Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 10070–10091.
DOI:10.3934/math.2021585
Received: 24 February 2021
Accepted: 05 July 2021
Published: 07 July 2021

Research article

Pseudo almost periodic solutions for quaternion-valued high-order Hopfield neural networks with time-varying delays and leakage delays on time scales

Xiaofang Meng and Yongkun Li*

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China

* **Correspondence:** Email: yklie@ynu.edu.cn.

Abstract: This paper deals with a class of quaternion-valued high-order Hopfield neural networks with time-varying delays and leakage delays on time scales. Based on the Banach fixed point theorem and the theory of calculus on time scales, some sufficient conditions are obtained for the existence and global exponential stability of pseudo almost periodic solutions for the considered networks. The results of this paper are completely new. Finally, an example is presented to illustrate the effectiveness of the obtained results.

Keywords: Hopfield neural networks; quaternion; pseudo almost periodic solutions; leakage delays; time scales
Mathematics Subject Classification: 34K14, 34K20, 34N05, 92B20

---

1. Introduction

Because high-order Hopfield neural networks have more extensive applications than Hopfield neural networks, various dynamical behaviours of high-order Hopfield neural networks such as the existence and stability of equilibrium points [1, 2], anti-periodic solutions [3], almost periodic solutions [4–6] and pseudo almost periodic solutions [7] have been studied by many scholars.

On the one hand, due to the limited switching speed of neurons and amplifiers, time delays are inevitably introduced into neural network models [8–12]. Among all kinds of time delays, the leakage delay, that is, the time delay in the leakage term, has been proved to have a great influence on the dynamics of the system. Therefore, it is significant to consider neural networks with time delays in leakage terms [13–16].

On the other hand, both continuous-time and discrete-time neural networks have equally importance in various applications. Therefore, it is necessary to consider both continuous time neural networks and discrete time neural networks. Fortunately, the theory of time scale calculus [17] can unify the study of continuous analysis and discrete analysis, so the study of neural network models on time scale can unify the study of continuous-time and discrete-time neural networks [18–21].

In addition, quaternion-valued neural networks, as an extension of real-valued neural networks and complex-valued neural networks, have been extensively applied in many fields such as robotics, satellite attitude control, computer graphics, ensemble control and so on [22–24]. Currently, the study quaternion-valued neural networks have received much attention of many scholars.

Moreover, although non-autonomous neural networks are more general and practical than autonomous neural networks, so far, there are still few results about the dynamic behavior of non-autonomous quaternion-valued neural networks [25–29]. It is well known that periodicity, almost periodicity and pseudo almost periodicity are very important dynamic behaviors of non-autonomous systems. Besides, almost periodicity is more reasonable than periodicity. Also, pseudo almost periodicity is more complex than almost periodicity [30, 31]. Therefore, for non-autonomous neural networks, pseudo almost periodic oscillation is a very important dynamics [32–37].

However, up to now, there has been no paper published on the pseudo almost periodic oscillation of quaternion-valued high-order Hopfield neural networks. Besides, the pseudo almost periodic oscillation of quaternion-valued neural networks with quaternion leakage coefficients on time scales has not been reported. Consequently, it is necessary to study the pseudo almost periodic solutions of quaternion-valued high-order Hopfield neural networks on time scales whose leakage coefficients are also quaternions.

Motivated by the above statement, in this paper, we consider the following quaternion-valued high-order Hopfield neural network with time-varying delays and leakage delays on time scales:

$$\begin{aligned} x_p^{\Delta}(t) &= -a_p(t)x_p(t-\eta_p(t)) + \sum_{q=1}^{n} b_{pq}(t)f_q(x_q(t)) + \sum_{q=1}^{n} c_{pq}(t)g_q(x_q(t-\tau_{pq}(t))) \\ &\quad + \sum_{q=1}^{n}\sum_{l=1}^{n} T_{pql}(t)h_q(x_q(t-\delta_{pql}(t)))h_l(x_l(t-\vartheta_{pql}(t))) + u_p(t), \quad t \geq t_0, \quad t \in \mathbb{T}, \end{aligned} \tag{1.1}$$

where $p \in \{1, 2, \ldots, n\} =: \mathcal{S}$, n is the number of neurons in layers; $x_p(t)$ denotes the activation of the pth neuron at time t; $a_p(t) \in \mathbb{Q}$ represents the rate with the pth unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t; $b_{pq}(t), c_{pq}(t) \in \mathbb{Q}$ are the delay connection weights from neuron q to neuron p at time t, respectively; $T_{pql}(t) \in \mathbb{Q}$ denotes the second-order connection weights of the neural network; $f_q, g_q, h_q : \mathbb{Q} \to \mathbb{Q}$ are the activation functions of signal transmission; $u_p(t) \in \mathbb{Q}$ is the external input on the pth unit at time t; η_p denotes the leakage delay at time t and satisfies $t - \eta_p(t) \in \mathbb{T}$; τ_{pq}, δ_{pql} and ϑ_{pql} are transmission delays at time t and satisfy $t - \tau_{pq}(t) \in \mathbb{T}$, $t - \delta_{pql}(t) \in \mathbb{T}$ and $t - \vartheta_{pql}(t) \in \mathbb{T}$ for $t \in \mathbb{T}$.

The skew field of quaternions is denoted by

$$\mathbb{Q} := \{q = q^R + iq^I + jq^J + kq^K\},$$

where q^R, q^I, q^J, q^K are real numbers, the three imaginary units i, j and k obey the Hamilton's multiplication rules:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1.$$

Throughout this paper, for $x = x^R + ix^I + jx^J + kx^K \in \mathbb{Q}$, we denote $\hat{x} = x^R - x$, $\|x\|_{\mathbb{Q}} = \max\{|x^R|, |x^I|, |x^J|, |x^K|\}$, and for $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{Q}^n$, we denote $\|x\|_{\mathbb{Q}^n} = \max_{p\in\mathcal{S}}\{\|x_p\|_{\mathbb{Q}}\}$.

Also, for convenience, we introduce the following notation:

$$a_p^- = \inf_{t\in\mathbb{T}} \{a_p^R(t)\}, \quad a_p^+ = \sup_{t\in\mathbb{T}} \{a_p^R(t)\}, \quad \hat{a}_p^+ = \sup_{t\in\mathbb{T}} \left\|\hat{a}_p(t)\right\|_{\mathbb{Q}},$$

$$b_{pq}^+ = \sup_{t\in\mathbb{T}} \|b_{pq}(t)\|_{\mathbb{Q}}, \quad c_{pq}^+ = \sup_{t\in\mathbb{T}} \|c_{pq}(t)\|_{\mathbb{Q}}, \quad T_{pql}^+ = \sup_{t\in\mathbb{T}} \|T_{pql}(t)\|_{\mathbb{Q}},$$

$$\eta^+ = \max_{p\in\mathcal{S}} \{ \sup_{t\in\mathbb{T}} \eta_p(t)\}, \quad \tau^+ = \max_{p,q\in\mathcal{S}} \{ \sup_{t\in\mathbb{T}} \tau_{pq}(t)\}, \quad \delta^+ = \max_{p,q,l\in\mathcal{S}} \{ \sup_{t\in\mathbb{T}} \delta_{pql}(t)\},$$

$$\vartheta^+ = \max_{p,q,l\in\mathcal{S}} \{ \sup_{t\in\mathbb{T}} \vartheta_{pql}(t)\}, \quad \theta = \max \{\eta^+, \tau^+, \delta^+, \vartheta^+\}.$$

The initial condition of system (1.1) is of the form

$$x_p(s) = \phi_p(s), \quad x_p^{\Delta}(s) = \psi_p(s), \quad s \in [t_0 - \theta, t_0]_{\mathbb{T}},$$

where $\phi_p, \psi_p^{\Delta} \in C([t_0 - \theta, t_0]_{\mathbb{T}}, \mathbb{Q})$, $p \in \mathcal{S}$.

Our main aim of this paper is to study the existence and stability of pseudo almost periodic solutions of (1.1). The main contributions of this paper are listed as follows. Firstly, this is the first time to consider quaternion-valued neural networks on time scales with all the coefficients are quaternions except time delays. Secondly, this is the first paper to study the pseudo almost periodic solutions for quaternion-valued high-order Hopfield neural networks with time-varying delays and leakage delays on time scales. Finally, our method of this paper can be used to study pseudo almost periodic solutions for other types of quaternion-valued neural networks on time scales.

This paper is organized as follows: In Section 2, we introduce some definitions, preliminary lemmas. In Section 3, we establish some sufficient conditions for the existence and global exponential stability of pseodo almost periodic solutions of system (1.1). In Section 4, we give an example to demonstrate the feasibility of our results. This paper ends with a brief conclusion in Section 5.

2. Preliminaries

Definition 2.1. *[38, 39] A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the forward graininess $\mu : \mathbb{T} \to [0, \infty)$ are defined, respectively, by*

$$\sigma(t) = \inf \{s \in \mathbb{T}, s > t\}, \quad \rho(t) = \sup \{s \in \mathbb{T}, s < t\}, \quad \mu(t) = \sigma(t) - t.$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense or right-scattered if $\rho(t) = t$, $\rho(t) < t, \sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m, define $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered maximum m, define $\mathbb{T}_{\kappa} = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_{\kappa} = \mathbb{T}$.

Definition 2.2. *[38, 39] Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t. Moreover, we say that f is delta (or Hilger) differentiable (or in short: differentiable) on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^k$.

The derivative of function $f(t) = f^R(t) + if^I(t) + jf^J(t) + kf^K(t) : \mathbb{T} \to \mathbb{Q}$ is given by

$$f^\Delta(t) = (f^R)^\Delta(t) + i(f^I)^\Delta(t) + j(f^J)^\Delta(t) + k(f^K)^\Delta(t),$$

where $f^R, f^I, f^J, f^K : \mathbb{T} \to \mathbb{R}$.

Definition 2.3. *[38, 39] A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided*

$$1 + \mu(t)p(t) \neq 0, \quad \forall\, t \in \mathbb{T}^\kappa.$$

The set of all positive regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ are denoted by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \quad \forall\, t \in \mathbb{T}\}.$$

Definition 2.4. *[38, 39] If $p \in \mathcal{R}^+$, then we define the exponential function by*

$$e_p(t, s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big), \quad \forall\, t, s \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Definition 2.5. *[38, 39] Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define*

$$p \oplus q = p + q + \mu pq, \quad \ominus p = -\frac{p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q).$$

Lemma 2.1. *[38, 39] Let $p \in \mathcal{R}$, and $t, s, r \in \mathbb{T}$. Then*

(*i*) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(*ii*) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(*iii*) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(*iv*) $(e_{\ominus p}(t, s))^\Delta = (\ominus p)(t)e_{\ominus p}(t, s)$.

Definition 2.6. *[18] A time scale $\mathbb{T}$ is called an almost periodic time scale if*

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall\, t \in \mathbb{T}\} \neq \{0\}.$$

We denote by $BC(\mathbb{T}, \mathbb{Q}^n)$ the set of all bounded continuous functions from $\mathbb{T}$ to $\mathbb{Q}^n$. Similar to Definition in [18], we give the following definition.

