AIMS Mathematics, 6(9): 10055–10069.
DOI: 10.3934/math.2021584
Received: 06 April 2021
Accepted: 28 June 2021
Published: 07 July 2021

http://www.aimspress.com/journal/Math

Research article

Computation of traveling wave solution for nonlinear variable-order fractional model of modified equal width equation

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Abstract: The Variable-order fractional operators (VO-FO) have considered mathematically formalized recently. The opportunity of verbalizing evolutionary leading equations has led to the effective application to the modeling of composite physical problems ranging from mechanics to transport processes, to control theory, to biology. In this paper, find the closed form traveling wave solutions for nonlinear variable-order fractional evolution equations reveal in all fields of sciences and engineering. The variable-order evolution equation is an impressive mathematical model describes the complex dynamical problems. Here, we discuss space-time variable-order fractional modified equal width equation (MEWE) and used $\exp(-\phi(\xi))$ method in the sense of Caputo fractional-order derivative. Based on variable order derivative and traveling wave transformation convert equation into nonlinear ordinary differential equation (ODE). As a result, constructed new exact solutions for nonlinear space-time variable-order fractional MEWE. It clearly shows that the nonlinear variable-order evolution equations are somewhat natural and efficient in mathematical physics.

Keywords: space-time fractional modified equal width equation; Caputo derivative of fractional order; $\exp(-\phi(\xi))$ method
Mathematics Subject Classification: 35C07, 35C08, 35K05, 35P99

1. Introduction

Fractional order differential and integral operator is a generalization of traditional integer order integration and derivation gain more attention from the last two decades because of physical interpretation in different fields such as biology, economics, biochemistry, medicine, and engineering science. The idea of fractional calculus is old as classical calculus; first time was discussed by Leibniz and L'Hospital in 1965. Nonlinear fractional evolution equations describe the complex phenomena in different areas such as biology, acoustics, physics, finance, and fractional dynamics [1, 2]. Many researchers have solved linear, nonlinear, integers and fractional order problems by various numerical and analytical methods. Numerical methods such as finite fourth order difference methods are used by Ali et al. [3]. They solved the non-integer order sub-diffusion model and find the theoretical analysis of stability and convergence. In another literature [4], they used 2D diffusion equation of fractional order by Crank-Nicolson scheme. Jiang and Jingtan [5] developed the high order finite element approach for fractional order differential equation and find the rate of convergence order $O(\tau^{2-\alpha}+N^{-r})$. Srivastava et al. [6] discussed the local meshless method for 3D convection diffusion equation. They approximated the space derivatives based on meshless procedure and fractional order time derivative are by Caputo derivative. The 2D time fractional order differential equation is solved in [7–9], they used numerical approximation and discussed the stability and convergence analysis for the diffusion model of fractional order. Ahmad et el. [10] studied new and simple numerical approach for the fifth order KdV equation. Also, compared the obtained values with Adomian's decomposition method and briefly explained the theoretical analysis to assess the accuracy. Ali and Abdullah [11] developed the Saul'yev technique explicitly for 2D diffusion model with stability analysis and provided test examples to demonstrate the accuracy. Ahmad et al. [12] suggested an efficient local differential quadrature technique for 2D hyperbolic telegraph equation. The time and space derivatives are approximated based on time integration technique and multiquadric radial basis, respectively. Balasim and Mohd Ali [13] worked on 2D fractional order Cable equation and find the solution by two numerical methods fully implicit and Crank-Nicolson method. Akgül [14] developed a novel approach based on reproducing kernel Hilbert space function and used Atanagana-Baleanu fractional derivatives. They solved fractional initial values problems to demonstrate the numerical results. Akgül et al. [15] considered the fractional order integrator circuit model and established a unique solution. They find out the stability analysis and numerical results of the proposed model by Atanaga-Toufik scheme. The new Caputo definition is discretized introduced by [16] having applications in control theory. Akgül [17] studied the solution of the fractional order differential model and used the Laplace transform to get the solution.

