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Research article

The extended generalized inverted Kumaraswamy Weibull distribution: Properties and applications

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Abstract: In this paper we introduce a new six-parameters extension of the Weibull distribution. It will be called the extended generalized inverted Kumaraswamy Weibull (EGIKw-Weibull), that is commonly used to model lifetime data. Various useful properties of the new distribution are derived. A Monte Carlo simulation is employed to investigate the maximum likelihood estimator (MLE) for the parameters. Two real applications are presented.

Keywords: generalized inverted Kumaraswamy Weibull distribution; Monte Carlo simulation; MLEs; structural properties
Mathematics Subject Classification: 62E10, 62E15

1. Introduction

The Weibull distribution is one of the most important lifetime model. It has many applications in statistics, bioscience, chemistry, engineering, economics and finance. The Exponential and Rayleigh, among other, distributions are specials. It is suitable for modeling data with hazard functions of different forms. It is proper in the cases where an item consists of multiple components and each component has an identical failure time distribution and the item fails when the weakest part fails [15]. A random variable X is said to have Weibull distribution if its cdf and pdf are respectively defined by

$$G(x) = 1 - \exp(-\delta x^{\varphi}), \tag{1.1}$$

$$g(x) = \delta\varphi x^{\varphi-1}\exp(-\delta x^{\varphi}), \tag{1.2}$$

where $\delta > 0$ and $\varphi > 0$ are respectively the scale and shape parameters for $x > 0$. Because of the widespread study and applications of the Weibull distribution, there is a need for new generalizations.

Several generalizations distributions of Eq (1.1) have been studied in the literature. Kumaraswamy generalized power Weibull distribution has been discussed by Selim and Badar [23]. A new three-parameter lifetime model, the Truncated Weibull Lomax (TWL) distribution has been proposed by Al-marzoki and Al-said [3]. The exponentiated power generalized Weibull distribution has been investigated by Pena-ramirez et al. [17]. A recurrence relations for the single and product moments of order statistics for power generalized Weibull (PGW) distribution have been established by Kumar and Dey [14]. Using these recurrence relations, they obtained the means, variances and covariances of all order statistics for different sample sizes in an efficient manner. Some useful generalizations of the standard Weibull distribution have been introduced by Ramos et al. [18] to describe the lifetime of two important components of sugarcane harvesting machines. The mathematical background of the considered model was discussed and different discrimination procedures were used to obtain the best fit for each component. The inverse Weibull generated (IW-G) family with two extra positive parameters was generated from inverse Weibull random variable [10]. Four special models for the new family, some mathematical properties, the estimation of the model parameters and an applications to real data were offered. A new two-parameter model, the inverse weighted Lindley (IWL) distribution with upside-down bathtub hazard rate was introduced by Ramos et al. [19]. A detailed account of useful mathematical properties of the new distribution, a numerical simulation and an application using a real data set were offered. The alpha power inverse Weibull (APIW) distribution was proposed by Basheer [6]. He explored various useful properties along with the estimation of the APIW parameters and an application of the new model to a real data representing the waiting time before customer service in the bank was provided. Moreover, Ramos et al. [20] introduced an extended Poisson family of life distribution via a new approach to generate flexible parametric families of distributions. They discussed several mathematical properties and inferential procedures of the proposed model. The applicability of considered model to real situation was illustrated by an important data set. Further, the exponentiated power generalized Weibull power series (EPGWPS) family of distributions, has been obtained by compounding the exponentiated power generalized Weibull and power series distributions by Aldahlan et al. [2]. Bayesian inferences for the inverse generalized gamma (IGG) distribution parameters under non-informative priors, namely, the Jeffreys prior and the reference prior was discussed by Ramos et al. [21], and the potentiality of the IGG model was analysed by employing real environmental data. The GIKum and its distribution function [12], is given by

$$F(x) = [1 - (1 + x^{\gamma})^{-\alpha}]^{\beta}, \tag{1.3}$$

where $x > 0$ and $\alpha > 0, \beta > 0, \gamma > 0$ are shape parameters. Let $s(t)$ be the pdf of a random variable $T \in [a, b]$, where $-\infty \leq a < b < \infty$ and consider $\xi[G(x)]$ be a function of the cdf of a random variable X, the $T - X$ family of distributions [4], is defined as

$$F(x) = \int_{a}^{\xi[G(x)]} s(t)dt, \tag{1.4}$$

assuming the following conditions are satisfied.

(1) $\xi[G(x)] \in [a, b]$.
(2) $\xi[G(x)]$ is differentiable and monotonically non decreasing function.

(3) $\xi[G(x)] \rightarrow a,\ as\ x \rightarrow -\infty$ and $\xi[G(x)] \rightarrow b,\ as\ x \rightarrow \infty.$

Our motivation here is to introduce a new more flexible model so called the extended generalized inverted Kumaraswamy Weibull (EGIKw-Weibull) distribution. It is rarly to get closed forms of statistical quntities of distributions, here we derive closed formes for most statistical quantities for the new model, including moments, moment generating function, reliability function, Rényi Entropy etc. The new model shows higher flexibility as compared to other commonly used standard distributions. Its hazard function shows different shapes that makes it a preferable choice for modeling the monotonic and non-monotonic hazard behaviors which are more likely to be encountered in practical situations like, human mortality, reliability analysis and biomedical applications. The remaining of this paper is organized as follows: the formation of the EGIKw-Weibull distribution and its reliability measures are provided in section 2; the density and distribution functions of EGIKw-Weibull distribution and some mathematical properties of the proposed model are derived in section 3; parameters are estimated using the maximum likelihood method (MLE) method in section 4; Monte Carlo simulation is employed in section 5, to investigate the model; two real applications are given in section 6 to demonstrate the properties; finally, the concluding remarks are given in section 7.

2. The EGIKw-Weibull distribution

Using $G(x,\vartheta)$ and $g(x,\vartheta)$ from Eq (1.1) and Eq (1.2), in Eq (1.4), the cdf of the EGIKw-Weibull distribution is given by

$$\begin{aligned} F_{EGIKw-W}(x) &= \alpha\beta\gamma \int_0^{\frac{[1-\exp(-\delta x^{\varphi})]^{\lambda}}{1-[1-\exp(-\delta x^{\varphi})]^{\lambda}}} t^{\gamma-1}\left(1+t^{\gamma}\right)^{-\alpha-1}\left[1-\left(1+t^{\gamma}\right)^{-\alpha}\right]^{\beta-1} dt \\ &= \left[1-\left\{1+\left(\frac{[1-\exp(-\delta x^{\varphi})]^{\lambda}}{1-[1-\exp(-\delta x^{\varphi})]^{\lambda}}\right)^{\gamma}\right\}^{-\alpha}\right]^{\beta}, \end{aligned} \tag{2.1}$$

where $x \geq 0,\ \alpha > 0,\ \beta > 0,\ \gamma > 0$ and $\lambda > 0$ are shape parameters. For $\varphi = 1$ we obtain the EGIKw-Exponential distribution. The corresponding pdf is given by

$$\begin{aligned} f_{EGIKw-W}(x) &= \alpha\beta\gamma\lambda\delta\varphi x^{\varphi-1}\exp(-\delta x^{\varphi})\left[1-\exp(-\delta x^{\varphi})\right]^{\lambda\gamma-1} \\ &\times\left[1-[1-\exp(-\delta x^{\varphi})]^{\lambda}\right]^{-\gamma-1} \\ &\times\left\{1+\left(\frac{[1-\exp(-\delta x^{\varphi})]^{\lambda}}{1-[1-\exp(-\delta x^{\varphi})]^{\lambda}}\right)^{\gamma}\right\}^{-\alpha-1} \\ &\times\left[1-\left\{1+\left(\frac{[1-\exp(-\delta x^{\varphi})]^{\lambda}}{1-[1-\exp(-\delta x^{\varphi})]^{\lambda}}\right)^{\gamma}\right\}^{-\alpha}\right]^{\beta-1}. \end{aligned} \tag{2.2}$$

This extension gives a highly flexible life distribution which admits different degrees of kurtosis and asymmetry. Figure 1(a) shows the unimodality and positivity skewed. The graphical representation of the cdf of the EGIKw-Weibull distribution is given in Figure 1(b). The plot at other parametric values produces similar shapes. In insurance problems and biomedical applications, it is often general to use the survival function to depict the distribution of survival time. Let the random variable X denotes the survival time and $F_X(x)$ be the cdf or the failure probability by time x, then the survival function is given by

$$S_x(x) = \mathbb{P}(X > x) = 1 - F_x(x).$$

The survival function is the probability of survival beyond time x. The survival function of $X \sim$ EGIKw-Weibull is given by

$$S_x(x) = 1 - \left[1 - \left\{1 + \left(\frac{[1 - \exp(-\delta x^{\varphi})]^{\lambda}}{1 - [1 - \exp(-\delta x^{\varphi})]^{\lambda}}\right)^{\gamma}\right\}^{-\alpha}\right]^{\beta}. \tag{2.3}$$

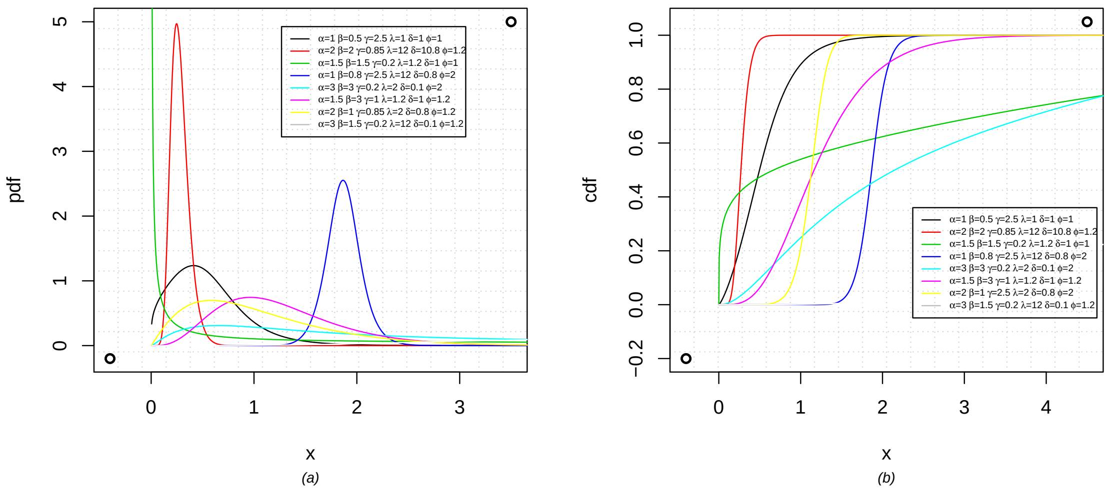

Figure 1. The graphs for EGIKw-Weibull distribution with selected parameters.

