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Research article

Complete solutions of the simultaneous Pell equations $(a^2 + 1)y^2 - x^2 = y^2 - bz^2 = 1$

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Abstract: In this paper, we consider the simultaneous Pell equations $(a^2 + 1)y^2 - x^2 = y^2 - bz^2 = 1$ where $a > 0$ is an integer and $b > 1$ is squarefree and has at most three prime divisors. We obtained the necessary and sufficient conditions that the above simultaneous Pell equations have positive integer solutions by using only the elementary methods of factorization, congruence, the quadratic residue and fundamental properties of Lucas sequence and the associated Lucas sequence. Moreover, we prove that these simultaneous Pell equations have at most one solution. When a solution exists, assuming the positive solutions of the Pell equation $x^2(a^2 + 1) - y^2 = -1$ are $x = x_m$ and $y = y_m$ with $m \geq 1$ odd, then the only solution of the system is given by $m = 3$ or $m = 5$ or $m = 7$ or $m = 9$.

Keywords: Diophantine equations; simultaneous Pell equations; minimal solutions; Lucas sequences
Mathematics Subject Classification: 11D25, 11B37, 11B39

1. Introduction

There have been several papers written that deal with the simultaneous Pell equations

$$x^2 - ay^2 = y^2 - bz^2 = 1 \tag{1.1}$$

where a and b are distinct positive integers. In [3], Bennett gave a result that says (1.1) possess at most three positive integer solutions (x, y, z). In [25], Yuan conjectured that for any positive integers a and b, the system of Pell equations (1.1) has at most one solution in positive integers. Yuan's conjecture has been confirmed for several classes of coefficients. For instance, by using a result of Ljunggren [17], results of Charmichael [6] and Voutier [23], and certain results on primitive prime factors of Lucas sequences of the first kind and properties of Lucas sequences, Yuan [25] showed that (1.1) possess at most one solution (x, y, z) in positive integers for $a = 4m(m + 1)$. Yuan gave the proof by an elementary argument. In a similar manner, Cipu [7] confirmed Yuan's conjecture for $a = 4m^2 - 1$. Hence, it was shown by Yuan and Cipu that the system of Pell equations

$$x^2 - (a^2 - 1)y^2 = y^2 - bz^2 = 1 \tag{1.2}$$

(independently of the parity of $a > 1$) has at most one solution in positive integers. Then, a substantial improvement was provided by Cipu and Mignotte. The authors [8] demonstrated that the system (1.1) has at most two positive integer solutions (x, y, z). Recently, in [2], the authors considered the system (1.2) in the case more specifically. They solved the system (1.2) in the case of $a = 5$ and b prime and showed that $(x, y, z, b) = (49, 10, 3, 11)$ and $(x, y, z, b) = (485, 99, 70, 2)$ are the only solutions. Right after, Irmak [14] considered (1.2), where $a \geq 2$ is an integer and b is prime. Assuming the positive solutions of the Pell equation $x^2 - (a^2 - 1)y^2 = 1$ are $x = x_m$ and $y = y_m$ $(m \geq 1)$, he proved that if $m \geq 3$ is an odd integer, then there is no positive integer solutions of (1.2). Moreover, he claims to give the complete list of solutions for $5 \leq a \leq 14$ in the cases of $m = 1$ and $m \geq 2$ even integer. But, when we look at the paper of Irmak, we see that the first part of the proof of Theorem 1 is not full, in particular, the case $\gcd(x_k, y_{k+1}) = a$ is missing. In [15], the solution $(x, y, z, b) = (485, 99, 70, 2)$ is found in the case of $m = 3$. Also, there is a missing solution $(x, y, z, b) = (287, 24, 5, 23)$ for $a = 12$. In [15], the authors considered the simultaneous Pell equations

$$x^2 - (a^2 - 1)y^2 = y^2 - pz^2 = 1 \tag{1.3}$$

where p is prime and $a > 1$. Assuming the positive solutions of the Pell equation $x^2 - (a^2 - 1)y^2 = 1$ are $x = x_m$ and $y = y_m$ with $m \geq 1$, they proved that system (1.3) has solutions only when $m = 2$ or $m = 3$. In the case of $m = 3$, they showed that $p = 2$ and gave the solutions of (1.3) in terms of Pell and Pell-Lucas sequences. When $m = 2$ and $p \equiv 3 \pmod 4$, the values of a, x, y, and z have been determined. They also showed that (1.3) has no solutions when $p \equiv 1 \pmod 4$. The case $a = 5$ of the system (1.3) has been considered by the authors [2]. In [9], the author obtained all the positive integer solutions of the system (1.2) when b is a product of two distinct primes or three distinct primes and additional conditions are fulfilled. In [13], the author obtained all the positive integer solutions of the system (1.2) when b is a product of 2 and three distinct odd primes.

In this paper, we consider the simultaneous Pell equations

$$(a^2 + 1)y^2 - x^2 = y^2 - bz^2 = 1 \tag{1.4}$$

where $a > 0$ is an integer and $b > 1$ is squarefree and has at most three prime divisors. Assuming the positive integer solutions of the Pell equation $x^2 - (a^2 + 1)y^2 = -1$ are $x = x_m$ and $y = y_m$ with $m \geq 1$ an odd integer, we prove that system (1.4) has solutions only when $m = 3$ or $m = 5$ or $m = 7$ or $m = 9$. We prove the following

Theorem 1.1. *Let p be a prime and let a be a positive integer. Then the simultaneous Pell equations*

$$(a^2 + 1)y^2 - x^2 = y^2 - pz^2 = 1 \tag{1.5}$$

have positive integer solutions if and only if $p = 2$ and $2a^2 + 1$ is a square. When a solution exists there is exactly one solution. The only one solution is given by

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{y_3^2 - 1}{2}}\right) = \left(4a^3 + 3a, 4a^2 + 1, 2a\sqrt{2a^2 + 1}\right).$$

Theorem 1.2. *Let p and q be two distinct primes and let a be a positive integer. Then the simultaneous Pell equations*

$$(a^2+1)y^2 - x^2 = y^2 - pqz^2 = 1 \tag{1.6}$$

have positive integer solutions if and only if $p = 2$ and $2a^2 + 1$ is a product of q and a square integer. When a solution exists there is exactly one solution. The only one solution is given by

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{y_3^2 - 1}{2q}}\right) = \left(4a^3 + 3a, 4a^2 + 1, 2a\sqrt{\frac{2a^2+1}{q}}\right).$$

We shall denote by $\square$ an unspecified perfect square.

Theorem 1.3. *Let p, q and r be prime and let a be a positive integer. Then the simultaneous Pell equations*

$$(a^2+1)y^2 - x^2 = y^2 - pqrz^2 = 1 \tag{1.7}$$

have at most one positive integer solution. Moreover, the solution exists if and only if one of the conditions holds:

α) $p = 2$ and $2a^2 + 1 = qr\square$;

β) $p = 2$ and $2a^2 + 1 = \square, 4a^2 + 1 = q\square, 4a^2 + 3 = r\square$;

γ) $2a^2 + 1 = \square, 8a^4 + 8a^2 + 1 = p\square, 4a^2 + 1 = q\square, 4a^2 + 3 = r\square$;

δ) $2a^2 + 1 = \square, 8a^4 + 8a^2 + 1 = p\square, 16a^4 + 12a^2 + 1 = q\square, 16a^4 + 20a^2 + 5 = r\square$.

