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Research article

A new reverse Mulholland-type inequality with multi-parameters

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Abstract: In this paper, we present a new reverse Mulholland-type inequality with multi-parameters and deal with its equivalent forms. Based on the obtained inequalities, the equivalent statements of the best possible constant factor related to several parameters are discussed. As an application, some interesting inequalities for double series are derived from the special cases of our main results.

Keywords: Mulholland-type inequality; double series inequalities; weight coefficient; multi-parameters; equivalent statement
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1. Introduction

In 1925, Hardy [1] gave a generalization of the Hilbert's inequality by introducing one pair of conjugated exponents (p,q) which satisfies $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$, as follows:

If $a_m \geq 0, b_n \geq 0, 0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}(\sum_{m=1}^{\infty}a_m^p)^{1/p}(\sum_{n=1}^{\infty}b_n^q)^{1/q}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. The inequality (1) is called the Hardy-Hilbert inequality. In particular, when $p=q=2$, the Hardy-Hilbert inequality reduces to the Hilbert's inequality (see [2]). As is known to us, the Hardy-Hilbert inequality plays an important role in analysis number theory, real analysis and divergent series theory (see [3]).

For the continuous case, the integral version of Hardy-Hilbert inequality can be stated as follows (see [3], Theorem 316):

If $f(x)\ge 0,\ g(y)\ge 0,\ 0<\int_0^\infty f^p(x)dx<\infty$ and $0<\int_0^\infty g^q(y)dy<\infty$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(y)dy\right)^{1/q}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

The Hardy-Hilbert inequalities (1) and (2) has been studied extensively, numerous variants, generalizations, and extensions can be found in the literatures (see [4–8]).

Motivated by the Hardy-Hilbert inequality, in 1929, Mulholland [9] proposed a similar version of inequality (1), which contains the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ as in (1), i.e.,

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{mn\ln mn}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{1}{m}a_m^p\right)^{1/p}\left(\sum_{n=2}^{\infty}\frac{1}{n}b_n^q\right)^{1/q}. \tag{3}$$

Obviously, the Mulholland’s inequality (3) can be rewritten in an equivalent form as:

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{1}{m^{1-p}}a_m^p\right)^{1/p}\left(\sum_{n=2}^{\infty}\frac{1}{n^{1-q}}b_n^q\right)^{1/q}. \tag{4}$$

In recent years, the Mulholland inequality has been generalized by various methods of constructing parameters, see [10–15] and the references cited therein.

Let us recall some results which are connected with the current investigation. Hong and Wen [16] and Hong [17] studied the Hilbert type series inequalities and the Hilbert type integral inequalities with homogeneous kernel, respectively. They established the necessary and sufficient condition for which the inequalities hold under the best constant factor. Subsequently, by using the quasi-homogeneous kernels instead of the homogeneous kernel in the Hilbert type integral inequalities, Hong, He and Yang [18] established the necessary and sufficient condition for which the inequalities hold under the best constant factor. Recently, with the help of the Euler-Maclaurin summation formula, Yang, Wu and Liao [19] and gave the extension of Hardy-Hilbert’s inequality and its equivalent forms. In [20], Yang, Wu and Chen investigated the generalization of Hardy-Littlewood-Polya’s inequality and its equivalent forms.

In this paper, following the ideas of [16–20], we will study the Mulholland-type inequalities. The present research objects are structurally different from Hilbert type inequalities, and this will involve new techniques in dealing with the inequalities. Specifically, we will establish a reverse Mulholland-type inequality with multi-parameters. And then, we discuss the equivalent statements of the best possible constant factor associated with several parameters. Finally, we illustrate that some new inequalities of Mulholland-type can be derived from the equivalent expressions of the reverse Mulholland-type inequality.

2. Preliminaries

In this section, we present some preliminary results which are essential for establishing our main results in subsequent sections. We begin with introducing the notations $k_\lambda^{(\eta)}(x,y)$ and $k_{\lambda,\eta}(\gamma)$ with their associated formulas.

(i) In view of the following expression (see [21])

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{x-\pi k} + \frac{1}{x+\pi k}\right) \ (x \in (0,\pi)),$$

for $b \in (0,1)$, we have

$$\begin{aligned} A_b &:= \int_0^\infty \frac{u^{b-1}}{1-u}du = \int_0^1 \frac{u^{b-1}}{1-u}du + \int_1^\infty \frac{u^{b-1}}{1-u}du \\ &= \int_0^1 \frac{u^{b-1}}{1-u}du - \int_0^1 \frac{v^{-b}}{1-v}dv = \int_0^1 \frac{u^{b-1}-u^{-b}}{1-u}du \\ &= \int_0^1 \sum_{k=0}^{\infty}(u^{k+b-1} - u^{k-b})du = \sum_{k=0}^{\infty}\int_0^1 (u^{k+b-1} - u^{k-b})du \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{k+b} - \frac{1}{k+1-b}\right) = \pi\left[\frac{1}{\pi b} + \sum_{k=1}^{\infty}\left(\frac{1}{\pi b-\pi k} + \frac{1}{\pi b+\pi k}\right)\right] \\ &= \pi \cot \pi b \quad \in \mathrm{R} := (-\infty,+\infty). \end{aligned}$$

Moreover, it is easy to observe that $A_b > 0$ for $b \in (0,\frac{1}{2})$; $A_b < 0$ for $b \in (\frac{1}{2},1)$; $A_b = 0$ for $b = \frac{1}{2}$.

(ii) For $\lambda,\eta > 0,$ we set the homogeneous function of order $-\lambda$ as follows:

$$k_\lambda^{(\eta)}(x,y) := \frac{x^\eta - y^\eta}{x^{\lambda+\eta} - y^{\lambda+\eta}} \ \ (x, y > 0),$$

which satisfies $k_\lambda^{(\eta)}(ux,uy) = u^{-\lambda}k_\lambda^{(\eta)}(x,y)$ $(u,x,y>0)$, and $k_\lambda^{(\eta)}(v,v) := \frac{\eta}{(\lambda+\eta)v^\lambda}$ $(v>0)$.