Definition 2.7. *Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in BC(\mathbb{T}, \mathbb{Q}^n)$ is called an almost periodic on $\mathbb{T}$ if the ε-translation set of*

$$T(\varepsilon, f) = \{\tau \in \Pi : \big\|f(t + \tau) - f(t)\big\|_0 < \varepsilon, \ \forall\, t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{R}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains at least one $\tau(\varepsilon) \in T(\varepsilon, f)$ such that

$$\big\|f(t + \tau) - f(t)\big\|_0 < \varepsilon, \ \forall\, t \in \mathbb{T}.$$

We denote by $AP(\mathbb{T},\mathbb{Q}^n)$ the set of all almost periodic functions defined on $\mathbb{T}$.

Define the class of functions $PAP_0(\mathbb{T},\mathbb{Q}^n)$ as follows:

$$PAP_0(\mathbb{T},\mathbb{Q}^n) = \Big\{f\in BC(\mathbb{T},\mathbb{Q}^n): f \ is\ \Delta-\text{measurable such that}\ \lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^{r}\big\|f(t)\big\|_0\Delta t=0,\ \text{where}\ r\in\mathbb{T}\Big\}.$$

Similar to Definition in [35], we give the following definition.

Definition 2.8. *A function $f \in BC(\mathbb{T},\mathbb{Q}^n)$ is called pseudo almost periodic if $f = g + h$, where $g \in AP(\mathbb{T},\mathbb{Q}^n)$ and $h \in PAP_0(\mathbb{T},\mathbb{Q}^n)$.*

We denote by $PAP(\mathbb{T},\mathbb{Q}^n)$ the set of all pseudo almost periodic functions from $\mathbb{T}$ to $\mathbb{Q}^n$.

Similar to the proofs in [21], it is not difficult to prove the following lemmas.

Lemma 2.2. *If $f, g \in PAP(\mathbb{T},\mathbb{Q}^n)$, then $f + g, fg \in PAP(\mathbb{T},\mathbb{Q}^n)$; if $f \in PAP(\mathbb{T},\mathbb{Q}^n)$, $g \in AP(\mathbb{T},\mathbb{Q}^n)$, then $fg \in PAP(\mathbb{T},\mathbb{Q}^n)$.*

Lemma 2.3. *If $f \in C(\mathbb{Q},\mathbb{Q})$ satisfies the Lipschitcz condition, $\varphi \in PAP(\mathbb{T},\mathbb{Q})$ and $\tau \in C^1(\mathbb{T},\Pi)\cap AP(\mathbb{T},\mathbb{R}^+)$ with $\inf_{t\in\mathbb{T}}\{1-\tau^\Delta(t)\}>0$, then $f(\varphi(\cdot-\tau(\cdot)))\in PAP(\mathbb{T},\mathbb{Q})$.*

Throughout this paper, we assume that the following conditions hold:

(H_1) $a_p^R\in AP(\mathbb{T},\mathbb{R}^+)$ with $-a_p^R\in\mathcal{R}^+$, $a_p\in AP(\mathbb{T},\mathbb{Q})$, $b_{pq}, c_{pq}, T_{pql}, u_p\in PAP(\mathbb{T},\mathbb{Q})$, $\eta_p,\tau_{pq},\delta_{pql},\vartheta_{pql}\in C^1(\mathbb{T},\Pi)\cap AP(\mathbb{T},\mathbb{R}^+)$ with $\inf_{t\in\mathbb{T}}\{1-\eta_p^\Delta(t)\}>0$, $\inf_{t\in\mathbb{T}}\{1-\tau_{pq}^\Delta(t)\}>0$, $\inf_{t\in\mathbb{T}}\{1-\delta_{pql}^\Delta(t)\}>0$, $\inf_{t\in\mathbb{T}}\{1-\vartheta_{pql}^\Delta(t)\}>0$, where $p,q,l\in\mathcal{S}$.

(H_2) There exist positive constants $L_q^f, L_q^g, L_q^h, M_q^h$ such that for any $x,\ y\in\mathbb{Q}$,

$$\big\|f_q(x)-f_q(y)\big\|_{\mathbb{Q}}\le L_q^f\big\|x-y\big\|_{\mathbb{Q}},\quad \big\|g_q(x)-g_q(y)\big\|_{\mathbb{Q}}\le L_q^g\big\|x-y\big\|_{\mathbb{Q}},$$

$$\big\|h_q(x)-h_q(y)\big\|_{\mathbb{Q}}\le L_q^h\big\|x-y\big\|_{\mathbb{Q}},\quad \big\|h_q(x)\big\|_{\mathbb{Q}}\le M_q^h$$

and $f_q(0)=g_q(0)=h_q(0)=0$, where $q\in\mathcal{S}$.

(H_3) $\max_{p\in\mathcal{S}}\Big\{\frac{\Xi_p}{a_p^-},\Big(1+\frac{a_p^+}{a_p^-}\Big)\Xi_p\Big\}=:\rho<1$, where

$$\Xi_p=a_p^+\eta_p^+ +\hat{a}_p^+ +\sum_{q=1}^n b_{pq}^+L_q^f+\sum_{q=1}^n c_{pq}^+L_q^g+\sum_{q=1}^n\sum_{l=1}^n T_{pql}^+\Big(M_q^hL_l^h+L_q^hM_l^h\Big).$$

3. Main results

Let $\mathbb{E}=\{\phi=(\phi_1,\phi_2,\ldots,\phi_n)^T\big|\phi,\phi^\Delta\in PAP(\mathbb{T},\mathbb{Q}^n)\}$ with the norm

$$\big\|\phi\big\|_{\mathbb{E}}=\max_{p\in\mathcal{S}}\Big\{\big\|\phi\big\|_0,\ \big\|\phi^\Delta\big\|_0\Big\},$$

where $\big\|\phi\big\|_0=\sup_{t\in\mathbb{T}}\max_{p\in\mathcal{S}}\Big\{\big\|\phi_p\big\|_\infty\Big\}$ and $\big\|\phi_p\big\|_\infty=\sup_{t\in\mathbb{T}}\big\|\phi_p(t)\big\|_{\mathbb{Q}}$, then $\mathbb{E}$ is a Banach space.

Set $\phi^0 = (\phi_1^0, \phi_2^0, \dots, \phi_n^0)^T$, where

$$\phi_p^0(t) = \int_{-\infty}^{t} e_{-a_p^R}(t, \sigma(s))u_p(s)\mathrm{d}s, \quad t \in \mathbb{T}, \quad p \in \mathcal{S}$$

and κ is a constant satisfying $\kappa \geq \left\|\phi^0\right\|_{\mathbb{E}}$.

Lemma 3.1. *Let* (H_1) *hold, then every bounded solution* $x = (x_1, x_2, \dots, x_n)^T$ *of system* (1.1) *is a solution of the following system:*

$$\begin{aligned} x_p(t) \;=\; & \int_{-\infty}^{t} e_{-a_p^R}(t, \sigma(s))\Big[a_p^R(s)\int_{s-\eta_p(s)}^{s} x_p^{\Delta}(u)\Delta u + \hat{a}_p(s)x_p(s-\eta_p(s)) \\ & + \sum_{q=1}^{n} b_{pq}(s)f_q(x_q(s)) + \sum_{q=1}^{n} c_{pq}(s)g_q(x_q(s-\tau_{pq}(s))) \\ & + \sum_{q=1}^{n}\sum_{l=1}^{n} T_{pql}(s)h_q(x_q(s-\delta_{pql}(s)))h_l(x_l(s-\vartheta_{pql}(s))) + u_p(s)\Big]\Delta s, \end{aligned} \tag{3.1}$$

where $p \in \mathcal{S}$, $t \in \mathbb{T}$, *and vice versa.*

Proof. On the one hand, if $x = (x_1, x_2, \dots, x_n)^T$ is a solution of (3.1), then by Δ-differentiate both sides of (3.1), we see that $x = (x_1, x_2, \dots, x_n)^T$ is also a solution of (1.1).

On the other hand, let x be a bounded solution of (1.1), then for $p \in \mathcal{S}$,

$$\begin{aligned} x_p^{\Delta}(t) \;=\; & -a_p^R(t)x_p(t) + a_p^R(t)\int_{t-\eta_p(t)}^{t} x_p^{\Delta}(s)\Delta s + \hat{a}_p(t)x_p(t-\eta_p(t)) \\ & + \sum_{q=1}^{n} b_{pq}(t)f_q(x_q(t)) + \sum_{q=1}^{n} c_{pq}(t)g_q(x_q(t-\tau_{pq}(t))) \\ & + \sum_{q=1}^{n}\sum_{l=1}^{n} T_{pql}(t)h_q(x_q(t-\delta_{pql}(t)))h_l(x_l(t-\vartheta_{pql}(t))) + u_p(t), \quad t \in \mathbb{T}. \end{aligned} \tag{3.2}$$

Multiply both sides of (3.2) by $e_{-a_p^R}(t_0, \sigma(t))$, we can get

$$\begin{aligned} & [x_p(t)e_{-a_p^R}(t_0, t)]^{\Delta} \\ =\; & e_{-a_p^R}(t_0, \sigma(t))\Big[a_p^R(t)\int_{t-\eta_p(t)}^{t} x_p^{\Delta}(s)\Delta s + \hat{a}_p(t)x_p(t-\eta_p(t)) \\ & + \sum_{q=1}^{n} b_{pq}(t)f_q(x_q(t)) + \sum_{q=1}^{n} c_{pq}(t)g_q(x_q(t-\tau_{pq}(t))) \\ & + \sum_{q=1}^{n}\sum_{l=1}^{n} T_{pql}(t)h_q(x_q(t-\delta_{pql}(t)))h_l(x_l(t-\vartheta_{pql}(t))) + u_p(t)\Big], \end{aligned} \tag{3.3}$$

where $t \geq t_0$, $t_0 \in \mathbb{T}$. Integrating both sides of (3.3) from t_0 to t, we obtain

$$x_p(t) \;=\; e_{-a_p^R}(t, t_0)x_p(t_0) + \int_{t_0}^{t} e_{-a_p^R}(t, \sigma(s))\Big[a_p^R(s)\int_{s-\eta_p(s)}^{s} x_p^{\Delta}(u)\Delta u$$

$$+\hat{a}_p(s)x_p(s-\eta_p(s))+\sum_{q=1}^{n}b_{pq}(s)f_q(x_q(s))+\sum_{q=1}^{n}c_{pq}(s)g_q(x_q(s-\tau_{pq}(s)))$$
$$+\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}(s)h_q(x_q(s-\delta_{pql}(s)))h_l(x_l(s-\vartheta_{pql}(s)))+u_p(s)\Big]\Delta s.$$

Letting $t_0 \to -\infty$, then we obtain that (1.1) holds. The proof is complete. □

Theorem 3.1. *Let* (H_1)-(H_3) *hold. Then system* (1.1) *has a unique pseudo almost periodic solution in* $\mathbb{E}^* = \{\phi \mid \phi \in \mathbb{E}\big| \|\phi-\phi^0\|_{\mathbb{E}} \le \frac{\kappa\rho}{1-\rho}\}$.