When it exists, the solution is given by formula

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{y_3^2 - 1}{2qr}}\right) = \left(4a^3 + 3a, 4a^2 + 1, 2a\sqrt{\frac{2a^2+1}{qr}}\right) \quad \textit{in case} \quad \alpha),$$

$$(x, y, z) = \left(x_5, y_5, 2a\sqrt{\frac{(2a^2+1)(4a^2+1)(4a^2+3)}{qr}}\right) \quad \textit{in case} \quad \beta),$$

$$(x, y, z) = \left(x_7, y_7, 4a\sqrt{\frac{(2a^2+1)(8a^4+8a^2+1)(4a^2+1)(4a^2+3)}{pqr}}\right) \quad \textit{in case} \quad \gamma),$$

$$(x, y, z) = \left(x_9, y_9, 4a\sqrt{\frac{(2a^2+1)(8a^4+8a^2+1)(16a^4+12a^2+1)(16a^4+20a^2+5)}{pqr}}\right) \quad \textit{in case} \quad \delta).$$

We organize this paper as follows. In Section 2, we present some basic definitions and some lemmas which are needed in the proofs of our main results. Consequently, in Sections 3 to 5, we give the proofs of Theorem 1.1 to 1.3, respectively.

2. Some tools and basic definitions and some Lemmas

In the proof of our main result, generalized Lucas sequences and associated Lucas sequences play an essential role. So, we need to recall them. Let P, Q be nonzero coprime integers, let $D = P^2-4Q$ be called discriminant, and assume that $D > 0$. Consider the polynomial $x^2 - Px + Q$, called characteristic polynomial, which has the roots

$$\alpha = \frac{P + \sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{D}}{2}.$$

For each $n \geq 0$, define the Lucas sequence $U_n = U_n(P, Q)$ and the associated Lucas sequence $V_n = V_n(P, Q)$ as follows:

$$U_0 = 0, U_1 = 1, U_{n+1} = PU_n - QU_{n-1} \quad (\text{for} \quad n \geq 1),$$

$$V_0 = 2, V_1 = P, V_{n+1} = PV_n - QV_{n-1} \quad (\text{for} \quad n \geq 1).$$

It is easy to see

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

and

$$V_n = \alpha^n + \beta^n.$$

For $(P, Q) = (1, -1), (U_n)$ and (V_n) are the sequences of Fibonacci and Lucas numbers, respectively. When $P = 2$ and $Q = 1, (U_n) = (P_n)$ and $(V_n) = (Q_n)$ are the familiar sequences of Pell and Pell-Lucas numbers. When $Q = -1$, we represent (U_n) and (V_n) by $(U_n(P, -1))$ and $(V_n(P, -1))$. Consider the Pell equation

$$x^2 - Dy^2 = 1. \tag{2.1}$$

If $x_1 + y_1\sqrt{D}$ is the fundamental solution of the Eq (2.1), then all positive integer solutions of this equation are given by

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$$

with $n \geq 1$. If $x_1 + y_1\sqrt{D}$ is the fundamental solution of the equation

$$x^2 - Dy^2 = -1, \tag{2.2}$$

then all positive integer solutions of this equation are given by

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$$

with n an odd integer.

Lemma 2.1. ([19]) *Let $x_1 + y_1\sqrt{D}$ be the fundamental solution of the Eq (2.1). Then all positive integer solutions of the Eq (2.1) are given by*

$$x_n = \frac{V_n(2x_1, 1)}{2} \quad \textit{and} \quad y_n = y_1 U_n(2x_1, 1)$$

with $n \geq 1$. If $x_1 + y_1\sqrt{d}$ is the fundamental solution of the Eq (2.2), then all positive integer solutions of (2.2) are given by

$$x_n = \frac{V_n(2x_1, -1)}{2} \quad \textit{and} \quad y_n = y_1 U_n(2x_1, -1)$$

with n an odd integer. If $x_1\sqrt{a} + y_1\sqrt{b}$ is the minimal positive integer solution of the equation $ax^2 - by^2 = 1$, where $a > 1$, ab is not a square, then all positive integer solutions of $ax^2 - by^2 = 1$ are given by

$$x_n = x_1 U_n(2y_1\sqrt{b}, -1) \quad \textit{and} \quad y_n = y_1 U_n(2x_1\sqrt{a}, 1)$$

with n an odd integer.

The following identities are fairly well known and valid for the numbers $U_n = U_n(P, -1)$ and $V_n = V_n(P, -1)$:

$$\text{If} \quad d = \gcd(m, n), \quad \text{then} \quad \gcd(U_m, U_n) = U_d, \tag{2.3}$$

$$U_{2n} = U_n V_n. \tag{2.4}$$

Let $m = 2^a k$, $n = 2^b l$, k and l odd, $a, b \geq 0$, and $d = \gcd(m, n)$.

$$\gcd(U_m, V_n) = \begin{cases} V_d, & \text{if} \quad a > b, \\ 1 \quad \text{or} \quad 2, & \text{if} \quad a \leq b \end{cases} \tag{2.5}$$

If P is even, then V_n is always even and U_m is even iff m is even. When P is even and $a \leq b$, we get $\gcd(U_m, V_n) = 2$ if m is even and $\gcd(U_m, V_n) = 1$ if m is odd. Moreover, we have

$$U_n^2 - 1 = U_{n-1}U_{n+1} \tag{2.6}$$

and

$$V_{2n} = V_n^2 + 2 \tag{2.7}$$

for all $n \geq 1$. We omit the proofs of the following lemmas, as they are based on straightforward induction. The details can be also seen in the references [12,16,18,20,24].

Lemma 2.2. *If k is even, then $P|U_k(P, -1)$ and if k is odd, then $P|V_k(P, -1)$.*

Lemma 2.3. *If $P \equiv 0 \pmod 2$, then*

$$v_2(U_n(P, -1)) = \begin{cases} 0, & \text{if} \quad n \equiv 1 \pmod 2, \\ v_2(P) + v_2(n) - 1, & \text{if} \quad n \equiv 0 \pmod 2 \end{cases}$$

and

$$v_2(V_n(P, -1)) = \begin{cases} v_2(P), & \text{if} \quad n \equiv 1 \pmod 2, \\ 1, & \text{if} \quad n \equiv 0 \pmod 2 \end{cases}$$

Lemma 2.4. ([17]) *Let the minimal positive integer solution of the equation $Ax^2 - By^2 = 1$ be $\varepsilon = x_0\sqrt{A} + y_0\sqrt{B}$, where $A > 1$ and B are coprime positive integers with $d = AB$ not a square. Then the only possible solution of the equation $Ax^2 - By^4 = 1$ is given by $x\sqrt{A} + y^2\sqrt{B} = \varepsilon^l$ where $y_0 = lf^2$ for some odd squarefree integer l.*

Lemma 2.5. ([4]) *Let b, d be positive integers with $b > 1$, then the Diophantine equation $b^2X^4 - dY^2 = 1$ has at most one solution in positive integers X, Y, which can be given by $X^2b + Y\sqrt{d} = ab + h\sqrt{d}$, where $ab + h\sqrt{d}$ is the minimal positive integer solution of the equation $b^2T^2 - dU^2 = 1$.*

Lemma 2.6. ([1,5]) *The Diophantine equation*

$$AX^4 - BY^2 = 1 \tag{2.8}$$

has at most two positive integer solutions. Moreover, (2.8) *is solvable if and only if x_0 is a square, where $x_0\sqrt{A} + y_0\sqrt{B} = \varepsilon$ is the minimal positive integer solution of $AU^2 - BV^2 = 1$. And if $x^2\sqrt{A} + y\sqrt{B} = \varepsilon^k$, then $k = 1$ or $k = p \equiv 3 \pmod 4$ is a prime.*