It follows that $k_\lambda^{(\eta)}(x,y)$ is a positive and continuous function with respect to $x,y>0$. For $x \neq y,$ we obtain

$$\frac{\partial}{\partial x}k_\lambda^{(\eta)}(x,y) = -x^{\eta-1}(x^{\lambda+\eta} - y^{\lambda+\eta})^{-2}\varphi(x,y),$$

where $\varphi(x,y) := \lambda x^{\lambda+\eta} - (\lambda+\eta)y^\eta x^\lambda + \eta y^{\lambda+\eta}$ $(x, y > 0).$

We find that for $0 < x < y$,

$$\frac{\partial}{\partial x}\varphi(x,y) = \lambda(\lambda+\eta)x^{\lambda-1}(x^\eta - y^\eta) < 0;$$

for $x > y$, $\frac{\partial}{\partial x}\varphi(x,y) > 0$. It follows that $\varphi(x,y)$ is strictly decreasing (resp. increasing) with respect to $x < y$ (resp. $x > y$). Since $\varphi(y,y) = \min_{x>0}\varphi(x,y) = 0 \ (y > 0)$, then $\varphi(x,y) > 0 \ (x \neq y)$, namely, $\frac{\partial}{\partial x}k_{\lambda}^{(\eta)}(x,y) < 0 \ (x \neq y)$. Note that $k_{\lambda}^{(\eta)}(x,y)$ is continuous at $x = y$, we conform that $k_{\lambda}^{(\eta)}(x,y)$ is strictly decreasing with respect to $x > 0$. In the same way, we can show that $k_{\lambda}^{(\eta)}(x,y)$ is also strictly decreasing with respect to $y > 0$.

(iii) For $\gamma \in (0,\lambda)$, since $k_{\lambda}^{(\eta)}(x,y) > 0$, by (i), we obtain (see [22])

$$\begin{aligned} k_{\lambda,\eta}(\gamma) &:= \int_0^\infty k_{\lambda}^{(\eta)}(u,1)u^{\gamma-1}du = \int_0^\infty \frac{u^{\eta}-1}{u^{\lambda+\eta}-1}u^{\gamma-1}du \\ &\overset{v=u^{\lambda+\eta}}{=} \frac{1}{\lambda+\eta}\left(\int_0^\infty \frac{v^{\frac{\gamma}{\lambda+\eta}-1}}{1-v}dv - \int_0^\infty \frac{v^{\frac{\gamma+\eta}{\lambda+\eta}-1}}{1-v}dv\right) \\ &= \frac{\pi}{\lambda+\eta}\left[\cot(\tfrac{\pi\gamma}{\lambda+\eta}) - \cot(\tfrac{\pi(\gamma+\eta)}{\lambda+\eta})\right] \\ &= \frac{\pi}{\lambda+\eta}\left[\cot(\tfrac{\pi\gamma}{\lambda+\eta}) + \cot(\tfrac{\pi(\lambda-\gamma)}{\lambda+\eta})\right] \in \mathrm{R}_+ := (0,\infty). \end{aligned}$$

In what follows, we suppose that

$$p < 0 \ \ (0 < q < 1), \frac{1}{p} + \frac{1}{q} = 1, \alpha, \beta, \eta > 0, 0 < \lambda_1, \lambda_2 < \lambda, \lambda_1 \le \frac{1}{\alpha},\ \lambda_2 \le \frac{1}{\beta},$$

$$k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) = \frac{\ln^{\alpha\eta} m - \ln^{\beta\eta} n}{\ln^{\alpha(\lambda+\eta)} m - \ln^{\beta(\lambda+\eta)} n} \ \ (m,n \in \mathrm{N}\setminus\{1\} := \{2,3,\cdots\});$$

for $\gamma = \lambda_1, \lambda - \lambda_2 \ (\in (0,\lambda))$,

$$k_{\lambda,\eta}(\gamma) = \frac{\pi}{\lambda+\eta}\left[\cot(\tfrac{\pi\gamma}{\lambda+\eta}) + \cot(\tfrac{\pi(\lambda-\gamma)}{\lambda+\eta})\right]. \tag{5}$$

Also, we assume $a_m, b_n \ge 0 \ (m,n \in \mathrm{N}\setminus\{1\})$ such that

$$0 < \sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p < \infty \ \ and \ \ 0 < \sum_{n=2}^{\infty} \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} n}{n^{1-p}} b_n^q < \infty.$$

Definition 1. We define the following weight functions:

$$\omega_{\lambda,\eta}(\lambda_2, m) := \ln^{\alpha(\lambda-\lambda_2)} m \sum_{n=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) \frac{\ln^{\beta\lambda_2-1} n}{n} \ \ (m \in \mathrm{N}\setminus\{1\}), \tag{6}$$

$$\varpi_{\lambda,\eta}(\lambda_1, n) := \ln^{\beta(\lambda-\lambda_1)} n \sum_{m=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) \frac{\ln^{\alpha\lambda_1-1} m}{m} \ \ (n \in \mathrm{N}\setminus\{1\}). \tag{7}$$

Lemma 1. For $\beta\lambda_2 \le 1$, we have

$$\omega_{\lambda,\eta}(\lambda_2, m) < \frac{1}{\beta} k_{\lambda,\eta}(\lambda - \lambda_2) \in \mathrm{R}_+ \ \ (m \in \mathrm{N} \setminus \{1\}); \tag{8}$$

For $\alpha\lambda_1 \le 1$, we have

$$0 < \frac{1}{\alpha} k_{\lambda,\eta}(\lambda_1)(1 - \theta_{\lambda,\eta}(\lambda_1, n)) < \varpi_{\lambda,\eta}(\lambda_1, n) < \frac{1}{\alpha} k_{\lambda,\eta}(\lambda_1) \ \ (n \in \mathrm{N} \setminus \{1\}), \tag{9}$$

where $\theta_{\lambda,\eta}(\lambda_1, n)\ (>0)$ is indicated as

$$\theta_{\lambda,\eta}(\lambda_1, n) := \frac{1}{k_{\lambda,\eta}(\lambda_1)} \int_0^{\frac{\ln^\alpha 2}{\ln^\beta n}} k_\lambda^{(\eta)}(u,1) u^{\lambda_1 - 1} du = O\left(\frac{1}{\ln^{\beta\lambda_1} n}\right). \tag{10}$$