Proof. For any $\phi \in \mathbb{E}$, we define a mapping $\Phi : \mathbb{E} \to PAP(\mathbb{T}, \mathbb{Q}^n)$ by setting

$$(\phi_1, \phi_2, \ldots, \phi_n) \to (x_1^{\phi}, x_2^{\phi}, \ldots, x_n^{\phi}),$$

where

$$\begin{aligned} x_p^{\phi}(t) &= \int_{-\infty}^{t} e_{-a_p^R}(t,\sigma(s))\Big[a_p^R(s)\int_{s-\eta_p(s)}^{s}\phi_p^{\Delta}(u)\Delta u+\hat{a}_p(s)\phi_p(s-\eta_p(s)) \\ &\quad +\sum_{q=1}^{n}b_{pq}(s)f_q(\phi_q(s))+\sum_{q=1}^{n}c_{pq}(s)g_q(\phi_q(s-\tau_{pq}(s))) \\ &\quad +\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}(s)h_q(\phi_q(s-\delta_{pql}(s)))h_l(\phi_l(s-\vartheta_{pql}(s)))+u_p(s)\Big]\Delta s, \quad p\in\mathcal{S}. \end{aligned}$$

First, we will prove that Φ maps $\mathbb{E}$ into itself. To this end, let

$$\begin{aligned} \mathcal{F}_p(s) &= a_p^R(s)\int_{s-\eta_p(s)}^{s}\phi_p^{\Delta}(u)\Delta u+\hat{a}_p(s)\phi_p(s-\eta_p(s))+\sum_{q=1}^{n}b_{pq}(s)f_q(\phi_q(s)) \\ &\quad +\sum_{q=1}^{n}c_{pq}(s)g_q(\phi_q(s-\tau_{pq}(s)))+\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}(s)h_q(\phi_q(s-\delta_{pql}(s))) \\ &\quad \times h_l(\phi_l(s-\vartheta_{pql}(s)))+u_p(s), \quad p\in\mathcal{S}. \end{aligned}$$

Then, by Lemmas 2.2 and 2.3, we find that $\mathcal{F}_p(s) \in PAP(\mathbb{T},\mathbb{Q})$. So, for all $p \in \mathcal{S}$, we can set $\mathcal{F}_p(s) = \mathcal{F}_p^1(s)+\mathcal{F}_p^0(s)$, where $\mathcal{F}_p^1 \in AP(\mathbb{T},\mathbb{Q})$ and $\mathcal{F}_p^0 \in PAP_0(\mathbb{T},\mathbb{Q})$. We shall show that $x_p^{\phi} \in PAP(\mathbb{T},\mathbb{Q})$, that is, x_p^{ϕ} can be expressed as

$$\begin{aligned} x_p^{\phi}(t) &= \int_{-\infty}^{t} e_{-a_p^R}(t,\sigma(s))\mathcal{F}_p^1(s)\Delta s+\int_{-\infty}^{t} e_{-a_p^R}(t,\sigma(s))\mathcal{F}_p^0(s)\Delta s \\ &:= \Omega_p^1(t)+\Omega_p^0(t), \quad p\in\mathcal{S}, \end{aligned}$$

where $\Omega_p^1 \in AP(\mathbb{T},\mathbb{Q})$ and $\Omega_p^0 \in PAP_0(\mathbb{T},\mathbb{Q})$.

In fact, since $a_p^R \in AP(\mathbb{T},\mathbb{R}^+)$ and $\mathcal{F}_p^1 \in AP(\mathbb{T},\mathbb{Q})$, for every $\varepsilon > 0$, there exists $l > 0$ such that every interval of length l contains a number $\tau \in \Pi$ satisfying

$$\left|a_p^R(t+\tau)-a_p^R(t)\right| < \varepsilon, \quad \left\|\mathcal{F}_p^1(t+\tau)-\mathcal{F}_p^1(t)\right\|_{\mathbb{Q}} < \varepsilon, \quad p\in\mathcal{S}, \quad t\in\mathbb{T}.$$

Consequently, we have

$$\begin{aligned}
& \left\|\mathcal{F}_p^1(t+\tau)-\mathcal{F}_p^1(t)\right\|_{\mathbb{Q}} \\
= & \left\|\int_{-\infty}^t e_{-a_p^R}(t+\tau,\sigma(s+\tau))\mathcal{F}_p^1(s+\tau)\Delta s-\int_{-\infty}^t e_{-a_p^R}(t,\sigma(s))\mathcal{F}_p^1(s)\Delta s\right\|_{\mathbb{Q}} \\
\leq & \left\|\int_{-\infty}^t e_{-a_p^R}(t+\tau,\sigma(s+\tau))\mathcal{F}_p^1(s+\tau)\Delta s-\int_{-\infty}^t e_{-a_p^R}(t+\tau,\sigma(s+\tau))\mathcal{F}_p^1(s)\Delta s\right\|_{\mathbb{Q}} \\
& +\left\|\int_{-\infty}^t e_{-a_p^R}(t+\tau,\sigma(s+\tau))\mathcal{F}_p^1(s)\Delta s-\int_{-\infty}^t e_{-a_p^R}(t,\sigma(s))\mathcal{F}_p^1(s)\Delta s\right\|_{\mathbb{Q}} \\
\leq & \int_{-\infty}^t\left|e_{-a_p^R}(t+\tau,\sigma(s+\tau))\right|\left\|\mathcal{F}_p^1(s+\tau)-\mathcal{F}_p^1(s)\right\|_{\mathbb{Q}}\Delta s \\
& +\int_{-\infty}^t\left|e_{-a_p^R}(t+\tau,\sigma(s+\tau))-e_{-a_p^R}(t,\sigma(s))\right|\left\|\mathcal{F}_p^1(s)\right\|_{\mathbb{Q}}\Delta s \\
< & \frac{\varepsilon}{a_p^-}+\frac{\left\|\mathcal{F}_p^1\right\|_{\infty}}{(a_p^-)^2}\varepsilon, \quad p\in\mathcal{S},
\end{aligned}$$

which implies that $\Omega_p^1 \in AP(\mathbb{T},\mathbb{Q})$. Then, we will prove that $\Omega_p^0 \in PAP_0(\mathbb{T},\mathbb{Q})$. In addition, from $\mathcal{F}_p^0 \in PAP_0(\mathbb{T},\mathbb{Q})$, $r\in\mathbb{T}$, we have

$$\begin{aligned}
& \lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^r\left\|\Omega_p^0(t)\right\|_{\mathbb{Q}}\Delta t \\
= & \lim_{r\to+\infty}\frac{1}{2r}\int_{-r}^r\left\|\int_{-\infty}^t e_{-a_p^R}(t,\sigma(s))\mathcal{F}_p^0(s)\Delta s\right\|_{\mathbb{Q}}\Delta t \\
\leq & \lim_{r\to+\infty}\frac{1}{2r}\int_{-\infty}^t e_{-a_p^R}(t,\sigma(s))\Big(\int_{-r}^r\left\|\mathcal{F}_p^0(s)\right\|_{\mathbb{Q}}\Delta s\Big)\Delta t, \quad p\in\mathcal{S},
\end{aligned}$$

which implies that $\Omega_p^0 \in PAP_0(\mathbb{T},\mathbb{Q})$. Therefore, $x_p^\phi \in PAP(\mathbb{T},\mathbb{Q})$, that is, Φ maps $\mathbb{E}$ into $PAP(\mathbb{T},\mathbb{Q}^n)$.

Next, we will show that Φ is a self-mapping from $\mathbb{E}^*$ to $\mathbb{E}^*$. In fact, for each $\phi\in\mathbb{E}^*$, we have

$$\begin{aligned}
& \left\|(\Phi\phi)(t)-\phi^0(t)\right\|_{\mathbb{Q}^n} \\
= & \max_{p\in\mathcal{S}}\Big\{\Big\|\int_{-\infty}^t e_{-a_p^R}(t,\sigma(s))\Big[a_p^R(s)\int_{s-\eta_p(s)}^s\phi_p^\Delta(u)\Delta u+\hat{a}_p(s)\phi_p(s-\eta_p(s)) \\
& +\sum_{q=1}^n b_{pq}(s)f_q(\phi_q(s))+\sum_{q=1}^n c_{pq}(s)g_q(\phi_q(s-\tau_{pq}(s))) \\
& +\sum_{q=1}^n\sum_{l=1}^n T_{pql}(s)h_q(\phi_q(s-\delta_{pql}(s)))h_l(\phi_l(s-\vartheta_{pql}(s)))\Big]\Delta s\Big\|_{\mathbb{Q}}\Big\} \\
\leq & \max_{p\in\mathcal{S}}\Big\{\int_{-\infty}^t e_{-a_p^-}(t,\sigma(s))\Big[a_p^+\eta_p^+\left\|\phi_p^\Delta\right\|_{\infty}+\left\|\hat{a}_p(s)\right\|_{\mathbb{Q}}\left\|\phi_p(s-\eta_p(s))\right\|_{\mathbb{Q}} \\
& +\sum_{q=1}^n\left\|b_{pq}(s)\right\|_{\mathbb{Q}}L_q^f\left\|\phi_q(s)\right\|_{\mathbb{Q}}+\sum_{q=1}^n\left\|c_{pq}(s)\right\|_{\mathbb{Q}}L_j^g\left\|\phi_q(s-\tau_{pq}(s))\right\|_{\mathbb{Q}}
\end{aligned}$$

$$+\sum_{q=1}^{n}\sum_{l=1}^{n}\left\|T_{pql}(s)\right\|_{\mathbb{Q}}M_l^hL_q^h\left\|\phi_q(s-\delta_{pql}(s)))\right\|_{\mathbb{Q}}\Big]\Delta s\Big\}$$

$$\leq \max_{p\in\mathcal{S}}\Big\{\frac{1}{a_p^-}\Big[a_p^+\eta_p^+ + \hat{a}_p^+ + \sum_{q=1}^{n}b_{pq}^+L_q^f + \sum_{q=1}^{n}c_{pq}^+L_q^g$$

$$+\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}^+M_l^hL_q^h\Big]\Big\}\left\|\phi\right\|_E.$$

Thus, we have

$$\begin{aligned}
&\left\|\Phi\phi-\phi^0\right\|_0\\
=\ &\sup_{t\in\mathbb{T}}\left\|(\Phi\phi)(t)-\phi^0(t)\right\|_{\mathbb{Q}^n}\\
\leq\ &\max_{p\in\mathcal{S}}\Big\{\frac{1}{a_p^-}\Big[a_p^+\eta_p^+ + \hat{a}_p^+ + \sum_{q=1}^{n}b_{pq}^+L_q^f + \sum_{q=1}^{n}c_{pq}^+L_q^g + \sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}^+M_l^hL_q^h\Big]\Big\}\left\|\phi\right\|_{\mathbb{E}}.
\end{aligned}\tag{3.4}$$