For brevity purpose, a graphical representation of the survival function of the EGIKw-Weibull distribution at selected parameter values is shown in Figure 2(a). The hazard rate function commonly used in lifetime modeling because it gives the amount of risk to fail. The hazard rate for EGIKw-Weibull is given as

$$\begin{aligned} h(x,\phi) &= \alpha\beta\gamma\lambda\delta\varphi x^{\varphi-1}\exp(-\delta x^{\varphi})\,[1-\exp(-\delta x^{\varphi})]^{\lambda\gamma-1} \\ &\quad \times \left[1 - [1-\exp(-\delta x^{\varphi})]^{\lambda}\right]^{-\gamma-1} \\ &\quad \times \left\{1 + \left(\frac{[1 - \exp(-\delta x^{\varphi})]^{\lambda}}{1 - [1 - \exp(-\delta x^{\varphi})]^{\lambda}}\right)^{\gamma}\right\}^{-\alpha-1} \\ &\quad \times \left[1 - \left\{1 + \left(\frac{[1 - \exp(-\delta x^{\varphi})]^{\lambda}}{1 - [1 - \exp(-\delta x^{\varphi})]^{\lambda}}\right)^{\gamma}\right\}^{-\alpha}\right]^{\beta-1} \\ &\quad \times \left\{1 - \left[1 - \left\{1 + \left(\frac{[1 - \exp(-\delta x^{\varphi})]^{\lambda}}{1 - [1 - \exp(-\delta x^{\varphi})]^{\lambda}}\right)^{\gamma}\right\}^{-\alpha}\right]^{\beta}\right\}^{-1}, \end{aligned} \tag{2.4}$$

where $x > 0$ and $\phi = \alpha, \beta, \gamma, \lambda, \delta, \varphi$. The EGIKw-Weibull model shows versatility and high flexibility. Its hazard rate function follows an upside down bathtub shape over time, when $\alpha < \beta < \gamma < \lambda$ and $\delta > \varphi$. In this situation, hazard rate decreases as proportion parameters increase. The hazard rate offers a J-shaped trend over time, when $\alpha > \beta > \gamma > \lambda$ and $\delta > \varphi$. In this scenario, the hazard rate increases as proportion parameters increase. Further, the hazard rate exhibits an exponential decreasing trend with increase in time, when $\alpha \geqslant \beta \geqslant \gamma \geqslant \lambda$ and $\delta \leqslant \varphi$. Similarly, the hazard rate function of the considered model offers various shapes, such as monotonically increasing, bathtub shape, constant and increasing-decreasing almost linearly, monotonically decreasing, constant and exponential increasing, and upside down bathtub shapes, for different parametric values. These attractive features render the EGIKw-Weibull distribution suitable for modeling the monotonic and non-monotonic hazard behaviors which are more likely to be encountered in practical situations like, human mortality, reliability analysis and biomedical applications thus enhancing its adaptability to fit diverse lifetime data, Figure 2(b). The quantile function is given by

$$\begin{aligned} Q(u) &= x_u = F^{-1}(u) \\ &= [-\frac{1}{\delta}\log\{1-(1+((1-u^{\frac{1}{\beta}})^{-\frac{1}{\alpha}}-1)^{-\frac{1}{\gamma}})^{-\frac{1}{\lambda}}\}]^{\frac{1}{\varphi}}, \end{aligned} \tag{2.5}$$

the random numbers from EGIKw-Weibull distribution can be simulated using the expression Eq (2.5), where $U \sim$ Uniform $(0, 1)$. In particular, the median of the EGIKw-Weibull distribution can be derived by substituting $u = 0.5$ in Eq (2.5), we have

$$Median = [-\frac{1}{\delta}\log\{1-(1+((1-0.5^{\frac{1}{\beta}})^{-\frac{1}{\alpha}}-1)^{-\frac{1}{\gamma}})^{-\frac{1}{\lambda}}\}]^{\frac{1}{\varphi}}.$$

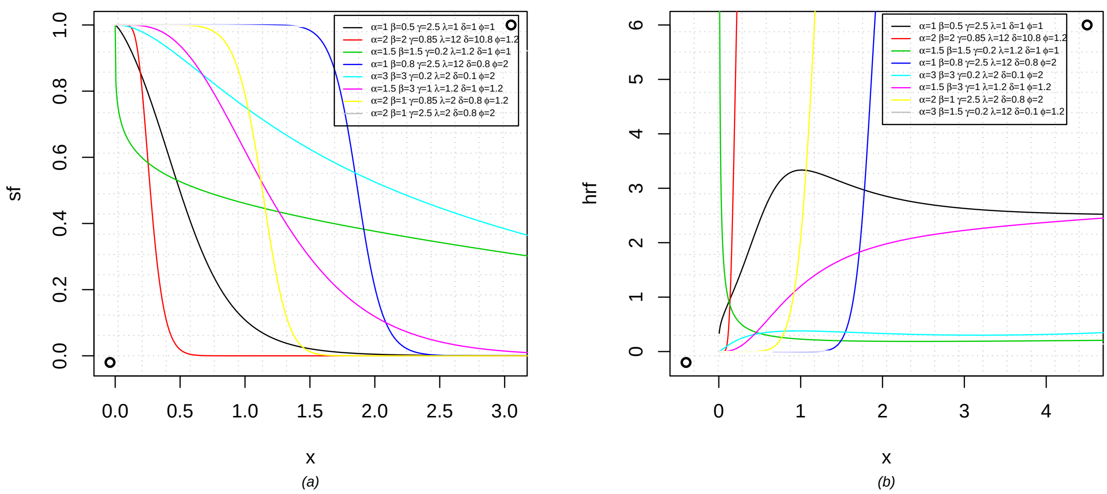

Figure 2. The survival and hazard rate function graphs for EGIKw-Weibull distribution with selected parameters.

Note that the EGIKw-Weibull distribution of models outlined above reduces to GIKw-Weibull distribution for $\gamma = 1$ and for $\gamma = 1, \lambda = 1$ we obtain the exponentinated generalized Weibull distribution. Hence the parameter γ of the EGIKw-Weibull distribution offers more flexibility to the extremes for the pdf curves. That's why the new distribution becomes more appropriate for analyzing data sets that exhibit heavy-tail.

3. Properties of EGIKw-Weibull distribution

Here properties of EGIKw-Weibull distribution are discussed.

3.1. Density functions

By using the binomial expansions in Eq (2.1), we obtain the linear combination for EGIKw-Weibull cdf (for $\gamma > 0$ integer) as

$$F(x) = \sum_{i,j,k=0}^{\infty} w_{i,j,k} \left[1 - \exp(-\delta x^{\varphi})\right]^{\lambda(\gamma j+k)}, \tag{3.1}$$

where $w_{i,j,k} = (-1)^{i+j} \binom{\beta}{i}\binom{\alpha i+j-1}{j}\binom{\gamma j+k-1}{k}$. Otherwise, for $\gamma > 0$ real non-integer, we have

$$F(x) = \sum_{r=0}^{\infty} z_r \left[1 - \exp(-\delta x^{\varphi})\right]^{r}, \tag{3.2}$$

where the coefficient $z_r = \sum_{i,j,k=0}^{\infty} \sum_{l=r}^{\infty} (-1)^{l+r} \binom{\lambda(\gamma j+k)}{l} \binom{l}{r} w_{i,j,k}$ is sum of constants. Moreover the EGIKw-Weibull cdf can be expressed in terms of Weibull Exponential-G cdf,s as

$$F(x) = \sum_{r=0}^{\infty} z_r V_r(x), \tag{3.3}$$

where $V_r(x) = \left[1 - \exp(-\delta x^{\varphi})\right]^{r}$ is the Weibull Exponential-G cdf with power parameter r. The corresponding expansions for the EGIKw-Weibull density fucntion are respectively obtained by differentiation of Eq (3.1) for $\gamma > 0$ integer and of Eq (3.2) and Eq (3.3) for $\gamma > 0$ real non-integer, as

$$f(x) = \delta\varphi x^{\varphi-1} \exp(-\delta x^{\varphi}) \sum_{i,j,k=0}^{\infty} \overset{''}{w}_{i,j,k} \left[1 - \exp(-\delta x^{\varphi})\right]^{\lambda(\gamma j+k)-1}, \tag{3.4}$$

$$f(x) = \delta\varphi x^{\varphi-1} \exp(-\delta x^{\varphi}) \sum_{r=0}^{\infty} \breve{z}_r \left[1 - \exp(-\delta x^{\varphi})\right]^{r}, \tag{3.5}$$

$$f(x) = \sum_{r=0}^{\infty} \overset{''}{z}_r v_{r+1}(x), \tag{3.6}$$

where $\overset{''}{w}_{i,j,k} = \lambda(\gamma j + k) w_{i,j,k}$, $\breve{z}_r = (r+1) z_{r+1}$, $\overset{''}{z}_r = z_{r+1}$ for $r = 0, 1, 2, ...$, and $v_{r+1}(x) = (r + 1)\delta\varphi x^{\varphi-1} \exp(-\delta x^{\varphi}) \left[1 - \exp(-\delta x^{\varphi})\right]^{r}$ is the Weibull Exponential-G density with parameter $(r + 1)$.