Proof. For fixed $m \in \mathrm{N} \setminus \{1\}$, $\beta\lambda_2 - 1 \le 0$, $k_\lambda^{(\eta)}(\ln^\alpha m, \ln^\beta t) \frac{\ln^{\beta\lambda_2 - 1} t}{t}$ is a strictly decreasing function with respect to $t > 1$. By the decreasing property of series, setting $u = \frac{\ln^\alpha m}{\ln^\beta t}$, we find that

$$\omega_{\lambda,\eta}(\lambda_2, m) < \ln^{\alpha(\lambda-\lambda_2)} m \int_1^\infty k_\lambda^{(\eta)}(\ln^\alpha m, \ln^\beta t) \frac{\ln^{\beta\lambda_2 - 1} t}{t} dt$$
$$= \frac{1}{\beta} \int_0^\infty k_\lambda^{(\eta)}(u,1) u^{(\lambda-\lambda_2)-1} du = \frac{1}{\beta} k_{\lambda,\eta}(\lambda - \lambda_2).$$

Hence, we get the inequality (8).

For $\alpha\lambda_1 - 1 \le 0$, it is evident that $k_\lambda^{(\eta)}(\ln^\alpha t, \ln^\beta n) \frac{\ln^{\alpha\lambda_1 - 1} t}{t}$ is a strictly decreasing function with respect to $t > 1$. By the decreasing property of series, setting $u = \frac{\ln^\alpha t}{\ln^\beta n}$, we find that

$$\varpi_{\lambda,\eta}(\lambda_1, n) < \ln^{\beta(\lambda-\lambda_1)} n \int_1^\infty k_\lambda^{(\eta)}(\ln^\alpha t, \ln^\beta n) \frac{\ln^{\alpha\lambda_1 - 1} t}{t} dt$$
$$= \frac{1}{\alpha} \int_0^\infty k_\lambda^{(\eta)}(u,1) u^{\lambda_1 - 1} du = \frac{1}{\alpha} k_{\lambda,\eta}(\lambda_1).$$

By the decreasing property of series and (10), we obtain

$$\varpi_{\lambda,\eta}(\lambda_1, n) > \ln^{\beta(\lambda-\lambda_1)} n \int_2^\infty k_\lambda^{(\eta)}(\ln^\alpha t, \ln^\beta n) \frac{\ln^{\alpha\lambda_1 - 1} t}{t} dt$$
$$= \frac{1}{\alpha} \int_{\frac{\ln^\alpha 2}{\ln^\beta n}}^\infty k_\lambda^{(\eta)}(u,1) u^{\lambda_1 - 1} du = \frac{1}{\alpha} k_{\lambda,\eta}(\lambda_1)(1 - \theta_{\lambda,\eta}(\lambda_1, n)) > 0,$$
$$0 < \theta_{\lambda,\eta}(\lambda_1, n) \le \frac{1}{k_{\lambda,\eta}(\lambda_1)} \int_0^{\frac{\ln^\alpha 2}{\ln^\beta n}} u^{\lambda_1 - 1} du = \frac{1}{\lambda_1 k_{\lambda,\eta}(\lambda_1)} \left(\frac{\ln^\alpha 2}{\ln^\beta n}\right)^{\lambda_1}.$$

Hence, we deduce the inequality (9). The Lemma 1 is proved.

Lemma 2. We have the following inequality:

$$
\begin{aligned}
I &:= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) a_m b_n \\
&> \frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2) k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) \{\sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p\}^{\frac{1}{p}} \\
&\times \{\sum_{n=2}^{\infty} (1-\theta_{\lambda,\eta}(\lambda_1, n)) \frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} n}{n^{1-q}} b_n^q\}^{\frac{1}{q}}.
\end{aligned} \tag{11}
$$

Proof. By the reverse Hölder inequality [23], we obtain

$$
\begin{aligned}
I &= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) [\frac{\ln^{(\beta\lambda_2-1)/p} n}{n^{1/p}} \frac{\ln^{(1-\alpha\lambda_1)/q} m}{m^{-1/q}} a_m] \\
&\quad \times [\frac{\ln^{(\alpha\lambda_1-1)/q} m}{m^{1/q}} \frac{\ln^{(1-\beta\lambda_2)/p} n}{n^{-1/p}} b_n] \\
&\geq \{\sum_{m=2}^{\infty} [\ln^{\alpha\lambda_1} m \sum_{n=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) \frac{\ln^{\beta\lambda_2-1} n}{n}] \frac{\ln^{p(1-\alpha\lambda_1)-1} m}{m^{1-p}} a_m^p\}^{\frac{1}{p}} \\
&\quad \times \{\sum_{n=2}^{\infty} [\ln^{\beta\lambda_2} n \sum_{m=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) \frac{\ln^{\alpha\lambda_1-1} m}{m}] \frac{\ln^{q(1-\beta\lambda_2)-1} n}{n^{1-q}} b_n^q\}^{\frac{1}{q}} \\
&= \{\sum_{m=2}^{\infty} \omega_{\lambda,\eta}(\lambda_2, m) \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p\}^{\frac{1}{p}} \{\sum_{n=2}^{\infty} \varpi_{\lambda,\eta}(\lambda_1, n) \frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} n}{n^{1-q}} b_n^q\}^{\frac{1}{q}}.
\end{aligned}
$$

By (8) and (9), we get the inequality (11). This completes the proof of Lemma 2.

