AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 10037–10054.
DOI: 10.3934/math.2021583
Received: 26 April 2021
Accepted: 23 June 2021
Published: 07 July 2021

Research article

Gevrey regularity for the generalized Kadomtsev-Petviashvili I (gKP-I) equation

Aissa Boukarou[1], **Kaddour Guerbati**[1], **Khaled Zennir**[2,*] **and Mohammad Alnegga**[2]

[1] Laboratoire de Mathématiques et Sciences appliquées Université de Ghardaia, Algerie

[2] Department of Mathematics, College of Sciences and Arts, Qassim University, Ar-Rass, Saudi Arabia

* **Correspondence:** Email: k.zennir@qu.edu.sa.

Abstract: The task of our work is to consider the initial value problem based on the model of the generalized Kadomtsev-Petviashvili I equation and prove the local well-posedness in an anisotropic Gevrey spaces and then global well-posedness which improves the recent results on the well-posedness of this model in anisotropic Sobolev spaces [17]. Also, wide information about the regularity of the solution in the time variable is provided.

Keywords: generalized Kadomtsev-Petviashvili I equation; anisotropic Gevrey space; radius of spatial analyticity
Mathematics Subject Classification: 35Q35, 35Q53

1. Introduction and position of problem

The study of nonlinear wave processes in real media with dispersion, despite the significant progress in this area in recent years, for example, [1–3] and numerous references in these works are still relevant. This, in particular, concerns the dynamics of oscillations in cases where high-energy particle fluxes occur in the medium, which significantly change such parameters of propagating wave structures, such as their phase velocity, amplitude and characteristic length. In recent and earlier years, a fairly large number of works have been devoted to studies of this kind of relativistic effects (see [12–14]).

Recently, the great interest on the KP equation has led to the construction and the study of many extensions to the KP equation . These new extended models propelled greatly the research that directly resulted in many promising findings and gave an insight into some novel physical features of scientific and engineering applications. Moreover, lump solutions, and interaction solutions between lump waves and solitons, have attracted a great amount of attentions aiming to make more progress in solitary waves theory. Lump solutions, have been widely studied by researchers for their significant features in physics and many other nonlinear fields [18–20].

Let $u = u(x, y, t), (x, y, t) \in \mathbb{R}^3$ and $\alpha \geq 4$. We consider the initial value problem for the generalized Kadomtsev-Petviashvili I equation,

$$\begin{cases} \partial_t u + |D_x|^\alpha \partial_x u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \\ u(x, y, 0) = f(x, y). \end{cases} \tag{1.1}$$

with

$$D_x^\alpha u(x, y, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |\xi|^\alpha \mathfrak{F}u(\xi, \mu, \tau) e^{ix\xi + iy\mu + it\tau} d\xi d\mu d\tau$$

This equation belongs to the class of Kadomtsev-Petviashvili equations, which are models for the propagation of long dispersive nonlinear waves which are essentially unidirectional and have weak transverse effects. Due to the asymmetric nature of the equation with respect to the spatial derivatives, it is natural to consider the Cauchy problem for (1.1) with initial data in the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$, defined by the norm

$$\|u\|_{H^{s_1,s_2}(\mathbb{R}^2)} = \left(\int \langle\xi\rangle^{2s_1} \langle\eta\rangle^{2s_2} |\widehat{u}(\xi, \eta)|^2 d\xi d\eta \right)^{1/2}.$$

Many authors have investigated the Cauchy problem for Kadomtsev-Petviashvili equations as in, for instance [4, 8, 16]. Yan et al. [17] established the local well-posedness of the Cauchy problem for the Kadomtsev-Petviashvili I equation in anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R})$ with $s_1 > -\frac{\alpha-1}{4}$, $s_2 \geq 0$ with $\alpha \geq 4$ and globally well-posed in $H^{s_1,0}(\mathbb{R})$ with $s_1 > -\frac{(\alpha-1)(3\alpha-4)}{4(5\alpha+3)}$ if $4 \leq \alpha \leq 5$ also proved that the Cauchy problem is globally well-posed in $H^{s_1,0}(\mathbb{R})$ with $s_1 > -\frac{\alpha(3\alpha-4)}{4(5\alpha+4)}$ if $\alpha > 5$. The authors in [8] proposed the problem

$$\begin{cases} \partial_t u - \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \\ u(x, y, 0) = u_0(x, y), \end{cases} \tag{1.2}$$

and proved that it is globally well-posed for given data in an anisotropic Gevrey space $G^{\sigma_1,\sigma_2}(\mathbb{R}^2), \sigma_1, \sigma_2 \geq 0$, with respect to the norm

$$\|f\|_{G^{\sigma_1,\sigma_2}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\sigma_1|\xi|} e^{2\sigma_2|\eta|} |\widehat{f}(\xi, \eta)|^2 d\xi d\eta \right)^{1/2}.$$

With initial data in anisotropic Gevrey space

$$G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}(\mathbb{R}^2) = G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}, \sigma_1, \sigma_2 \geq 0, s_1, s_2 \in \mathbb{R},$$

and $\kappa \geqslant 1$, we will consider the problem (1.1). The spaces $G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}$ can be defined as the completion of the Schwartz functions with respect to the norm

$$\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\sigma_1|\xi|^{\frac{1}{\kappa}}} e^{2\sigma_2|\eta|^{\frac{1}{\kappa}}} \mu^2(s_1, s_2) |\widehat{f}(\xi, \eta)|^2 d\xi d\eta \right)^{1/2},$$

where

$$\mu(s_1, s_2) = \langle\xi\rangle^{s_1} \langle\eta\rangle^{s_2}.$$

In addition to the holomorphic extension property, Gevrey spaces satisfy the embeddings $G^{\sigma_1,\sigma_2}_{s_1} \hookrightarrow G^{\sigma_1',\sigma_2'}_{s_1'}$ for $s_1, s_1' \in \mathbb{R}$ and $\sigma_i' < \sigma_i$ where $G^{\sigma_1,\sigma_2,1}_{s_1,0} = G^{\sigma_1,\sigma_2}_{s_1}$, which follow from the corresponding estimates

$$\|f\|_{G^{\sigma_1',\sigma_2'}_{s_1'}} \lesssim \|f\|_{G^{\sigma_1,\sigma_2}_{s_1}}.$$

The main aim to consider initial data in these spaces is because of the Paley-Wiener Theorem.

Proposition 1.1. *Let $\sigma_1 > 0, s \in \mathbb{R}$. Then $f \in G^{\sigma_1}_x(\mathbb{R})$ if and only if it is the restriction to the to the real line of a function F which is holomorphic in the strip $\{x + iy \in \mathbb{C} : |y| < \sigma\}$, and satisfies*

$$\sup_{|y|<\sigma_1} \|F(x + iy)\|_{H^s_x} < \infty.$$

Notation

We will also need the full space time Fourier transform denoted by

$$\hat{f}(\xi, \eta, \tau) = \int_{\mathbb{R}^3} f(x, y, t)e^{-i(x\xi+y\eta+t\tau)}\, dxdydt.$$

In both cases, we will denote the corresponding inverse transform of a function

$$f = f(\xi, \eta) \text{ or } f = f(\xi, \eta, \tau) \text{ by } \mathfrak{F}^{-1}(f).$$

To simplify the notation, we introduce some operators. We first introduce the operator $A^{\sigma_1,\sigma_2}_\kappa$, which we define as

$$A^{\sigma_1,\sigma_2}_\kappa f = \mathfrak{F}^{-1}\left(e^{\sigma_1|\xi|^{\frac{1}{\kappa}}} e^{\sigma_2|\eta|^{\frac{1}{\kappa}}} \hat{f}\right). \tag{1.3}$$

Then, we may then define another useful operator

$$N^{\sigma_1,\sigma_2}_\kappa(f) = \partial_x\left[(A^{\sigma_1,\sigma_2}_\kappa f)^2 - A^{\sigma_1,\sigma_2}_\kappa(f^2)\right]. \tag{1.4}$$

For $x \in \mathbb{R}^n$, we denote $\langle x\rangle = (1 + |x|^2)^{1/2}$. Finally, we write $a \lesssim b$ if there exists a constant $C > 0$ such that $a \le Cb$, and $a \sim b$ if $a \lesssim b \lesssim a$. If the constant C depends on some quantity q, we denote this by $a \lesssim_q b$.