On the other hand, we have

$$\begin{aligned}
&\left\|[\Phi\phi-\phi^0]^\Delta\right\|_0\\
=\ &\sup_{t\in\mathbb{T}}\left\|[(\Phi\phi)(t)-\phi^0(t)]^\Delta\right\|_{\mathbb{Q}^n}\\
=\ &\max_{p\in\mathcal{S}}\Big\{\sup_{t\in\mathbb{T}}\Big\|\mathcal{F}_p(t)-a_p^R(t)\int_{-\infty}^{t}e_{-a_p^R}(t,\sigma(s))\mathcal{F}_p(s)\Delta s\Big\|_{\mathbb{Q}}\Big\}\\
\leq\ &\max_{pq\in\mathcal{S}}\Big\{\Big[a_p^+\eta_p^+ + \hat{a}_p^+ + \sum_{q=1}^{n}b_{pq}^+L_q^f + \sum_{q=1}^{n}c_{pq}^+L_q^g + \sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}^+M_l^hL_q^h\Big]\\
&+\frac{a_p^+}{a_p^-}\Big[a_p^+\eta_p^+ + \hat{a}_p^+ + \sum_{q=1}^{n}b_{pq}^+L_q^f + \sum_{q=1}^{n}c_{pq}^+L_q^g + \sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}^+M_l^hL_q^h\Big]\Big\}\left\|\phi\right\|_{\mathbb{E}}.
\end{aligned}\tag{3.5}$$

Noting the fact that for $\phi\in\mathbb{E}^*$, we have

$$||\phi||_{\mathbb{E}}\leq||\phi^0||_{\mathbb{E}}+||\phi-\phi^0||_{\mathbb{E}}\leq\kappa+\frac{\kappa\rho}{1-\rho}\leq\frac{\kappa}{1-\rho}.$$

It follows from (3.4)-(3.5), and (H_3) that

$$\left\|\Phi\phi-\phi^0\right\|_{\mathbb{E}}\leq\frac{\kappa\rho}{1-\rho},$$

thus, we have $\Phi\phi\in\mathbb{E}^*$.

Finally, we will show that Φ is a contraction mapping in $\mathbb{E}^*$. For any $\phi,\psi\in\mathbb{E}^*$, we can get

$$\begin{aligned}
&\left\|(\Phi\phi)(t)-(\Phi\psi)(t)\right\|_{\mathbb{Q}^n}\\
=\ &\max_{p\in\mathcal{S}}\Big\{\Big\|\int_{-\infty}^{t}e_{-a_p^R}(t,\sigma(s))\Big(a_p^R(s)\int_{s-\eta_p(s)}^{s}[\phi_p^\Delta(u)-\psi_p^\Delta(u)]\Delta u
\end{aligned}$$

$$
\begin{aligned}
&+\hat{a}_p(s)[\phi_p(s-\eta_p(s))-\psi_p(s-\eta_p(s))]+\sum_{q=1}^{n}b_{pq}(s)[f_q(\phi_q(s))-f_q(\psi_q(s))]\\
&+\sum_{q=1}^{n}c_{pq}(s)[g_q(\phi_q(s-\tau_{pq}(s)))-g_q(\psi_q(s-\tau_{pq}(s)))]\\
&+\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}(s)[h_q(\phi_q(s-\delta_{pql}(s)))h_l(\phi_l(s-\vartheta_{pql}(s)))\\
&-h_q(\phi_q(s-\delta_{pql}(s)))h_l(\phi_l(s-\vartheta_{pql}(s)))]\Big)\Delta s\Big\|_{\mathbb{Q}}\Big\}\\
\leq\ &\max_{p\in\mathcal{S}}\Big\{\int_{-\infty}^{t}e_{-a_p^-}(t,\sigma(s))\Big(a_p^+\eta_p^+\big\|\phi_p^\Delta-\psi_p^\Delta\big\|_\infty+\big\|\hat{a}_p(s)\big\|_{\mathbb{Q}}\big\|\phi_p(s-\eta_p(s))\\
&-\psi_p(s-\eta_p(s))\big\|_{\mathbb{Q}}+\sum_{q=1}^{n}\big\|b_{pq}(s)\big\|_{\mathbb{Q}}L_q^f\big\|\phi_q(s)-\psi_q(s)\big\|_{\mathbb{Q}}+\sum_{q=1}^{n}\big\|c_{pq}(s)\big\|_{\mathbb{Q}}L_q^g\\
&\times\big\|\phi_q(s-\tau_{pq}(s))-\psi_q(s-\tau_{pq}(s))\big\|_{\mathbb{Q}}+\sum_{q=1}^{n}\sum_{l=1}^{n}\big\|T_{pql}(s)\big\|_{\mathbb{Q}}\\
&\times\Big[\big\|h_q(\phi_q(s-\delta_{pql}(s)))\big\|_{\mathbb{Q}}\big\|h_l(\phi_l(s-\vartheta_{pql}(s)))-h_l(\psi_l(s-\vartheta_{pql}(s)))\big\|_{\mathbb{Q}}\\
&+\big\|h_q(\phi_q(s-\delta_{pql}(s)))-h_q(\psi_q(s-\delta_{pql}(s)))\big\|_{\mathbb{Q}}\big\|h_l(\psi_l(s-\vartheta_{pql}(s)))\big\|_{\mathbb{Q}}\Big]\Big)\Delta s\Big\}\\
\leq\ &\max_{p\in\mathcal{S}}\Big\{\frac{1}{a_p^-}\Big[a_p^+\eta_p^++\hat{a}_p^++\sum_{q=1}^{n}b_{pq}^+L_q^f+\sum_{q=1}^{n}c_{pq}^+L_q^g\\
&+\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}\Big(M_q^hL_l^h+L_q^hM_l^h\Big)\Big]\Big\}\big\|\phi-\psi\big\|_{\mathbb{E}}.
\end{aligned}
\tag{3.6}
$$

It follows from (3.6) that

$$
\begin{aligned}
&\big\|\Phi\phi-\Phi\psi\big\|_0\\
\leq\ &\max_{p\in\mathcal{S}}\Big\{\frac{1}{a_p^-}\Big[a_p^+\eta_p^++\hat{a}_p^++\sum_{q=1}^{n}b_{pq}^+L_q^f+\sum_{q=1}^{n}c_{pq}^+L_q^g\\
&+\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}^+\Big(M_q^hL_l^h+L_q^hM_l^h\Big)\Big]\Big\}\big\|\phi-\psi\big\|_{\mathbb{E}}=\max_{p\in\mathcal{S}}\Big\{\frac{\Xi_p}{a_p^-}\Big\}\big\|\phi-\psi\big\|_{\mathbb{E}}.
\end{aligned}
\tag{3.7}
$$

On the other hand, we can derive that

$$
\big\|[\Phi\phi-\Phi\psi]^\Delta\big\|_0\leq\max_{p\in\mathcal{S}}\Big\{\Xi_p+\frac{a_p^+}{a_p^-}\Xi_p\Big\}\big\|\phi-\psi\big\|_{\mathbb{E}}. \tag{3.8}
$$

By (3.7), (3.8) and (H_3), we have

$$
\big\|\Phi\phi-\Phi\psi\big\|_{\mathbb{E}}\leq\rho\big\|\phi-\psi\big\|_{\mathbb{E}},
$$

in view of the definition of ρ, which implies that Φ is a contraction mapping. Therefore, Φ has a unique fixed point in $\mathbb{E}^*$, that is, (1.1) has a unique pseudo almost periodic solution in $\mathbb{E}^*$. The proof is complete. □

Definition 3.1. *Let* $x = (x_1, x_2, \dots, x_n)^T$ *be a solution of* (1.1) *with the initial value* $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$. *If there exist positive constants* $\lambda > 0$ *and* $M > 0$ *such that every solution* $y = (y_1, y_2, \dots, y_n)^T$ *of* (1.1) *with initial value* $\psi = (\psi_1, \psi_2, \dots, \psi_n)^T$ *satisfies*

$$\left\|y(t) - x(t)\right\|_1 \le M e_{\ominus\lambda}(t, t_0)\left\|\psi - \phi\right\|_\theta, \quad \forall\, t \in (t_0, +\infty)_{\mathbb{T}},$$

where

$$\left\|y(t) - x(t)\right\|_1 = \max\left\{\left\|y(t) - x(t)\right\|_{\mathbb{Q}^n}, \left\|[y(t) - x(t)]^\Delta\right\|_{\mathbb{Q}^n}\right\},$$

$$\left\|\psi - \phi\right\|_\theta = \max\Big\{\sup_{s\in[t_0-\theta,t_0]_{\mathbb{T}}}\left\|\psi(s) - \phi(s)\right\|_{\mathbb{Q}^n}, \sup_{s\in[t_0-\theta,t_0]_{\mathbb{T}}}\left\|[\psi(s) - \phi(s)]^\Delta\right\|_{\mathbb{Q}^n}\Big\},$$

then the solution x is said to be globally exponentially stable.

Theorem 3.2. *Assume that* (H_1)–(H_3) *hold, then system* (1.1) *has a unique pseudo almost periodic solution that is globally exponentially stable.*

Proof. From Theorem 3.1, we see that system (1.1) has a pseudo almost periodic solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ with initial value $\phi(s) = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^T$. Suppose that $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$ is an arbitrary solution of system (1.1) with initial value $\psi(s) = (\psi_1(s), \psi_2(s),$
$\dots, \psi_n(s))^T$ and let $z(t) = y(t) - x(t)$, then we have

$$\begin{aligned}
z_p^\Delta(t) \;=\; & -a_p^R(t)[y_p(t) - x_p(t)] + a_p^R(t)\int_{t-\eta_p(t)}^{t}[y_p^\Delta(s) - x_p^\Delta(s)]\Delta s \\
& + \hat{a}_p(t)[y_p(t - \eta_p(t)) - x_p(t - \eta_p(t))] + \sum_{q=1}^{n} b_{pq}(t)[f_q(y_q(t)) - f_q(x_q(t))] \\
& + \sum_{q=1}^{n} c_{pq}(t)[g_q(y_q(t - \tau_{pq}(t))) - g_q(x_q(t - \tau_{pq}(t)))] \\
& + \sum_{q=1}^{n}\sum_{l=1}^{n} T_{pql}(t)[h_q(y_q(t - \delta_{pql}(t)))h_l(y_l(t - \vartheta_{pql}(t))) \\
& - h_q(x_q(t - \delta_{pql}(t)))h_l(x_l(t - \vartheta_{pql}(t)))], \quad p \in \mathcal{S}, \quad t \in \mathbb{T}.
\end{aligned} \tag{3.9}$$