Remark 1. By (11), for $\lambda_1 + \lambda_2 = \lambda$, we find

$$
0 < \sum_{m=2}^{\infty} \frac{\ln^{p(1-\alpha\lambda_1)-1} m}{m^{1-p}} a_m^p < \infty, \ 0 < \sum_{n=2}^{\infty} \frac{\ln^{q(1-\beta\lambda_2)-1} n}{n^{1-p}} b_n^q < \infty,
$$

and obtain the following inequality:

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) a_m b_n \\
&> \frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}(\lambda_1) [\sum_{m=2}^{\infty} \frac{\ln^{p(1-\alpha\lambda_1)-1} m}{m^{1-p}} a_m^p]^{\frac{1}{p}} [\sum_{n=2}^{\infty} (1-\theta_{\lambda,\eta}(\lambda_1, n)) \frac{\ln^{q(1-\beta\lambda_2)-1} n}{n^{1-q}} b_n^q]^{\frac{1}{q}}.
\end{aligned} \tag{12}
$$

In particular, for $\alpha = \beta = 1$, we have

$$
\tilde{\theta}_{\lambda,\eta}(\lambda_1, n) := \frac{1}{k_{\lambda,\eta}(\lambda_1)} \int_0^{\frac{\ln 2}{\ln n}} k_{\lambda}^{(\eta)}(u,1) u^{\lambda_1-1} du = O(\frac{1}{\ln^{\lambda_1} n}),
$$

$$
0 < \sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p < \infty, \ 0 < \sum_{n=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-p}} b_n^q < \infty,
$$

and the following inequality:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\ln^{\eta} m-\ln^{\eta} n}{\ln^{\lambda+\eta} m-\ln^{\lambda+\eta} n}a_m b_n$$

$$> k_{\lambda,\eta}(\lambda_1)[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}}a_m^p]^{\frac{1}{p}}[\sum_{n=2}^{\infty}(1-\widetilde{\theta}_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}}b_n^q]^{\frac{1}{q}}. \quad (13)$$

Hence, inequality (12) is an extension of inequality (13).

Lemma 3. For any $\varepsilon > 0$, we have

$$L := \sum_{n=2}^{\infty} O(\frac{1}{n\ln^{\beta(\lambda_1+\varepsilon)+1} n}) = O(1). \quad (14)$$

Proof. There exist constants $m > 0, M > 0$ such that

$$0 < m\sum_{n=2}^{\infty}\frac{1}{n\ln^{\beta(\lambda_1+\varepsilon)+1} n} \le L \le M[\frac{1}{2\ln^{\beta(\lambda_1+\varepsilon)+1} 2}+\sum_{n=3}^{\infty}\frac{1}{n\ln^{\beta(\lambda_1+\varepsilon)+1} n}].$$

By the decreasing property of series, it follows that

$$\begin{aligned} 0 < L &\le M[\frac{1}{2\ln^{\beta(\lambda_1+\varepsilon)+1} 2}+\int_2^{\infty}\frac{1}{y\ln^{\beta(\lambda_1+\varepsilon)+1} y}dy] \\ &= M[\frac{1}{2\ln^{\beta(\lambda_1+\varepsilon)+1} 2}+\frac{1}{\beta(\lambda_1+\varepsilon)}\ln^{-\beta(\lambda_1+\varepsilon)} 2] \\ &\le M(\frac{1}{2\ln^{\beta\lambda_1+1} 2}+\frac{1}{\beta\lambda_1}\ln^{-\beta\lambda_1} 2) < \infty. \end{aligned}$$

Hence, inequality (14) follows. The proof of Lemma 3 is complete.

Lemma 4. For $\delta_0 := \frac{1}{2}\min_{i=1,2}\{\lambda_i, \lambda-\lambda_i\}$, $0 < \varepsilon < q\delta_0$, we have

$$k_{\lambda,\eta}(\lambda_1+\frac{\varepsilon}{q}) \to k_{\lambda,\eta}(\lambda_1) \ (\varepsilon \to 0^+). \quad (15)$$

Proof. By Levi Theorem (see [24]), it follows that

$$\int_0^1 k_\lambda^{(\eta)}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du \to \int_0^1 k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du \ (\varepsilon \to 0^+).$$

For $0 < \varepsilon < q\delta_0$, we have $0 < \frac{\varepsilon}{q} < \delta_0$, and then $\lambda_1+\delta_0 < \lambda, \lambda-\lambda_1+\delta_0 < \lambda$,

$$0 \le \int_1^{\infty} k_\lambda^{(\eta)}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du \le \int_1^{\infty} k_\lambda^{(\eta)}(u,1)u^{\lambda_1+\delta_0-1}du \le k_{\lambda,\eta}(\lambda_1+\delta_0) < \infty.$$

By using Lebesgue control convergence theorem [24], we have

$$\int_1^{\infty} k_\lambda^{(\eta)}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du \to \int_1^{\infty} k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du \ (\varepsilon \to 0^+).$$

Then we find

$$k_{\lambda,\eta}(\lambda_1+\tfrac{\varepsilon}{q})=\int_0^1 k_\lambda^{(\eta)}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du+\int_1^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du$$
$$\to\int_0^1 k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du+\int_1^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du=k_{\lambda,\eta}(\lambda_1)\ (\varepsilon\to 0^+).$$

The Lemma 4 is proved.

Lemma 5. If $\lambda_1+\lambda_2=\lambda$, then the constant factor $k_{\lambda,\eta}(\lambda_1)$ in (12) is the best possible.