Function spaces

Since our proofs rely heavily on the theory developed by Yan et al., let us state the function spaces they used explicitly, so that we can state their useful properties which we will exploit in our modifications of their spaces. The main function spaces they used are the so-called anisotropic Bourgain spaces,adapted to the generalized Kadomtsev-Petviashvili I, whose norm is given by

$$\|u\|_{X_{s_1,s_2,b}} = \left(\int_{\mathbb{R}^3} \theta^2(s_1, s_2, b)|\hat{u}(\xi, \eta, \tau)|^2 d\xi d\eta d\tau\right)^{\frac{1}{2}},$$

where

$$\theta(s_1, s_2, b) = \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\langle\tau + m(\xi, \eta)\rangle^b,$$

with $m(\xi,\eta) = \xi|\xi|^{\alpha} + \frac{\eta^2}{\xi}$.

Furthermore, we will also need a hybrid of the anlytic Gevrey and anisotropic Bourgain spaces, designated $X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}(\mathbb{R}^3)$ and defined by the standard

$$\|u\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} = \left(\int_{\mathbb{R}^3} e^{2\sigma_1|\xi|^{\frac{1}{\kappa}}} e^{2\sigma_2|\eta|^{\frac{1}{\kappa}}} \theta^2(s_1,s_2,b)|\hat{u}(\xi,\eta,\tau)|^2 d\xi d\eta d\tau\right)^{\frac{1}{2}},$$

It is well-known that these spaces satisfy the embedding

$$X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b} \hookrightarrow C\left(\mathbb{R}; G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}(\mathbb{R}^2)\right).$$

Thus, solutions constructed in $X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}$ belong to the natural solution space.

When considering local solutions, it is useful to consider localized versions of these spaces. For a time interval I and a Banach space $\mathcal{Y}$, we define the localized space $\mathcal{Y}(I)$ by the norm

$$\|u\|_{\mathcal{Y}(I)} = \inf\{\|v\|_{\mathcal{Y}} : \ v = u \text{ on } I\}.$$

2. Main results

The first result related to the short-term persistence of analyticity of solutions is given in the next Theorem.

Theorem 2.1. *Let $s_1 > -\frac{\alpha-1}{4}$, $s_2 \geq 0$, $\alpha \geq 4$, $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ and $\kappa \geq 1$. Then for all initial data $f \in G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}$ and $|\xi|^{-1}\widehat{f}(\xi,\mu) \in L^2(\mathbb{R}^2)$, there exists $\delta = \delta(\| f \|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}}) > 0$ and a unique solution u of* (1.1) *on the time interval* $[0,\delta]$ *such that*

$$u \in C\left([0,\delta]; G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}(\mathbb{R}^2)\right).$$

Moreover the solution depends continuously on the data f. In particular, the time of existence can be chosen to satisfy

$$\delta = \frac{c_0}{(1 + \|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}})^{\gamma}},$$

for some constants $c_0 > 0$ and $\gamma > 1$. Moreover, the solution u satisfies

$$\sup_{t\in[0,\delta]} \|u(t)\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}} \leq 4\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}}.$$

The second main result concerns the evolution of the radius of analyticity for the x-direction is given in the next Theorem. Here

$$X^{\sigma_1,0,1}_{s_1,0,b} = X^{\sigma_1,0}_{s_1,b},\ s_2, \sigma_2 = 0 \text{ and } \kappa = 1.$$

Theorem 2.2. *Let $\sigma_1 > 0$, $s_1 > -\frac{\alpha-1}{4}$, $\alpha = 4,6,8,...$ and assume that $f \in G^{\sigma_1,0}_{s_1}$, $|\xi|^{-1}\widehat{f}(\xi,\mu) \in L^2(\mathbb{R}^2)$. Then the solution u given by Theorem 2.1 extends globally in time, and for any $T > 0$, we have*

$$u \in C\left([0,T], G^{\sigma_1(T),0}_{s_1}(\mathbb{R}^2)\right) \quad \text{with} \quad \sigma_1(T) = \min\{\sigma_1, CT^{-\rho}\},$$

with $\rho = \frac{4}{\alpha-1} + \varepsilon$ for $\varepsilon > 0$ when $\alpha = 4$ and $\rho = 1$ when $\alpha = 6,8,10,...$ and the constant C is a positive.

The method used here for proving lower bounds on the radius of analyticity was introduced in [15] in the study of the non-periodic KdV equation. It was applied to the the higher order nonlinear dispersive equation in [9] and the system of mKdV equation in [10].

Our last aim is to show the regularity of the solution in the time. A non-periodic function $\phi(x)$ is the Gevrey class of order κ i.e, $\phi(x) \in G^{\kappa}$, if there exists a constant $C > 0$ such that

$$|\partial_x^k \phi(x)| \leq C^{k+1}(k!)^{\kappa} \quad k = 0, 1, 2, \dots. \tag{2.1}$$

Here we will show that for $x, y \in \mathbb{R}$, for every $t \in [0, \delta]$ and $j, l, n \in \{0, 1, 2, \dots\}$, there exist $C > 0$ such that,

$$|\partial_t^j \partial_x^l \partial_y^n u(x, y, t)| \leq C^{j+l+n+1}(j!)^{(\alpha+1)\kappa}(l!)^{\kappa}(n!)^{\kappa}. \tag{2.2}$$

i.e, $u(\cdot, \cdot, t) \in G^{\sigma}(\mathbb{R}) \times G^{\sigma}(\mathbb{R})$ in x, y and $u(x, y, \cdot) \in G^{(\alpha+1)\kappa}([0, \delta])$ in time variable.

Theorem 2.3. *Let $s_1 > -\frac{\alpha-1}{4}$, $s_2 \geq 0, \alpha \geq 4$, $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ and $\kappa \geq 1$.*
If $f \in G^{\sigma_1, \sigma_2, \kappa}_{s_1, s_2}$ then the solution

$$u \in C\left([0, \delta], G^{\sigma_1, \sigma_2, \kappa}_{s_1, s_2}\right),$$

given by Theorem 2.1, belongs to the Gevrey class $G^{(\alpha+1)\kappa}$ in time variable.

Corollary 2.4. *Let $\sigma_1 > 0$, $s_1 > -\frac{\alpha-1}{4}$, $\alpha = 4, 6, 8, \dots$. If $f \in G^{\sigma_1, 0}_{s_1}$ then the solution*

$$u \in C\left([0, T], G^{\sigma_1(T), 0}_{s_1}(\mathbb{R}^2)\right),$$

given by Theorem 2.2, belongs to the Gevrey class $G^{(\alpha+1)}$ in time variable.

The rest of the paper is organized as follows: In section 3, we present all the auxiliary estimates that will be employed in the remaining sections. We prove Theorem 2.1 in subsection 4.1 using the standard contraction method and Theorem 2.2 in subsection 4.2. Finally, in section 5, we prove $G^{(\alpha+1)}$ regularity in time.