Lemma 2.7. ([11]) *Let the fundamental solution of the equation $v^2 - du^2 = -1$ be $\varepsilon = x_0 + y_0\sqrt{d}$. Then the only possible solution of the equation $X^4 - dY^2 = -1$ is given by $X^2 + y\sqrt{d} = \varepsilon^l$ where $x_0 = lf^2$ for some odd squarefree integer l.*

Lemma 2.8. ([10,21]) *Let the fundamental solution of the equation $v^2 - du^2 = 1$ be $a + b\sqrt{d}$. Then the only possible solutions of the equation $X^4 - dY^2 = 1$ are given by $X^2 = a$ and $X^2 = 2a^2 - 1$; both solutions occur in the following cases: $d = 1785, 7140, 28560$.*

Lemma 2.9. ([22]) *Let $D > 0$ be a nonsquare integer. Define*

$$T_n + U_n\sqrt{D} = (T_1 + U_1\sqrt{D})^n,$$

where $T_1 + U_1\sqrt{D}$ is the fundamental solution of the Pell equation

$$X^2 - DY^2 = 1. \tag{2.9}$$

There are at most two positive integer solutions (X, Y) to the equation

$$X^2 - DY^4 = 1. \tag{2.10}$$

1. If two solutions $Y_1 < Y_2$ exist, then $Y_1^2 = U_1$ and $Y_2^2 = U_2$, except only if $D = 1785$ or $D = 16 \cdot 1785$, in which case $Y_1^2 = U_1$ and $Y_2^2 = U_4$.

2. If only one positive integer solution (X, Y) to Eq (2.10) exists, then $Y^2 = U_l$ where $U_1 = lv^2$ for some squarefree integer l, and either $l = 1$, $l = 2$ or $l = p$ for some prime $p \equiv 3 \pmod 4$.

Lemma 2.10. *The Diophantine equation*

$$x^2 - 4a^2(a^2 + 1)y^4 = 1 \tag{2.11}$$

has a unique positive integer solution $(x, y) = (2a^2 + 1, 1)$.

Proof. It is easy to see that $(x, y) = (2a^2 + 1, 1)$ is the fundamental solution of (2.9) with $D = 4a^2(a^2 + 1)$ as well as a positive solution of (2.11). The result immediately follows by Lemma 2.9 since $4a^2(2a^2 + 1) \neq 1785, 7140, 28560$. □

Lemma 2.11. *The Diophantine equation*

$$8x^4 + 8x^2 + 1 = y^2 \tag{2.12}$$

has no positive integer solutions.

Proof. Assume that (a, b) is a positive integer solution of (2.12). Then we have that

$$(2a^2 + 1)\sqrt{2} + b = (\sqrt{2} + 1)^n, \tag{2.13}$$

for some odd n. Write $n = m + 1$ with m even, we have

$$(2a^2 + 1)\sqrt{2} + b = (\sqrt{2} + 1)^m(\sqrt{2} + 1) = (x_m + y_m)\sqrt{2} + (x_m + 2y_m),$$

where $x_m + y_m\sqrt{2} = (\sqrt{2} + 1)^m$ is a positive integer solution of $X^2 - 2Y^2 = 1$.

This implies that either

$$x_m + 1 = 2u^2, x_m - 1 = v^2, y_m = uv, \gcd(u, v) = 1, 2|v \tag{2.14}$$

or

$$x_m + 1 = u^2, x_m - 1 = 2v^2, y_m = uv, \gcd(u, v) = 1, 2|u \tag{2.15}$$

for some positive integers u and v. From (2.14) we get that $v(v + u) = 2a^2$. And so

$$x_m = v^2 + 1 = (2a_1^2)^2 + 1, u^2 - 2a_1^4 = 1,$$

which is impossible by Lemma 2.9. Similarly, from (2.15) we get that

$$v(2v + u) = 2a^2.$$

And so

$$x_m = 2v^2 + 1 = 2a_1^4 + 1, a_1^4 - 2(u/2)^2 = -1.$$

This implies that $a_1 = 1, u = 2$ by Lemma 2.7. Hence

$$x_m = 3, y_m = 2, 2a^2 + 1 = x_m + y_m = 5,$$

which is impossible. This completes the proof of Lemma 2.11. □

The first equation of (1.5)

$$x^2 - (a^2 + 1)y^2 = -1 \tag{2.16}$$

has the fundamental solution $\alpha = a + \sqrt{a^2 + 1}$. Then all positive integer solutions of the Eq (2.16) are given by

$$x = \frac{V_{2m+1}(2a, -1)}{2}, y = U_{2m+1}(2a, -1)(m \geq 0)$$

by Lemma 2.1. In the sequel, we write V_m and U_m instead of $V_m(2a, -1)$ and $U_m(2a, -1)$, respectively.

3. Proof of Theorem 1.1

Assume that (x, y, z) is a positive integer solution of (1.5). By Lemma 2.1 we know that

$$x = \frac{V_{2m+1}}{2}, y = U_{2m+1} \tag{3.1}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. Since $U_{4k+1}^2 - 1 = pz^2$, it follows from (2.6) that $U_{4k}U_{4k+2} = pz^2$. Using the fact that $\gcd(4k, 4k+2) = 2$, we get $\gcd(U_{4k}, U_{4k+2}) = U_2 = 2a$ by (2.3). Then either

$$U_{4k} = 2ab^2, U_{4k+2} = 2apc^2 \tag{3.2}$$

or

$$U_{4k} = 2apb^2, U_{4k+2} = 2ac^2 \tag{3.3}$$

for some integers b and c.

If (3.2) holds, then $\left(\frac{V_{4k}}{2}, b\right)$ is a solution of (2.11). Thus we get by Lemma 2.10 that $b = 1$, so $U_{4k} = 2a = U_2$, which leads to a contradiction.

If (3.3) holds, then $\left(\frac{V_{4k+2}}{2}, c\right)$ is a solution of (2.11). We get by Lemma 2.10 that $c = 1$, which leads to a contradiction again. Hence, both of these are impossible.

The case m is odd, say $m = 2k+1$ for some nonnegative integer k. Since $U_{4k+3}^2 - 1 = pz^2$, it follows from (2.6) that $U_{4k+4}U_{4k+2} = pz^2$. Using the fact that $\gcd(4k+4, 4k+2) = 2$, we get $\gcd(U_{4k+4}, U_{4k+2}) = U_2 = 2a$ by (2.3). Then either

$$U_{4k+4} = 2ab^2, U_{4k+2} = 2apc^2 \tag{3.4}$$

or

$$U_{4k+4} = 2apb^2, U_{4k+2} = 2ac^2 \tag{3.5}$$

for some integers b and c.

If (3.4) holds, then $\left(\frac{V_{4k+4}}{2}, b\right)$ is a solution of (2.11). Thus we get by Lemma 2.10 that $b = 1$, so $U_{4k+4} = 2a = U_2$, which leads to a contradiction.

If (3.5) holds, then $\left(\frac{V_{4k+2}}{2}, c\right)$ is a solution of (2.11). We get by Lemma 2.10 that $c = 1$, which leads to $U_{4k+2} = 2a = U_2$. Hence $m = 1$ and $4a(2a^2+1) = U_2V_2 = U_4 = 2apb^2$, which implies that $p = 2$ and $2a^2 + 1 = b^2$. Conversely, if $p = 2$ and $2a^2 + 1 = b^2$, then by calculation one can easily find that

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{y_3^2 - 1}{2}}\right) = (4a^3 + 3a, 4a^2 + 1, 2ab)$$

is a solution of (1.5).