3.2. Moments

For p and q non-negative integers, the ordinary moments are defined by

$$\begin{aligned}\tau_{p,q} = E[X^pG(X)^q] &= \int x^pG(x)^q\,g(x)\,dx \\ &= \delta\varphi\int_0^\infty x^{p+\varphi-1}\exp(-\delta x^\varphi)\left[1-\exp(-\delta x^\varphi)\right]^q dx \\ &= \frac{1}{\delta^{\frac{p}{\varphi}}}\sum_{l=0}^{q}\binom{q}{l}(-)^l\frac{\Gamma\left(\frac{p}{\varphi}+1\right)}{(l+1)^{\frac{p}{\varphi}+1}}. \end{aligned} \tag{3.7}$$

The p^{th} ordinary moment for an integer for $\gamma > 0$ can be expressed as

$$\mu_p^{'} = E(X^p) = \sum_{i,j,k=0}^{\infty}\overset{''}{w}_{i,j,k}\;\tau_{p,\lambda(\gamma j+k)-1}, \tag{3.8}$$

where $\overset{''}{w}_{i,j,k}$, is defined in Eq (3.4) and $\tau_{p,\lambda(\gamma j+k)-1}$, is the $(p,\lambda(\gamma j+k)-1)^{th}$ PWM of Weibull distribution given in Eq (3.7). For a non-integer $\gamma > 0$, we can write

$$\mu_p^{'} = E(X^p) = \sum_{r=0}^{\infty}\breve{z}_r\;\tau_{p,r}, \tag{3.9}$$

where $\breve{z}_r$, is defined in Eq (3.5) and $\tau_{p,r}$, is the $(p,r)^{th}$ PWM of Weibull distribution. Moreover, we can also provide the moments of the EGIKw-Weibull distribution in terms of Weibull Exponential-G moments. Let X_{r+1} has Weibull Exponential-G distribution with cdf, $V_{r+1}(x) = \left[1-\exp(-\delta x^\varphi)\right]^r$ and pdf, $v_{r+1}(x) = (r+1)\delta x^{\varphi-1}\exp(-\delta x^\varphi)\left[1-\exp(-\delta x^\varphi)\right]^r$ with power parameter $(r+1)$, the p^{th} ordinary moment of Weibull Exponential-G distribution (for $\gamma > 0$ non-integer) is given as

$$E(X_{r+1}^p) = \int_0^\infty x^p v_{r+1}(x)dx.$$

Hence the p^{th} moment of the EGIKw-Weibull distribution can be expressed in terms of infinite weighted sum of Weibull Exponential-G moments as

$$\mu_p^{'} = \sum_{r=0}^{\infty}\overset{''}{z}_r\int_0^\infty x^p v_{r+1}(x)dx, \tag{3.10}$$

where $\overset{''}{z}_r$, is defined in Eq (3.6).

3.3. Moment generating function

The moment generating function (MGF) for an integer $\gamma > 0$, can be derived using Eq (3.4) as

$$M(t) = \delta\sum_{i,j,k=0}^{\infty}\overset{''}{w}_{i,j,k}\int_0^\infty x^{\varphi-1}\exp(tx)\exp(-\delta x^\varphi)\left[1-\exp(-\delta x^\varphi)\right]^{\lambda(\gamma j+k)-1}dx,$$

$$
\begin{aligned}
&= \sum_{i,j,k,l,u=0}^{\infty} \overset{''}{w}_{i,j,k,l,u} \int_0^{\infty} x^{\varphi(u+1)-1} \exp(tx)dx, \\
&= \sum_{i,j,k,l,u=0}^{\infty} \overset{''}{w}_{i,j,k,l,u} \frac{\Gamma(\varphi(u+1))}{(-t)^{\varphi(u+1)}}, \qquad (3.11)
\end{aligned}
$$

where $\overset{''}{w}_{i,j,k,l,u} = w_{i,j,k}\binom{\lambda(\gamma j+k)-1}{l}\frac{\delta^{u+1}(l+1)^u}{u!}(-1)^{l+u}$. For a non-integer $\gamma > 0$ an alternative representation for M(t) can be derived from Eq (3.5) as

$$
\begin{aligned}
M(t) &= \sum_{r=0}^{\infty} \breve{z}_r \int \exp(tx) g(x) G(x)^r \, dx, \\
&= \delta \sum_{r=0}^{\infty} \breve{z}_r \int_0^{\infty} x^{\varphi-1} \exp(tx) \exp(-\delta x^{\varphi}) \left[1-\exp(-\delta x^{\varphi})\right]^r dx, \\
&= \sum_{r,u=0}^{\infty} \sum_{l=0}^{r} \breve{z}_r \binom{r}{l} \frac{\delta^{u+1}(l+1)^u}{u!} (-1)^{l+u} \frac{\Gamma(\varphi(u+1))}{(-t)^{\varphi(u+1)}}. \qquad (3.12)
\end{aligned}
$$

Another representation for $M(t)$ in terms of Weibull Exponential-G MGF, for a non-integer $\gamma > 0$ is obtained from Eq (3.6) as

$$
\begin{aligned}
M(t) &= \sum_{r=0}^{\infty} \overset{''}{z}_r \int_0^{\infty} \exp(tx) \, v_{r+1}(t) \, dx, \\
&= \sum_{r=0}^{\infty} \overset{''}{z}_r M_{r+1}(t), \qquad (3.13)
\end{aligned}
$$

where $M_{r+1}(t)$ is the mgf of X ~Weibull Exponential-G random variable with power parameter $(r+1)$.

3.4. Mean deviations

Let X be a EGIKw-Weibull random variable with mean $\mu = E(X)$ and median M. The mean deviation about the mean $\delta_\mu(X)$ and about the median $\delta_M(X)$ are respectively defined by

$$
\delta_\mu(X) = E(|X-\mu_1'|) = 2\mu_1' F(\mu_1') - 2T(\mu_1'), \qquad (3.14)
$$

$$
\delta_M(X) = E(|X-M|) = \mu_1' - 2T(M), \qquad (3.15)
$$

where $T(z) = \delta\varphi \int_0^z x^{\varphi} \exp(-\delta x^{\varphi}) dx. = \delta^{-1/\varphi} \curlyvee(1/\varphi+1, \delta z^{\varphi})$ is first GIKw-Weibull incomplete moment with $\curlyvee(.,.)$ the incomplete gamma function, $\mu_1' = E(X)$ is the first ordinary moment, M =Median(X) denotes the median determined from the Eq (2.5) for $u = 1/2$, and $F(\mu_1')$ comes from Eq (2.1). Using the quantile function, two additional forms for $T(x)$ are obtained. Firstly, when $\gamma > 0$ an integer,

$$
T(z) = \sum_{i,j,k=0}^{\infty} \overset{''}{w}_{i,j,k} \int_0^{\left[1-\exp(-\delta z^{\varphi})\right]} u^{\lambda(\gamma j+k)-1} Q(u) du,
$$

where $Q(u)$ is the EGIKw-Weibull quantile functin given in Eq (2.5) and the second representation for $\gamma > 0$ is derived as

$$
T(z) = \sum_{r=0}^{\infty} \breve{z}_r \int_0^{\left[1-\exp(-\delta z^{\varphi})\right]} u^r Q(u) du.
$$

Alternatively using EGIKw-Weibull density Eq (3.6), in terms of Exponential-G distribution we obtain,

$$T(z) = \sum_{r=0}^{\infty} \overset{''}{z}_r \int_0^z x\, v_{r+1}(x)\, dx,$$

where $\overset{''}{w}_{i,j,k}, \breve{z}_r, \overset{''}{z}_r$ are given in Eqs (3.4–3.6).

3.5. Rényi entropy

The Rényi [22], is one of the most popular measures of entropy and for EGIKw-Weibull distribution, it is defined as

$$\begin{aligned} I_\zeta(x) &= \frac{1}{1-\zeta} \log\Big((\alpha\beta\gamma\lambda\varphi\delta)^\zeta \\ &\quad \sum_{i,j,k=0}^{\infty} \tilde{w}_{i,j,k} \int_0^\infty [1-\exp(-\delta x^\varphi)]^{\lambda(\gamma j+k)+\zeta(\gamma\lambda-1)}\, x^{\zeta(\varphi-1)} \exp(-\zeta\delta x^\varphi) dx \Big), \\ &= \frac{1}{1-\zeta} \log\Big((\alpha\beta\gamma\lambda\varphi\delta)^\zeta \sum_{i,j,k=0}^{\infty} \tilde{w}_{i,j,k,t} \int_0^\infty x^{\zeta(\varphi-1)} \exp(-\delta x^\varphi (\zeta+t)) dx \Big), \\ &= \frac{1}{1-\zeta} \log\Big((\alpha\beta\gamma\lambda\varphi\delta)^\zeta \sum_{i,j,k=0}^{\infty} \tilde{w}_{i,j,k,t} \frac{\Gamma\left(\zeta - \frac{\zeta-1}{\varphi}\right)}{\varphi\, [\delta\,(\zeta+t)]^{\left(\zeta-\frac{\zeta-1}{\varphi}\right)}} \Big), \end{aligned} \tag{3.16}$$

where $\tilde{w}_{i,j,k,t} = (-1)^{i+j+t} \binom{\zeta(\beta-1)}{i} \binom{\alpha i+\zeta(\alpha+1)+j-1}{j} \binom{\gamma j+\zeta(\gamma+1)+k-1}{k} \binom{\lambda(\gamma j+k)+\zeta(\gamma\lambda-1)}{t}$.