3. Auxiliary estimates

To begin with, let us consider the related linear problem

$$\begin{aligned} &\partial_t u + |D_x|^{\alpha} \partial_x u + \partial_x^{-1} \partial_y^2 u = F, \\ &u(0) = f. \end{aligned}$$

By Duhamel's principle the solution can be written as

$$u(t) = S(t)f - \frac{1}{2} \int_0^t S(t - t')F(t')dt', \tag{3.1}$$

where

$$\widehat{S(t)f}(\xi, \eta) = e^{itm(\xi,\eta)} \widehat{f}(\xi, \eta).$$

We localize it in t by using a cut-off function satisfying $\psi \in C_0^{\infty}(\mathbb{R})$, with $\psi = 1$ in $[-1, 1]$ and supp$\psi \subset [-2, 2]$.

We consider the operator Φ given by

$$\Phi(u) = \psi(t)S(t)f - \frac{\psi_\delta(t)}{2}\int_0^t S(t-t')\left(\partial_x u^2(t')\right)dt', \tag{3.2}$$

where $\psi_\delta(t) = \psi(\frac{t}{\delta})$. To this operator, we apply the following estimates.

Lemma 3.1. *(Linear estimate) Let $s_1, s_2 \in \mathbb{R}, -\frac{1}{2} < b' \le 0 \le b \le b' + 1, \sigma_1 \ge 0, \sigma_2 \ge 0, \kappa \ge 1$ and $\delta \in (0,1)$. Then*

$$\|\psi(t)S(t)f\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \le C\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}}, \tag{3.3}$$

$$\left\|\psi_\delta(t)\int_0^t S(t-t')F(x,y,t')\mathrm{d}t'\right\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \le C\delta^{1-b+b'}\|F\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b'}}. \tag{3.4}$$

Proof. The proofs of (3.3) and (3.4) for $\sigma_1 = \sigma_2 = 0$ can be found in Lemma 2.1 of [17]. These inequalities clearly remain valid for $\sigma_1, \sigma_2 > 0$, as one merely has to replace f by $A^{\sigma_1,\sigma_2}_\kappa f$, F by $A^{\sigma_1,\sigma_2}_\kappa F$. □

The final preliminary fact we must state is the following bilinear estimate, which is Lemma 3.1 of [17].

Lemma 3.2. *(Bilinear estimate in Bourgain space.)*
Let $s_1 \ge -\frac{\alpha-1}{4} + 4\alpha\epsilon, s_2 \ge 0$, $\alpha \ge 4$, $b = \frac{1}{2} + \epsilon$ and $b' = -\frac{1}{2} + 2\epsilon$. Then, we have

$$\|\partial_x(u_1u_2)\|_{X^{s_1,s_2,b'}} \lesssim \|u_1\|_{X^{s_1,s_2,b}}\|u_2\|_{X^{s_1,s_2,b}}.$$

To this result, we apply the following Lemma, which is a corollary of Lemma 3.2.

Lemma 3.3. *(Bilinear estimate in Gevrey-Bourgain space.)*
Let $s_1 > -\frac{\alpha-1}{4} + 4\alpha\epsilon, s_2 \ge 0, \alpha \ge 4, \sigma_1 \ge 0, \sigma_2 \ge 0, \kappa \ge 1, b = \frac{1}{2} + \epsilon$ and $b' = -\frac{1}{2} + 2\epsilon$. Then, we have

$$\|\partial_x(u_1u_2)\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b'}} \lesssim \|u_1\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}\|u_2\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}.$$

Proof. It is not hard to see that

$$\begin{aligned}
&e^{2(\sigma_1|\xi|^{\frac{1}{\kappa}}+\sigma_2|\eta|^{\frac{1}{\kappa}})}\left|\widehat{u_1u_2}(\xi,\eta,\tau)\right|^2 \\
&= e^{2(\sigma_1|\xi|^{\frac{1}{\kappa}}+\sigma_2|\eta|^{\frac{1}{\kappa}})}\left|\int \hat{u}_1(\xi-\xi_1,\eta-\eta_1,\tau-\tau_1)\hat{u}_2(\xi_1,\eta_1,\tau_1)\,d\xi_1d\eta_1d\tau_1\right|^2 \\
&\le \left|\int e^{\sigma_1|\xi-\xi_1|^{\frac{1}{\kappa}}+\sigma_2|\eta-\eta_1|^{\frac{1}{\kappa}}}\hat{u}_1(\xi-\xi_1,\eta-\eta_1,\tau-\tau_1)e^{\sigma_1|\xi_1|^{\frac{1}{\kappa}}+\sigma_2|\eta_1|^{\frac{1}{\kappa}}}\hat{u}_2(\xi_1,\eta_1,\tau_1)\,d\xi_1d\eta_1d\tau_1\right|^2 \\
&= \left|\widehat{A^{\sigma_1,\sigma_2}_\kappa u A^{\sigma_1,\sigma_2}_\kappa v}\right|^2.
\end{aligned}$$

By Lemma 3.2, we get

$$\begin{aligned}
\|\partial_x(u_1u_2)\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b'}} &\le \|\partial_x(A^{\sigma_1,\sigma_2}_\kappa u_1 A^{\sigma_1,\sigma_2}_\kappa u_2)\|_{X^{s_1,s_2,b'}} \\
&\lesssim \|A^{\sigma_1,\sigma_2}_\kappa u_1\|_{X^{s_1,s_2,b}}\|A^{\sigma_1,\sigma_2}_\kappa v\|_{X^{s_1,s_2,b}} \\
&= \|u_1\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}\|u_2\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b'}}.
\end{aligned}$$

□

4. Proof of main results regarding the existence

4.1. Local well-posedness in an anisotropic Gevrey space

The above Lemmas will be used without somtimes mention to prove Theorem 2.1.

Lemma 4.1. *Let* $s_1 > -\frac{\alpha-1}{4} + 4\alpha\epsilon, s_2 \geq 0, \alpha \geq 4, \sigma_1 \geq 0, \sigma_2 \geq 0, \kappa \geq 1, b = \frac{1}{2} + \epsilon, b' = -\frac{1}{2} + 2\epsilon$ *and* $0 < \delta < 1$. *Then*

$$\|\Phi(u)\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \leq C\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}} + C\delta^{\epsilon}\|u\|^2_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}},$$

and

$$\|\Phi(u_1) - \Phi(u_2)\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \leq \frac{1}{2}\|u_1 - u_2\|^2_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}.$$

Proof. Combining Lemma 3.3 and Lemma 3.1 with the fixed point Theorem. We define

$$\mathbb{B}(0, 2C\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}}) = \left\{u : \|u\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \leq 2C\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}}\right\}.$$

Then, we have

$$\begin{aligned}
\|\Phi(u)\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} &\leq \|\psi(t)S(t)f\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} + \|\frac{1}{2}\psi_{\delta}(t)\int_0^t S(t-t')\partial_x u^2 \mathrm{d}t'\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \\
&\leq C\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}} + C\delta^{\epsilon}\|\partial_x u^2\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b'}} \\
&\leq C\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}} + C\delta^{\epsilon}\|u\|^2_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}.
\end{aligned}$$

We choose δ such that

$$\delta < \frac{1}{(C^2\|f\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}})^{\frac{1}{\epsilon}}}. \tag{4.1}$$

We have

$$\|\Phi(u)\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \leq 2C\|f\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}.$$

Thus, $\Phi(u)$ maps $\mathbb{B}(0, 2C\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}})$ into $\mathbb{B}(0, 2C\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}})$ which is a contraction, since

$$\begin{aligned}
\|\Phi(u_1) - \Phi(u_2)\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} &\leq C\left\|\frac{1}{2}\psi_{\delta}(t)\int_0^t S(t-t')(\partial_x u_1^2 - \partial_x u_1^2)\mathrm{d}t'\right\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \\
&\leq C\delta^{\epsilon}\|u_1 - u_2\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}\left[\|u_1\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} + \|u_2\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}\right] \\
&\leq 4C^2\delta^{\epsilon}\|f\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}}\|u_1 - u_2\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}} \\
&\leq \frac{1}{2}\|u_1 - u_2\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}.
\end{aligned}$$

Here we choose δ such that

$$\delta < \frac{1}{(8C^2\|f\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}})^{\frac{1}{\epsilon}}}. \tag{4.2}$$

We choose the time of existence where

$$\delta = \frac{c_0}{(1 + \|f\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}})^{\frac{1}{\epsilon}}}.$$

For appropriate choice of c_0, this will satisfy inequalities (4.1) and (4.2). □

From Lemma 4.1, we see that for initial data $f(x,y) \in G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}(\mathbb{R}^2)$ if the lifespan $\delta = c_0/(1 + \|f\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}})^{\frac{1}{\epsilon}}$ then the map $\Phi(u)$ is a contraction on a small ball centered at the origin in $X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}$. Hence, the map $\Phi(u)$ has a unique fixed point u in a neighborhood of 0 with respect to the norm $\|\cdot\|_{X^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2,b}}$.