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Assume that (x, y, z) is a positive integer solution of (1.6). By Lemma 2.1 we know that

$$x = \frac{V_{2m+1}}{2}, y = U_{2m+1} \tag{4.1}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. Since $U_{4k+1}^2 - 1 = pqz^2$, it follows from (2.6) that $U_{4k}U_{4k+2} = pqz^2$. Using the fact that $\gcd(4k, 4k+2) = 2$, we get $\gcd(U_{4k}, U_{4k+2}) = U_2 = 2a$ by (2.3). Then

$$U_{4k} = 2ab^2, U_{4k+2} = 2apqc^2 \tag{4.2}$$

or

$$U_{4k} = 2apqb^2, U_{4k+2} = 2ac^2 \tag{4.3}$$

or

$$U_{2k}V_{2k} = U_{4k} = 2apb^2, U_{2k+1}V_{2k+1} = U_{4k+2} = 2aqc^2 \tag{4.4}$$

for some integers b and c.

If (4.2) holds, then $\left(\frac{V_{4k}}{2}, b\right)$ is a solution of (2.11). Thus we get by Lemma 2.10 that $b = 1$, so $U_{4k} = 2a = U_2$, which leads to a contradiction.

If (4.3) holds, then $\left(\frac{V_{4k+2}}{2}, c\right)$ is a solution of (2.11). We get by Lemma 2.10 that $c = 1$, so $U_{4k+2} = 2a = U_2$, which leads to a contradiction again. Hence, both of (4.2) and (4.3) are impossible.

If (4.4) holds, then we get from the former equation that either

$$U_{2k} = apb_1^2, V_{2k} = 2b_2^2 \tag{4.5}$$

or

$$U_{2k} = ab_1^2, V_{2k} = 2pb_2^2 \tag{4.6}$$

for some integers b_1 and b_2 since $\gcd(\frac{U_{2k}}{a}, \frac{V_{2k}}{2}) = 1$ from Eq (2.5) and Lemma 2.3.

If (4.5) holds, then we know from the latter equation that (b_2, U_{2k}) is a solution of equation $X^4 - (a^2+1)Y^2 = 1$. Since $(2a^2+1, 2a)$ is the fundamental solution of the equation $X^2 - (a^2+1)Y^2 = 1$, by Lemma 2.8 we get that either $2a^2 + 1 = b_2^2$ or $2(2a^2+1)^2 - 1 = b_2^2$. The latter equation leads to $8a^4 + 8a^2 + 1 = b_2^2$, which is impossible by Lemma 2.11. We get from the former equation that $V_2/2 = 2a^2 + 1 = b_2^2 = V_{2k}/2$. Hence $k = 1$. Substituting the value into the latter equation of (4.4) gives $U_3V_3 = 2aqc^2$. It follows that either $4a^2 + 1 = U_3 = \square$ or $4a^2 + 3 = \frac{V_3}{2a} = \square$ since $\gcd(U_3, \frac{V_3}{2a}) = 1$ from Eq (2.5) and Lemma 2.3. However, the former equation is obviously not true. By taking the latter equation modulo 4, we see that it is impossible.

If (4.6) holds, then we know from the former equation that $(V_{2k}/2, b_1)$ is a solution of equation $X^2 - a^2(a^2+1)Y^4 = 1$. Since $(2a^2+1, 2)$ is the fundamental solution of the equation $X^2 - a^2(a^2+1)Y^2 = 1$, by Lemma 2.9 we get that $U_{2k} = ab_1^2 = 4a(2a^2+1) = U_2V_2$. Hence $k = 2$. Substituting the value into the latter equation of (4.4) gives $U_5\frac{V_5}{2a} = qc^2$. It follows that either $16a^4 + 20a^2 + 5 = \frac{V_5}{2a} = \square$ or $U_5 = \square$ and $V_5/2 = aq\square$ since $\gcd(U_5, \frac{V_5}{2a}) = 1$ from Eq (2.5) and Lemma 2.3. By taking the former equation modulo 8, we get that $4a^2 + 5 \equiv 1 \pmod 8$. Hence a is odd. But from equation $2a^2 + 1 = \left(\frac{b_1}{2}\right)^2$, we have that $3 \equiv 2a^2 + 1 \equiv 1 \pmod 4$, which leads to a contradiction. We get from the latter two equations that $(\sqrt{U_5}, \frac{V_5}{2})$ is a solution of equation $(a^2+1)X^4 - Y^2 = 1$. Since $(U_1 = 1, a)$ is the minimal positive integer solution of the equation $(a^2+1)X^2 - Y^2 = 1$, by Lemma 2.6 we get that $U_5 = U_P$, and so $5 = P \equiv 3 \pmod 4$, which leads to a contradiction.

The case m is odd, say $m = 2k+1$ for some nonnegative integer k. Since $U_{4k+3}^2 - 1 = pqz^2$, it follows from (2.6) that $U_{4k+4}U_{4k+2} = pqz^2$. Using the fact that $\gcd(4k+4, 4k+2) = 2$, we get $\gcd(U_{4k+4}, U_{4k+2}) = U_2 = 2a$ by (2.3). Then

$$U_{4k+4} = 2ab^2, U_{4k+2} = 2apqc^2 \tag{4.7}$$

or

$$U_{4k+4} = 2apqb^2, U_{4k+2} = 2ac^2 \tag{4.8}$$

or

$$U_{2k+2}V_{2k+2} = U_{4k+4} = 2apb^2, U_{2k+1}V_{2k+1} = U_{4k+2} = 2aqc^2 \tag{4.9}$$

for some integers b and c.

If (4.7) holds, then $\left(\frac{V_{4k+4}}{2}, b\right)$ is a solution of (2.11). Thus we get by Lemma 2.10 that $b = 1$, so $U_{4k+4} = 2a = U_2$, which leads to a contradiction.

If (4.8) holds, then $(\frac{V_{4k+2}}{2}, c)$ is a solution of (2.11). We get by Lemma 2.10 that $c = 1$, which leads to $U_{4k+2} = 2a = U_2$. Hence $k = 0$. Substituting the value into the former Eq (4.8) gives $4a(2a^2+1) = 2apqb^2$. It follows that $p = 2$ and $2a^2+1 = qb^2$. Clearly, when $p = 2$ and $2a^2+1 = qb^2$, then $(x, y, z) = \left(x_3, y_3, \sqrt{\frac{y_3^2-1}{2q}}\right)$ is a solution of (1.6).

If (4.9) holds, then we get from the former equation that either

$$U_{2k+2} = apb_1^2, V_{2k+2} = 2b_2^2 \tag{4.10}$$

or

$$U_{2k+2} = ab_1^2, V_{2k+2} = 2pb_2^2 \tag{4.11}$$

for some integers b_1 and b_2 since $\gcd(U_{2k+2}/a, V_{2k+2}/2) = 1$ from Eq (2.5) and Lemma 2.3.

If (4.10) holds, then we know from the latter equation that (b_2, pb_1^2) is a solution of equation $X^4 - a^2(a^2+1)Y^2 = 1$. Since $(2a^2+1, 2)$ is the fundamental solution of the equation $X^2 - a^2(a^2+1)Y^2 = 1$, by Lemma 2.8 we get that either $2a^2+1 = b_2^2$ or $2(2a^2+1)^2 - 1 = b_2^2$. The latter equation leads to $8a^4 + 8a^2 + 1 = b_2^2$, which is impossible by Lemma 2.11. We get from the former equation that $V_2/2 = 2a^2+1 = b_2^2 = V_{2k+2}/2$. Hence $k = 0$. Substituting the value into the former Eq (4.9) gives $2a = U_2 = 2aqc^2$, which is a contradiction.