3.6. Stress-strength reliability

Let X_1 be a random variable having EGIKw-Weibull distribution with pdf, $f_1(x)$ given in Eq (2.2) with parameters $\alpha_1, \beta_1, \gamma_1, \lambda_1, \delta, \varphi$ and X_2 be a random variable having the cdf $F_2(x)$ given in Eq (2.1) with parameters $\alpha_2, \beta_2, \gamma_2, \lambda_2, \delta, \varphi$. Assuming X_1 and X_2 to be independent, the reliability function R is defined by

$$\begin{aligned} R &= P(Y<X) = \int f_1(x) F_2(x) dx \\ &= \alpha_1\beta_1\gamma_1\lambda_1\varphi\delta \int_0^\infty x^{\varphi-1} \exp(-\delta x^\varphi)\, [1-\exp(-\delta x^\varphi)]^{\gamma_1\lambda_1-1} \\ &\quad \times \left[1 - [1-\exp(-\delta x^\varphi)]^{\lambda_1}\right]^{-\gamma_1-1} \\ &\quad \times \left[1 + \left(\frac{[1-\exp(-\delta x^\varphi)]^{\lambda_1}}{1-[1-\exp(-\delta x^\varphi)]^{\lambda_1}}\right)^{\gamma_1}\right]^{-\alpha_1-1} \\ &\quad \times \left[1 - \left[1 + \left(\frac{[1-\exp(-\delta x^\varphi)]^{\lambda_1}}{1-[1-\exp(-\delta x^\varphi)]^{\lambda_1}}\right)^{\gamma_1}\right]^{-\alpha_1}\right]^{\beta_1-1} \\ &\quad \times \left[1 - \left[1 + \left(\frac{[1-\exp(-\delta x^\varphi)]^{\lambda_2}}{1-[1-\exp(-\delta x^\varphi)]^{\lambda_2}}\right)^{\gamma_2}\right]^{-\alpha_2}\right]^{\beta_2} dx. \end{aligned} \tag{3.17}$$

Alternatively, with the change of variable $x = Q_1(u)$, where Q_1(u) denotes the EGIKw-Weibull quantile function Eq (2.5) corresponding to $f_1(x)$, we have

$$\begin{aligned} R &= \int_0^1 F_2(Q_1(u))du \qquad (3.18) \\ &= \int_0^1 \left\{1-\left[1+\left\{\left[1+\left\{\left(1-u^{\frac{1}{\beta_1}}\right)^{-\frac{1}{\alpha_1}}-1\right\}^{-\frac{1}{\gamma_1}}\right]^{\frac{\lambda_2}{\lambda_1}}-1\right\}^{-\gamma_2}\right]^{-\alpha_2}\right\}^{\beta_2} du. \end{aligned}$$

In particular, from this expression we see that R does not depend on the baseline distribution characterized by the cdf $[1-\exp(-\delta x^{\varphi})]$. Various forms of R for $\gamma_1, \gamma_2 > 0$ integer, by using linear expression can be obtained as

$$\begin{aligned} f_1(x) &= \varphi\delta x^{\varphi-1}\exp(-\delta x^{\varphi}) \sum_{t,u,v=0}^{\infty} \overset{''}{w}_{t,u,v} \left[1-\exp(-\delta x^{\varphi})\right]^{\lambda_1(\gamma_1 u+v)-1}, \\ F_2(x) &= \sum_{i,j,k=0}^{\infty} \bar{w}_{i,j,k} \left[1-\exp(-\delta x^{\varphi})\right]^{\lambda_2(\gamma_2 j+k)}, \end{aligned}$$

where $\overset{''}{w}_{t,u,v} = \lambda_1(\gamma_1 u+v)(-1)^{t+u}\binom{\beta_1}{t}\binom{\alpha_1 t+u-1}{u}\binom{\gamma_1 u+v-1}{v}$ *and* $\bar{w}_{i,j,k} = (-1)^{i+j}$ $\binom{\beta_2}{i}\binom{\alpha_2 i+j-1}{j}\binom{\gamma_2 j+k-1}{k}$. Thus, we have

$$\begin{aligned} R &= \varphi\,\delta \sum_{i,j,k,t,u,v=0}^{\infty} \bar{w}_{i,j,k}\overset{''}{w}_{t,u,v} \\ &\quad \int_0^{\infty} x^{\varphi-1}\exp(-\delta x^{\varphi})\left[1-\exp(-\delta x^{\varphi})\right]^{\lambda_1(\gamma_1 u+v)+\lambda_2(\gamma_2 j+k)-1} dx \\ &= \sum_{i,j,k,t,u,v=0}^{\infty} \frac{\bar{w}_{i,j,k}\overset{''}{w}_{t,u,v}}{\lambda_1(\gamma_1 u+v)+\lambda_2(\gamma_2 j+k)}. \qquad (3.19) \end{aligned}$$

Similar expressions can be obtained for the case $\gamma_1, \gamma_2 > 0$ non-integers.

3.7. Lorenz and Bonferroni curves

Various expressions for EGIKw-Weibull Lorenz *L(p)* and Bonferroni *B(p)* curves for $\gamma > 0$ integer, are given as

$$\begin{aligned} L(p) &= \frac{E_{X\le x}}{E(X)} = \frac{1}{E(X)}\int_0^x t f(t)dt \\ &= \frac{\delta\varphi}{\mu}\sum_{i,j,k=0}^{\infty} \overset{''}{w}_{i,j,k}\int_0^x t^{\varphi}\exp(-\delta t^{\varphi})\left[1-\exp(-\delta t^{\varphi})\right]^{\lambda(\gamma j+k)-1} dt \\ &= \frac{\delta\varphi}{\mu}\sum_{i,j,k=0}^{\infty} \overset{''}{w}_{i,j,k}\binom{\lambda(\gamma j+k)-1}{l}(-1)^l\int_0^x t^{\varphi}\exp(-\delta t^{\varphi}(l+1))dt \end{aligned}$$

$$= \frac{\delta}{\mu} \sum_{i,j,k,l=0}^{\infty} \overset{''}{w}_{i,j,k} \binom{\lambda(\gamma j+k)-1}{l} (-1)^l \frac{\curlyvee(1/\varphi+1, \delta(l+1)x^{\varphi})}{[\delta(l+1)]^{1/\varphi+1}}. \tag{3.20}$$

Equivalently based upon EGIKw-Weibull quantile function Eq (2.5) we have

$$L(p) = \frac{1}{\mu} \sum_{i,j,k=0}^{\infty} \overset{''}{w}_{i,j,k} \int_0^{[1-\exp(-\delta t^{\varphi})]} u^{\lambda(\gamma j+k)-1} Q(u) du.$$

Alternatively using the expression given in Eq (3.6) in terms of Exponential-G density $v_r(t)$ we have

$$L(p) = \frac{1}{\mu} \sum_{i,j,k=0}^{\infty} \frac{\overset{''}{w}_{i,j,k}}{\lambda(\gamma j+k)} \int_0^x t v_{\lambda(\gamma j+k)}(t) \, dt.$$

The corresponding expressions for the Bonferroni Curve are given by

$$\begin{aligned} B(p) &= \frac{E_{X\le x}}{F(X)E(X)} = \frac{L(X)}{F(X)} = \frac{1}{F(X)E(X)} \int_0^x t f(t) dt \qquad (3.21) \\ &= \frac{\delta}{\mu F(x)} \sum_{i,j,k,l=0}^{\infty} \overset{''}{w}_{i,j,k} \binom{\lambda(\gamma j+k)-1}{l} (-1)^l \frac{\curlyvee(1/\varphi+1, \delta(l+1)x^{\varphi})}{[\delta(l+1)]^{1/\varphi+1}}. \end{aligned}$$

and

$$\begin{aligned} B(p) &= \frac{1}{\mu F(X)} \sum_{i,j,k=0}^{\infty} \overset{''}{w}_{i,j,k} \int_0^{G(x)} u^{\lambda(\gamma j+k)-1} Q(u) du \\ &= \frac{1}{\mu F(X)} \sum_{i,j,k=0}^{\infty} \frac{\overset{''}{w}_{i,j,k}}{\lambda(\gamma j+k)} \int_0^x t v_{\lambda(\gamma j+k)}(t) \, dt, \qquad (3.22) \end{aligned}$$

where $\curlyvee(.,.)$ is the upper incomplete gamma function. Similar expressions can be obtained for the case of $\gamma > 0$ non-integer using Eq (3.5).

3.8. Moments of residual life function

The residual life plays an important role in life testing situations and reliability theory. The n^{th} moment of the residual life is defined as

$$\begin{aligned} m_n(t) &= E\left[(X-t)^n / X > t\right] = \frac{1}{R(t)} \int_t^{\infty} (x-t)^n f(x) \, dx \\ &= \frac{1}{R(t)} \sum_{a=0}^{n} \tbinom{n}{a} (-t)^{n-a} \int_t^{\infty} x^a f(x) \, dx. \qquad (3.23) \end{aligned}$$

For $\gamma > 0$ integer, using pdf Eq (3.4) we have

$$\begin{aligned} m_n(t) &= \frac{\delta\varphi}{S(t)} \sum_{i,j,k=0}^{\infty} \sum_{a=0}^{n} \overset{''}{w}_{i,j,k} \tbinom{n}{a} (-t)^{n-a} \\ &\times \int_t^{\infty} x^{a+\varphi-1} \exp(-\delta x^{\varphi}) \left[1-\exp(-\delta x^{\varphi})\right]^{\lambda(\gamma j+k)-1} dx \end{aligned}$$

$$= \frac{\delta\varphi}{S(t)} \sum_{i,j,k,l=0}^{\infty} \sum_{a=0}^{n} \overset{''}{w}_{i,j,k,l} \binom{n}{a} (-t)^{n-a} \int_t^{\infty} x^{a+\varphi-1} \exp(-\delta x^{\varphi}(l+1)) dx$$

$$= \frac{\delta}{S(t)} \sum_{i,j,k,l=0}^{\infty} \sum_{a=0}^{n} \overset{''}{w}_{i,j,k,l} \binom{n}{a} (-t)^{n-a} \frac{\Gamma(a/\varphi+1, \delta(l+1) t^{\varphi})}{[\delta(l+1)]^{a/\varphi+1}}, \tag{3.24}$$

where $\overset{''}{w}_{i,j,k,l} = \overset{''}{w}_{i,j,k} \binom{\lambda(\gamma j+k)-1}{l} (-1)^l$ and $\Gamma(.,.)$ is lower incomplete gamma function, similarly using the EGIKw-Weibull quantile function for $\gamma > 0$ non-integer, we have

$$m_n(t) = \frac{1}{S(t)} \sum_{r=0}^{\infty} \sum_{a=0}^{n} \breve{z}_r \binom{n}{a} (-t)^{n-a} \int_{[1-\exp(-\delta t^{\varphi})]}^{1} u^r Q(u)^a du. \tag{3.25}$$

An alternative representation can be derived from Weibull Exponential-G distribution, as

$$m_n(t) = \frac{1}{S(t)} \sum_{r=0}^{\infty} \sum_{a=0}^{n} \overset{''}{z}_r \binom{n}{a} (-t)^{n-a} \int_t^{\infty} x^a v_{r+1}(x) dx, \tag{3.26}$$

where $S(x) = 1 - F(x)$ is the EGIKw-Weibull survival function and v_{r+1} is the Weibull Exponential-G density function as given in Eq (3.6).