The analytical effective methods to construct the solitary wave for differential equations such as Shang and Zheng [18] constructed all possible exact solutions by (G'/G) method. Yokus et al. [19] solved the Bogoyavlenskii equation and used (G′/G, 1/G)-expansion and (1/G′)-expansion to find the exact traveling wave solution. Barman et al. [20] studied the interesting nonlinear equations Riemann wave and Novikov-Veselov that describes the tidal and tsunami types of waves in the ocean. The author's implemented the generalized Kudryashov technique for the exact solution of the proposed equations and obtained various solitons. Jawad et al. [21] discussed the nonlinear evolution equations which describes the nerve propagation in biology and genetics. They applied the simple equation approach for the suggested equations and discussed the physical phenomena. In [22–24] authors presented the (G′/G)-expansion to find the solitons solutions for the evolution equations and used the Jumaire's definition for fractional order derivative. Bashan et al. [25] combined the finite difference procedure (FDP) with quadratic differential scheme (QDS) to discuss the solution of modified wave type physical phenomenon. They obtained and discussed the solitary wave nature solution. They recorded and listed the error norms and solution is displayed against several emerging parameters in the form of graphs. Modified spline technique (MST) has been adopted by Bashan et al. [26] to compute the soliton solution of nonlinear Schrodinger equation. They examined the efficiency and effectiveness of proposed procedure for five different problems and found an excellent agreement while computing the error norms. Few important contributions are covered in [27–30]. The three models of shallow water wave equations are determined by Wazwaz [31]. The Hirota bilinear approach was used for multiple solitons solutions and the coth-tanh for single soliton solution. Hosseini et al. [32] considered the special type of mathematical model (3+1)-dimensional breaking solitons equation and used the linear superposition method. The method shown high efficiency and strongly handled the nonlinear model. In the said literature partial and fractional order differential equations are solved successfully. Fractional order is sometime a function of dependent or independent variables which are more appropriate to discuss the diffusion processes in porous medium and medium structure [33]. The reaction kinetics of proteins has been originated to show simple mechanisms that are accurately defined by fractional order changes with temperature [34]. These examples show that the variable-order operator describes some classes of physical models better than fractional order. In the review article Sun et al. [35] provided basics definitions, models, numerical techniques, and their applications. So far, in the previous literature many researchers have solved the variable-order fractional evolution equations by various numerical methods such as Shekari [36] solved the 2D time fractional variable-order wave equation base on numerical moving least squares approach for different domain. The resulted solution confirmed the efficiency and easy implementation on variable order models. Chen et al. [37] focused on the variable order Stokes first problem and found the solution numerically. Also, discussed the theoretical analysis via Fourier series. The theoretical analysis supported the obtained numerical solution. The advection-diffusion equation of variable-order are solved with nonlinear forcing term explicitly and implicitly by Zhuang [38]. Chen et al. [39] considered the anomalous diffusion of variable order equation with numerical algorithm. The theoretical analysis of stability, convergence and solvability via Fourier were discussed. The numerical solutions were effective, and the proposed scheme is powerful for such types of variable order models. The studies reported in [46–48] discussed the chaotic analysis by using the fractional operators.

The aim of this paper to extend the closed form traveling waves solution to the nonlinear variable-order fractional evolution equations. Here, we solve nonlinear space-time variable-order fractional MEWE based on variable-order Caputo derivative by $\exp\left(-\phi(\xi)\right)$ method. The variable-order problems are apparently more complicated than a constant fractional order problem, and evolution of a system can be more accurately described. This contribution seems natural and modeled many systems with variable-order [40]. The closed form solutions for variable-order evolution equations are to the author's knowledge unavailable and we hope that it a good contribution to the literature. Few important contributions relating the concepts of variable order fractional operators and other related studies are reported in [49–70].

2. Caputo fractional derivative

The Caputo fractional derivative operator is discussed many times in the literature. Moreover, the variable-order fractional operators [49–52] are progressively discussed and established more definitions by researchers [41–44]. Here, the variable-order Caputo fractional derivative as follows:

$$D_t^{\delta(x,t)}u(x,t) = \begin{cases} \frac{1}{\Gamma(1+\delta(x,t))}\int_0^t (t-\xi)^{-\delta(x,t)}\frac{d}{dt}u(x,t)d\xi, 0<\delta(x,t)<1, \\ \frac{d}{dt}u(x,t), \qquad \delta(x,t)=1. \end{cases} \tag{1}$$