If (4.11) holds, then we know from the former equation that $(V_{2k+2}/2, b_1)$ is a solution of equation $X^2 - a^2(a^2+1)Y^4 = 1$. Since $(2a^2+1, 2)$ is the fundamental solution of the equation $X^2 - a^2(a^2+1)Y^2 = 1$, by Lemma 2.9 we get that $U_{2k+2} = 4a(2a^2+1) = U_2V_2$. Hence $k = 1$. Substituting the value into the latter equation of (4.4) gives $2a(4a^2+3)(4a^2+1) = V_3U_3 = 2aqc^2$. It follows that either $4a^2+3 = c_1^2$ or $4a^2+1 = c_2^2$ since $\gcd(4a^2+3, 4a^2+1) = 1$. By taking the former equation modulo 4, we get that it is impossible. The latter equation is obviously not true. Thus we proved that if and only if $p = 2$ and $2a^2+1 = qb^2$, the Eq (1.6) has a unique solution

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{y_3^2-1}{2q}}\right) = \left(4a^3+3a, 4a^2+1, 2ab\right).$$

This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

Assume that (x, y, z) is a positive integer solution of (1.7). By Lemma 2.1 we know that

$$x = \frac{V_{2m+1}}{2}, y = U_{2m+1} \tag{5.1}$$

for some positive integer m. We shall discuss separately two cases.

The case m is even, say $m = 2k$ for some positive integer k. Since $U_{4k+1}^2 - 1 = pqrz^2$, it follows from (2.6) that $U_{4k}U_{4k+2} = pqrz^2$. Using the fact that $\gcd(4k, 4k + 2) = 2$, we get $\gcd(U_{4k}, U_{4k+2}) = U_2 = 2a$ by (2.3). Then

$$U_{4k} = 2ab^2, U_{4k+2} = 2apqrc^2 \tag{5.2}$$

or

$$U_{4k} = 2apqrb^2, U_{4k+2} = 2ac^2 \tag{5.3}$$

or

$$U_{2k}V_{2k} = U_{4k} = 2apqb^2, U_{2k+1}V_{2k+1} = U_{4k+2} = 2arc^2 \tag{5.4}$$

or

$$U_{2k}V_{2k} = U_{4k} = 2apb^2, U_{2k+1}V_{2k+1} = U_{4k+2} = 2aqrc^2 \tag{5.5}$$

for some integers b and c.

If (5.2) holds, then $\left(\frac{V_{4k}}{2}, b\right)$ is a solution of (2.11). Thus we get by Lemma 2.10 that $b = 1$, which leads to $U_{4k} = 2a = U_2$, a contradiction.

If (5.3) holds, then $\left(\frac{V_{4k+2}}{2}, c\right)$ is a solution of (2.11). We get by Lemma 2.10 that $c = 1$, which leads to $U_{4k+2} = 2a = U_2$ a contradiction again. Hence, both of (5.2) and (5.3) are impossible.

If (5.4) holds, we distinguish two subcases.

Subcase 1: k is even. Then we get from Lemmas 2.2, 2.3 and (2.5) that $\gcd(U_{2k}/a, V_{2k}/2) = 1$. Hence we get from the former equation of (5.4) that

$$U_kV_k = U_{2k} = apqb_1^2, V_{2k} = 2b_2^2 \tag{5.6}$$

or

$$U_kV_k = U_{2k} = ab_1^2, V_{2k} = 2pqb_2^2 \tag{5.7}$$

or

$$U_kV_k = U_{2k} = apb_1^2, V_{2k} = 2qb_2^2 \tag{5.8}$$

for some integers b_1 and b_2.

If (5.6) holds, then we know from the latter equation that (b_2, pqb_1^2) is a solution of equation $X^4 - a^2(a^2 + 1)Y^2 = 1$. Since $(2a^2 + 1, 2)$ is the fundamental solution of the equation $X^2 - a^2(a^2 + 1)Y^2 = 1$, by Lemma 2.8 we get that either $2a^2 + 1 = b_2^2$ or $2(2a^2 + 1)^2 - 1 = b_2^2$. The latter equation leads to $8a^4 + 8a^2 + 1 = b_2^2$, which is impossible by Lemma 2.11. We get from the former equation that $V_2/2 = 2a^2 + 1 = b_2^2 = V_{2k}/2$, which contradicts the condition that k is even.

If (5.7) holds, then we know from the former equation that $(V_{2k}/2, b_1)$ is a solution of equation $X^2-a^2(a^2+1)Y^4 = 1$. Since $(2a^2+1, 2)$ is the fundamental solution of the equation $X^2-a^2(a^2+1)Y^2 = 1$, by Lemma 2.9 we get that $U_{2k} = 4a(2a^2 + 1) = U_2V_2$. Hence $k = 2$. Substituting the value into the latter equation of (5.4) gives $U_5V_5 = 2arc^2$. Since $\gcd(U_5, V_5) = 1$ by (2.5) and Lemma 2.3, and $2a|V_5$ by Lemma 2.2, then either

$$2a(16a^4 + 20a^2 + 5) = V_5 = 2ac_1^2, U_5 = rc_2^2 \tag{5.9}$$

or

$$2a(16a^4 + 20a^2 + 5) = V_5 = 2arc_1^2, U_5 = c_2^2 \tag{5.10}$$

for some integers c_1 and c_2.

If (5.9) holds, then we get that $16a^4 + 20a^2 + 5 = c_1^2$. By taking modulo 8, we get that $4a^2 + 5 \equiv c_1^2 \equiv 1 \pmod 8$. Hence a is odd. But from Eq (5.7), we have that $4a(2a^2 + 1) = U_4 = ab_1^2$. So $3 \equiv 2a^2 + 1 \equiv (b_1/2)^2 \equiv 1 \pmod 4$, which leads to a contradiction.

If (5.10) holds, then (c_2, rc_1^2) is a solution of equation $(a^2+1)X^4 - a^2Y^2 = 1$. Since $\sqrt{a^2+1}+a$ is the fundamental solution of the equation $(a^2 + 1)X^2 - a^2Y^2 = 1$, by Lemma 2.6 we get that $U_5\sqrt{a^2+1} + a\frac{V_5}{2a} = (\sqrt{a^2+1} + a)^P$ with $P \equiv 3 \pmod 4$ a prime. Hence

$$16a^4 + 12a^2 + 1 = U_5 = \sum_{r=0}^{(P-1)/2} \binom{P}{2r} (a^2 + 1)^{\frac{P-2r-1}{2}} a^{2r}.$$

It is easy to see that the above equation is impossible. Hence (5.7) is impossible.

If (5.8) holds, then we get from Lemmas 2.2, 2.3 and (2.5) that $\gcd(U_k/2a, V_k/2) = 1$. Then from the former equation we know that either

$$U_k = 2au^2, V_k = 2pv^2 \tag{5.11}$$

or

$$U_k = 2apu^2, V_k = 2v^2 \tag{5.12}$$

for some integers u and v.

If (5.11) holds, then we know from the former equation that $(V_k/2, u)$ is a solution of Eq (2.11). Thus we get by Lemma 2.10 that $u = 1$. Hence $k = 2$. Substituting the value into the latter equation of (5.4) gives $U_5V_5 = 2arc^2$. We already proved that this equation is impossible in our discussion of (5.7).