3.9. Order statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from the EGIKw-Weibull distribution and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are the corresponding order statistics, then the pdf of i^{th} order statistic can be obtained as

$$\begin{aligned} f_{i:n}(x) &= \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1} [1 - F(x)]^{n-i} \\ &= \frac{f(x)}{B(i, n-i+1)} \sum_{h=0}^{n-i} (-1)^h \binom{n-i}{h} F(x)^{h+i-1}, \end{aligned} \tag{3.27}$$

where $B(.,.)$ is the beta function and $F(x)$ is the EGIKw-Weibull cdf. Replacing Eq (3.2) in above expression, we have

$$\begin{aligned} F(x)^{h+i-1} &= \left[\sum_{t=0}^{\infty} z_t [1 - \exp(-\delta t^{\varphi})]^t \right]^{h+i-1} \\ &= \sum_{t=0}^{\infty} c_{t,h+i-1} [1 - \exp(-\delta t^{\varphi})]^t, \end{aligned} \tag{3.28}$$

where $c_{0,h+i-1} = (z_0)^{h+i-1}$, $c_{t,h+i-1} = (tz_0)^{-1} \sum_{m=1}^{t} [m(h+i) - t] z_m c_{t-m,h+i-1}$. Using Eq (3.28) in Eq (3.27), with $f(x)$ given in Eq (3.4) for $\gamma > 0$ integer, and with $f(x)$ Eq (3.5) for $\gamma > 0$ non-integer, we respectively obtain

$$f_{i:n}(x) = \frac{\delta\varphi t^{\varphi-1} \exp(-\delta t^{\varphi})}{B(i, n-i+1)} \sum_{l,j,k,t=0}^{\infty} \sum_{h=0}^{n-i} \frac{\overset{''}{w}_{l,j,k} c_{t,h+i-1} (-1)^h \binom{n-i}{h}}{[1 - \exp(-\delta t^{\varphi})]^{-(\lambda(\gamma j+k)+t-1)}}, \tag{3.29}$$

and

$$f_{i:n}(x) = \frac{\delta\varphi t^{\varphi-1}\exp(-\delta t^{\varphi})}{B(i,n-i+1)} \sum_{r,t=0}^{\infty} \sum_{h=0}^{n-i} \breve{z}_r c_{t,h+i-1} (-1)^h \binom{n-i}{h} [1-\exp(-\delta t^{\varphi})]^{r+t}.$$

The corresponding expressions for moments, the mgf and other properties of the EGIKw-Weibull order statistics can be obtained likewise.

4. Estimation

In this section, we employ the method of MLE to estimate the unknown parameters of EGIKw-Weibull distribution. We consider independent random variables $X_1, X_2..., X_n$, from an EGIKw-Weibull distribution with parameter vector $\Theta = (\alpha, \beta, \gamma, \lambda, \varphi, \delta)'$. The log-likelihood $l(\Theta) = \log L(\Theta)$ for the model parameters obtained from Eq (2.2) is

$$\begin{aligned} l(\Theta) = {} & n\log(\alpha\beta\gamma\lambda\delta\varphi) + (\varphi-1)\sum_{i=1}^{n}\log x - \delta\sum_{i=1}^{n} x^{\varphi} - (\lambda+1) \\ & \times \sum_{i=1}^{n} \log[1-\exp(-\delta x^{\varphi})] - (\gamma+1) \\ & \times \sum_{i=1}^{n} \log\left[[1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right] - (\alpha+1) \\ & \times \sum_{i=1}^{n} \log\left[1 + \left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right)^{-\gamma}\right] + (\beta-1) \\ & \times \sum_{i=1}^{n} \log\left[1 - \left[1 + \left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right)^{-\gamma}\right]^{-\alpha}\right]. \end{aligned} \tag{4.1}$$

The components of score vector $U = (U_\alpha, U_\beta, U_\gamma, U_\lambda, U_\varphi)'$ are given by

$$\begin{aligned} U_\alpha = {} & \frac{n}{\alpha} - \sum_{i=1}^{n} \log\left[1 + \left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right)^{-\gamma}\right] + (\beta-1) \\ & \times \sum_{i=1}^{n} \frac{\log\left[1 + \left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right)^{-\gamma}\right]}{\left[1 + \left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right)^{-\gamma}\right]^{\alpha} - 1}, \\ U_\beta = {} & \frac{n}{\beta} + \sum_{i=1}^{n} \log\left[1 - \left[1 + \left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right)^{-\gamma}\right]^{-\alpha}\right], \\ U_\gamma = {} & \frac{n}{\gamma} - \sum_{i=1}^{n} \log\left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right) \\ & + (\alpha+1)\sum_{i=1}^{n} \frac{\log\left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right)}{1 + \left([1-\exp(-\delta x^{\varphi})]^{-\lambda} - 1\right)^{\gamma}} - (\beta-1)\,\alpha \end{aligned}$$

$$\times \sum_{i=1}^{n} \frac{\log\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)\left[1+\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{\gamma}\right]^{-1}}{\left[\left(1+\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{-\gamma}\right)^{\alpha}-1\right]},$$

$$\begin{aligned}
U_{\lambda} &= \frac{n}{\lambda}-\sum_{i=1}^{n}\log[1-\exp(-\delta x^{\varphi})]-(\gamma+1)\sum_{i=1}^{n}\frac{\log[1-\exp(-\delta x^{\varphi})]}{\left([1-\exp(-\delta x^{\varphi})]^{\lambda}-1\right)} \\
&\quad -(\alpha+1)\,\gamma\sum_{i=1}^{n}\frac{\log[1-\exp(-\delta x^{\varphi})]}{\left(1-[1-\exp(-\delta x^{\varphi})]^{\lambda}\right)\left[1+\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{\gamma}\right]} \\
&\quad +\sum_{i=1}^{n}\frac{\log[1-\exp(-\delta x^{\varphi})]}{\left[1+\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{-\gamma}\right]^{\alpha}-1} \\
&\quad \times\frac{\alpha\gamma(\beta-1)}{\left(1-[1-\exp(-\delta x^{\varphi})]^{\lambda}\right)\left[1+\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{\gamma}\right]}, \\
U_{\delta} &= \frac{n}{\delta}-\sum_{i=1}^{n}x^{\varphi}-(\lambda+1)\sum_{i=1}^{n}\frac{x^{\varphi}}{\exp(\delta x^{\varphi})-1}-\lambda(\gamma+1) \\
&\quad \times\sum_{i=1}^{n}\frac{x^{\varphi}}{\left[[1-\exp(-\delta x^{\varphi})]^{\lambda}-1\right][\exp(\delta x^{\varphi})-1]}+\gamma\lambda(\alpha+1) \\
&\quad \times\sum_{i=1}^{n}\frac{x^{\varphi}\left[\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{\gamma}+1\right]^{-1}}{\left[[1-\exp(-\delta x^{\varphi})]^{\lambda}-1\right][\exp(\delta x^{\varphi})-1]}-\alpha\gamma\lambda(\beta-1) \\
&\quad \times\sum_{i=1}^{n}\frac{x^{\varphi}[\exp(\delta x^{\varphi})-1]^{-1}\left[\left(\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{-\gamma}+1\right)^{\alpha}-1\right]^{-1}}{\left[\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{\gamma}+1\right]\left[[1-\exp(-\delta x^{\varphi})]^{\lambda}-1\right]},
\end{aligned}$$

$$\begin{aligned}
U_{\varphi} &= \frac{n}{\varphi}+\sum_{i=1}^{n}\log x-\delta\sum_{i=1}^{n}x^{\varphi}\log x-\delta(\lambda+1)\sum_{i=1}^{n}\frac{x^{\varphi}\log x}{\exp(\delta x^{\varphi})-1}-\delta\lambda(\gamma+1) \\
&\quad \times\sum_{i=1}^{n}\frac{x^{\varphi}\log x}{\left[[1-\exp(-\delta x^{\varphi})]^{\lambda}-1\right][\exp(\delta x^{\varphi})-1]}+\delta\gamma\lambda(\alpha+1) \\
&\quad \times\sum_{i=1}^{n}\frac{x^{\varphi}\log x\left[\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{\gamma}+1\right]^{-1}}{\left[[1-\exp(-\delta x^{\varphi})]^{\lambda}-1\right][\exp(\delta x^{\varphi})-1]}-\alpha\delta\gamma\lambda(\beta-1) \\
&\quad \times\sum_{i=1}^{n}\frac{x^{\varphi}\log x\left[\left(\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{-\gamma}+1\right)^{\alpha}-1\right]^{-1}}{\left[\left([1-\exp(-\delta x^{\varphi})]^{-\lambda}-1\right)^{\gamma}+1\right]\left[[1-\exp(-\delta x^{\varphi})]^{\lambda}-1\right][\exp(\delta x^{\varphi})-1]}.
\end{aligned}$$

Setting these equations to zero and solving them simultaneously yields the MLEs of the GKw-E parameters. Since there are no close form for these MLEs, a numerical solution to these equations can be determined by using a standard statistical software.

