left(\frac{3\alpha^\tau+\beta^\tau}{4\tau}\right)\right|-\left|f''\left(\frac{3\beta^\tau+\alpha^\tau}{4\tau}\right)\right|\Bigg]$$

where $\varkappa^\tau \in [\alpha^\tau, \beta^\tau]$, $\zeta \in (0,1)$.

Proof. Using Lemma 3 and the Jensen integral inequality, we get

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$

$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1\left((1-k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$+\int_0^1\left((k^\tau)^{\zeta+1}-1\right)k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk\Bigg]$$

$$= \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$-\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|dk$$

$$+\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk-\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|dk\Bigg]$$

$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}dk\right)$$

$$\left|f''\left(\frac{\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)dk}{\int_0^1(1-k^\tau)^{\zeta+1}k^{\tau-1}dk}\right)\right|$$

$$-\left(\int_0^1 k^0dk\right)\left|f''\left(\frac{\int_0^1 k^0k^{\tau-1}\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)dk}{\int_0^1 k^0dk}\right)\right|$$

$$+\left(\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}dk\right)\left|f''\left(\frac{\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)dk}{\int_0^1(k^\tau)^{\zeta+1}k^{\tau-1}dk}\right)\right|$$

$$-\left(\int_0^1 k^0dk\right)\left|f''\left(\frac{\int_0^1 k^0k^{\tau-1}\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)dk}{\int_0^1 k^0dk}\right)\right|\Bigg].$$

By computing the above integrals we have

$$\Bigg|\,{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$
$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$
$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\frac{1}{\tau(\zeta+2)}\right)\left|f''\left(\frac{\alpha^\tau+\beta^\tau}{2(\zeta+3)}+\frac{\zeta+2}{\zeta+3}\alpha^\tau\right)\right|$$
$$+\left(\frac{1}{\tau(\zeta+2)}\right)\left|f''\left(\frac{\zeta+2}{\zeta+3}\beta^\tau+\frac{\alpha^\tau+\beta^\tau}{2(\zeta+3)}\right)\right|-\left|f''\left(\frac{3\alpha^\tau+\beta^\tau}{4\tau}\right)\right|-\left|f''\left(\frac{3\beta^\tau+\alpha^\tau}{4\tau}\right)\right|\Bigg].$$

□

Remark 19. *For $\tau\to 1$ in Theorem 12 , we will get Theorem 3.5 in [29].*

Theorem 13. *Let $\zeta\in(0,1)$ and $\tau>0$ and $f:[\alpha^\tau,\beta^\tau]\to\mathbb{R}$ be a twice differentiable mapping on (α^τ,β^τ) with $0\leq\alpha<\beta$ and $f''\in X^\tau_c(\alpha^\tau,\beta^\tau)$. If $|f''|^s$ is a concave mapping, then*

$$\Bigg|\,{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$
$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$
$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left|f''\left(\frac{3\alpha^\tau+\beta^\tau}{4\tau}\right)\right|-\left|f''\left(\frac{3\alpha^\tau+\beta^\tau}{4\tau}\right)\right|$$
$$+\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left|f''\left(\frac{3\beta^\tau+\alpha^\tau}{4\tau}\right)\right|-\left|f''\left(\frac{3\beta^\tau+\alpha^\tau}{4\tau}\right)\right|\Bigg].$$

where $r^{-1}+s^{-1}=1$, $\varkappa^\tau\in[\alpha^\tau,\beta^\tau]$, $\zeta\in(0,1)$, $s>1$.

Proof. From Lemma 3 and Hölder integral inequality, we get

$$\Bigg|\,{}^{ABK\tau}_{\alpha^+}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^{\zeta}_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$
$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$
$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\int_0^1(1-k^\tau)^{(\zeta+1)r}k^{\tau-1}dk\right)^{\frac{1}{r}}$$
$$\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|^s dk\right)^{\frac{1}{s}}$$
$$-\left(\int_0^1(k^0)^r dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|^s dk\right)^{\frac{1}{s}}$$

$$+\left(\int_0^1 (k^\tau)^{(\zeta+1)r}k^{\tau-1}dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|^s dk\right)^{\frac{1}{s}}$$

$$-\left(\int_0^1 (k^0)^r dk\right)^{\frac{1}{r}}\left(\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\frac{\alpha^\tau+\beta^\tau}{2}\right)\right|^s dk\right)^{\frac{1}{s}}\Bigg].$$