If (5.12) holds, then we know from the latter equation that (v, U_k) is a solution of equation $X^4 - (a^2 + 1)Y^2 = 1$. Thus we get by Lemma 2.8 that either $V_k = 2v^2 = 2(2a^2 + 1) = V_2$ or $v^2 = 2(2a^2 + 1)^2 - 1$. The former equation leads to $k = 2$, which is impossible by the above result that has been proved. The latter equation will lead to $8a^4 + 8a^2 + 1 = v^2$, which is also impossible by Lemma 2.11.

Subcase 2: k is odd. Then we get from Lemmas 2.2, 2.3 and (2.5) that $\gcd(U_{2k}/a, V_{2k}/2) = 1$. Hence we get from the former equation of (5.4) that

$$U_kV_k = U_{2k} = apqb_1^2, V_{2k} = 2b_2^2 \tag{5.13}$$

or

$$U_kV_k = U_{2k} = ab_1^2, V_{2k} = 2pqb_2^2 \tag{5.14}$$

or

$$U_kV_k = U_{2k} = apb_1^2, V_{2k} = 2qb_2^2 \tag{5.15}$$

for some integers b_1 and b_2.

If (5.13) holds, then we know from the latter equation that (b_2, pqb_1^2) is a solution of equation $X^4 - a^2(a^2+1)Y^2 = 1$. Since $(2a^2+1, 2)$ is the fundamental solution of the equation $X^2-(a^2+1)Y^2 = 1$, by Lemma 2.8 we get that either $2a^2 + 1 = b_2^2$ or $2(2a^2 + 1)^2 - 1 = b_2^2$. The latter equation leads to $8a^4 + 8a^2 + 1 = b_2^2$, which is impossible by Lemma 2.11. We get from the former equation that $V_2/2 = 2a^2 + 1 = b_2^2 = V_{2k}/2$, which leads to $k = 1$. Substituting the value into the former equation of (5.13) gives $2a = U_2 = apqb_1^2$, which is a contradiction.

If (5.14) holds, then we know from the former equation that $(V_{2k}/2, b_1)$ is a solution of equation $X^2 - a^2(a^2 + 1)Y^4 = 1$. Thus we get by Lemma 2.9 that $U_4 = U_2V_2 = 4a(2a^2 + 1) = ab_1^2 = U_{2k}$, which is impossible since k is odd.

If (5.15) holds, then either

$$U_k = pu^2, V_k = 2av^2 \tag{5.16}$$

or

$$U_k = u^2, V_k = 2apv^2 \tag{5.17}$$

for some integers u and v since $\gcd(U_k, V_k) = 1$ by Lemma 2.3 and $2a|V_k$ by Lemma 2.2.

If (5.16) holds, then we know from the latter equation that (U_k, v) is a solution of equation $(a^2 + 1)X^2 - a^2Y^4 = 1$. Since $(1, 1)$ is the minimal positive integer solution of the equation $(a^2+1)X^2-a^2Y^2 = 1$, by Lemma 2.4 we get that $U_k = 1 = U_1$. Hence $k = 1$. Substituting the value into the former equation of (5.16) gives $1 = U_1 = pu^2$, which is a contradiction.

If (5.17) holds, then we know from the former equation that $(u, V_k/2)$ is a solution of equation $(a^2+1)X^4 - Y^2 = 1$. Thus we get by Lemmas 2.1 and 2.6 that either $k = 1$, which is impossible because the latter equation of (5.17) becomes $V_1 = 2a = 2apv^2$, or $k \equiv 3 \pmod 4$ is a prime. Substituting the value into the latter equation of (5.4) gives $U_{2k+1}V_{2k+1} = U_{4k+2} = 2arc^2$. It follows that either U_{2k+1} is a square or $V_{2k+1}/2$ is a product of a and a square. If U_{2k+1} is a square, then $(\sqrt{U_{2k+1}}, V_{2k+1}/2)$ is also a solution of equation $(a^2 + 1)X^4 - Y^2 = 1$, which is impossible by Lemma 2.6. If $V_{2k+1}/2$ is a product of a and a square, then $(U_{2k+1}, \sqrt{V_{2k+1}/2a})$ is a solution of equation $(a^2 + 1)X^2 - a^2Y^4 = 1$. Then we get by Lemma 2.4 that $U_{2k+1} = 1 = U_1$, which is impossible since k is a prime. Hence (5.4) is impossible.

If (5.5) holds, then we get from the former equation of (5.5) that

$$U_{2k} = apb_1^2, V_{2k} = 2b_2^2 \tag{5.18}$$

or

$$U_{2k} = ab_1^2, V_{2k} = 2pb_2^2 \tag{5.19}$$

for some integers b_1 and b_2 since $\gcd(U_{2k}/a, V_{2k}/2) = 1$ from (2.5) and Lemma 2.3.

If (5.18) holds, then we know from the latter equation that (b_2, U_{2k}) is a solution of equation $X^4 - (a^2+1)Y^2 = 1$. Thus we get by Lemma 2.8 that either $V_{2k}/2 = b_2^2 = 2a^2+1 = V_2/2$ or $b_2^2 = 2(2a^2+1)^2-1$. The latter equation is impossible by Lemma 2.11. The former equation leads to $k = 1$. Substituting the value into the former equation of (5.18) gives $2a = U_2 = apb_1^2$. So $p = 2$ and $b_1 = 1$. Substituting the value $k = 1$ into the latter equation of (5.5) gives $(4a^2+3)(4a^2+1) = qrc^2$. It follows that $4a^2+3 = qc_1^2$ and $4a^2 + 1 = rc_2^2$. Thus in this case, we prove that

$$(x, y, z) = \left(x_5, y_5, \sqrt{\frac{y_5^2 - 1}{2qr}}\right)$$

is a solution of (1.7) with

$$p = 2, 2a^2 + 1 = b_2^2, 4a^2 + 3 = qc_1^2 \quad \text{and} \quad 4a^2 + 1 = rc_2^2.$$

If (5.19) holds, then we know from the former equation that $(V_{2k}/2, b_1)$ is a solution of equation $X^2 - a^2(a^2 + 1)Y^4 = 1$. Since $(2a^2 + 1, 2)$ is the fundamental solution of $X^2 - a^2(a^2 + 1)Y^2 = 1$, by Lemma 2.9 we get that

$$V_{2k}/2 + b_1\sqrt{a^2(a^2 + 1)} = (2a^2 + 1 + 2\sqrt{a^2(a^2 + 1)})^2 = V_4/2 + 4(2a^2 + 1)\sqrt{a^2(a^2 + 1)}.$$

Hence $k = 2$. Substituting the value into the latter equation of (5.5) gives $U_5V_5 = 2aqrc^2$. We have already proved that U_5 is not a square and that $V_5/2$ is not a product of a and a square, so $16a^4 + 12a^2 + 1 = U_5 = qc_1^2$ and $16a^4 + 20a^2 + 5 = V_5/2a = rc_2^2$. Thus in this case, we proved that

$$(x, y, z) = \left(x_9, y_9, \sqrt{\frac{y_9^2 - 1}{pqr}}\right)$$

is a solution of (1.7) with

$$2a^2 + 1 = (b_1/2)^2, 16a^4 + 12a^2 + 1 = qc_1^2 \quad \text{and} \quad 16a^4 + 20a^2 + 5 = rc_2^2.$$

The case m is odd, say $m = 2k + 1$ for some nonnegative integer k. Since $U_{4k+3}^2 - 1 = pqrz^2$, it follows from (2.6) that $U_{4k+4}U_{4k+2} = pqrz^2$. Using the fact that $\gcd(4k + 4, 4k + 2) = 2$, we get $\gcd(U_{4k+4}, U_{4k+2}) = U_2 = 2a$ by (2.3). Then

$$U_{4k+4} = 2ab^2, U_{4k+2} = 2apqrc^2 \tag{5.20}$$

or

$$U_{4k+4} = 2apqrb^2, U_{4k+2} = 2ac^2 \tag{5.21}$$

or

$$U_{2k+2}V_{2k+2} = U_{4k+4} = 2apqb^2, U_{2k+1}V_{2k+1} = U_{4k+2} = 2arc^2 \tag{5.22}$$

or

$$U_{2k+2}V_{2k+2} = U_{4k+4} = 2apb^2, U_{2k+1}V_{2k+1} = U_{4k+2} = 2aqrc^2 \tag{5.23}$$

for some integers b and c.