And the properties are as follows:

$$D_t^{\delta(x,t)}u(x,t) = \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\delta(x,t))}t^{\gamma-\delta(x,t)},\ 0<\delta(x,t)<1 \tag{2}$$

3. Methodology of $\exp(-\phi(\xi))$ method

Consider the following nonlinear variable order FPDE of order $\gamma(x,y,..,t), \delta(x,y,..,t)$ and $\rho(x,y,..,t)$ is given by the form:

$$H\left(D_x^{\gamma(x,y,..,t)}u, D_y^{\delta(x,y,..,t)}D_t^{\rho(x,y,..,t)}u, D_x^{2\gamma(x,y,..,t)}D_y^{3\delta(x,y,..,t)}u, \ldots\right) = 0. \tag{3}$$

Where Eq (3) represents a polynomial H in u and the fractional derivatives represented by the variable order $\gamma(x,y,..,t), \delta(x,y,..,t)$ and $\delta(x,y,..,t)$. The linear and nonlinear of highest order terms are involved in the FPDE. The $\exp(-\phi(\xi))$ method explained briefly as following [45].

Taking the variable order fractional transformation equation.

$$u(x,y,z,t) = U(\xi), \xi = \frac{kx^{\gamma(x,y,..,t)}}{\Gamma(1+\gamma(x,y,..,t))} + \frac{ly^{\rho(x,y,..,t)}}{\Gamma(1+\rho(x,y,..,t))} + \cdots - \frac{\omega t^{\delta(x,y,..,t)}}{\Gamma(1+\delta(x,y,..,t))}. \tag{4}$$

Transforms Eq (3) into an ODE with respect to ξ as follows:

$$\mathrm{H}(U, \omega U', kU'', lU''', \omega lU''', klU''', \ldots) = 0, \tag{5}$$

Here k, l and ω are constants. Let the solution is in the form:

$$U(\xi) = a_0 + a_1 exp(-\phi(\xi)) + \cdots + a_n exp(-n\phi(\xi)). \tag{6}$$

Where n is calculated by using homogenous balance principle and $\phi(\xi)$ is a function that satisfies a first order equation as

$$\phi'(\xi) = exp(-\phi(\xi)) + \mu exp(-\phi(\xi)) + \lambda, \tag{7}$$

The all-possible solution of Eq (7) is as following:

Case 1. If $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, then

$$\phi_1(\xi) = \ln\left(\frac{1}{2\mu}\left(-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + C)\right) - \lambda\right)\right).$$

Case 2. If $\lambda^2 - 4\mu > 0,\ \mu = 0$, then

$$\phi_2(\xi) = -\ln\left(\frac{\lambda}{\cosh\big(\lambda(\xi + C)\big) + \sinh\big(\lambda(\xi + C)\big) - 1}\right).$$

Case 3. If $\lambda^2 - 4\mu < 0$, then

$$\phi_3(\xi) = \ln\left(\frac{1}{2\mu}\left(\sqrt{4\mu - \lambda^2}\tanh\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + C)\right) - \lambda\right)\right).$$

Case 4. If $\lambda^2 - 4\mu = 0,\ \mu \neq 0,$ and $\lambda \neq 0$, then

$$\phi_4(\xi) = \ln\left(-\frac{2\lambda(\xi + C) + 4}{\lambda^2(\xi + C)}\right).$$

Case 5. If $\lambda^2 - 4\mu = 0,\ \mu = 0,$ and $\lambda = 0$, then

$$\phi_5(\xi) = \ln(\xi + C).$$

Substituting the values of constants and solution of Eq (7) in (6), we obtained the exact solutions of (3).

4. Formulation for the solutions of modified equal width equation

The MEWEs are utilized for the reproduction of wave transmission in the nonlinear media and represents soliton wave solutions with both $+$ve and $-$ve amplitudes and having equal width. Here, we apply the proposed method to study nonlinear space-time variable order fractional MEWE, and construct traveling wave solution based on $\exp\big(-\phi(\xi)\big)$ method.