For $p \in \mathcal{S}$, let Θ_p and Ψ_p be defined as follows:

$$\begin{aligned}
\Theta_p(\omega) \;=\; & a_p^- - \omega - \exp\big(\omega \sup_{s\in\mathbb{T}} \mu(s)\big)\Big[a_p^+\eta_p^+ \exp(\omega\eta_p^+) + \hat{a}_p^+ \exp(\omega\eta_p^+) \\
& + \sum_{q=1}^{n} b_{pq}^+ L_q^f + \sum_{q=1}^{n} c_{pq}^+ L_q^g \exp(\omega\tau_{pq}^+) + \sum_{q=1}^{n}\sum_{l=1}^{n} T_{pql}^+\Big(M_q^h L_l^h \exp(\omega\delta_{pql}^+) \\
& + L_q^h M_l^h \exp(\omega\vartheta_{pql}^+)\Big)\Big]
\end{aligned}$$

and

$$\Psi_p(\omega) \;=\; a_p^- - \omega - (a_p^+ \exp(\omega \sup_{s\in\mathbb{T}} \mu(s)) + a_p^-)\Big[a_p^+\eta_p^+ \exp(\omega\eta_p^+)$$

$$+\hat{a}_p^+ \exp(\omega\eta_p^+) + \sum_{q=1}^n b_{pq}^+ L_q^f + \sum_{q=1}^n c_{pq}^+ L_q^g \exp(\omega\tau_{pq}^+)$$
$$+ \sum_{q=1}^n \sum_{l=1}^n T_{pql}^+\Big(M_q^h L_l^h \exp(\omega\delta_{pql}^+) + L_q^h M_l^h \exp(\omega\vartheta_{pql}^+)\Big)\Big].$$

By (H_3), we have

$$\Theta_p(0) = a_p^- - \Xi_p > 0$$

and

$$\Psi_p(0) = a_p^- - (a_p^+ + a_p^-)\Xi_p > 0.$$

Based on the continuities of functions Θ_p and Ψ_p on $[0, +\infty)$, and the fact that $\Theta_p(\omega), \Psi_p(\omega) \to -\infty$, as $\omega \to +\infty$, there exist $\zeta_p, \xi_p > 0$ such that $\Theta_p(\zeta_p) = \Psi_p(\xi_p) = 0$ and $\Theta_p(\omega) > 0$ for $\omega \in (0, \zeta_p)$, $\Psi_p(\omega) > 0$ for $\omega \in (0, \xi_p)$. Take $\gamma = \min_{p\in\mathcal{S}} \{\zeta_p, \xi_p\}$, we have $\Theta_p(\gamma) \geq 0$, $\Psi_p(\gamma) \geq 0$. So, we can choose a positive constant $0 < \lambda < \min\{\gamma, \min_{p\in\mathcal{S}}\{a_p^-\}\}$ with $\lambda \in \mathcal{R}^+$ such that

$$\Theta_p(\lambda) > 0, \quad \Psi_p(\lambda) > 0, \quad p \in \mathcal{S},$$

which implies that

$$\frac{\exp(\lambda \sup_{s\in\mathbb{T}} \mu(s))}{a_p^- - \lambda}\Big[a_p^+\eta_p^+ \exp(\lambda\eta_p^+) + \hat{a}_p^+ \exp(\lambda\eta_p^+)$$
$$+ \sum_{q=1}^n b_{pq}^+ L_q^f + \sum_{q=1}^n c_{pq}^+ L_q^g \exp(\lambda\tau_{pq}^+) + \sum_{q=1}^n \sum_{l=1}^n T_{pql}^+\Big(M_q^h L_l^h \exp(\lambda\delta_{pql}^+)$$
$$+L_q^h M_l^h \exp(\lambda\vartheta_{pql}^+)\Big)\Big] < 1$$

and

$$\Big(1 + \frac{a_p^+ \exp(\lambda \sup_{s\in\mathbb{T}} \mu(s))}{a_p^- - \lambda}\Big)\Big[a_p^+\eta_p^+ \exp(\lambda\eta_p^+) + \hat{a}_p^+ \exp(\lambda\eta_p^+)$$
$$+ \sum_{q=1}^n b_{pq}^+ L_q^f + \sum_{q=1}^n c_{pq}^+ L_q^g \exp(\lambda\tau_{pq}^+) + \sum_{q=1}^n \sum_{l=1}^n T_{pql}^+\Big(M_q^h L_l^h \exp(\lambda\delta_{pql}^+)$$
$$+L_q^h M_l^h \exp(\lambda\vartheta_{pql}^+)\Big)\Big] < 1, \quad p \in \mathcal{S}.$$

Let $M = \max_{p\in\mathcal{S}} \Big\{\frac{a_p^-}{\Xi_p}\Big\}$, then by (H_3), we have $M > 1$. Thus,

$$\frac{1}{M} - \min_{p\in\mathcal{S}}\Big\{\frac{\exp(\lambda \sup_{s\in\mathbb{T}} \mu(s))}{a_p^- - \lambda}\Big[a_p^+\eta_p^+ \exp(\lambda\eta_p^+) + \hat{a}_p^+ \exp(\lambda\eta_p^+)$$

$$+\sum_{q=1}^{n} b_{pq}^{+} L_q^f + \sum_{q=1}^{n} c_{pq}^{+} L_q^g \exp(\lambda \tau_{pq}^{+}) + \sum_{q=1}^{n}\sum_{l=1}^{n} T_{pql}^{+}\Big(M_q^h L_l^h \exp(\lambda \delta_{pql}^{+})$$
$$+L_q^h M_l^h \exp(\lambda \vartheta_{pql}^{+})\Big)\Big]\Big\} < 0.$$

Since $e_{\ominus\lambda}(t,t_0) > 1$ for $t \in [t_0-\theta, t_0]_{\mathbb{T}}$, it is obvious that, for any $\varepsilon > 0$,

$$\left\|z(t_0)\right\|_1 < (\|\psi-\phi\|_\theta + \varepsilon)$$

and

$$\begin{aligned}\left\|z(t)\right\|_1 &< (\|\psi-\phi\|_\theta + \varepsilon)e_{\ominus\lambda}(t,t_0)\\ &< M(\|\psi-\phi\|_\theta + \varepsilon)e_{\ominus\lambda}(t,t_0), \quad \forall\, t \in [t_0-\theta, t_0]_{\mathbb{T}}.\end{aligned}$$

We claim that

$$\left\|z(t)\right\|_1 < M(\|\psi-\phi\|_\theta + \varepsilon)e_{\ominus\lambda}(t,t_0), \quad \forall\, t \in (t_0, +\infty)_{\mathbb{T}}. \tag{3.10}$$

If (3.10) is not true, then there must be some $t_1 \in (t_0, +\infty)_{\mathbb{T}}$ such that

$$\begin{cases}\left\|z(t_1)\right\|_1 \geq M(\|\psi-\phi\|_\theta + \varepsilon)e_{\ominus\lambda}(t_1,t_0),\\ \left\|z(t)\right\|_1 \leq M(\|\psi-\phi\|_\theta + \varepsilon)e_{\ominus\lambda}(t,t_0), \quad t \in (t_0, t_1)_{\mathbb{T}}.\end{cases}$$

Hence, there must exist a constant $\mathcal{P} \geq 1$ such that

$$\begin{cases}\left\|z(t_1)\right\|_1 = \mathcal{P}M(\|\psi-\phi\|_\theta + \varepsilon)e_{\ominus\lambda}(t_1,t_0),\\ \left\|z(t)\right\|_1 \leq \mathcal{P}M(\|\psi-\phi\|_\theta + \varepsilon)e_{\ominus\lambda}(t,t_0), \quad t \in (t_0, t_1)_{\mathbb{T}}.\end{cases} \tag{3.11}$$

Multiplying both sides of (3.9) by $e_{-a_p^R}(t_0, \sigma(t))$ and integrating over $[t_0, t]_{\mathbb{T}}$, we get

$$\begin{aligned}z_p(t) &= e_{-a_p^R}(t,t_0)z_p(t_0) + \int_{t_0}^{t} e_{-a_p^R}(t,\sigma(s))\Big(a_p^R(s)\int_{s-\eta_p(s)}^{s}[y_p^{\Delta}(u) - x_p^{\Delta}(u)]\Delta u\\ &\quad +\hat{a}_p(s)[y_p(s-\eta_p(s)) - x_p(s-\eta_p(s))] + \sum_{q=1}^{n} b_{pq}(s)[f_q(y_q(s))\\ &\quad -f_q(x_q(s))] + \sum_{q=1}^{n} c_{pq}(s)[g_q(y_q(s-\tau_{pq}(s))) - g_q(x_q(s-\tau_{pq}(s)))]\\ &\quad +\sum_{q=1}^{n}\sum_{l=1}^{n} T_{pql}(s)[h_q(y_q(s-\delta_{pql}(s)))h_l(y_l(s-\vartheta_{pql}(s)))\\ &\quad -h_q(x_q(s-\delta_{pql}(s)))h_l(x_l(s-\vartheta_{pql}(s)))]\Big)\Delta s.\end{aligned} \tag{3.12}$$

In view of (3.12) and $M > 1$, we have

$$\left\|z(t_1)\right\|_{\mathbb{Q}^n} = \max_{p\in\mathcal{S}}\left\{\left\|z_p(t_1)\right\|_{\mathbb{Q}}\right\}$$