The rest of the proof follows the standard argument.

4.2. Global well-posedness

In this section, we prove Theorem 2.2. The first step is to obtain estimates on the growth of the norm of the solutions. For this end, we need to prove the following approximate conservation law.

Theorem 4.2. *Let $\sigma_1 > 0$ and δ be as in Theorem 2.1, there exist $b \in (1/2, 1)$ and $C > 0$, such that for any solution $u \in X^{\sigma_1,0}_{0,b}(I)$ to the Cauchy problem (1.1) on the time interval $I \subset [0,\delta]$, we have the estimate*

$$\sup_{t\in[0,\delta]} \|u(t)\|^2_{G^{\sigma_1,0}_0} \leq \|f\|^2_{G^{\sigma_1,0}_0} + C\sigma_1^{\varrho}\|u\|^3_{X^{\sigma_1,0}_{0,b}(I)}. \tag{4.3}$$

with $\varrho \in [0, \frac{3}{4})$ if $\alpha = 4$, and $\varrho = 1$ if $\alpha = 6, 8, 10, \ldots$.

Before we may show the proof, let us first state some preliminary Lemmas. The first one is an immediate consequence of Lemma 12 in [15].

Lemma 4.3. *For $\sigma > 0$, $0 \leq \theta \leq 1$ and $\xi, \xi_1 \in \mathbb{R}$, we have*

$$e^{\sigma|\xi-\xi_1|}e^{\sigma|\xi_1|} - e^{\sigma|\xi|} \lesssim \sigma^{\theta}\frac{\langle\xi-\xi_1\rangle\langle\xi_1\rangle}{\langle\xi\rangle}e^{\sigma|\xi-\xi_1|}e^{\sigma|\xi_1|}.$$

This will be used to prove the following key estimate.

Lemma 4.4. *Let $N^{\sigma_1,0}_1(u)$ be as in Eq (1.4) for $\sigma_1 \geq 0$ and $\sigma_2 = 0$. Then for b as in Lemma 3.2, we have*

$$\|N^{\sigma_1,0}_1(u)\|_{X_{0,b-1}} \leq C\sigma_1^{\varrho}\|u\|^2_{X^{\sigma_1,0}_{0,b}},$$

with $\varrho \in [0, \frac{3}{4})$ if $\alpha = 4$ and $\varrho = 1$ if $\alpha = 6, 8, 10, \ldots$

Proof. We first observe that the inequality in Lemma 3.2, is equivalent to

$$\left\|\xi\theta(s_1,s_2,b-1)\int \frac{\hat{f}(\xi-\xi_1,\eta-\eta_1,\tau-\tau_1)}{\langle\xi-\xi_1\rangle^{s_1}\langle\eta-\eta_1\rangle^{s_2}\langle\phi(\xi-\xi_1,\eta-\eta_1,\tau-\tau_1)\rangle^{b-1}} \times \right.$$
$$\left. \times \frac{\hat{g}(\xi_1,\eta_1,\tau_1)}{\langle\xi_1\rangle^{s_1}\langle\eta_1\rangle^{s_2}\langle\phi(\xi_1,\eta_1,\tau_1)\rangle^{b-1}}\, d\xi_1 d\eta_1 d\tau_1\right\|_{L^2_{\xi,\eta}} \lesssim \|f\|_{L^2_{x,y}}\|g\|_{L^2_{x,y}},$$

where we denote $\phi(\tau,\xi,\eta)=\langle\tau+m(\xi,\eta)\rangle$. With this, we observe that the left side of the inequality in Lemma 4.4 can be estimated by Lemma 4.3 as

$$\|N_1^{\sigma_1,0}(u)\|_{X_{0,b-1}}\lesssim\sigma\left\|\frac{\xi\langle\xi\rangle^{-1}}{\langle\phi(\tau,\xi,\eta)\rangle^{\beta}}\int\frac{e^{\sigma_1|\xi-\xi_1|}\hat{u}(\xi-\xi_1,\eta-\eta_1,\tau-\tau_1)}{\langle\xi-\xi_1\rangle^{-1}}\times\right.$$
$$\left.\times\,\frac{e^{\sigma_1|\xi_1|}\hat{u}(\xi_1,\eta_1,\tau_1)}{\langle\xi_1\rangle^{-1}}\,d\xi_1 d\eta_1\right\|_{L^2_{\xi,\eta}}.$$

If we apply Lemma 3.2 with $s_1=-\varrho$, $s_2=0$, it will follow, from the comments above, that

$$\|N_1^{\sigma_1,0}(u)\|_{X_{0,b-1}}\lesssim\sigma_1^{\varrho}\|u\|^2_{X^{\sigma_1,0}_{0,b}}.$$

□

Proof of Theorem 4.2. Begin by applying the operator $A_1^{\sigma_1,0}$ to Eq (1.1). If we let $U=A_1^{\sigma_1,0}u$, then Eq (1.1) becomes

$$\partial_t U-\partial_x^{\alpha}U+\partial_x^{-1}\partial_y^2U+U\partial_xU=N_1^{\sigma_1,0}(u),\tag{4.4}$$

where $\alpha=4,6,8,..$ and $N_1^{\sigma_1,0}(u)$ is defined in Lemma 4.4. Multiplying (4.4) by U and integrating with respect to the spatial variables, we obtain

$$\int U\partial_tU-U\partial_x^{\alpha}U+U\partial_x^{-1}\partial_y^2U+U^2\partial_xU\;dxdy=\int UN_1^{\sigma_1,0}(u)\;dxdy.$$

If we apply integration by parts, we may rewrite the left-hand side as

$$\partial_t\int\frac{1}{2}U^2\;dxdy+\int\partial_x^{\frac{\alpha}{2}}U\partial_x^{\frac{\alpha}{2}+1}Udxdy-\int\partial_yU\partial_x^{-1}\partial_yU\;dxdy+\int U^2\partial_xU\;dxdy=\int UN_1^{\sigma_1,0}(u)\;dxdy,$$

which can then be rewritten as

$$\partial_t\int\frac{1}{2}U^2\;dxdy+\int\frac{1}{2}\partial_x((\partial_x^{\frac{\alpha}{2}}U)^2)dxdy-\int\frac{1}{2}\partial_x[(\partial_x^{-1}\partial_xU)^2]\;dxdy+\int\frac{1}{3}\partial_x(U^3)\;dxdy=\int UN_1^{\sigma_1,0}(u)\;dxdy.$$

By noticing that $\partial_x^jU(x,t)\to0$ as $|x|\to\infty$ (see [15]) we obtaining

$$\partial_t\int U^2(x,y,t)\;dxdy=2\int U(x,y,t)N_1^{\sigma_1,0}(u)(x,y,t)\;dxdy.$$