Consider the variable-order fractional MEWE as follows:

$$D_t^{\alpha(x,t)}u + \sigma u^2\, D_x^{\beta(x,t)}u - \alpha D_x^{2\beta(x,t)}\left(D_t^{\alpha(x,t)}u\right) = 0. \tag{8}$$

Using as $\xi = \frac{kx^{\beta(x,t)}}{\Gamma\left(1+\beta(x,t)\right)} - \frac{\omega t^{\alpha(x,t)}}{\Gamma\left(1+\alpha(x,t)\right)}$, Eq (8) reduces to

$$-\omega U + \sigma k U^3 + a k^2 \omega \frac{d^2}{d\xi^2}U = 0. \tag{9}$$

By balancing the uppermost nonlinear term $(U)^3$ with the uppermost linear term with U'' we obtain $M = 1$. Therefore Eq (6) becomes:

$$U = a_0 + a_1 exp\big(-\phi(\xi)\big). \tag{10}$$

Inserting (10), into (9) in term of $\left(e^{(-\phi(\xi))}\right)$. Equating the like powers of $\left(exp\big(-\phi(\xi)\big)\right)^n$, obtained the system of equations given in the following.

$$\left(exp\left(-\phi(\xi)\right)\right)^0 : ka_1^3 + 2ak^2\omega a_1 = 0,$$

$$\left(exp\left(-\phi(\xi)\right)\right)^1 : \sigma k a_0 a_1^2 + \sigma a k^2 \omega a_1 \lambda = 0,$$

$$\left(exp\left(-\phi(\xi)\right)\right)^2 : -\omega a_1 + \sigma k a_0^2 a_1 + 2ak^2\omega a_1 \mu + ak^2\omega a_1 \lambda^2 = 0,$$

$$\left(exp\left(-\phi(\xi)\right)\right)^3 : -\omega a_1 + k a_0^3 + a k^3 \omega a_1 \mu \lambda = 0,$$

Solving the above system of equations by using computer algebra and obtained

$$a_0 = \frac{1}{2}a_1\lambda, a_1 = a_1,\ \omega = \pm\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}}{4a}, k = \pm\frac{2}{\sqrt{8a\mu-2a\lambda^2}}, \quad (11)$$

where ω, k, α, β and σ are arbitrary constants.

Substituting (11) into (10), we obtained

$$U_1 = \frac{1}{2}a_1\lambda + a_1 exp\left(-\phi(\xi)\right). \quad (12)$$

Now, substituting the solutions of Eq (7), we can obtain five kinds of distinct traveling wave solutions for the variable order fractional MEWE (8).

Here, for variable-order factional modified equal width equation we are writing three cases for Eq (8), we obtain.

Case 1. If $\lambda^2 - 4\mu > 0$, and $\mu \neq 0$. Then

$$u_1(x,t) = \frac{a_1\left(\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}+\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)-\lambda\right)-4\mu\right)}{2\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}+\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)-\lambda\right)},$$

Case 2. If $\lambda^2 - 4\mu > 0, \mu = 0$, then

$$u_2(x,t) = \frac{a_1\lambda\left(\begin{array}{c} cosh\left(\lambda\left(\frac{2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}+\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)\right)+ \\ \sinh\left(\lambda\left(\frac{2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}+\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)\right)+1\end{array}\right)}{\begin{array}{c} cosh\left(\lambda\left(\frac{2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}+\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)\right)+ \\ \sinh\left(\lambda\left(\frac{2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}+\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)\right)-1\end{array}}$$

Case 3. If $\lambda^2 - 4\mu < 0$, then

$$u_3 = \frac{a_1\left(\lambda\sqrt{4\mu-\lambda^2}\tanh\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}+\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)-\lambda\right)-4\mu\right)}{2\sqrt{4\mu-\lambda^2}\tanh\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}+\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)-\lambda\right)}.$$

Now for

$$a_0 = \frac{1}{2}a_1\lambda, a_1 = a_1,\ \omega = \frac{a_1^2\sqrt{8a\mu-2a\lambda^2}}{4a},\ k = -\frac{2}{\sqrt{8a\mu-2a\lambda^2}}.$$

Case 4. If $\lambda^2 - 4\mu > 0$, and $\mu \neq 0$. Then

$$u_4 = \frac{a_1\left(\lambda\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{-2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}-\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)-\lambda\right)-4\mu\right)}{2\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{-2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}-\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)-\lambda\right)},$$