Proof. For $0<\varepsilon<q\delta_0$, we set

$$\widetilde{a}_m:=\frac{\ln^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}m}{m},\widetilde{b}_n:=\frac{\ln^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}n}{n}\ \ (m,n\in\mathrm{N}\setminus\{1\}).$$

If there exists a constant $M\ge\frac{1}{\beta^{1/p}\alpha^{1/q}}k_{\lambda,\eta}(\lambda_1)$ such that the inequalities (12) is valid when replacing $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_{\lambda,\eta}(\lambda_1)$ by M, then, in particular, we have

$$\widetilde{I}:=\sum_{n=2}^\infty\sum_{m=2}^\infty k_\lambda^{(\eta)}(\ln^\alpha m,\ln^\beta n)\widetilde{a}_m\widetilde{b}_n$$

$$>M[\sum_{m=2}^\infty\frac{\ln^{p(1-\alpha\lambda_1)-1}m}{m^{1-p}}\widetilde{a}_m^p]^{\frac{1}{p}}[\sum_{n=2}^\infty(1-\theta_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q(1-\beta\lambda_2)-1}n}{n^{1-q}}\widetilde{b}_n^q]^{\frac{1}{q}}.$$

In view of (14), we obtain

$$\widetilde{I}>M[\sum_{m=2}^\infty\frac{\ln^{p(1-\alpha\lambda_1)-1}m}{m^{1-p}}\frac{\ln^{p\alpha\lambda_1-\alpha\varepsilon-p}m}{m^p}]^{\frac{1}{p}}$$

$$\times[\sum_{n=2}^\infty(1-\theta_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q(1-\beta\lambda_2)-1}n}{n^{1-q}}\frac{\ln^{q\beta\lambda_2-\beta\varepsilon-q}n}{n^q}]^{\frac{1}{q}}$$

$$=M(\frac{\ln^{-\alpha\varepsilon-1}2}{2}+\sum_{m=3}^\infty\frac{\ln^{-\alpha\varepsilon-1}m}{m})^{\frac{1}{p}}(\sum_{n=2}^\infty\frac{\ln^{-\beta\varepsilon-1}n}{n}-\sum_{n=2}^\infty O(\frac{1}{\ln^{\beta\lambda_1}n})\frac{\ln^{-\beta\varepsilon-1}n}{n})^{\frac{1}{q}}$$

$$>M(\frac{\ln^{-\alpha\varepsilon-1}2}{2}+\int_2^\infty\frac{\ln^{-\alpha\varepsilon-1}x}{x}dx)^{\frac{1}{p}}(\int_2^\infty\frac{\ln^{-\beta\varepsilon-1}y}{y}dy-\sum_{n=2}^\infty O(\frac{1}{n\ln^{\beta(\lambda_1+\varepsilon)+1}n}))^{\frac{1}{q}}$$

$$=\frac{M}{\varepsilon}(\frac{\varepsilon\ln^{-\alpha\varepsilon-1}2}{2}+\frac{1}{\alpha}\ln^{-\alpha\varepsilon}2)^{\frac{1}{p}}(\frac{1}{\beta}\ln^{-\beta\varepsilon}2-\varepsilon O(1))^{\frac{1}{q}}.$$

By (8), for $\hat{\lambda}_2=\lambda_2-\frac{\varepsilon}{q}(\in(0,\frac{1}{\beta}))\ (\hat{\lambda}_1=\lambda_1+\frac{\varepsilon}{q}<\lambda)$, we find

$$\widetilde{I}=\sum_{m=2}^{\infty}[\ln^{\alpha(\lambda_1+\frac{\varepsilon}{q})} m\sum_{n=2}^{\infty}k_{\lambda}^{(\eta)}(\ln^{\alpha} m,\ln^{\beta} n)\frac{\ln^{\beta(\lambda_2-\frac{\varepsilon}{q})-1} n}{n}]\frac{\ln^{-\alpha\varepsilon-1} m}{m}$$
$$=\sum_{m=2}^{\infty}\omega_{\lambda,\eta}(\hat{\lambda}_2,m)\frac{\ln^{-\alpha\varepsilon-1} m}{m} \quad <\frac{1}{\beta}k_{\lambda,\eta}(\hat{\lambda}_1)(\frac{\ln^{-1-\alpha\varepsilon} 2}{2}+\sum_{m=3}^{\infty}\frac{\ln^{-1-\alpha\varepsilon} m}{m})$$
$$\le\frac{1}{\beta}k_{\lambda,\eta}(\hat{\lambda}_1)(\frac{\ln^{-1-\alpha\varepsilon} 2}{2}+\int_2^{\infty}\frac{\ln^{-1-\alpha\varepsilon} x}{x}dx)$$
$$=\frac{1}{\varepsilon\beta\alpha}k_{\lambda,\eta}(\hat{\lambda}_1)(\frac{\varepsilon\alpha\ln^{-1-\alpha\varepsilon} 2}{2}+\ln^{-\alpha\varepsilon} 2).$$

Then we have

$$\frac{1}{\beta\alpha}k_{\lambda,\eta}(\lambda_1+\frac{\varepsilon}{q})(\frac{\varepsilon\alpha\ln^{-1-\alpha\varepsilon} 2}{2}+\ln^{-\alpha\varepsilon} 2)$$
$$\ge\varepsilon\widetilde{I}>M(\frac{\varepsilon\ln^{-\alpha\varepsilon-1} 2}{2}+\frac{1}{\alpha}\ln^{-\alpha\varepsilon} 2)^{\frac{1}{p}}(\frac{1}{\beta}\ln^{-\varepsilon} 2-\varepsilon O(1))^{\frac{1}{q}}.$$

For $\varepsilon\to 0^+$, by (15), we get

$$\frac{1}{\beta\alpha}k_{\lambda,\eta}(\lambda_1)\ge M(\frac{1}{\alpha})^{\frac{1}{p}}(\frac{1}{\beta})^{\frac{1}{q}},$$

namely, $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_{\lambda,\eta}(\lambda_1)\ge M$. Hence, $M=\frac{1}{\beta^{1/p}\alpha^{1/q}}k_{\lambda,\eta}(\lambda_1)$ is the best possible constant factor of (12). The proof of Lemma 5 is completed.