5. Monte Carlo simulation

In this section, an extensive numerical investigation is carried out to examine the performance of MLEs for EGIKw-Weibull model. The performance of estimators is evaluated through their average bias (AB) and root mean square errors (RMSE) for different sample sizes. The quantile function is given in Eq (2.5) is used to generate random samples from the EGIKw-Weibull distribution. The simulations experiment is repeated for $N = 2,000$ times, for each set of parameters with sample sizes $n = 20, 50, 100$ and 120 and assumed parametric values $I : (\alpha = 0.8, \beta = 2.0, \gamma = 1.5, \lambda = 3.0, \delta = 2.5, \varphi = 1.25)$, $II : (\alpha = 2.0, \beta = 1.5, \gamma = 1.2, \lambda = 2.5, \delta = 2.2, \varphi = 1.75)$, $III : (\alpha = 1.4, \beta = 0.5, \gamma = 0.75, \lambda = 2.5, \delta = 0.7, \varphi = 3.5)$ and $IV : (\alpha = 0.2, \beta = 0.85, \gamma = 0.15, \lambda = 2.25, \delta = 0.3, \varphi = 1.5)$. The AB and RMSE values of the parameters $\alpha, \beta, \gamma, \lambda, \delta$ and φ for different sample sizes are presented in Table 1 and Table 2. From the results of these Tables, it is clear that the AB and RMSE for the estimators of the parameters are showing decreasing pattern as the sample size increases. The results indicate that the method of MLE performs quite well in estimating the model parameters of the proposed distribution.

6. Applications

This section provides two real applications to show how the proposed distribution can be applied in practice. The importance and potentiality of the EGIKw-Weibull distribution are examined and compared with the other fitted models namely the exponentiated Kumaraswamy-Weibull (ExKu-weibuII) distribution [9], generalized Inverted Kumaraswamy Weibull (GIKw-Weibull) distribution [11], the inverse Weibull Weibull (IW-weibuII) distribution [10], the Kumaraswamy-Weibull (Ku-weibuII) distribution [7], Type II Half Logistic Weibull (TyIIKwHL-weibuII) distribution [11], exponentiated Weibull (Ex-weibuII) distribution [16], generalized inverse Weibull distribution[8] (GIWD) and the well known Weibull distribution. To do so, we consider two real applications: first, the life of fatigue fracture of Kevlar data [1], and secondly, the gauge lengths data [13].

6.1. Fatigue fracture of Kevlar data

The first data set has 76 observations and represents the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90 percent stress level until all had failed. Among other applications, the data set has been used to assess the superiority of the Transmuted Gompertz distribution over the Gompertz distribution [1].

6.2. Gauge lengths data

The second data set consists of 63 observations and represents the gauge lengths of 10 mm as reported in [13]. For each model, we estimate the parameters by using the method of MLE and adopt the maximum value of log(likelihood) evaluated at MLEs $(-l)$, minimum value of the Cram'er-vonMises (W^*) statistics, Anderson-Darling (A^*) and Kolmogorov-Smirnov (K-S) test statistics for model comparison purposes. In general, the smaller the values of these statistics, the better the fit to the data. The TTT transformation curves of these data sets are depicted in Figure 3(a) and Figure 3(b) respectively, which suggest an increasing frf for both data sets and therefore, indicate that the proposed model is suitable for fitting these data [5]. Furthermore, the key descriptive statistics of the data set 1 are listed in Table 3. Table 4 gives the MLEs of parameters with there corresponding standard errors in parenthesis. To compare goodness-of-fit of considered models, the computed goodness-of-fit measures are provided in Table 5. The estimated pdf, cdf, PP-plots and QQ-plots of the various models are respectively plotted in Figures 4–7, for the first data set. The key descriptive statistics, estimates of the parameters as well as the values of the goodness of fit statistics for data set 2 are listed in Tables 6–8. The estimated pdf, cdf, PP-plots and QQ-plots of the different models are plotted in Figures 8–11 for the second data set. We note that the EGIKw-Weibull distribution provides the best fit for both data sets. Hence, the proposed six parameter Weibull distribution is superior to other well known models in term of empirical model fitting to real data.

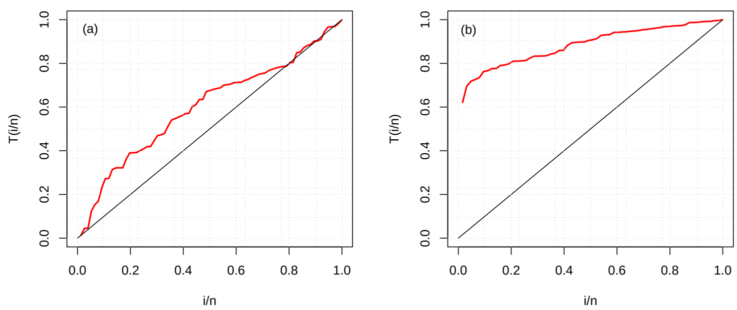

Figure 3. TTT-transform plot for the data.

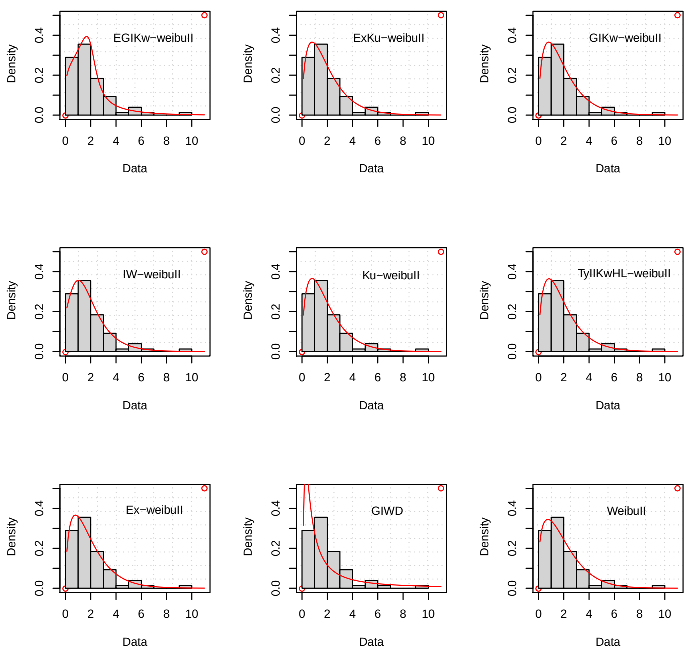

Figure 4. Estimated pdf of the considered models for the first data set.

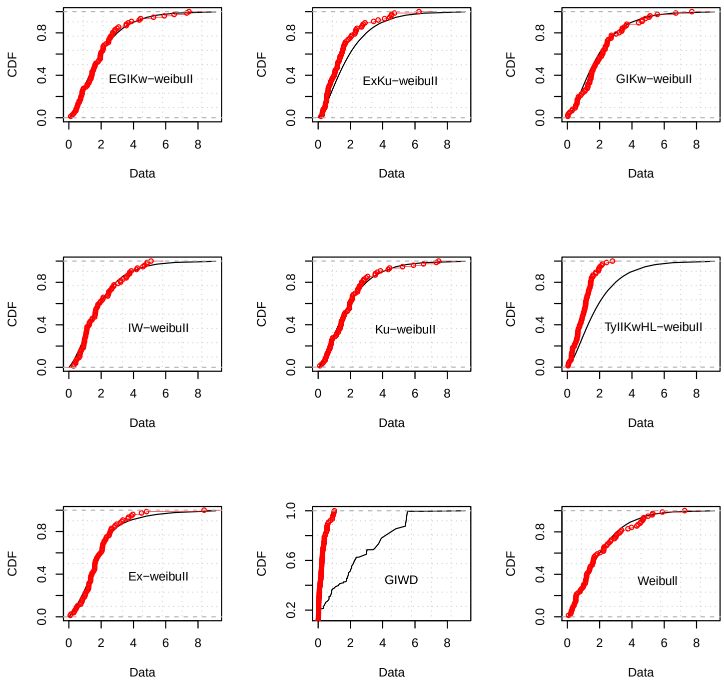

Figure 5. Esimated cdf of the considered models for the first data set.

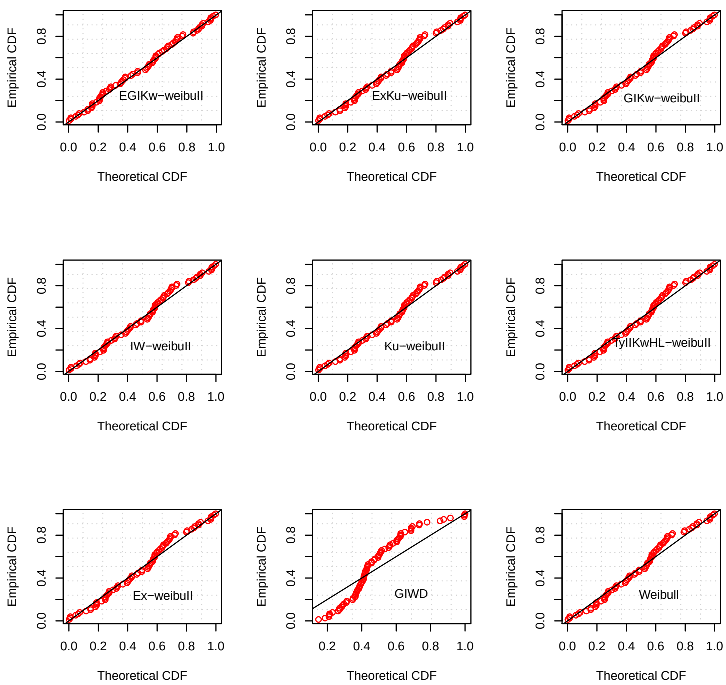

Figure 6. The sample pp-plots of the considered models for the first data set.