Integrating with respect to time yields

$$\int U^2(x,y,t)\;dxdy=\int U^2(x,y,0)\;dxdy+2\int_0^t\int U(x,y,t')\partial_xN_1^{\sigma_1,0}(u)(x,y,t')\;dxdydt'.$$

Applying Cauchy-Schwarz and the definition of U, we obtain

$$\|u(t)\|^2_{G_0^{\sigma_1,0}}\le\|f\|^2_{G_0^{\sigma_1,0}}+\|u\|_{X^{\sigma_1,0}_{0,b}}\|N_1^{\sigma_1,0}(u)\|_{X^{0,0}_{0,b-1}(I)}.$$

We now apply Lemma 4.4 and the fact that $b=\frac{1}{2}+\epsilon$, we can further estimate this by

$$\|u(t)\|^2_{G_0^{\sigma_1,0}}\le\|f\|^2_{G_0^{\sigma_1,0}}+C\sigma_1^{\varrho}\|u\|^3_{X^{\sigma_1,0}_{0,b}},\tag{4.5}$$

as desired. □

Proof of Theorem 2.2. With the tools established in the previous subsection, we may begin the proof of Theorem 2.2. Let us first suppose that T^* is the supremum of the set of times T for which

$$u \in C([0,T]; G_x^{\sigma_1,0}).$$

If $T^* = \infty$, there is nothing to prove, so let us assume that $T^* < \infty$. In this case, it suffices to prove that

$$u \in C\left([0,T], G_x^{\sigma_1(T),0}\right), \tag{4.6}$$

for all $T > T^*$. To show that this is the case, we will use Theorem 2.1 and Theorem 4.2 to construct a solution which exists over subintervals of width δ, using the parameter σ_1 to control the growth of the norm of the solution. We first prove the case $s = 0$ and then we will generalize the case.

4.2.1. The case $s = 0$

The desired result will follow from the following proposition.

Proposition 4.5. *Let $T > 0$, $x = 0, 0 < \sigma_1 \leq \sigma_0$ and $\delta > 0$ be numbers such that $n\delta \leq T < (n+1)\delta$. Then the solution u to the Cauchy problem* (1.1) *satisfies*

$$\sup_{t\in[0,n\delta]} \|u(t)\|^2_{G_0^{\sigma_1,0}} \leq \|f\|^2_{G_0^{\sigma_1,0}} + 2^3 C\sigma_0^{\varrho} n \|f\|^3_{G_0^{\sigma_0,0}}, \tag{4.7}$$

and

$$\sup_{t\in[0,n\delta]} \|u(t)\|^2_{G_0^{\sigma_1,0}} \leq 4\|u(t)\|^2_{G_0^{\sigma_0,0}}, \tag{4.8}$$

if

$$\sigma_1 = C_1 T^{-\frac{1}{\varrho}}, \text{ and } C_1 = \left(\frac{c_0}{C2^5 \parallel f \parallel_{G_0^{\sigma_0,0}} (1 + 2 \parallel f \parallel_{G_0^{\sigma_0,0}})^{\frac{1}{\epsilon}}}\right)^{\frac{1}{\varrho}},$$

for some constant $C > 0$.

Proof. For fixed $T \geq T^*$, we will prove, for sufficiently small $\sigma_1 > 0$, that

$$\sup_{t\in[0,T]} \|u(0)\|^2_{G_0^{\sigma_1,0}} \leq 4\|u(0)\|^2_{G_0^{\sigma_0,0}}. \tag{4.9}$$

We will use the Theorem 2.1 and Theorem 4.2 with the time step

$$\delta = \frac{c_0}{(1 + 4 \parallel f \parallel_{G_0^{\sigma_0,0}})^{\frac{1}{\epsilon}}}. \tag{4.10}$$

The smallness conditions on σ_1 will be

$$\sigma_1 \leq \sigma_0 \;\; and \;\; \frac{2T}{\delta} C\sigma_1^{\varrho} 4^{\frac{3}{2}} \parallel f \parallel_{G_0^{\sigma_0,0}} \leq 1, \tag{4.11}$$

where $C > 0$ is the constant in Theorem 4.2. Proceeding by induction, we will verify that

$$\sup_{t\in[0,\delta]} \|u(t)\|^2_{G_0^{\sigma_1,0}} \leq \|f\|^2_{G_0^{\sigma_1,0}} + nC\sigma_1^{\varrho} 2^3 \|f\|^3_{G_0^{\sigma_0,0}}, \tag{4.12}$$

$$\sup_{t\in[0,\delta]} \|u(t)\|^2_{G_0^{\sigma_1,0}} \leq 4\|f\|^2_{G_0^{\sigma_0,0}}, \tag{4.13}$$

for $n \in \{1, \cdots, m+1\}$, where $m \in \mathbb{N}$ is chosen, so that $T \in [m\delta, (m+1)\delta)$. This m does exist, since by Theorem 2.1 and the definition of T^*, we have

$$\delta < \frac{c_0}{(1+ \parallel f \parallel_{G_0^{\sigma_0,0}})^{\frac{1}{\epsilon}}} < T^*, \ \ hence \ \ \delta < T.$$

We cover now, the interval $[0, \delta]$, and by Theorem 4.2, we have

$$\sup_{t\in[0,\delta]} \|u(t)\|^2_{G_0^{\sigma_1,0}} \leq \|f\|^2_{G_0^{\sigma_1,0}} + C\sigma_1^{\varrho}\|f\|^3_{G_0^{\sigma,0}} \leq \|f\|^2_{G_0^{\sigma_1,0}} + C\sigma_1^{\varrho}\|f\|^3_{G_0^{\sigma_0,0}},$$

where we used that

$$\|f\|_{G_0^{\sigma_1,0}} \leq \|f\|_{G_0^{\sigma_0,0}},$$

since $\sigma_1 \leq \sigma_0$. This verifies (4.12) for $n = 1$ and now, (4.13) follows using again

$$\|f\|_{G_0^{\sigma_1,0}} \leq \|f\|_{G_0^{\sigma_0,0}},$$

as well as $C\sigma_1^{\varrho}\|f\|_{G_0^{\sigma_0,0}} \leq 1$. Next, assuming that (4.12) and (4.13) hold for some $n \in \{1, \cdots, m\}$, we will prove that they hold for $n + 1$. We estimate

$$\begin{aligned}
\sup_{t\in[n\delta,(n+1)\delta]} \|u(t)\|^2_{G_0^{\sigma_1,0}} &\leq \|u(n\delta)\|^2_{G_0^{\sigma_1,0}} + C\sigma_1^{\varrho}\|u(n\delta)\|^3_{G_0^{\sigma_1,0}} \\
&\leq \|u(n\delta)\|^2_{G_0^{\sigma_1,0}} + C\sigma_1^{\varrho}2^3\|f\|^3_{G_0^{\sigma_0,0}} \\
&\leq \|f\|^2_{G_0^{\sigma_1,0}} + nC\sigma_1^{\varrho}2^3\|f\|^3_{G_0^{\sigma_0,0}} + C\sigma_1^{\varrho}2^3\|f\|^3_{G_0^{\sigma_0,0}},
\end{aligned}$$

verifying (4.12) with n replaced by $n + 1$. To get (4.13) with n replaced by $n + 1$, it is then enough to have

$$(n+1)C\sigma_1^{\varrho}2^3\|f\|_{G_0^{\sigma_0,0}} \leq 1.$$

But this holds by (4.11), since

$$n+1 \leq m+1 \leq \frac{T}{\delta} + 1 < \frac{2T}{\delta}.$$

Finally, the condition (4.11) is satisfied for $\sigma_1 \in (0, \sigma_0)$ such that

$$\frac{2T}{\delta} C\sigma_1^{\varrho} 2^3 \parallel f \parallel_{G_0^{\sigma_0,0}} = 1.$$