Remark 2. Setting $\widetilde{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\lambda_1+\frac{\lambda-\lambda_1-\lambda_2}{p},\widetilde{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, we find

$$\widetilde{\lambda}_1+\widetilde{\lambda}_2=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda.$$

For $|\lambda-\lambda_1-\lambda_2|<|p|\delta_0$, it means that $-\delta_0<\frac{\lambda-\lambda_1-\lambda_2}{p}<\delta_0$, $0<\widetilde{\lambda}_1<\lambda, 0<\widetilde{\lambda}_2=\lambda-\widetilde{\lambda}_1<\lambda$, and then $k_{\lambda,\eta}(\widetilde{\lambda}_1)\in\mathrm{R}_+$. We reduce (11) as follows:

$$I>\frac{1}{\beta^{1/p}\alpha^{1/q}}k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\alpha\widetilde{\lambda}_1)-1} m}{m^{1-p}}a_m^p]^{\frac{1}{p}}$$
$$\times[\sum_{n=2}^{\infty}(1-\theta_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q(1-\beta\widetilde{\lambda}_2)-1} n}{n^{1-q}}b_n^q]^{\frac{1}{q}}. \tag{16}$$

Lemma 6. If $|\lambda-\lambda_1-\lambda_2|<|p|\delta_0$, and the constant factor $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ in (11) is the best possible, then we have $\lambda=\lambda_1+\lambda_2$.

Proof. If the constant factor $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ in (11) (or (16)) is the best possible, then by (12) (for $\lambda_i = \tilde{\lambda}_i$ $(i = 1,2)$), we have

$$\frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) \geq \frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}(\tilde{\lambda}_1) \ (\in \mathrm{R}_+),$$

namely, $k_{\lambda,\eta}(\tilde{\lambda}_1) \leq k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$.

By the reverse Hölder inequality, we find

$$\begin{aligned} k_{\lambda,\eta}(\tilde{\lambda}_1) &= k_{\lambda,\eta}\left(\tfrac{\lambda-\lambda_2}{p} + \tfrac{\lambda_1}{q}\right) \\ &= \int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du = \int_0^\infty k_\lambda^{(\eta)}(u,1)(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du \\ &\geq \left(\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda-\lambda_2-1}du\right)^{\frac{1}{p}}\left(\int_0^\infty k_\lambda^{(\eta)}(u,1)u^{\lambda_1-1}du\right)^{\frac{1}{q}} \\ &= k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1). \end{aligned} \tag{17}$$

It follows that (17) keeps the form of equality.

We observe that (17) keeps the form of equality if and only if there exist constants A and B such that they are not both zero and

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \ a.e. \text{ in } \mathrm{R}_+ = (0,\infty).$$

Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{A}{B}$ *a.e.* in R_+, and then $\lambda-\lambda_2-\lambda_1 = 0$.

Namely, $\lambda = \lambda_1 + \lambda_2$. This completes the proof of Lemma 6.

3. Main results

Theorem 1. Inequality (11) is equivalent to the following inequalities:

$$\begin{aligned} J &:= \left[\sum_{n=2}^\infty \frac{\ln^{p\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1} n}{(1-\theta_{\lambda,\eta}(\lambda_1,n))^{p-1} n}\left(\sum_{m=2}^\infty k_\lambda^{(\eta)}(\ln^\alpha m, \ln^\beta n)a_m\right)^p\right]^{\frac{1}{p}} \\ &> \frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=2}^\infty \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p\right\}^{\frac{1}{p}}, \end{aligned} \tag{18}$$

$$\begin{aligned} J_1 &:= \left[\sum_{m=2}^\infty \frac{\ln^{q\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1} m}{m}\left(\sum_{n=2}^\infty k_\lambda^{(\eta)}(\ln^\alpha m, \ln^\beta n)b_n\right)^q\right]^{\frac{1}{q}} \\ &> \frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)\left\{\sum_{n=2}^\infty (1-\theta_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} n}{n^{1-q}} b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{19}$$

Proof. Suppose that (18) is valid. By the reverse Hölder inequality, we have

$$I=\sum_{n=2}^{\infty}\Big[\frac{\ln^{\frac{-1}{p}+\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}n}{(1-\theta_{\lambda,\eta}(\lambda_1,n))^{1/q}n^{1/p}}\sum_{m=2}^{\infty}k_{\lambda}^{(\eta)}(\ln^{\alpha}m,\ln^{\beta}n)a_m\Big]$$
$$\times\Big[(1-\theta_{\lambda,\eta}(\lambda_1,n))^{\frac{1}{q}}\frac{\ln^{\frac{1}{p}-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}n}{n^{-1/p}}b_n\Big]$$
$$\geq J\Big\{\sum_{n=2}^{\infty}(1-\theta_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}n}{n^{1-q}}b_n^q\Big\}^{\frac{1}{q}}. \tag{20}$$

Then by (18), we obtain (11). On the other hand, assuming that (11) is valid, we set

$$b_n:=\frac{\ln^{p\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}n}{(1-\theta_{\lambda,\eta}(\lambda_1,n))^{p-1}n}\Big(\sum_{m=2}^{\infty}k_{\lambda}^{(\eta)}(\ln^{\alpha}m,\ln^{\beta}n)a_m\Big)^{p-1}, n\in\mathbf{N}\backslash\{1\}.$$

It follows that

$$J^p=\sum_{n=2}^{\infty}(1-\theta_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}n}{n^{1-q}}b_n^q=I. \tag{21}$$

If $J=\infty$, then (18) is naturally valid; if $J=0$, then it is impossible that makes (18) valid, namely, $J>0$. Suppose that $0<J<\infty$. By (11), we find

$$\sum_{n=2}^{\infty}(1-\theta_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}n}{n^{1-q}}b_n^q=J^p=I$$
$$>k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)\Big\{\sum_{m=2}^{\infty}\frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}m}{m^{1-p}}a_m^p\Big\}^{\frac{1}{p}}J^{p-1},$$
$$J=\Big\{\sum_{n=2}^{\infty}(1-\theta_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}n}{n^{1-q}}b_n^q\Big\}^{\frac{1}{p}}$$
$$>k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)\Big\{\sum_{m=2}^{\infty}\frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}m}{m^{1-p}}a_m^p\Big\}^{\frac{1}{p}},$$

thus, (18) follows, which is equivalent to (11).