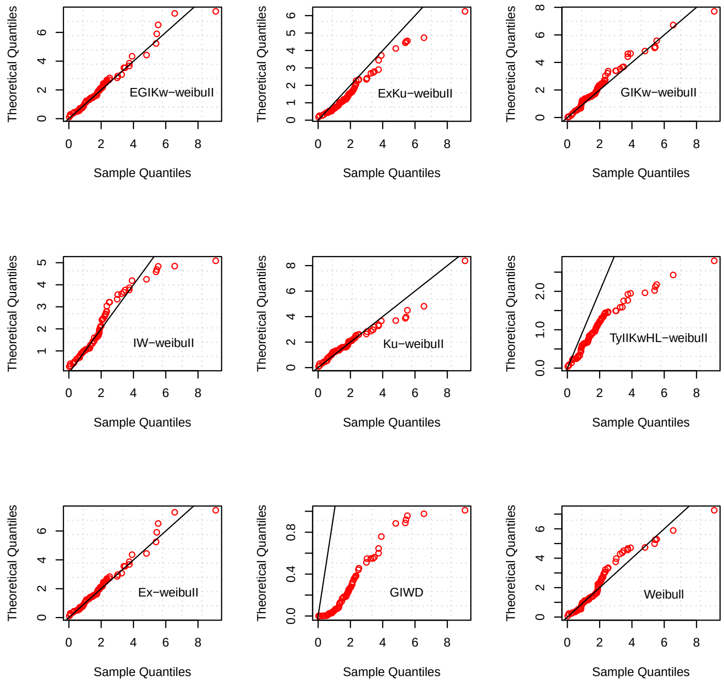

Figure 7. The sample QQ-plots of the considered models for the first data set.

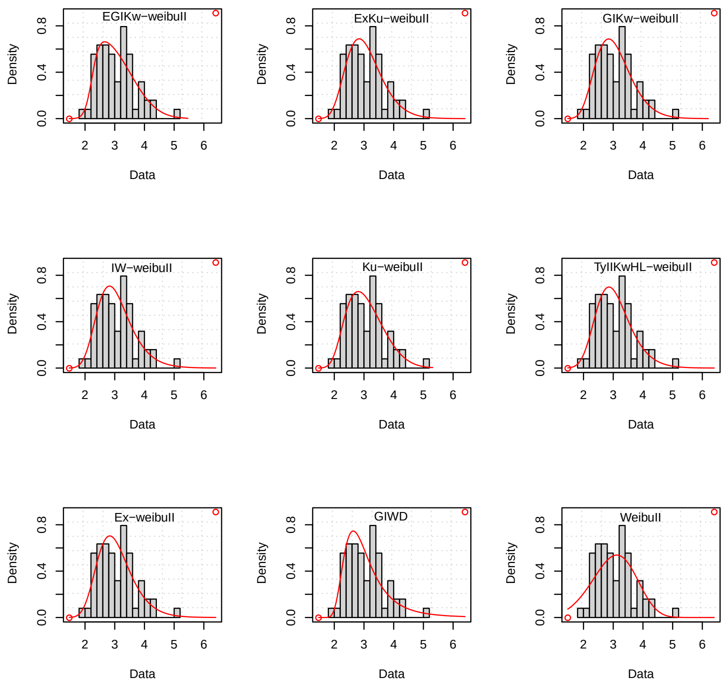

Figure 8. Estimated pdf of the considered models for the second data set.

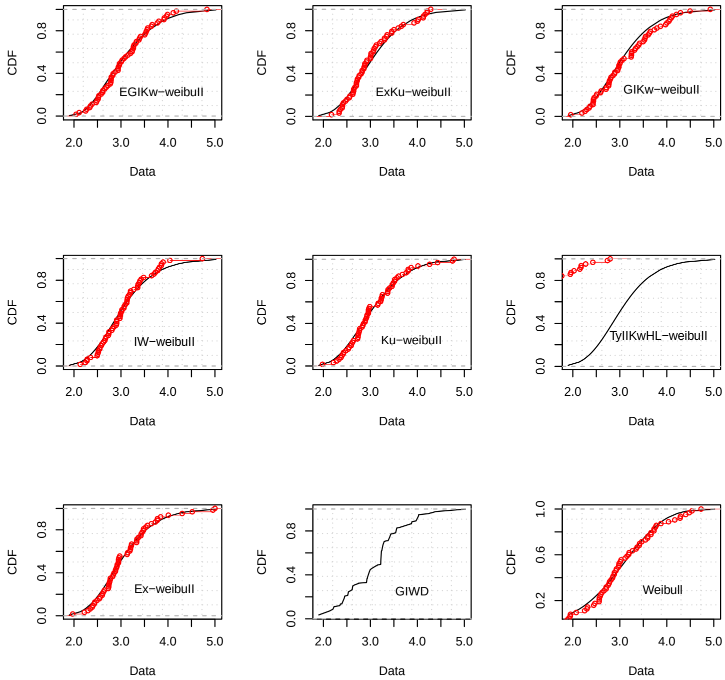

Figure 9. Estimated cdf of the considered models for the second data set.

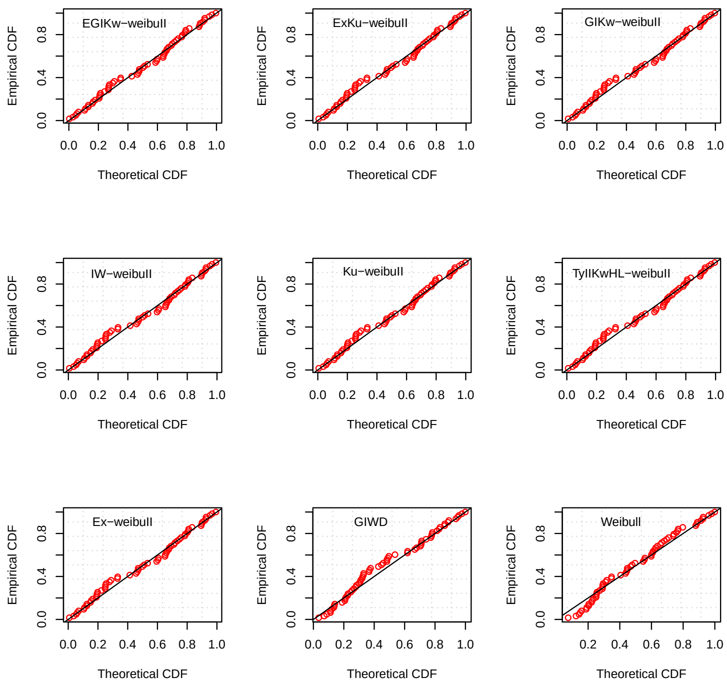

Figure 10. The sample pp-plots of the considered models for the second data set.

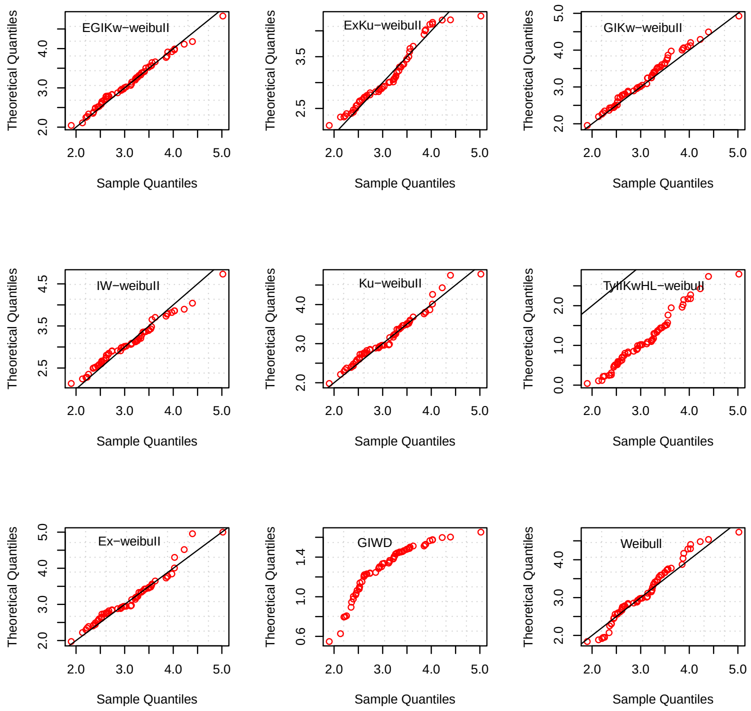

Figure 11. The sample QQ-plots of the considered models for the second data set.

Table 1. Mean estimates, AB and RMSEs of EGIKw-Weibull distribution for some parameters values.

n	Par	*I*			*II*		
		MLE	RMSE	AB	MLE	RMSE	AB
20	α	0.9713689	0.4806459	-0.1713689	2.014639	0.067753	-0.01464
	β	1.797029	0.5646082	0.202971	1.361224	0.387118	0.138776
	γ	2.814135	0.5174388	0.1858648	2.378661	0.338917	0.121339
	λ	1.549498	0.2232983	-0.04949762	1.281762	0.275996	-0.08176
	δ	2.197286	0.8419829	0.3027142	1.938168	0.730219	0.261832
	φ	1.976742	2.061288	-0.726742	2.390088	1.819486	-0.64009
50	α	0.8735313	0.3081877	-0.07353128	2.00552	0.031534	-0.00552
	β	1.910288	0.3747256	0.0897124	1.438533	0.255304	0.061467
	γ	2.916386	0.3493424	0.08361364	2.444653	0.229937	0.055347
	λ	1.512552	0.07593448	-0.01255213	1.231942	0.144346	-0.03194
	δ	2.364755	0.5649153	0.135245	2.08021	0.497489	0.11979
	φ	1.583688	1.408957	-0.3336881	2.066649	1.32863	-0.31665
100	α	0.8558051	0.2669619	-0.05580507	2.004274	0.02959	-0.00427
	β	1.93124	0.3278556	0.06875985	1.452371	0.225401	0.047629
	γ	2.935567	0.3072824	0.06443255	2.457278	0.202285	0.042722
	λ	1.508123	0.06450604	-0.008123143	1.224952	0.129524	-0.02495
	δ	2.395748	0.4970771	0.104252	2.108008	0.435321	0.091992
	φ	1.515932	1.282937	-0.2659319	1.986238	1.125242	-0.23624
120	α	0.8420048	0.2306365	-0.04200483	2.004662	0.030125	-0.00466
	β	1.947733	0.2863216	0.05226719	1.441842	0.248806	0.058158
	γ	2.951091	0.2679509	0.04890942	2.447648	0.224043	0.052352
	λ	1.505399	0.04747133	-0.005399344	1.228734	0.134517	-0.02873
	δ	2.421266	0.4313021	0.0787338	2.087134	0.482818	0.112866
	φ	1.445015	1.077493	-0.1950147	2.047259	1.285779	-0.29726