$$
\begin{aligned}
&= \max_{p\in\mathcal{S}}\bigg\{\bigg\|e_{-a_p^R}(t_1,t_0)z_p(t_0)+\int_{t_0}^{t_1}e_{-a_p^R}(t_1,\sigma(s))\Big(a_p^R(s)\int_{s-\eta_p(s)}^{s}[y_p^{\Delta}(u)-x_p^{\Delta}(u)]\Delta u \\
&\quad +\hat{a}_p(s)[y_p(s-\eta_p(s))-x_p(s-\eta_p(s))]+\sum_{q=1}^{n}b_{pq}(s)[f_q(y_q(s)) \\
&\quad -f_q(x_q(s))]+\sum_{q=1}^{n}c_{pq}(s)[g_q(y_q(s-\tau_{pq}(s)))-g_q(x_q(s-\tau_{pq}(s)))] \\
&\quad +\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}(s)[h_q(y_q(s-\delta_{pql}(s)))h_l(y_l(s-\vartheta_{pql}(s))) \\
&\quad -h_q(x_q(s-\delta_{pql}(s)))h_l(x_l(s-\vartheta_{pql}(s)))]\Big)\Delta s\bigg\|_{\mathbb{Q}}\bigg\} \\
&\le \max_{p\in\mathcal{S}}\bigg\{e_{-a_p^R}(t_1,t_0)\big\|z_p(t_0)\big\|_{\mathbb{Q}}+\mathcal{P}Me_{\ominus\lambda}(t_1,t_0)(\|\psi-\phi\|_1+\varepsilon) \\
&\quad \times\int_{t_0}^{t_1}e_{-a_p^R\oplus\lambda}(t_1,\sigma(s))\Big[a_p^+\int_{s-\eta_p(s)}^{s}e_\lambda(\sigma(u),u)\Delta u+\hat{a}_p^+e_\lambda(\sigma(s),s-\eta_p(s)) \\
&\quad +\sum_{q=1}^{n}b_{pq}^+L_q^f e_\lambda(\sigma(s),s)+\sum_{q=1}^{n}c_{pq}^+L_q^g e_\lambda(\sigma(s),s-\tau_{pq}(s)) \\
&\quad +\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}^+\Big(M_q^hL_l^h e_\lambda(\sigma(s),s-\delta_{pql}(s))+L_q^hM_l^h e_\lambda(\sigma(s),s-\vartheta_{pql}(s))\Big)\Big]\Delta s\bigg\} \\
&\le \max_{p\in\mathcal{S}}\bigg\{e_{-a_p^R}(t_1,t_0)\big\|z_p(t_0)\big\|_{\mathbb{Q}}+\mathcal{P}Me_{\ominus\lambda}(t_1,t_0)(\|\psi-\phi\|_1+\varepsilon) \\
&\quad \times\int_{t_0}^{t_1}e_{-d_p\oplus\lambda}(t_1,\sigma(s))\Big[a_p^+\eta_p^+\exp\,[\lambda(\eta_p^+ +\sup_{s\in\mathbb{T}}\mu(s))] \\
&\quad +\hat{a}_p^+\exp\,[\lambda(\eta_p^+ +\sup_{s\in\mathbb{T}}\mu(s))]+\sum_{q=1}^{n}b_{pq}^+L_q^f\exp\,(\lambda\sup_{s\in\mathbb{T}}\mu(s)) \\
&\quad +\sum_{q=1}^{n}c_{pq}^+L_q^g\exp\,[\lambda(\tau_{pq}^+ +\sup_{s\in\mathbb{T}}\mu(s))]+\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}^+\Big(M_q^hL_l^h \\
&\quad \times\exp\,[\lambda(\delta_{pql}^+ +\sup_{s\in\mathbb{T}}\mu(s))]+L_q^hM_l^h\exp\,[\lambda(\vartheta_{pql}^+ +\sup_{s\in\mathbb{T}}\mu(s))]\Big)\Big]\Delta s\bigg\} \\
&\le \max_{p\in\mathcal{S}}\bigg\{\mathcal{P}Me_{\ominus\lambda}(t_1,t_0)(\|\psi-\phi\|_1+\varepsilon)\bigg\{\frac{e_{-a_p^R\oplus\lambda}(t_1,t_0)}{\mathcal{P}M}+\int_{t_0}^{t_1}e_{-a_p^R\oplus\lambda}(t_1,\sigma(s)) \\
&\quad \times\exp\,(\lambda\sup_{s\in\mathbb{T}}\mu(s))\Big[a_p^+\eta_p^+\exp\,(\lambda\eta_p^+)+\hat{a}_p^+\exp\,(\lambda\eta_p^+) \\
&\quad +\sum_{q=1}^{n}b_{pq}^+L_q^f+\sum_{q=1}^{n}c_{pq}^+L_q^g\exp\,(\lambda\tau_{pq}^+)+\sum_{q=1}^{n}\sum_{l=1}^{n}T_{pql}^+\Big(M_q^hL_l^h\exp\,(\lambda\delta_{pql}^+) \\
&\quad +L_q^hM_l^h\exp\,(\lambda\vartheta_{pql}^+)\Big)\Big]\Delta s\bigg\}\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
< \quad & \max_{p\in\mathcal{S}}\Big\{\mathcal{P}Me_{\ominus\lambda}(t_1,t_0)(\|\psi-\phi\|_1+\varepsilon)\Big\{\frac{e_{-a_p^R\oplus\lambda}(t_1,t_0)}{M}+\frac{1-e_{-a_p^R\oplus\lambda}(t_1,t_0)}{a_p^- - \lambda} \\
& \times\exp\big(\lambda\sup_{s\in\mathbb{T}}\mu(s)\big)\Big[a_p^+\eta_p^+\exp(\lambda\eta_p^+)+\hat{a}_p^+\exp(\lambda\eta_p^+)+\sum_{q=1}^n b_{pq}^+L_q^f \\
& +\sum_{q=1}^n c_{pq}^+L_q^g\exp(\lambda\tau_{pq}^+)+\sum_{q=1}^n\sum_{l=1}^n T_{pql}^+\Big(M_q^hL_l^h\exp(\lambda\delta_{pql}^+) \\
& +L_q^hM_l^h\exp(\lambda\vartheta_{pql}^+)\Big)\Big]\Big\}\Big\} \\
= \quad & \max_{p\in\mathcal{S}}\Big\{\mathcal{P}Me_{\ominus\lambda}(t_1,t_0)(\|\psi-\phi\|_1+\varepsilon)\Big\{\Big(\frac{1}{M}-\frac{\exp(\lambda\sup_{s\in\mathbb{T}}\mu(s))}{a_p^- - \lambda} \\
& \times\Big[a_p^+\eta_p^+\exp(\lambda\eta_p^+)+\hat{a}_p^+\exp(\lambda\eta_p^+)+\sum_{q=1}^n b_{pq}^+L_q^f+\sum_{q=1}^n c_{pq}^+L_q^g\exp(\lambda\tau_{pq}^+) \\
& +\sum_{q=1}^n\sum_{l=1}^n T_{pql}^+\Big(M_q^hL_l^h\exp(\lambda\delta_{pql}^+)+L_q^hM_l^h\exp(\lambda\vartheta_{pql}^+)\Big)\Big]\Big) \\
& \times e_{-a_p^R\oplus\lambda}(t_1,t_0)+\frac{\exp(\lambda\sup\limits_{s\in\mathbb{T}}\mu(s))}{a_p^- - \lambda}\Big[a_p^+\eta_p^+\exp(\lambda\eta_p^+)+\hat{a}_p^+\exp(\lambda\eta_p^+) \\
& +\sum_{q=1}^n b_{pq}^+L_q^f+\sum_{q=1}^n c_{pq}^+L_q^g\exp(\lambda\tau_{pq}^+)+\sum_{q=1}^n\sum_{l=1}^n T_{pql}^+\Big(M_q^hL_l^h\exp(\lambda\delta_{pql}^+) \\
& +L_q^hM_l^h\exp(\lambda\vartheta_{pql}^+)\Big)\Big]\Big\}\Big\} \\
< \quad & \mathcal{P}Me_{\ominus\lambda}(t_1,t_0)(\|\psi-\phi\|_\theta+\varepsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
\big\|z^\Delta(t_1)\big\|_{\mathbb{Q}^n} \quad = \quad & \max_{p\in\mathcal{S}}\Big\{\big\|z_p^\Delta(t_1)\big\|_{\mathbb{Q}}\Big\} \\
\leq \quad & \max_{p\in\mathcal{S}}\Big\{a_p^+e_{-a_p^R}(t_1,t_0)(\|\psi-\phi\|_1+\varepsilon)+\mathcal{P}Me_{\ominus\lambda}(t_1,t_0)(\|\psi-\phi\|_1+\varepsilon) \\
& \times\Big[a_p^+\int_{t_1-\eta_p(t_1)}^{t_1}e_\lambda(\sigma(s),s)\Delta s+\hat{a}_p^+e_\lambda(\sigma(t_1),t_1-\eta_p(t_1)) \\
& +\sum_{q=1}^n b_{pq}^+L_q^fe_\lambda(\sigma(t_1),t_1)+\sum_{q=1}^n c_{pq}^+L_q^ge_\lambda(\sigma(t_1),t_1-\tau_{pq}(t_1)) \\
& +\sum_{q=1}^n\sum_{l=1}^n T_{pql}^+\Big(M_q^hL_l^he_\lambda(\sigma(t_1),t_1-\delta_{pql}(t_1))+L_q^hM_l^he_\lambda(\sigma(t_1),t_1-\vartheta_{pql}(t_1))\Big)\Big] \\
& +a_p^+\mathcal{P}Me_{\ominus\lambda}(t_1,t_0)(\|\psi-\phi\|_1+\varepsilon)\int_{t_0}^{t_1}e_{-a_p^R\oplus\lambda}(t_1,\sigma(s)) \\
& \times\Big[a_p^+\int_{s-\eta_p(s)}^{s}e_\lambda(\sigma(u),u)\Delta u+\hat{a}_p^+e_\lambda(\sigma(s),s-\eta_p(s))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{q=1}^{n} b_{pq}^{+} L_{q}^{f} e_{\lambda}(\sigma(s), s)+\sum_{q=1}^{n} c_{pq}^{+} L_{q}^{g} e_{\lambda}(\sigma(s), s-\tau_{pq}(s)) \\
&+\sum_{q=1}^{n} \sum_{l=1}^{n} T_{pql}^{+}\Big(M_{q}^{h} L_{l}^{h} e_{\lambda}(\sigma(s), s-\delta_{pql}(s))+L_{q}^{h} M_{l}^{h} e_{\lambda}(\sigma(s), s-\vartheta_{pql}(s))\Big)\Big]\Delta s\Big\} \\
\leq\ & \max_{p \in \mathcal{S}}\Big\{a_{p}^{+} e_{-a_{p}^{R}}(t_{1}, t_{0})(\|\psi-\phi\|_{1}+\varepsilon)+\mathcal{P} M e_{\ominus \lambda}(t_{1}, t_{0})(\|\psi-\phi\|_{1}+\varepsilon) \\
&\times\Big[a_{p}^{+} \eta_{p}^{+} \exp(\lambda \eta_{p}^{+})+\hat{a}_{p}^{+} \exp(\lambda \eta_{p}^{+})+\sum_{q=1}^{n} b_{pq}^{+} L_{q}^{f}+\sum_{q=1}^{n} c_{pq}^{+} L_{q}^{g} \exp(\lambda \tau_{pq}^{+}) \\
&+\sum_{q=1}^{n} \sum_{l=1}^{n} T_{pql}^{+}\Big(M_{q}^{h} L_{l}^{h} \exp(\lambda \delta_{pql}^{+})+L_{q}^{h} M_{l}^{h} \exp(\lambda \vartheta_{pql}^{+})\Big)\Big] \\
&\times\Big[1+a_{p}^{+} \exp(\lambda \sup_{s \in \mathbb{T}} \mu(s)) \int_{t_{0}}^{t_{1}} e_{-a_{p}^{R} \oplus \lambda}(t_{1}, \sigma(s)) \Delta s\Big]\Big\} \\
\leq\ & \max_{p \in \mathcal{S}}\Big\{\mathcal{P} M e_{\ominus \lambda}(t_{1}, t_{0})(\|\psi-\phi\|_{1}+\varepsilon)\Big\{\Big[\frac{1}{M}-\frac{\exp(\lambda \sup_{s \in \mathbb{T}} \mu(s))}{a_{p}^{-}-\lambda} \\
&\times\Big[a_{p}^{+} \eta_{p}^{+} \exp(\lambda \eta_{p}^{+})+\hat{a}_{p}^{+} \exp(\lambda \eta_{p}^{+})+\sum_{q=1}^{n} b_{pq}^{+} L_{q}^{f}+\sum_{q=1}^{n} c_{pq}^{+} L_{q}^{g} \exp(\lambda \tau_{pq}^{+}) \\
&+\sum_{q=1}^{n} \sum_{l=1}^{n} T_{pql}^{+}\Big(M_{q}^{h} L_{l}^{h} \exp(\lambda \delta_{pql}^{+})+L_{q}^{h} M_{l}^{h} \exp(\lambda \vartheta_{pql}^{+})\Big)\Big] e_{-a_{p}^{R} \oplus \lambda}(t_{1}, t_{0}) \\
&+\Big(1+\frac{a_{p}^{+} \exp(\lambda \sup_{s \in \mathbb{T}} \mu(s))}{a_{p}^{-}-\lambda}\Big)\Big[a_{p}^{+} \eta_{p}^{+} \exp(\lambda \eta_{p}^{+})+\hat{a}_{p}^{+} \exp(\lambda \eta_{p}^{+}) \\
&+\sum_{q=1}^{n} b_{pq}^{+} L_{q}^{f}+\sum_{q=1}^{n} c_{pq}^{+} L_{q}^{g} \exp(\lambda \tau_{pq}^{+})+\sum_{q=1}^{n} \sum_{l=1}^{n} T_{pql}^{+}\Big(M_{q}^{h} L_{l}^{h} \exp(\lambda \delta_{pql}^{+}) \\
&+L_{q}^{h} M_{l}^{h} \exp(\lambda \vartheta_{pql}^{+})\Big)\Big]\Big\}\Big\} \\
<\ & \mathcal{P} M e_{\ominus \lambda}(t_{1}, t_{0})(\|\psi-\phi\|_{\theta}+\varepsilon).
\end{aligned}
$$