By using concavity of $|f''|^s$ and Jensen integral inequality, we get

$$\begin{aligned}\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|^s dk &= \int_0^1 k^0k^{\tau-1}\left|f''\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)\right|^s dk\\ &\leq \left(\int_0^1 k^0dk\right)\left|f''\left(\frac{\int_0^1 k^{\tau-1}\left(k^\tau\frac{\alpha^\tau+\beta^\tau}{2}+(1-k^\tau)\alpha^\tau\right)dk}{\int_0^1 k^0dk}\right)\right|^s\\ &= \left|f''\left(\frac{3\alpha^\tau+\beta^\tau}{4\tau}\right)\right|^s.\end{aligned}$$

Similarly

$$\int_0^1 k^{\tau-1}\left|f''\left(k^\tau\beta^\tau+(1-k^\tau)\varkappa^\tau\right)\right|^s dk \leq \left|f''\left(\frac{3\beta^\tau+\alpha^\tau}{4\tau}\right)\right|^s.$$

So, we obtain

$$\Bigg|{}^{ABK\tau}_{\alpha^+}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)+{}^{ABK\tau}_{\beta^-}I^\zeta_{(\frac{\alpha^\tau+\beta^\tau}{2})^{1/\tau}}f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)-\frac{(\beta^\tau-\alpha^\tau)^\zeta}{2^\zeta\tau^\zeta B(\zeta)\Gamma(\zeta)}\big[f(\alpha^\tau)+f(\beta^\tau)\big]$$

$$-\frac{2(1-\zeta)f\left(\frac{\alpha^\tau+\beta^\tau}{2}\right)}{B(\zeta)}\Bigg|$$

$$\leq \frac{(\beta^\tau-\alpha^\tau)^{\zeta+2}}{2^{\zeta+2}\tau^{\zeta-1}(\zeta+1)B(\zeta)\Gamma(\zeta)}\Bigg[\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left|f''\left(\frac{3\alpha^\tau+\beta^\tau}{4\tau}\right)\right|-\left|f''\left(\frac{3\alpha^\tau+\beta^\tau}{4\tau}\right)\right|$$

$$+\left(\frac{1}{\tau(\zeta r+r+1)}\right)^{\frac{1}{r}}\left|f''\left(\frac{3\beta^\tau+\alpha^\tau}{4\tau}\right)\right|-\left|f''\left(\frac{3\beta^\tau+\alpha^\tau}{4\tau}\right)\right|\Bigg].$$

□

Remark 20. *For $\tau \to 1$ in Theorem 13 , we will get Theorem 3.6 in [29].*

5. Conclusions

In this paper, we formulate two new integral identities with the second-order derivatives involving ABK-fractional integrals. We obtain generalized fractional Mid-point and Trapezoid type inequalities via ABK-fractional integrals with the help of these identities. Some related bounds are presented by using Hölder's, Young, power-mean, and Jensen inequalities. Several applications of main findings can be provided for the special means of real numbers by choosing $\alpha = 1$ and $\rho = 1$.

Recently, researchers working in the field of fractional analysis have defined some integral operators that include new and general forms. The usefulness and effectiveness of these new operators in the solutions of the real world problems have been tried to be demonstrated with the help of various applications and simulations. The interested readers can find several illustrative simulations related to the $ABK-$fractional integral operators in [39]. Also, in that study, the authors have established some new integral inequalities via $ABK-$ fractional integral operators that have been provided general forms of the earlier studies.

Our main findings will be motivated to obtain new extensions by using different kinds of convex functions for the interested researchers.

Acknowledgements

We would like to express our gratitude to the editor and reviewers who helped to improve this article and reach its final form. Also, we would like to thank to Prof. Dr. Ahmet Ocak Akdemir for his contributions in the revision process of this article. The authors T. Abdeljawad and W. Shatanawi would like to thank Prince Sultan University for funding this work through the TAS Lab.

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