Table 2. Mean estimates, AB and RMSEs of EGIKw-Weibull distribution for some parameters values.

n	Par	*III*			*IV*		
		MLE	RMSE	AB	MLE	RMSE	AB
20	α	1.543784	0.3283173	-0.143784	0.591521	0.872937	-0.39152
	β	0.487087	0.03467822	0.01291299	0.767038	0.185238	0.082962
	γ	2.311938	0.4212962	0.1880618	2.11129	0.309592	0.13871
	λ	0.9680638	0.5161316	-0.2180638	0.502191	0.807812	-0.35219
	δ	0.5904005	0.2462409	0.1095995	0.26946	0.071814	0.03054
	φ	4.195821	1.624	-0.6958206	2.624829	2.548846	-1.12483
50	α	1.459047	0.2051262	-0.0590471	0.390737	0.602225	-0.19074
	β	0.49507	0.01889445	0.004930034	0.808772	0.130248	0.041228
	γ	2.418678	0.2786795	0.08132221	2.178767	0.224895	0.071233
	λ	0.8356844	0.3054641	-0.08568435	0.311112	0.51445	-0.16111
	δ	0.6516943	0.1658323	0.04830566	0.282696	0.056125	0.017304
	φ	3.819403	1.12277	-0.3194033	2.091998	1.890743	-0.592
100	α	1.453862	0.194879	-0.05386162	0.365697	0.559331	-0.1657
	β	0.4951045	0.019064	0.004895462	0.81373	0.122496	0.03627
	γ	2.425325	0.2675725	0.07467457	2.187578	0.210686	0.062422
	λ	0.8270703	0.2840287	-0.07707026	0.287341	0.467534	-0.13734
	δ	0.6559371	0.1581156	0.04406292	0.284998	0.051635	0.015002
	φ	3.787467	1.050978	-0.2874671	2.009697	1.735128	-0.5097
120	α	1.449761	0.1873277	-0.04976089	0.339417	0.515608	-0.13942
	β	0.4962008	0.0157394	0.003799213	0.820154	0.110474	0.029846
	γ	2.431216	0.2565162	0.06878395	2.198409	0.1908	0.051591
	λ	0.8202737	0.2678941	-0.07027365	0.267761	0.439284	-0.11776
	δ	0.6581765	0.1561381	0.04182355	0.287243	0.048087	0.012757
	φ	3.790683	1.103578	-0.2906829	1.932524	1.612671	-0.43252

Table 3. Descriptive statistics for the first data set.

n	Min	Max	Mean	Var	SD	CV	Skew	Kurt
76	0.0251	9.0960	1.9593	2.4774	1.5739	0.8034	3.9187	8.1608

Table 4. MLEs (standard errors in parentheses) of the considered models for the first data set.

Model	Estimates					
EGIKw-weibuII	0.11449	0.16608	3.94536	1.18814	0.27103	1.52339
	(0.00629)	(0.02175)	(0.00013)	(0.38551)	(0.02559)	(0.02524)
ExKu-weibuII	0.82736	0.31688	2.04223	4.57473	0.93841	-
	(0.63134)	(0.62384)	(3.54711)	(10.23424)	(1.12137)	
GIKw-weibuII	4.63961	0.94028	2.04370	0.31526	0.82531	-
	(10.55804)	(1.13866)	(3.62283)	(0.63603)	(0.64412)	
IW-weibuII	0.84909	4.29129	152.03867	0.26840	-	-
	(0.14019)	(1.16797)	(197.56958)	(0.0643)		
Ku-weibuII	0.88930	0.30474	1.78170	3.75457	-	-
	(0.69313)	(0.4527)	(1.60054)	(12.70693)		
TyIIKwHL-weibuII	1.41373	0.07720	2.66288	1.06894	-	-
	(0.4219)	(0.09838)	(3.76596)	(0.50664)		
Ex-weibuII	1.10106	1.44330	0.60971	-	-	-
	(0.26312)	(0.6446)	(0.21844)			
GIWD	0.46726	1.53337	0.75881	-	-	-
	(13.63017)	(33.9413)	(0.05409)			
WeibuII	1.32562	0.36638	-	-	-	-
	(0.11382)	(0.06197)				

Table 5. The goodness-of-fit statistics of considered models for the first data set.

Model	(-l)	AIC	CAIC	BIC	HQIC	W	A	K-S
EGIKw-weibuII	118.66240	249.32490	250.54230	253.30930	252.91370	0.03151	0.21232	0.05335(0.9741)
ExKu-weibuII	122.06510	254.13010	254.98730	265.78380	258.78750	0.11359	0.67302	0.09783(0.4335)
GIKw-weibuII	122.06490	254.12980	254.98700	265.78350	258.78720	0.11358	0.67298	0.09781(0.4338)
IW-weibuII	120.90400	249.80810	250.37140	259.13100	253.53400	0.08702	0.51589	0.08924(0.5502)
Ku-weibuII	122.07060	252.14120	252.70460	261.46420	255.86710	0.11372	0.67373	0.09729(0.4403)
TyIIKwHL-weibuII	121.87680	251.75360	252.31700	261.07660	255.47950	0.10938	0.64725	0.09688(0.4457)
Ex-weibuII	122.16360	250.32720	250.66060	257.31940	253.12160	0.11674	0.69121	0.0987(0.4225)
GIWD	153.53920	313.07840	313.41180	320.07060	315.87290	0.91679	5.33941	0.18931(0.00738)
WeibuII	122.52470	249.04940	249.21380	253.71080	250.91230	0.13060	0.76725	0.10993(0.2953)

Table 6. Descriptive statistics for the second data set.

n	Min	Max	Mean	Var	SD	CV	Skew	Kurt
63	1.9010	5.0200	3.0593	0.3855	0.6209	0.2029	0.4005	3.2863

Table 7. MLEs (standard errors in parentheses) of the considered models for the second data set.

Model	Estimates					
EGIKw-weibuII	0.11984	1.02573	1.63553	6.89710	0.15743	3.27713
	(0.06482)	(0.31323)	(0.66425)	(0.09642)	(0.02753)	(0.03203)
ExKu-weibuII	2.88268	0.30531	13.95201	0.20752	1.41919	-
	(0.03177)	(0.03358)	(0.09115)	(0.04015)	(0.30414)	
GIKw-weibuII	0.19163	1.26746	14.11254	0.26003	3.04698	-
	(0.02692)	(0.0732)	(0.02852)	(0.00307)	(0.00415)	
IW-weibuII	1.43479	1.03217	88.42556	0.82022	-	-
	(0.56986)	(0.02232)	(0.02115)	(0.02125)		
Ku-weibuII	3.23365	0.16491	11.12793	0.20890	-	-
	(0.01132)	(0.02347)	(0.0532)	(0.05046)		
TyIIKwHL-weibuII	1.68067	4.58986	0.11899	23.08838	-	-
	(0.62688)	(8.39343)	(0.21052)	(32.14521)		
Ex-weibuII	1.50781	31.80247	0.81865	-	-	-
	(0.63744)	(52.9703)	(0.54679)			
GIWD	1.88542	7.34656	5.43376	-	-	-
	(34.72914)	(735.31185)	(0.50783)			
WeibuII	4.73338	0.00363	-	-	-	-
	(0.22272)	(0.00099)				

Table 8. The goodness-of-fit statistics of considered models for the second data set.

Model	(-l)	AIC	CAIC	BIC	HQIC	W	A	K-S
EGIKw-weibuII	55.61034	118.62750	119.03430	125.05690	121.15620	0.03993	0.22349	0.06985(0.9183)
ExKu-weibuII	55.93587	121.87170	122.92440	132.58740	126.08630	0.04502	0.25666	0.07596(0.8604)
GIKw-weibuII	55.85388	121.70780	122.76040	132.42340	125.92230	0.04348	0.25233	0.08062(0.8075)
IW-weibuII	56.29762	120.59520	121.28490	129.16780	123.96690	0.06183	0.32767	0.0816(0.7956)
Ku-weibuII	55.98875	119.97750	120.66720	128.55000	123.34910	0.04507	0.25914	0.07311(0.8893)
TyIIKwHL-weibuII	56.36674	120.73350	121.42310	129.30600	124.10510	0.05971	0.32679	0.08212(0.7892)
Ex-weibuII	56.31374	123.22070	124.72070	136.07950	128.27810	0.06118	0.32698	0.07968(0.8187)
GIWD	58.90215	123.80430	124.21110	130.23370	126.33300	0.11497	0.64200	0.10012(0.5528)
WeibuII	62.22267	128.44530	128.64530	132.73160	130.13120	0.12003	0.83604	0.10977(0.4335)

7. Conclusions

There has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models to improve the modeling of survival data. As a result, significant progress has been made towards the generalization of the traditional Weibull model. In this article, a new six parameter Weibull extension named the EGIKw-Weibull distribution is proposed. The EGIKw-Weibull model is motivated by the fact that the generalization provides more flexibility to analyze positive real-life data. Graphs of the pdf, cdf, hrf and cumulative hrf of the distribution are presented. From Figure 1(a), it can be seen that the shape of the distribution is extremely left skewed, and Figure 2(b) shows that the hazard rate function of the EGIKw-Weibull distribution exhibits various shapes. That support using it in modeling the monotonic and non-monotonic hazard behaviors which are more likely to be encountered in practical situations like, human mortality, reliability analysis and biomedical applications. Various properties of the new model have been derived and explicit expressions for order statistics have been provided that makes analysis of data available. Parameter estimation is done by the method of MLE. Finally, a Monte Carlo Simulation study has been provided to assess the performance of the proposed model. The practical importance of the proposed distribution is demonstrated using two real applications, it is found that the EGIKw-Weibull model is well fitted as compared to its competing models, Tables 4, 5, 7 and 8.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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