If (5.20) holds, then $(\frac{V_{4k+4}}{2}, b)$ is a solution of (2.11). Thus we get by Lemma 2.10 that $b = 1$, which leads to $U_{4k+4} = 2a = U_2$, a contradiction.

If (5.21) holds, then $(\frac{V_{4k+2}}{2}, c)$ is a solution of (2.11). We get by Lemma 2.10 that $c = 1$, which leads to $U_{4k+2} = 2a = U_2$. Hence $k = 0$. Substituting the value into the former Eq (5.21) gives $4a(2a^2 + 1) = 2apqrb^2$. It follows that $p = 2$ and $2a^2 + 1 = qrb^2$. Thus in this case we proved that

$$(x, y, z) = \left(x_3, y_3, \sqrt{\frac{y_3^2 - 1}{2q}}\right)$$

is a solution of (1.7) with

$$p = 2 \quad \text{and} \quad 2a^2 + 1 = qrb^2.$$

If (5.22) holds, we distinguish two subcases.

Subcase 1: k is even. Then we get from Lemma 2.2, 2.3 and (2.5) that $\gcd(U_{2k+2}/a, V_{2k+2}/2) = 1$. Hence we get from the former equation of (5.22) that

$$U_{k+1}V_{k+1} = U_{2k+2} = apqb_1^2, V_{2k+2} = 2b_2^2 \tag{5.24}$$

or

$$U_{k+1}V_{k+1} = U_{2k+2} = ab_1^2, V_{2k+2} = 2pqb_2^2 \tag{5.25}$$

or

$$U_{k+1}V_{k+1} = U_{2k+2} = apb_1^2, V_{2k+2} = 2qb_2^2 \tag{5.26}$$

for some integers b_1 and b_2.

If (5.24) holds, then we know from the latter equation that (b_2, pqb_1^2) is a solution of equation $X^4-a^2(a^2+1)Y^2 = 1$. Since $(2a^2+1, 2)$ is the fundamental solution of the equation $X^2-a^2(a^2+1)Y^2 = 1$, by Lemma 2.8 we get that either $2a^2 + 1 = b_2^2$ or $2(2a^2 + 1)^2 - 1 = b_2^2$. The latter equation leads to $8a^4 + 8a^2 + 1 = b_2^2$, which is impossible by Lemma 2.11. We get from the former equation that $V_2/2 = 2a^2 + 1 = b_2^2 = V_{2k+2}/2$. Hence $k = 0$ and so $2a = U_2 = apqb_1^2$, which is a contradiction.

If (5.25) holds, then we know from the former equation that $(V_{2k+2}/2, b_1)$ is a solution of equation $X^2-a^2(a^2+1)Y^4 = 1$. Since $(2a^2+1, 2)$ is the fundamental solution of the equation $X^2-a^2(a^2+1)Y^2 = 1$, by Lemma 2.9 we get that $V_{2k+2}/2 = 2(2a^2 + 1)^2 - 1 = V_4/2$, which contradicts the condition that k is even.

If (5.26) holds, then we get from Lemma 2.2, 2.3 and (2.5) that $\gcd(U_{k+1}, V_{k+1}/a) = 1$. Then from the former equation, either

$$U_{k+1} = u^2, V_{k+1} = apv^2 \tag{5.27}$$

or

$$U_{k+1} = pu^2, V_{k+1} = av^2 \tag{5.28}$$

for some integers u and v.

If (5.27) holds, then we know from the former equation that $(u, V_{k+1}/2)$ is a solution of equation $(a^2+1)X^4-Y^2 = 1$. Thus we get by Lemma 2.6 that $u = 1$ or $k+1 = P \equiv 3 \pmod 4$ a prime. $u = 1$ will lead to $k = 0$. Substituting the value into the latter equation of (5.22) gives $2a = U_2 = 2arc^2$, which is impossible. Substituting the value $k+1 = P$ into the latter equation of (5.22) gives $U_{2P-1}V_{2P-1} = 2arc^2$. We already proved that such an equation is impossible (see discussion of (5.17)).

If (5.28) holds, by taking equation $\left(\frac{av^2}{2}\right)^2-(a^2+1)(pu^2)^2 = -1$ modulo 4 we get that $a \equiv 0 \pmod 2$. Thus Lemma 2.3 gives $v_2(V_{k+1}) = v_2(2a) = 1 + v_2(a) \neq v_2(av^2)$, which leads to a contradiction. We know from the latter equation that (U_{k+1}, v) is a solution of equation $(a^2 + 1)X^2 - (a/2)^2Y^4 = 1$, which is impossible by Lemma 2.4 since $(1, 2)$ is the minimal positive integer solution of equation $(a^2 + 1)X^2 - (a/2)^2Y^2 = 1$.

Subcase 2: k is odd. Then we get from Lemma 2.2, Lemma 2.3 and (2.5) that $\gcd(U_{2k+2}/a, V_{2k+2}/2) = 1$. Hence we get from the former equation of (5.22) that

$$U_{k+1}V_{k+1} = U_{2k+2} = apqb_1^2, V_{2k+2} = 2b_2^2 \tag{5.29}$$

or

$$U_{k+1}V_{k+1} = U_{2k+2} = ab_1^2, V_{2k+2} = 2pqb_2^2 \tag{5.30}$$

or

$$U_{k+1}V_{k+1} = U_{2k+2} = apb_1^2, V_{2k+2} = 2qb_2^2 \tag{5.31}$$

for some integers b_1 and b_2.

If (5.29) holds, then we know from the latter equation that (b_2, pqb_1^2) is a solution of equation $X^4 - a^2(a^2+1)Y^2 = 1$. Since $(2a^2+1, 2)$ is the fundamental solution of the equation $X^2-(a^2+1)Y^2 = 1$, by Lemma 2.8 we get that either $2a^2 + 1 = b_2^2$ or $2(2a^2 + 1)^2 - 1 = b_2^2$. The latter equation leads to $8a^4 + 8a^2 + 1 = b_2^2$, which is impossible by Lemma 2.11. We get from the former equation that $V_2/2 = 2a^2 + 1 = b_2^2 = V_{2k+2}/2$, which leads to $k = 0$. Substituting the value into the latter equation of (5.22) gives $2a = U_1V_1 = 2arc^2$, which is impossible.

If (5.30) holds, then we know from the former equation that $(V_{2k+2}/2, b_1)$ is a solution of equation $X^2 - a^2(a^2+1)Y^4 = 1$. Thus we get by Lemma 2.9 that $V_4/2 = 2(2a^2+1)^2-1 = V_{2k+2}/2$. Hence $k = 1$, which is impossible by what has been proved in our discussion of (4.5).

If (5.31) holds, then either

$$U_{k+1} = 2apu^2, V_{k+1} = 2v^2 \tag{5.32}$$

or

$$U_{k+1} = 2au^2, V_{k+1} = 2pv^2 \tag{5.33}$$

for some integers u and v since $\gcd(U_{k+1}/2a, V_{k+1}/2) = 1$ by (2.5) and Lemma 2.3.