The above two inequalities imply that

$$\left\|z(t_1)\right\|_1 < \mathcal{P} M e_{\ominus\lambda}(t_1, t_0)(\|\psi-\phi\|_\theta+\varepsilon),$$

which contradicts the first equation of (3.11). Therefore, (3.10) holds. Letting $\varepsilon \to 0^+$ leads to

$$\left\|z(t)\right\|_1 \leq M e_{\ominus\lambda}(t, t_0)\|\psi-\phi\|_\theta, \quad \forall\, t \in (t_0, +\infty)_{\mathbb{T}}.$$

Hence, the pseudo almost periodic solution of system (1.1) is globally exponentially stable. The proof is complete. □

4. Illustrative example

Example 4.1. *In system* (1.1)*, let* $n = 2, t_0 = 0$ *and take*

$$f_q(x_q) = \frac{1}{40}\sin(x_q^R + x_q^I + x_q^K) + i\frac{1}{50}\sin x_q^I + j\frac{1}{40}\sin x_q^J + k\frac{1}{45}\sin x_q^K,$$

$$g_q(x_q) = \frac{1}{50}\sin x_q^R + i\frac{1}{55}\sin(x_q^I + x_q^K) + j\frac{1}{60}\sin x_q^J + k\frac{1}{55}\sin(x_q^R + x_q^K),$$

$$h_q(x_q) = \frac{1}{30}\sin x_q^R + i\frac{1}{20}\sin x_q^I + j\frac{1}{25}\sin x_q^J + k\frac{1}{25}\sin(x_q^I + x_q^K),$$

$$a_1(t) = 0.4 + 0.1|\cos\sqrt{2}t| + 0.2i\cos 2t + (0.2 - 0.05\cos 3t)j + 0.1k\sin\sqrt{2}t,$$

$$a_2(t) = 0.6 - 0.1\sin\sqrt{3}t + 0.3i\sin\sqrt{2}t + 0.2j\sin\sqrt{3}t + 0.25k\cos 2t,$$

$$b_{11}(t) = b_{12}(t) = 0.2\cos\sqrt{3}t + 0.2i\cos 2t + 0.4j\sin\sqrt{2}t + 0.3k\sin 2t,$$

$$b_{21}(t) = b_{22}(t) = 0.4\sin 2t + 0.2i\cos 3t + 0.5j\sin\sqrt{3}t + 0.5k\sin\sqrt{2}t,$$

$$c_{11}(t) = c_{12}(t) = 0.5\sin\sqrt{2}t + 0.5i\sin t + 0.8j\cos t + 0.6k\cos 2t,$$

$$c_{21}(t) = c_{22}(t) = 0.8\cos t + \frac{1}{1+t^2} + i\sin 2t + 0.5j\sin t + 0.9k\cos\sqrt{3}t,$$

$$T_{111}(t) = T_{112}(t) = 0.4\sin\sqrt{2}t + 0.3i\sin t + 0.5j\sin\sqrt{3}t + 0.2k\cos\sqrt{2}t,$$

$$T_{121}(t) = T_{122}(t) = 1.2\sin t + 1.5i\cos t + 0.7j\cos\sqrt{2}t + k\cos t,$$

$$T_{211}(t) = T_{212}(t) = 0.5\cos 2t + 0.8i\sin t + 0.6j\sin\sqrt{2}t + 0.7k\sin\sqrt{2}t,$$

$$T_{221}(t) = T_{222}(t) = 2\cos 2t + 1.6i\cos t + 0.9j\cos 2t + k\sin 2t,$$

$$u_1(t) = 0.2\Big(\sin\sqrt{3}t + e^{-t^2\cos^2 t}\Big) + 0.4i\cos 2t + 0.3j\cos\sqrt{2}t + 0.3k\sin\sqrt{3}t,$$

$$u_2(t) = 0.3(\cos\sqrt{2}t + e^{-t^2\cos^2 t}) + 0.4i\sin\sqrt{2}t + 0.35j\sin 2t + 0.2k\cos\sqrt{3}t.$$

If $\mathbb{T} = \mathbb{R}$*, we take*

$$\eta_p(t) = 0.1|\sin 2t|, \quad \tau_{pq}(t) = 0.2|\cos t|, \quad \delta_{pql}(t) = \vartheta_{pql}(t) = 0.3|\sin 2t|.$$

If $\mathbb{T} = \mathbb{Z}$*, we take*

$$\eta_p(t) = 2|\cos(2\pi t)|, \quad \tau_{pq}(t) = 3|\sin \pi t|, \quad \delta_{pql}(t) = \vartheta_{pql}(t) = 2|\sin(2\pi t)|.$$

By a simple calculation, we have

$$L_q^f = \frac{1}{40}, \quad L_q^g = \frac{1}{50}, \quad L_q^h = L_l^h = \frac{1}{20}, \quad M_q^h = M_l^h = \frac{1}{20},$$

$$a_1^- = 0.4, \quad a_2^- = 0.5, \quad a_1^+ = 0.5, \quad a_2^+ = 0.7, \quad \hat{a}_1^+ = 0.2, \quad \hat{a}_2^+ = 0.3,$$

$$b_{11}^+ = b_{12}^+ = 0.4, \quad b_{21}^+ = b_{22}^+ = 0.5, \quad c_{11}^+ = c_{12}^+ = 0.8, \quad c_{21}^+ = c_{22}^+ = 1,$$

$$T_{111}^+ = T_{112}^+ = 0.5, \quad T_{121}^+ = T_{122}^+ = 1.5, \quad T_{211}^+ = T_{212}^+ = 0.8, \quad T_{221}^+ = T_{222}^+ = 2.$$

When $\mathbb{T} = \mathbb{R}$*, we have*

$$\eta_p^+ = 0.1, \quad \tau_{pq}^+ = 0.2, \quad \delta_{pql}^+ = \vartheta_{pql}^+ = 0.3, \quad \Xi_1 = 0.322, \quad \Xi_2 = 0.463$$

and

$$\max\left\{\frac{\Xi_1}{a_1^-}, \Big(1+\frac{a_1^+}{a_1^-}\Big)\Xi_1, \frac{\Xi_2}{a_2^-}, \Big(1+\frac{a_2^+}{a_2^-}\Big)\Xi_2\right\}$$
$$= \max\{0.805, 0.7245, 0.926, 0.6482\} = 0.926 = \rho < 1.$$

When $\mathbb{T} = \mathbb{Z}$*, we have*

$$\eta_p^+ = 0, \quad \tau_{pq}^+ = 3, \quad \delta_{pql}^+ = \vartheta_{pql}^+ = 2, \quad \Xi_1 = 0.272, \quad \Xi_2 = 0.393$$

and

$$\max\left\{\frac{\Xi_1}{a_1^-}, \Big(1+\frac{a_1^+}{a_1^-}\Big)\Xi_1, \frac{\Xi_2}{a_2^-}, \Big(1+\frac{a_2^+}{a_2^-}\Big)\Xi_2\right\}$$
$$= \max\{0.68, 0.34, 0.786, 0.5502\} = 0.786 = \rho < 1.$$

Hence, whether $\mathbb{T} = \mathbb{R}$ *or* $\mathbb{T} = \mathbb{Z}$*, all the conditions of Theorems 3.1 and 3.2 are satisfied. Consequently, we know that system* (1.1) *has a pseudo almost periodic solution, which is globally exponentially stable. Simulated by Matlab, when* $\mathbb{T} = \mathbb{R}$ *and* $\mathbb{T} = \mathbb{Z}$*, Figures 1 and 2 show the time responses of the variables of system* (1.1)*. Figure 1 has initial values*

$$(x_1^R(0), x_2^R(0))^T = (-0.5, -0.1)^T, (0.5, 0.3)^T, (0.1, -0.2)^T,$$
$$(x_1^I(0), x_2^I(0))^T = (-0.2, -0.5)^T, (-0.3, 0.1)^T, (0.4, 0.2)^T,$$
$$(x_1^J(0), x_2^J(0))^T = (-0.1, 0.1)^T, (0.3, -0.3)^T, (0.5, -0.5)^T,$$
$$(x_1^K(0), x_2^K(0))^T = (-0.25, 0.3)^T, (-0.5, 0.5)^T, (-0.15, 0.1)^T.$$

Figure 2 has initial values

$$(x_1^R(0), x_2^R(0))^T = (0.1, -0.2)^T, (0.2, -0.3)^T, (0.4, -0.5)^T,$$
$$(x_1^I(0), x_2^I(0))^T = (0.5, -0.4)^T, (-0.3, -0.1)^T, (0.2, 0.35)^T,$$
$$(x_1^J(0), x_2^J(0))^T = (-0.5, 0.1)^T, (-0.1, 0.3)^T, (-0.3, 0.5)^T,$$
$$(x_1^K(0), x_2^K(0))^T = (0.05, -0.2)^T, (-0.4, -0.5)^T, (0.25, 0.5)^T.$$

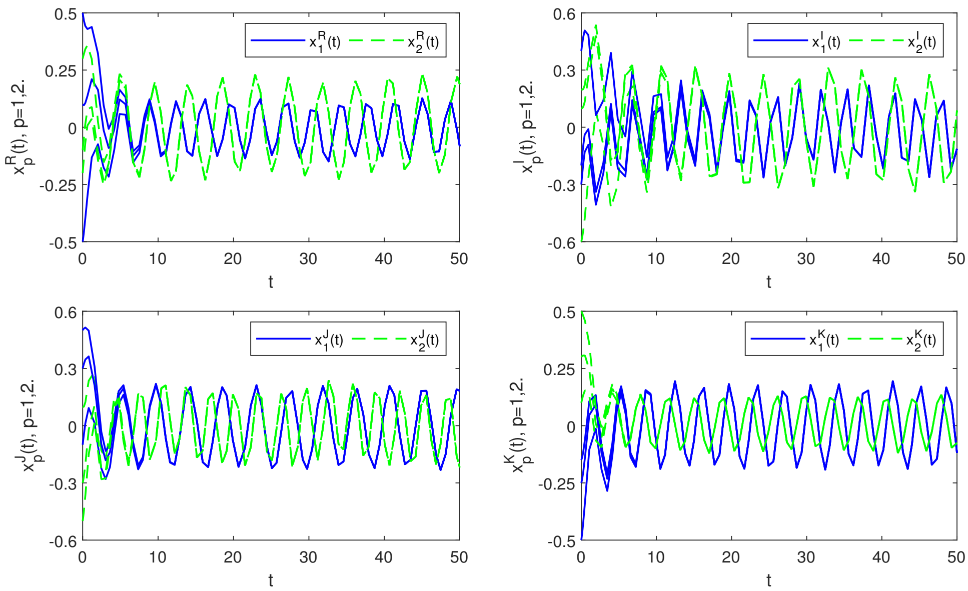

Figure 1. $\mathbb{T} = \mathbb{R}$. Transient states of the solutions $x_p^R(t)$, $x_p^I(t)$, $x_p^J(t)$ and $x_p^K(t)$, where $p = 1, 2$.