Case 5. If $\lambda^2 - 4\mu > 0, \mu = 0$, and $\lambda \neq 0$, Then

$$u_5 = \frac{a_1\lambda\left(\begin{array}{l}\cosh\left(\lambda\left(\frac{-2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}-\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)\right)+\\ sinh\left(\lambda\left(\frac{-2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}-\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)\right)+1\end{array}\right)}{\begin{array}{l}cosh\left(\lambda\left(\frac{-2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}-\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)\right)+\\ sinh\left(\lambda\left(\frac{-2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}-\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)\right)-1\end{array}},$$

Case 6. If $\lambda^2 - 4\mu < 0$, and $\mu \neq 0$, Then

$$u_6 = \frac{a_1\left(\lambda\sqrt{4\mu-\lambda^2}\tanh\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{-2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}-\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)-\lambda\right)-4\mu\right)}{2\sqrt{4\mu-\lambda^2}\tanh\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{-2x^{\alpha(x,t)}}{\Gamma(1+\alpha(x,t))\sqrt{8a\mu-2a\lambda^2}}-\frac{a_1^2\sqrt{8a\mu-2a\lambda^2}x^{\beta(x,t)}}{4a\Gamma(1+\beta(x,t))}+C\right)-\lambda\right)}.$$

5. Graphical representation

Lu et al. [71] constructed the exact traveling wave solution for modified equal width equation in the form of dark, bright, periodic and kink solitary wave solution. However, in the present study the graphical representation discussed for various values of $\alpha(x,y)$ and $\beta(x,y)$ as shown in Figures 1–5 for Eq (8), with the help of software Maple15. The obtained 3D plots are supported the soliton solution of variable-order fractional modified equal width equation for different values of the unknown parameters $k, \omega, \alpha, a_0, a_1, \lambda, \sigma$ and fractional order derivatives $0 < \alpha(x,t) \leq 1$ and $0 < \beta(x,t) \leq 1$, obtained Periodic, Kink and singular soliton type solutions for different values of parameters as following.

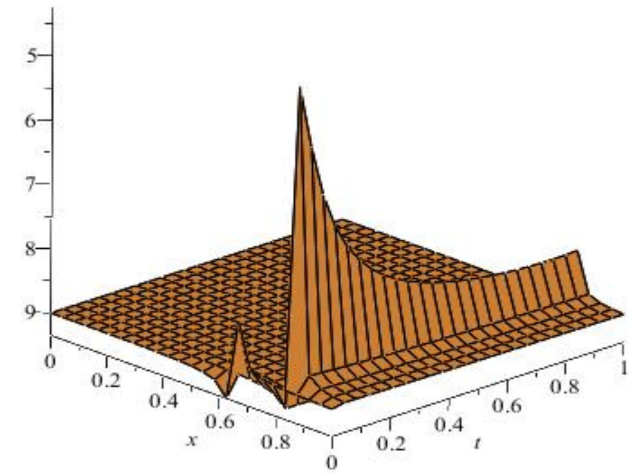


Figure 1. Soliton solution of Eq (8) for u_1 at $\omega = 0.1, k = 2, a_0 = 2, a_1 = 2, \alpha(x,t) = \frac{e^{(xt)} - 2\sin(xt)}{200}, \beta(x,t) = \frac{e^{(xt)} + cos(xt)}{10}, C = 0, \sigma = 3, \lambda = 9, \mu = 0.01,$ shown in 3D plot.