Thus, $\sigma_1 = C_1 T^{-\frac{1}{\varrho}}$, where

$$C_1 = \left(\frac{c_0}{C2^5 \parallel f \parallel_{G_0^{\sigma_0,0}} (1 + 2 \parallel f \parallel_{G_0^{\sigma_0,0}})^{\frac{1}{\epsilon}}} \right)^{\frac{1}{\varrho}}.$$

□

4.2.2. The general case

For general s, we have

$$u_0 \in G_s^{\sigma_0,0} \subset G_0^{\sigma_0/2,0}.$$

The case $s = 0$ already being proved, we know that there is a $T_1 > 0$, such hat

$$u \in C\left([0, T_1), G_0^{\sigma_0/2,0}\right),$$

and

$$u \in C\left([0, T], G_0^{2\varsigma T^{-1/\varrho},0}\right), \quad for\ T \geq T_1,$$

where $\varsigma > 0$ depends on f, σ_0 and ς. We now conclude that

$$u \in C\left([0, T_1), G_s^{\sigma_0/4,0}\right),$$

and

$$u \in C\left([0, T], G_s^{\varsigma T^{-1/\varrho},0}\right), \quad for\ T \geq T_1.$$

The proof of Theorem 2.2 is now completed. □

5. Gevrey's regularity in time

We follow the methods found in [5–7, 11] to treat the regularity in time in Gevrey sens for unique solution of (1.1).

Proposition 5.1. *Let* $\delta > 0, s_1 > -\frac{\alpha-1}{4}$, $s_2 \geq 0$ *and*

$$u \in C\left([0, \delta]; G_{s_1,s_2}^{\sigma_1,\sigma_2,\kappa}\right),$$

be the solution of (1.1). Then u belong in x, y to G^κ *for all times near the zero. In other words,*

$$|\partial_x^l \partial_y^n u(x, y, t)| \leq C^{l+n+1}(l!)^\kappa (n!)^\kappa, \tag{5.1}$$

for all $(x, y) \in \mathbb{R}^2 \quad , C > 0, t \in [0, \delta], l, n \in \{0, 1, ...\}$.

Proof. We have, for any $t \in [0, \delta]$

$$\begin{aligned}
\|\partial_x^l \partial_y^n u(\cdot, \cdot, t)\|_{H^{s_1,s_2}}^2 &= \int_{\mathbb{R}^2} \langle\xi\rangle^{2s_1} \langle\eta\rangle^{2s_2} |\widehat{\partial_x^l \partial_y^n u}(\xi, \eta, t)|^2 d\xi d\eta \\
&= \int_{\mathbb{R}^2} |\xi|^{2l} |\eta|^{2n} \langle\xi\rangle^{2s_1} \langle\eta\rangle^{2s_2} |\widehat{u}(\xi, \eta, t)|^2 d\xi d\eta \\
&= \int_{\mathbb{R}^2} |\xi|^{2l} |\eta|^{2n} e^{-2\sigma_1|\xi|^{\frac{1}{\kappa}}} e^{-2\sigma_2|\eta|^{\frac{1}{\kappa}}} \langle\xi\rangle^{2s_1} \langle\eta\rangle^{2s_2} e^{2\sigma_1|\xi|^{\frac{1}{\kappa}}} e^{2\sigma_2|\eta|^{\frac{1}{\kappa}}} |\widehat{u}(\xi, \eta, t)|^2 d\xi d\eta.
\end{aligned}$$

We observe that

$$e^{\frac{2\sigma_1}{\kappa}|\xi|^{\frac{1}{\kappa}}} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{2\sigma_1}{\kappa}|\xi|^{\frac{1}{\kappa}}\right)^j \geq \frac{1}{(2l)!} \left(\frac{2\sigma_1}{\kappa}\right)^{2l} |\xi|^{\frac{2l}{\kappa}}, \quad \forall l \in \{0, 1, ...\}, \xi \in \mathbb{R},$$

and

$$e^{\frac{2\sigma_2}{\kappa}|\eta|^{\frac{1}{\kappa}}} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{2\sigma_2}{\kappa}|\eta|^{\frac{1}{\kappa}}\right)^j \geq \frac{1}{(2n)!}\left(\frac{2\sigma_2}{\kappa}\right)^{2n}|\eta|^{\frac{2l}{\kappa}}, \quad \forall n \in \{0,1,...\}, \eta \in \mathbb{R}.$$

This implies that

$$\begin{aligned}|\xi|^{2l}e^{-2\sigma_1|\xi|^{\frac{1}{\kappa}}} &\leq C^{2l}_{\sigma_1,\kappa}(2l)!^{\kappa},\\ |\eta|^{2n}e^{-2\sigma_2|\eta|^{\frac{1}{\kappa}}} &\leq C^{2n}_{\sigma_2,\kappa}(2n)!^{\kappa}.\end{aligned}$$

Thus,

$$\begin{aligned}\|\partial_x^l\partial_y^n u(\cdot,\cdot,t)\|^2_{H^{s_1,s_2}} &\leq C^{2l+2n}_{\sigma_1,\sigma_2,\kappa}(2l)!^{\kappa}(2n)!^{\kappa}\int_{\mathbb{R}^2}\langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2}e^{2\sigma_1|\xi|^{\frac{1}{\kappa}}}e^{2\sigma_2|\eta|^{\frac{1}{\kappa}}}|\widehat{u}(\xi,\eta,t)|^2 d\xi d\eta\\ &= C^{2l+2n}_{\sigma_1,\sigma_2,\kappa}(2l)!^{\kappa}(2n)!^{\kappa}\|u(\cdot,\cdot,t)\|^2_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}}.\end{aligned}$$

Since $(2l)!^{\kappa} \leq c_1^{2l}(l!)^{2\kappa}$ and $(2n)!^{\kappa} \leq c_2^{2n}(n!)^{2\kappa}$, for some $c_1, c_2 > 0$, we have for all $l, n \in \{0,1,2,\dots\}$

$$|\partial_x^l\partial_y^n u| \lesssim \|\partial_x^l\partial_y^n u(\cdot,\cdot,t)\|_{H^{s_1,s_2}} \leq C_0 C_1^{l+n}(l)!^{\kappa}(n)!^{\kappa} \quad \text{for} \quad \forall t \in [0,\delta],$$

where $C_0 = \|u(\cdot,\cdot,t)\|_{G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}}$ and $C_1 = c_0^2 C_{\sigma_1,\sigma_2,\kappa}$ and $c_0 = \max(c_1, c_2)$, which implies that the solution u is analytic in x, y for all time near zero and $s_1, s_2 \geq 0$.