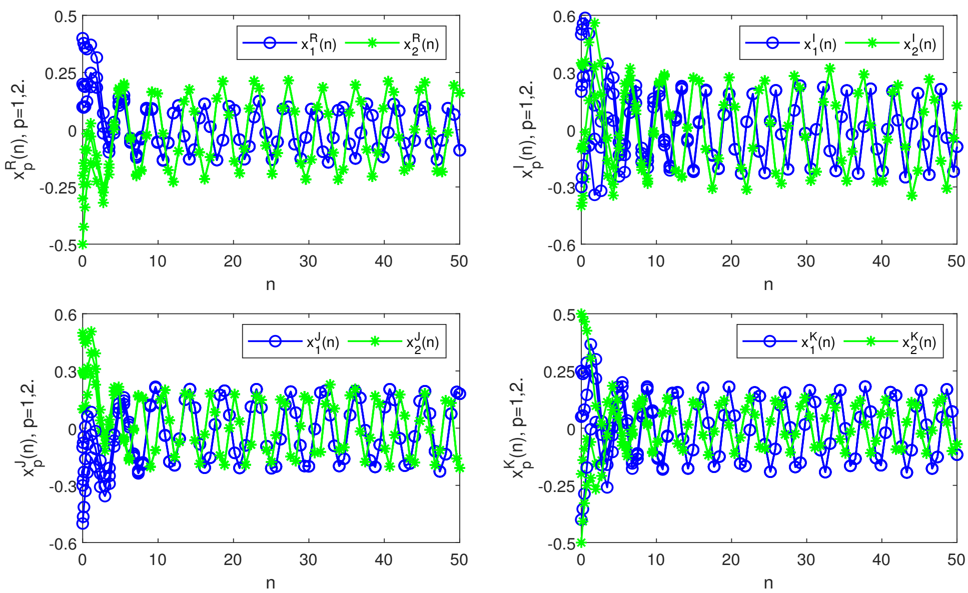

Figure 2. $\mathbb{T} = \mathbb{Z}$. Transient states of the solutions $x_p^R(n)$, $x_p^I(n)$, $x_p^J(n)$ and $x_p^K(n)$, where $p = 1, 2$.

5. Conclusions

In this paper, we have established the existence and global exponential stability of pseudo almost periodic solutions for quaternion-valued high-order Hopfield neural networks with time-varying delays and leakage delays on time scales. The results of this paper are essentially new. In addition, we expect to extend this work to study other types of quaternion-valued neural networks on time scales.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant No. 11861072.

Conflict of interest

Both authors declare no conflicts of interest in this paper.

References

1. B. Xu, X. Liu, X. Liao, Global asymptonic stability of high-order Hopfield neural networks with time delays, *Comput. Math. Appl.*, **45** (2003), 1729–1737.

2. X. Lou, B. Cui, Novel global stability criteria for high-order Hopfield-type neural networks with time-varying delays, *J. Math. Anal. Appl.*, **330** (2007), 144–158.

3. C. Ou, Anti-periodic solutions for high-order Hopfield neural networks, *Comput. Math. Appl.*, **56** (2008), 1838–1844.

4. Y. Yu, M. Cai, Existence and exponential stability of almost-periodic solutions for high-order Hopfield neural networks, *Math. Comput. Model.*, **47** (2008), 943–951.

5. B. Xiao, H. Meng, Existence and exponential stability of positive almost periodic solutions for high-order Hopfield neural networks, *Appl. Math. Model.*, **33** (2009), 532–542.

6. L. Duan, L. Huang, Z. Guo, Stability and almost periodicity for delayed high-order Hopfield neural networks with discontinuous activations, *Nonlinear Dynam.*, **77** (2014), 1469–1484.

7. C. Xu, P. Li, Pseudo almost periodic solutions for high-order Hopfield neural networks with time-varying leakage delays, *Neural Process. Lett.*, **46** (2017), 41–58.

8. C. Xu, P. Li, Y. Pang, Global exponential stability for interval general bidirectional associative memory (BAM) neural networks with proportional delays, *Math. Method. Appl. Sci.*, **39** (2016), 5720–5731.

9. F. Kong, Q. Zhu, K Wang, J. J. Nieto, Stability analysis of almost periodic solutions of discontinuous BAM neural networks with hybrid time-varying delays and D operator, *J. Franklin. I.*, **356** (2019), 11605–11637.

10. F. Kong, R. Rajan, Finite-time and fixed-time synchronization control of discontinuous fuzzy Cohen-Grossberg neural networks with uncertain external perturbations and mixed time, *Fuzzy Set. Syst.*, **411** (2021), 105–135.

11. W. Shen, X. Zhang, Y. Wang, Stability analysis of high order neural networks with proportional delays, *Neurocomputing*, **372** (2020), 33–39.

12. Z. Dong, X. Zhang, X. Wang, State estimation for discrete-time high-order neural networks with time-varying delays, *Neurocomputing*, **411** (2020), 282–290.

13. K. Gopalsamy, Kondalsamy, Leakage delays in BAM, *J. Math. Anal. Appl.*, **325** (2007), 1117–1132.

14. P. Balasubramaniam, V. Vembarasan, R. Rakkiyappan, Leakage delays in TCS fuzzy cellular neural networks, *Neural Process. Lett.*, **33** (2011), 111–136.

15. R. Sakthivel, P. Vadivel, K. Mathiyalagan, A. Arunkumar, M. Sivachitra, Design of state estimator for bidirectional associative memory neural networks with leakage delays, *Inform. Sciences*, **296** (2015), 263–274.

16. C. Xu, L. Chen, P. Li, Effect of proportional delays and continuously distributed leakage delays on global exponential convergence of CNNs, *Asian J. Control*, **21** (2019), 2476–2483.

17. S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18–56.

18. Y. Li, C. Wang, Uniformly almost periodic functions and almost periodic solutions to dynamic equations on time scales, *Abstr. Appl. Anal.*, **2011** (2011), 341520.

19. Y. Li, L. Yang, Almost automorphic solution for neutral type high-order Hopfield neural networks with delays in leakage terms on time scales, *Appl. Math. Comput.*, **242** (2014), 679–693.

20. W. Yang, W. Yu, J. Cao, F. Alsaadi, T. Hayat, Almost automorphic solution for neutral type high-order Hopfield BAM neural networks with time-varying leakage delays on time scales, *Neurocomputing*, **267** (2017), 241–260.

21. Y. Li, X. Meng, L. Xiong, Pseudo almost periodic solutions for neutral type high-order Hopfield neural networks with mixed time-varying delays and leakage delays on time scales, *Int. J. Mach. Learn. Cyb.*, **8** (2017), 1915–1927.

22. T. Isokawa, T. Kusakabe, N. Matsui, F. Peper, *Quaternion neural network and its application*, Berlin, Springer, 2003.

23. N. Matsui, T. Isokawa, H. Kusamichi, F. Peper, H. Nishimura, Quaternion neural network with geometrical operators, *J. Intell. Fuzzy Syst.*, **15** (2004), 149–164.

24. M. Yoshida, Y. Kuroe, T. Mori, Models of hopfield-type quaternion neural networks and their energy functions, *Int. J. Neural Syst.*, **15** (2005), 129–135.

25. Y. Li, J. Qin, B. Li, Existence and global exponential stability of anti-periodic solutions for delayed quaternion-valued cellular neural networks with impulsive effects, *Math. Method. Appl. Sci.*, **42** (2019), 5–23.

26. N. Huo, B. Li, Y. Li, Existence and exponential stability of anti-periodic solutions for inertial quaternion-valued high-order Hopfield neural networks with state-dependent delays, *IEEE Access*, **7** (2019), 60010–60019.

27. Y. Li, H. Wang, X. Meng, Almost automorphic synchronization of quaternion-valued high-order Hopfield neural networks with time-varying and distributed delays, *IMA J. Math. Control I.*, **36** (2019), 983–1013.
28. Y. Li, J. Xiang, Existence and global exponential stability of almost periodic solution for quaternion-valued high-order Hopfield neural networks with delays via a direct method, *Math. Method. Appl. Sci.*, **43** (2020), 6165–6180.
29. H. Wang, G. Wei, S. Wen, T, Huang, Impulsive disturbance on stability analysis of delayed quaternion-valued neural networks, *Appl. Math. Comput.*, **390** (2021), 125680.
30. C. Zhang, Pseudo almost periodic solutions of some differential equations, *J. Math. Anal. Appl.*, **151** (1994), 62–76.
31. T. Diagana, Pseudo almost periodic solutions to some differential equations, *Nonlinear Anal. Theor.*, **60** (2005), 1277–1286.
32. F. Kong, X. Fang, Pseudo almost periodic solutions of discrete-time neutral-type neural networks with delays, *Appl. Intell.*, **48** (2018), 3332–3345.
33. A. Zhang, Almost periodic solutions for SICNNs with neutral type proportional delays and D operators, *Neural Process. Lett.*, **47** (2018), 57–70.
34. A. Zhang, Pseudo almost periodic high-order cellular neural networks with complex deviating arguments, *Int. J. Mach. Learn. Cyb.*, **10** (2019), 301–309.
35. Y. Li, C. Wang, Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales, *Adv. Differ. Equ.*, **2012** (2012), 77.
36. A. Arbi, J. Cao, Pseudo-almost periodic solution on time-space scales for a novel class of competitive neutral-type neural networks with mixed time-varying delays and leakage delays, *Neural Process. Lett.*, **46** (2017), 719–745.
37. A. Zhang, Pseudo almost periodic solutions for neutral type SICNNs with D operator, *J. Exp. Theor. Artif. In.*, **29** (2017), 795–807.
38. M. Bohner, A. Peterson, *Dynamic equations on time scales: An introuduction with applications*, Springer Science & Business Media, 2001.
39. M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Springer Science & Business Media, 2002.