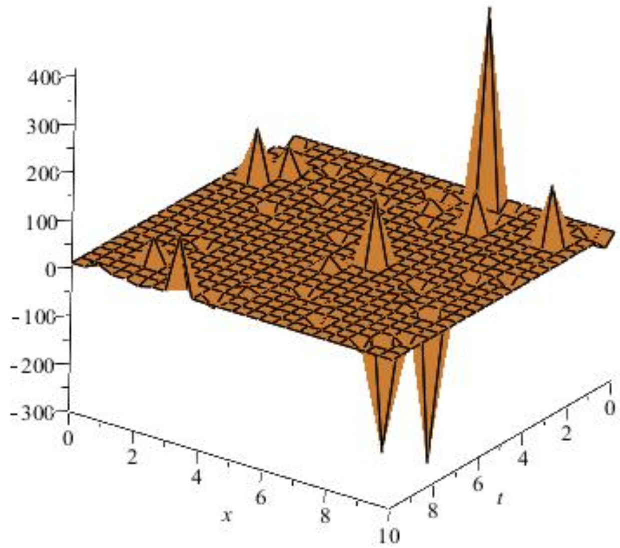
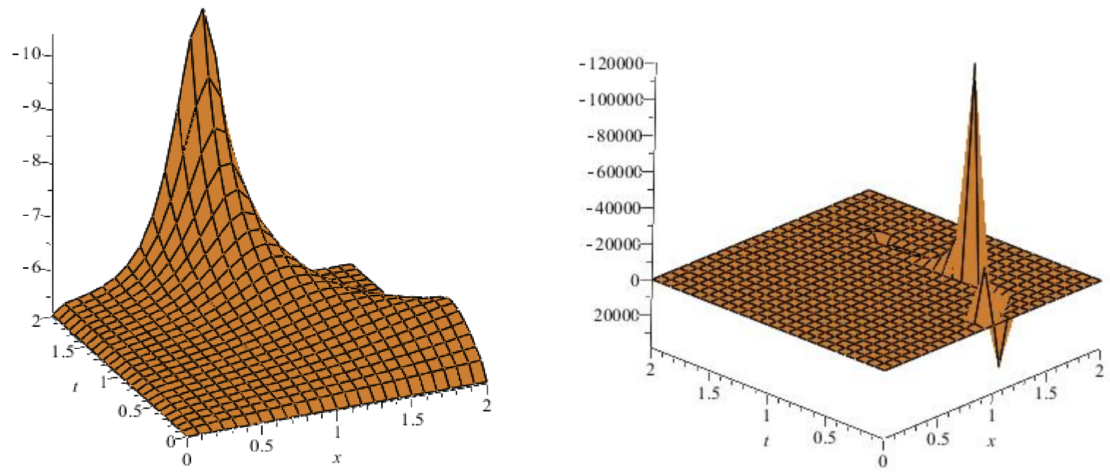
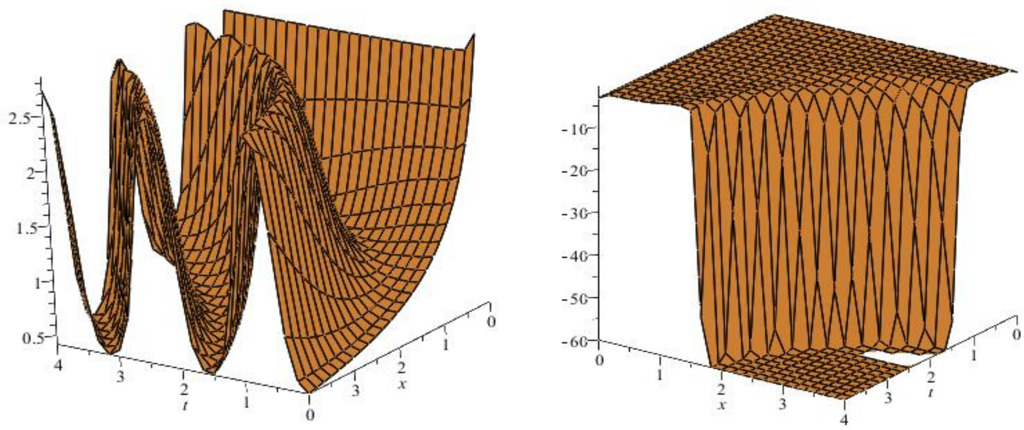


Figure 2. Singular shape solution of Eq (8) for u_2 at $\omega = 5, k = 2, a_0 = 2, a_1 = 4, \alpha(x,t) = \sin\left(xt + \frac{2\pi}{5}\right), \beta(x,t) = \sin\left(xt + \frac{1}{100}\right), C = 1, \sigma = 3, \lambda = 5, \mu = 6,$ shown in 3D plot.