Suppose that (19) is valid. By the reverse Hölder inequality, we have

$$I=\sum_{m=2}^{\infty}\Big[\frac{\ln^{\frac{1}{q}-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}m}{m^{-1/q}}a_m\Big]\Big[\frac{\ln^{\frac{-1}{q}+\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}m}{m^{1/q}}\sum_{n=2}^{\infty}k_{\lambda}^{(\eta)}(\ln^{\alpha}m,\ln^{\beta}n)b_n\Big]$$
$$\geq\Big\{\sum_{m=2}^{\infty}\frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}m}{m^{1-p}}a_m^p\Big\}^{\frac{1}{p}}J_1. \tag{22}$$

Then by (19), we obtain (11). On the other hand, assuming that (11) is valid, we set

$$a_m:=\frac{\ln^{q\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}m}{m}\Big(\sum_{n=2}^{\infty}k_{\lambda}^{(\eta)}(\ln^{\alpha}m,\ln^{\beta}n)b_n\Big)^{q-1},\ m\in\mathrm{N}\backslash\{1\}.$$

It follows that

$$J_1^q = \sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p = I. \tag{23}$$

If $J_1 = \infty$, then (19) is naturally valid; if $J_1 = 0$, then it is impossible that makes (19) valid, namely, $J_1 > 0$. Suppose that $0 < J_1 < \infty$. By (11), we have

$$\sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p = J_1^q = I$$
$$> k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2) k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) J_1^{q-1} \{\sum_{n=2}^{\infty} (1-\theta_{\lambda,\eta}(\lambda_1,n)) \frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} n}{n^{1-q}} b_n^q\}^{\frac{1}{q}},$$
$$J_1 = \{\sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p\}^{\frac{1}{q}}$$
$$> k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2) k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) \{\sum_{n=2}^{\infty} (1-\theta_{\lambda,\eta}(\lambda_1,n)) \frac{\ln^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} n}{n^{1-q}} b_n^q\}^{\frac{1}{q}},$$

thereby, (19) follows, which is equivalent to (11). Hence, inequalities (11), (18) and (19) are equivalent.

This completes the proof of Theorem 1.

Theorem 2. Suppose that $|\lambda-\lambda_1-\lambda_2| < |p|\delta_0$. The following statements (i), (ii), (iii), (iv) and (v) are equivalent:

(i) Both $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2) k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ and $k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are independent of p, q;

(ii) $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2) k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1) = k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$;

(iii) $\lambda = \lambda_1 + \lambda_2$;

(iv) $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2) k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (11);

(v) $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2) k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (18) (resp. (19)).

Moreover, if the statement (iii) follows, namely, $\lambda = \lambda_1 + \lambda_2$, then we have (12) and the following equivalent inequalities with the best possible constant factor $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}(\lambda_1)$:

$$[\sum_{n=2}^{\infty} \frac{\ln^{p\beta\lambda_2-1} n}{(1-\theta_{\lambda,\eta}(\lambda_1,n))^{p-1} n} (\sum_{m=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) a_m)^p]^{\frac{1}{p}}$$
$$> \frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}(\lambda_1) [\sum_{m=2}^{\infty} \frac{\ln^{p(1-\alpha\lambda_1)-1} m}{m^{1-p}} a_m^p]^{\frac{1}{p}}, \tag{24}$$

$$[\sum_{m=2}^{\infty} \frac{\ln^{q\alpha\lambda_1-1} m}{m} (\sum_{n=2}^{\infty} k_{\lambda}^{(\eta)}(\ln^{\alpha} m, \ln^{\beta} n) b_n)^q]^{\frac{1}{q}}$$
$$> \frac{1}{\beta^{1/p}\alpha^{1/q}} k_{\lambda,\eta}(\lambda_1) [\sum_{n=2}^{\infty} (1-\theta_{\lambda,\eta}(\lambda_1,n)) \frac{\ln^{q(1-\beta\lambda_2)-1} n}{n^{1-q}} b_n^q]^{\frac{1}{q}}. \tag{25}$$

Proof. (i)$\Rightarrow$(ii). By (i), we have

$$k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)=\lim_{p\to-\infty}\lim_{q\to1^-}k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)=k_{\lambda,\eta}(\lambda_1).$$

Then by (i), (15) and the above result, we find

$$k_{\lambda,\eta}(\tfrac{\lambda-\lambda_2}{p}+\tfrac{\lambda_1}{q})=\lim_{p\to-\infty}k_{\lambda,\eta}(\lambda_1+\tfrac{\lambda-\lambda_1-\lambda_2}{p})=k_{\lambda,\eta}(\lambda_1)=k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1).$$

(ii)$\Rightarrow$(iii). by (ii), (17) keeps the form of equality. In view of the proof of Lemma 6, it follows that $\lambda=\lambda_1+\lambda_2$.

(iii)$\Rightarrow$(i). By (iii), for $\lambda=\lambda_1+\lambda_2$, both $k_{\lambda,\eta}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda,\eta}^{\frac{1}{q}}(\lambda_1)$ and $k_{\lambda,\eta}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are equal to $k_{\lambda,\eta}(\lambda_1)$, which are independent of p,q. Hence, we have (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

(iii)$\Leftrightarrow$(iv). By Lemma 5 and Lemma 6, we have the conclusions.

(iv)$\Leftrightarrow$(v). If the constant factor in (11) is the best possible, then so is the constant factor in (18) (resp. (19)). Otherwise, by (20) (resp. (22)), we would reach a contradiction that the constant factor in (11) is not the best possible. On the other hand, if the constant factor in (18) (resp. (19)) is the best possible, then so is the constant factor in (11). Otherwise, by (21) (resp. (23)), we would reach a contradiction that the constant factor in (18) (resp. (19)) is not the best possible.

Hence, the statements (i), (ii), (iii), (iv) and (v) are equivalent. The proof of Theorem 2 is complete.

4. An application

In this section, we illustrate that some novel inequalities of Mulholland type can be derived from our main results as special cases.