Now, for $-\frac{\alpha-1}{4} < s_1 < 0$, $s_2 \geq 0$ and for any $0 < \epsilon < \sigma_1$, there exists a positive constant $C = C_{s,\epsilon} > 0$ such that

$$\begin{aligned}&\int_{\mathbb{R}^2} e^{2(\sigma_1-\epsilon)|\xi|^{\frac{1}{\kappa}}}e^{2\sigma_2|\eta|^{\frac{1}{\kappa}}}\langle\eta\rangle^{2s_2}|\widehat{u}(\xi,\eta,t)|^2 d\xi d\eta\\ &\leq C\int_{\mathbb{R}^2}\frac{e^{2\epsilon|\xi|^{\frac{1}{\kappa}}}}{\langle\xi\rangle^{-2s_1}}e^{2(\sigma_1-\epsilon)|\xi|^{\frac{1}{\kappa}}}e^{2\sigma_2|\eta|^{\frac{1}{\kappa}}}\langle\eta\rangle^{2s_2}|\widehat{u}(\xi,\eta,t)|^2 d\xi d\eta\\ &= C\int_{\mathbb{R}^2}\langle\xi\rangle^{2s_1}\langle\eta\rangle^{2s_2}e^{2\sigma_1|\xi|^{\frac{1}{\kappa}}}e^{2\sigma_2|\eta|^{\frac{1}{\kappa}}}|\widehat{u}(\xi,\eta,t)|^2 d\xi d\eta.\end{aligned}$$

This implies that if

$$u \in C\left([0,T]; G^{\sigma_1,\sigma_2,\kappa}_{s_1,s_2}\right) \textit{ and } s_1 < 0, s_2 \geq 0,$$

then

$$u \in C\left([0,T]; G^{\sigma_1-\epsilon,\sigma_2,\kappa}_{0,s_2}\right),$$

which allows us to conclude that u is in G^{κ} in x, y for all $s_1 > -\frac{\alpha-1}{4}$, $s_2 \geq 0$. □

In order to prove Theorem 2.3 it is enough to prove the following result.

Lemma 5.2. *For $j, l, n \in \{0,1,2,\dots\}$, the next inequality*

$$\left|\partial_t^j\partial_x^l\partial_y^n u\right| \leq C^{j+l+n+1}((l+n+(\alpha+1)j)!)^{\kappa}L^j, \tag{5.2}$$

holds, where $L = C^{\alpha} + \frac{1}{120^{\kappa}} + \frac{C}{40^{\kappa}}$, $\forall x, y \in \mathbb{R}$, $t \in [0,\delta]$.

In fact, taking $l = n = 0$ we obtain

$$\left|\partial_t^j u\right| \leq C^{j+1}((\alpha+1)j!)^{\kappa}L^j \leq K^{j+1}(j!)^{(\alpha+1)\kappa}.$$

Proof. We use the induction on j to prove Lemma 5.2.
For $j = 0$ and $l, n \in \{0, 1, 2, \dots\}$, we have, by (5.1)

$$\left|\partial_x^l \partial_y^n u(x, y, t)\right| \le C^{l+n+1}(l!)^\kappa (n!)^\kappa \le C^{l+n+1}(l+n)!^\kappa. \tag{5.3}$$

For $j = 1$ and $l, n \in \{0, 1, 2, \dots\}$, we have

$$\left|\partial_t \partial_x^l \partial_y^k u\right| \le \left||D_x|^\alpha \partial_x^{l+1} \partial_y^n u\right| + \left|\partial_x^{l-1} \partial_y^{n+2} u\right| + \left|\partial_x^l \partial_y^n (u \partial_x u)\right|. \tag{5.4}$$

The terms of (5.4) can be estimated as

$$\begin{aligned} \left||D_x|^\alpha \partial_x^{l+1} \partial_y^n u\right| &\le C^{l+1+\alpha+n+1}(l+1+\alpha+n)!^\kappa \\ &\le C^{l+n+1+1}(l+n+(\alpha+1)\cdot 1)!^\kappa C^\alpha, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \left|\partial_x^{l-1} \partial_y^{n+2} u\right| &\le C^{l-1+n+2+1}(l-1+n+2)!^\kappa \\ &\le C^{l+n+1+1}(l+n+5\cdot 1)! \frac{1}{(l+n+2)^\kappa (l+n+3)^\kappa (l+n+4)^\kappa (l+n+5)^\kappa} \\ &\le C^{l+n+1+1}(l+n+5\cdot 1)! \frac{1}{120^\kappa}. \end{aligned} \tag{5.6}$$

The nonlinear terms are treated as follows

$$\left|\partial_x^l \partial_y^n (u \partial_x u)\right| = \left|\sum_{p=0}^{l} \sum_{k=0}^{n} \binom{l}{p} \binom{n}{k} \left(\partial_x^{l-p} \partial_y^{n-k} u\right) \left(\partial_x^{p+1} \partial_y^k u\right)\right|.$$

Recalling that for $l \ge p$ and $n \ge k$, we have the next inequality

$$\binom{l}{p} \binom{n}{k} \le \binom{l+n}{p+k}. \tag{5.7}$$

By (5.7), we have

$$\begin{aligned} \left|\partial_x^l \partial_y^n (u \partial_x u)\right| &\le \left|\sum_{p=0}^{l} \sum_{k=0}^{n} \binom{l+n}{p+k} \left(\partial_x^{l-p} \partial_y^{n-k} u\right) \left(\partial_x^{p+1} \partial_y^k u\right)\right| \\ &\le \sum_{p=0}^{l} \sum_{k=0}^{n} \frac{((l+n)!)^\kappa}{((p+k)!)^\kappa ((l+n-p-k)!)^\kappa} C^{l-p+n-k+1}((l+n-p-k)!)^\kappa \\ &\quad C^{p+1+k+1}((p+1+k)!)^\kappa \\ &= C^{l+n+3}((l+n)!)^\kappa \sum_{p=0}^{l} \sum_{k=0}^{n} (p+1+k)^\kappa. \end{aligned}$$

At this stage, we use the fact that

$$\sum_{p=0}^{l}\sum_{k=0}^{n}(p+1+k)=\frac{(l+1)(n+1)(l+n+2)}{2}. \tag{5.8}$$

Then,

$$\begin{aligned}\left|\partial_x^l\partial_y^n(u\partial_x u)\right| &\le C^{l+n+3}((l+n)!)^\kappa\frac{(l+1)^\kappa(n+1)^\kappa(l+n+2)^\kappa}{2^\kappa}\\ &\le C^{l+n+1+1}((l+n)!)^\kappa(l+n+1)^\kappa(l+n+2)^\kappa(l+n+3)^\kappa\frac{C}{2^\kappa}\\ &= C^{l+n+1+1}((l+n+(\alpha+1))!)^\kappa\frac{1}{(l+n+4)^\kappa(l+n+(\alpha+1))^\kappa}\frac{C}{2^\kappa}\\ &\le C^{l+n+1+1}((l+n+(\alpha+1)\cdot 1)!)^\kappa\frac{C}{40^\kappa}.\end{aligned} \tag{5.9}$$

From (5.5), (5.6) and (5.9), it follows that

$$\left|\partial_t\partial_x^l\partial_y^k u\right|\le C^{l+n+1+1}((l+n+(\alpha+1)\cdot 1)!)^\kappa L^1, \forall x,y\in\mathbb{R}, t\in[0,\delta].$$

We assume that (5.2) is correct for $j\ge m\ge 1$ where $l,n\in\{0,1,2,\dots\}$ and then we prove it for $m=j+1$ and $l,n\in\{0,1,2,\dots\}$.