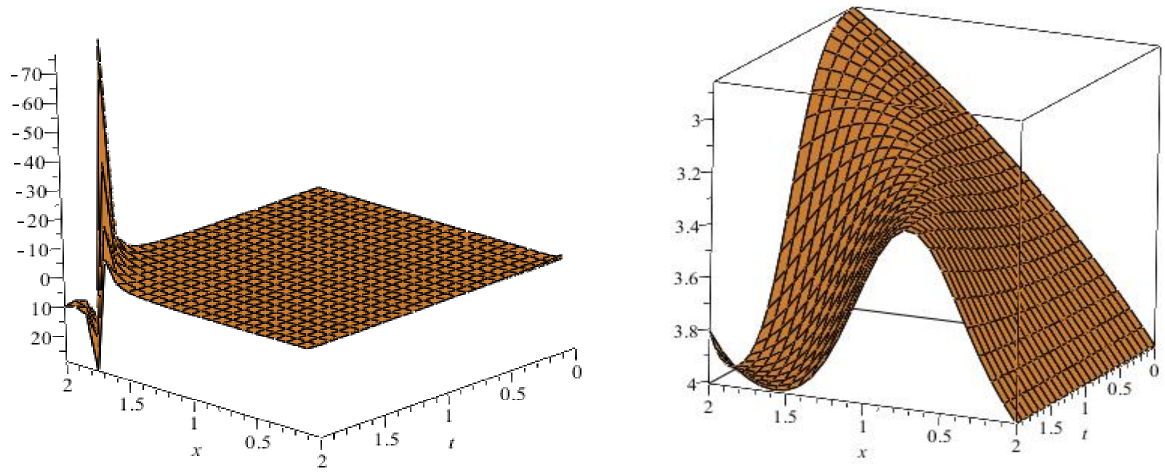
(a) (b)

Figure 3. Singular shape solution of Eq (8) for u_3 at $\omega = -0.5,\ k = 3, a_0 = 12, a_1 = 2.5,\ C = 0,\ \sigma = 3, \lambda = 5, \mu = 15,$ shown in 3D plot. (a) $\alpha(x,t) = \frac{e^{(xt)} - \sin(xt)}{20}, \beta(x,t) = \frac{e^{(xt)} + \cos(xt)}{100}$,(b) $\alpha(x,t) = \frac{e^{(xt)} - (xt)}{6}, \beta(x,t) = \frac{e^{(xt)} - (xt)^3}{5}$.



(a) (b)

Figure 4. Periodic and Kink types of soliton solution of Eq (8) for u_4 at $\omega = -0.1, k = 2, a_0 = -2, a_1 = 4, C = 1, \sigma = 3, \lambda = 5, \mu = 6,$ shown in 3D plot. (a) $\alpha(x,t) = \sin\left(3xt + \frac{2\pi}{60}\right),\ \beta(x,t) = \sin\left(xt + \frac{3}{80}\right)$, (b) $\alpha(x,t) = \frac{e^{(xt)} + 5\sin(xt)}{60},\ \beta(x,t) = \frac{e^{(xt)} + \cos(xt)}{150}$.



(a) (b)

Figure 5. Singular and Kink shape soliton solution of Eq (8) for u_5 at $\omega = -1.5, k = 6, a_0 = 2, a_1 = -5,\ C = 0, \sigma = 3, \lambda = 0, \mu = 0,$ shown in 3D plot. (a) $\alpha(x,t) = \frac{e^{(xt)} - (xt)^2}{8}, \beta(x,t) = \frac{e^{(xt)} - (xt)^3}{10},$ (b) $\alpha(x,t) = sin\left(2xt + \frac{2\pi}{5}\right), \beta(x,t) = cos\left(2xt + \right.$ *3100.*

6. Conclusions

In this paper, the nonlinear variable-order fractional evolution equation successfully solved and obtained new exact traveling wave solutions. This study shows that the variable-order fractional evolution equations are quite efficient and accurate. Here, nonlinear space-time variable-order fractional MEWE has solved successfully and constructed the possible exact solutions. Periodic, Kink and singular soliton type solutions are obtained for arbitrary values of variable order $\alpha(x,t)$ and $\beta(x,t)$ for the proposed variable-order model in fractional sense. This contribution is effective and seems more natural in the literature. The reported study will be extended by considering the fractional and variable order fractional operator for the modelling of conservation laws in different situations. So far no one considered the modeling of conservation laws in variable order. The future project will be a foundation in that direction.

Funding

This research received the Higher Education Commission (HEC) Start-up Research Grant Program (SRGP), ref no. 322/IPFP-II(Batch-I)/SRGP/NAHE/HEC/2020/126.

Conflicts of interest

All the authors declare no conflict of interest.

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