Remark 3. Taking $\alpha=\beta=1$ in (24) and (25) respectively, we have the following inequalities which is equivalent to (13) with the best possible constant factor $k_{\lambda,\eta}(\lambda_1)$:

$$[\sum_{n=2}^{\infty}\frac{\ln^{p\lambda_2-1}n}{(1-\tilde{\theta}_{\lambda,\eta}(\lambda_1,n))^{p-1}n}(\sum_{m=2}^{\infty}\frac{\ln^{\eta}m-\ln^{\eta}n}{\ln^{\lambda+\eta}m-\ln^{\lambda+\eta}n}a_m)^p]^{\frac{1}{p}}$$
$$>k_{\lambda,\eta}(\lambda_1)[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}m}{m^{1-p}}a_m^p]^{\frac{1}{p}}, \tag{26}$$

$$[\sum_{m=2}^{\infty}\frac{\ln^{q\lambda_1-1}m}{m}(\sum_{n=2}^{\infty}\frac{\ln^{\eta}m-\ln^{\eta}n}{\ln^{\lambda+\eta}m-\ln^{\lambda+\eta}n}b_n)^q]^{\frac{1}{q}}$$
$$>k_{\lambda,\eta}(\lambda_1)[\sum_{n=2}^{\infty}(1-\tilde{\theta}_{\lambda,\eta}(\lambda_1,n))\frac{\ln^{q(1-\lambda_2)-1}n}{n^{1-q}}b_n^q]^{\frac{1}{q}}. \tag{27}$$

In particular, putting $\lambda=1,\lambda_1=\lambda_2=\frac{1}{2}$ in (13), (26) and (27) respectively, we have the following equivalent reverse inequalities with the best possible constant factor $\frac{2\pi}{1+\eta}\cot\frac{\pi}{2(1+\eta)}$:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\ln^{\eta}m-\ln^{\eta}n}{\ln^{1+\eta}m-\ln^{1+\eta}n}a_m b_n$$

$$>\frac{2\pi}{1+\eta}\cot\frac{\pi}{2(1+\eta)}\Big(\sum_{m=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}m}{m^{1-p}}a_m^p\Big)^{\frac{1}{p}}\Big[\sum_{n=2}^{\infty}(1-\tilde{\theta}_{1,\eta}(\tfrac{1}{2},n))\frac{\ln^{\frac{q}{2}-1}n}{n^{1-q}}b_n^q\Big]^{\frac{1}{q}}, \tag{28}$$

$$\Big[\sum_{n=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}n}{(1-\tilde{\theta}_{1,\eta}(\frac{1}{2},n))^{p-1}n}\Big(\sum_{m=2}^{\infty}\frac{\ln^{\eta}m-\ln^{\eta}n}{\ln^{1+\eta}m-\ln^{1+\eta}n}a_m\Big)^p\Big]^{\frac{1}{p}}$$

$$>\frac{2\pi}{1+\eta}\cot\frac{\pi}{2(1+\eta)}\Big(\sum_{m=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}m}{m^{1-p}}a_m^p\Big)^{\frac{1}{p}}, \tag{29}$$

$$\Big[\sum_{m=2}^{\infty}\frac{\ln^{\frac{q}{2}-1}m}{m}\Big(\sum_{n=2}^{\infty}\frac{\ln^{\eta}m-\ln^{\eta}n}{\ln^{1+\eta}m-\ln^{1+\eta}n}b_n\Big)^q\Big]^{\frac{1}{q}}$$

$$>\frac{2\pi}{1+\eta}\cot\frac{\pi}{2(1+\eta)}\Big[\sum_{n=2}^{\infty}(1-\tilde{\theta}_{1,\eta}(\tfrac{1}{2},n))\frac{\ln^{\frac{q}{2}-1}n}{n^{1-q}}b_n^q\Big]^{\frac{1}{q}}, \tag{30}$$

where $\tilde{\theta}_{1,\eta}(\frac{1}{2},n):=\frac{1+\eta}{2\pi}\tan\frac{\pi}{2(1+\eta)}\int_0^{\frac{\ln 2}{\ln n}}\frac{u^{\eta}-1}{u^{1+\eta}-1}u^{-\frac{1}{2}}du=O(\frac{1}{\ln^{1/2}n})$.

Choosing $\eta=1$ in (28), (29) and (30) respectively, we have the reverse of inequality (3) and the equivalent forms with the best possible constant factor π:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{\ln mn}a_m b_n>\pi\Big(\sum_{m=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}m}{m^{1-p}}a_m^p\Big)^{\frac{1}{p}}\Big[\sum_{n=2}^{\infty}(1-\tilde{\theta}(n))\frac{\ln^{\frac{q}{2}-1}n}{n^{1-q}}b_n^q\Big]^{\frac{1}{q}}, \tag{31}$$

$$\Big[\sum_{n=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}n}{(1-\tilde{\theta}(n))^{p-1}n}\Big(\sum_{m=2}^{\infty}\frac{1}{\ln mn}a_m\Big)^p\Big]^{\frac{1}{p}}>\pi\Big(\sum_{m=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}m}{m^{1-p}}a_m^p\Big)^{\frac{1}{p}}, \tag{32}$$

$$\Big[\sum_{m=2}^{\infty}\frac{\ln^{\frac{q}{2}-1}m}{m}\Big(\sum_{n=2}^{\infty}\frac{1}{\ln mn}b_n\Big)^q\Big]^{\frac{1}{q}}>\pi\Big[\sum_{n=2}^{\infty}(1-\tilde{\theta}(n))\frac{\ln^{\frac{q}{2}-1}n}{n^{1-q}}b_n^q\Big]^{\frac{1}{q}}, \tag{33}$$

where $\tilde{\theta}(n):=\tilde{\theta}_{1,1}(\frac{1}{2},n)=\frac{1}{\pi}\int_0^{\frac{\ln 2}{\ln n}}\frac{1}{u+1}u^{-\frac{1}{2}}du=O(\frac{1}{\ln^{1/2}n})\in(0,1)$.

5. Conclusions

In this paper, by the use of the weight coefficients and the idea of introducing parameters, a new reverse Mulholland-type inequality with multi-parameters and the equivalent forms were given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to several parameters were obtained in Theorem 2. Some other inequalities associated with reverse Mulholland-type inequality were established in Remarks 1 and 3. The lemmas and theorems presented in this paper provided an extensive account of this type of inequalities.

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Conflict of interest

The authors declare that they have no competing interests.

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