We obtain

$$\left|\partial_t^{j+1}\partial_x^l\partial_y^k u\right|\le\left|\partial_t^j|D_x|^\alpha\partial_x^{l+1}\partial_y^n u\right|+\left|\partial_t^j\partial_x^{l-1}\partial_y^{n+2}u\right|+\left|\partial_t^j\partial_x^l\partial_y^n(u\partial_x u)\right|.$$

These terms are estimated as follows

$$\begin{aligned}\left|\partial_t^j|D_x|^\alpha\partial_x^{l+1}\partial_y^n u\right| &\le C^{j+l+(\alpha+1)+n+1}(l+n+((\alpha+1)(j+1))!)^\kappa L^j\\ &\le C^{(j+1)+l+n+1}(l+n+((\alpha+1)(j+1))!)^\kappa C^\alpha L^j,\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\left|\partial_t^j\partial_x^{l-1}\partial_y^{n+2}u\right| &\le C^{j+l-1+n+2+1}((j+l-1+n+2)!)^\kappa L^j\\ &\le C^{(j+1)+l+n+1}((l+n+(\alpha+1)(j+1))!)^\kappa\frac{L^j}{120^\kappa}.\end{aligned} \tag{5.11}$$

The nonlinear terms are treated as follows

$$\begin{aligned}\partial_t^j\partial_x^l\partial_y^n(u\partial_x u) &= \sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_t^j\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_x^{p+1}\partial_y^k u\right)\\ &+\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^j\partial_x^{p+1}\partial_y^k u\right)\\ &+\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{j}{q}\binom{l}{p}\binom{n}{k}\left(\partial_t^{j-q}\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^q\partial_x^{p+1}\partial_y^k u\right).\end{aligned} \tag{5.12}$$

Using (5.7) to estimate $(5.12)_1$

$$\left|\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_t^j\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_x^{p+1}\partial_y^k u\right)\right|$$
$$\leq \frac{1}{3}C^{(j+1)+l+n+1}((l+n+(\alpha+1)(j+1))!)^{\kappa}\frac{C}{40^{\kappa}}L^j. \tag{5.13}$$

We estimate $(5.12)_2$ as

$$\left|\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^j\partial_x^{p+1}\partial_y^k u\right)\right| \leq$$
$$\frac{1}{3}C^{(j+1)+l+n+1}((l+n+(\alpha+1)(j+1))!)^{\kappa}\frac{C}{40^{\kappa}}L^j. \tag{5.14}$$

To estimate $(5.12)_3$, we recall that for $j \geq q$, $l \geq p$ and $n \geq k$, we have the next inequality

$$\binom{j}{q}\binom{l}{p}\binom{n}{k} \leq \binom{j+l+n}{q+p+k}.$$

Then

$$\left|\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{j}{q}\binom{l}{p}\binom{n}{k}\left(\partial_t^{j-q}\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^q\partial_x^{p+1}\partial_y^k u\right)\right|$$
$$\leq \sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{j+l+n}{q+p+k}C^{j-q+l-p+n-k+1}((l-p+n-k+(\alpha+1)(j-q))!)^{\kappa}L^{j-q} \tag{5.15}$$
$$C^{q+p+1+k+1}((p+1+k+(\alpha+1)q)!)^{\kappa}L^q$$
$$\leq \frac{1}{3}C^{(j+1)+l+n+1}((l+n+(\alpha+1)(j+1))!)^{\kappa}\frac{C}{40^{\kappa}}L^j.$$

Finally by using (5.13)–(5.15) we arrive at

$$\left|\partial_t^{j+1}\partial_x^l\partial_y^k u\right| \leq C^{(j+1)+l+n+1}((l+n+(\alpha+1)(j+1))!)^{\kappa}L^{j+1},$$

for all $(x, y) \in \mathbb{R}^2$, $t \in [0, \delta]$.

The detailed proof of (5.12) for $\kappa = 1$ is given in [6]. □

6. Conclusions

We have discussed the local well-posedness for a generalized Kadomtsev-Petviashvili I equation in an anisotropic Gevrey space. We proved the existence of solutions using the Banach contraction mapping principle. This was done by using the bilinear estimates in anisotropic Gevrey-Bourgain. We used this local result and a Gevrey approximate conservation law to prove that global solutions exist. These solutions are Gevrey class of order $(\alpha + 1)\kappa$ in the time variable. The results of the present paper are new and significantly contribute to the existing literature on the topic.

Acknowledgments

The authors wish to thank deeply the anonymous referee for his/here useful remarks and his/here careful reading of the proofs presented in this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. A. V. Kempinski, An initial-boundary value problem for three-dimensional Zakharov-Kuznetsov equation, *J. Differ. Equations*, **260** (2016), 3029–3055.
2. A. V. Faminskii, Initial-boundary value problems in a half-strip for two-dimensional Zakharov-Kuznetsov equation, *Ann. Inst. Henri Poincaré C*, **35** (2018), 1235–1265.
3. V. Y. Belashov, S. V. Vladimirov, *Solitary waves in dispersive complex media, theory, simulation, applications*, Springer-Verlag Berlin Heidelberg, 2005.
4. M. Ben-Artzi, J. C. Saut, Uniform decay estimates for a class of oscillatory integrals and applications, *Differ. Integr. Equations*, **12** (1999), 137–145.
5. A. Boukarou, K. Guerbati, K. Zennir, S. Alodhaibi, S. Alkhalaf, Well-posedness and time regularity for a system of modified Korteweg-de Vries-type equations in analytic gevrey spaces, *Mathematics*, **8** (2020), 809.
6. A. Boukarou, K. Zennir, K. Guerbati, S. G. Georgiev, Well-posedness and regularity of the fifth order Kadomtsev-Petviashvili I equation in the analytic Bourgain spaces, *Ann. Univ. Ferrara*, **66** (2020), 255–272.
7. A. Boukarou, K. Zennir, K. Guerbati, S. G. Georgiev, Well-posedness of the Cauchy problem of Ostrovsky equation in analytic Gevrey spaces and time regularity, *Rend. Circ. Mat. Palermo, Ser. 2*, **70** (2020), 349–364.
8. A. Boukarou, D. O. da Silva, K. Guerbati, K. Zennir, Global well-posedness for the fifth-order Kadomtsev-Petviashvili II equation in anisotropic Gevrey Spaces, 2020. Available from: `https://arxiv.org/abs/2006.12859`.
9. A. Boukarou, K. Guerbati, K. Zennir, On the radius of spatial analyticity for the higher order nonlinear dispersive equation, *Math. Bohemica*, (2021), 1–14.
10. K. Zennir, A. Boukarou, R. N. Alkhudhayr, Global well-posedness for coupled system of mKdV equations in analytic spaces, *J. Funct. Spaces*, **2021** (2021), 6614375.
11. A. Boukarou, K. Guerbati, K. Zennir, Local well-posedness and time regularity for a fifth-order shallow water equations in analytic Gevrey-Bourgain spaces, *Monatsh. Math.*, **193** (2020) , 763–782.
12. C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier-Stokes equations, *J. Funct. Anal.*, **87** (1989), 359–369.

13. K. L. Jones, X. G, He, Y. K. Chen, Existence of periodic traveling wave solution to the forced generalized nearly concentric Korteweg-de Vries equation, *Internat. J. Math. Math. Sci.*, **24** (2000), 202387.

14. Y. Katznelson, *An introduction to harmonic analysis*, Cambridge University Press, 2004.

15. S. Selberg, D. O. da Silva, Lower bounds on the radius of a spatial analyticity for the KdV equation, *Ann. Henri Poincaré*, **18** (2017), 1009–1023.

16. N. Tzvetkov, On the Cauchy problem for the Kadomtsev-Petviashvili equation, *Commun. Partial Differ. Equations*, **24** (1999), 1367–1397.

17. W. Yan, Y. S. Li, J. H. Huang, J. Q. Duan, The Cauchy problem for a two-dimensional generalized Kadomtsev-Petviashvili I equation in anisotropic Sobolev spaces, *Anal. Appl.*, **18** (2020), 469–522.

18. Y. L. Ma, B. Q. Li, Mixed lump and soliton solutions for a generalized (3+1)-dimensional Kadomtsev-Petviashvili equation, *AIMS Math.*, **5** (2020), 1162–1176.

19. Y. L. Ma, A. M. Wazwaz, B. Q. Li, New extended Kadomtsev-Petviashvili equation: Multiple soliton solutions, breather, lump and interaction solutions, *Nonlinear Dyn.*, **104** (2021), 1581–1594.

20. Y. L. Ma, A. M. Wazwaz, B. Q. Li, A new (3+1)-dimensional Kadomtsev-Petviashvili equation and its integrability, multiple-solitons, breathers and lump waves, *Math. Comput. Simul.*, **187** (2021), 505–519.

AIMS Press

© 2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)