If (5.32) holds, then we know from the latter equation that (v, U_{k+1}) is a solution of equation $X^4-(a^2+1)Y^2 = 1$. Thus we get by Lemma 2.8 that either $v^2 = 2a^2+1$ or $v^2 = 8a^4+8a^2+1$. The latter equation is impossible by Lemma 2.11. We get from the former equation that $V_{k+1}/2 = v^2 = 2a^2 + 1 = V_2/2$. Hence $k = 1$. Substituting the value into the former equation of (5.32) gives $2a = U_2 = 2apu^2$, which is impossible.

If (5.33) holds, then we know from the former equation that $(V_{k+1}/2, u)$ is a solution of Eq (2.11). Thus we get by Lemma 2.10 that $u = 1$. So $U_{k+1} = 2au^2 = 2a = U_2$. It follows $k = 1$. Substituting the value into the latter equation of (5.22) gives that $2a(4a^2 + 3)(4a^2 + 1) = U_3V_3 = 2arc^2$, which is impossible by what has been proved in our discussion of (4.5). Hence (5.22) is impossible.

If (5.23) holds, then we get from the former equation of (5.23) that

$$U_{2k+2} = apb_1^2, V_{2k+2} = 2b_2^2 \tag{5.34}$$

or

$$U_{2k+2} = ab_1^2, V_{2k+2} = 2pb_2^2 \tag{5.35}$$

for some integers b_1 and b_2 since $\gcd(U_{2k+2}/a, V_{2k+2}/2) = 1$ from (2.5).

If (5.34) holds, then we know from the latter equation that (b_2, U_{2k+2}) is a solution of equation $X^4 - (a^2 + 1)Y^2 = 1$. Thus we get by Lemma 2.8 that either $V_{2k+2}/2 = b_2^2 = 2a^2 + 1 = V_2/2$ or $b_2^2 = 2(2a^2 + 1)^2 - 1$. The latter equation is impossible by Lemma 2.11. The former equation leads to $k = 0$. Substituting the value into the latter equation of (5.23) gives $2a = U_2 = 2aqrc^2$, which is a contradiction.

If (5.35) holds, then we know from the former equation that $(V_{2k+2}/2, b_1)$ is a solution of equation $X^2 - a^2(a^2 + 1)Y^4 = 1$. Since $(2a^2 + 1, 2)$ is the fundamental solution of $X^2 - a^2(a^2 + 1)Y^2 = 1$, by Lemma 2.9 we get that

$$V_{2k+2}/2 + b_1\sqrt{a^2(a^2+1)} = \left(2a^2 + 1 + 2\sqrt{a^2(a^2+1)}\right)^2 = V_4/2 + 4(2a^2+1)\sqrt{a^2(a^2+1)}.$$

Hence $k = 1$. Substituting the value $k = 1$ into the latter equation of (5.23) gives $2a(4a^2+3)(4a^2+1) = U_3V_3 = 2aqrc^2$. It follows that $4a^2 + 1 = U_3 = qc_1^2$ and $4a^2 + 3 = V_3/2a = rc_2^2$. Thus in this case, we proved that

$$(x, y, z) = \left(x_7, y_7, \sqrt{\frac{y_7^2 - 1}{pqr}}\right)$$

is a solution of (1.7) with

$$2a^2 + 1 = (b_1/2)^2, 8a^4 + 8a^2 + 1 = V_4/2 = pb_2^2, 4a^2 + 1 = U_3 = qc_1^2 \quad \text{and} \quad 4a^2 + 3 = V_3/2a = rc_2^2.$$

This completes the proof of Theorem 1.3.

6. Applications

In this section, we give some examples of applications of the results.

1. Let $a = a_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{2\sqrt{2}}$ for some positive integer n. Then $2a^2 + 1$ is a square, since $3 + 2\sqrt{2}$ is the fundamental solution of the Diophantine equation $X^2 - 2Y^2 = 1$. Thus

$$(x, y, z) = \left(4a^3 + 3a, 4a^2 + 1, 2a\sqrt{2a^2 + 1}\right)$$

is the only solution of the simultaneous Pell equations (1.5) (see the following Table 1).

Table 1. Some examples of applications of the Theorem 1.1.

n	$a = a_n$	$2a^2 + 1$	$x = 4a^3 + 3a$	$y = 4a^2 + 1$	$z = 2a\sqrt{2a^2 + 1}$
2	12	17^2	6948	577	408
3	70	99^2	1372210	19601	13860
4	408	577^2	271670472	665857	470832
5	2378	3363^2	53789263742	22619537	15994428

2. Let $p = 2$ and let q be an odd prime such that the Diophantine equation $qX^2 - 2Y^2 = 1$ has positive integer solutions. Assume that $x_0\sqrt{q} + y_0\sqrt{2}$ is the minimal positive integer solution of $qX^2 - 2Y^2 = 1$. Let $a = a_n = \frac{(x_0\sqrt{q}+y_0\sqrt{2})^n - (x_0\sqrt{q}-y_0\sqrt{2})^n}{2\sqrt{2}}$ for some odd integer n. Then $2a^2 + 1$ is a product of q and a square. Thus

$$(x, y, z) = \left(4a^3 + 3a, 4a^2 + 1, 2a\sqrt{\frac{2a^2 + 1}{q}}\right)$$

is the only solution of the simultaneous Pell equations (1.6) (see the following Table 2).

Table 2. Some examples of applications of the Theorem 1.2.

n	$a = a_n$	$2a^2 + 1$	$x = 4a^3 + 3a$	$y = 4a^2 + 1$	$z = 2a\sqrt{\frac{2a^2+1}{19}}$
1	3	19×1^2	117	37	6
3	225	19×73^2	45563175	202501	32850
5	16647	19×5401^2	18453040338033	1108490437	179820894
7	1231653	19×399601^2	7473518232827223407	6067876449637	984339540906

3. Let $p = 2$ and let q and r be two distinct odd primes such that the Diophantine equation $qrX^2 - 2Y^2 = 1$ has positive integer solutions. Assume that $x_0\sqrt{qr} + y_0\sqrt{2}$ is the minimal positive integer solution of $qrX^2 - 2Y^2 = 1$. Let $a = a_n = \frac{(x_0\sqrt{qr}+y_0\sqrt{2})^n-(x_0\sqrt{qr}-y_0\sqrt{2})^n}{2\sqrt{2}}$ for some odd integer n. Then $2a^2 + 1$ is a product of qr and a square. Thus

$$(x, y, z) = \left(4a^3 + 3a, 4a^2 + 1, 2a\sqrt{\frac{2a^2 + 1}{qr}}\right)$$

is the only solution of the simultaneous Pell equations (1.7) satisfying the condition α) of Theorem 1.3 (see the following Table 3).

Table 3. Some examples of applications of the Theorem 1.3 (satisfying the condition α).

n	$a = a_n$	$2a^2 + 1$	$x = 4a^3 + 3a$	$y = 4a^2 + 1$	$z = 2a\sqrt{\frac{2a^2+1}{3\times 11}}$
1	4	$3 \times 11 \times 1^2$	268	65	8
3	524	$3 \times 11 \times 129^2$	575512868	1098305	135192
5	68116	$3 \times 11 \times 16769^2$	1264175594543932	18559157825	2284474408
7	8854556	$3 \times 11 \times 2179841^2$	2776900752506148093952	313612647828545	38603048411192

4. Let $p = 2$ and let $a = 2$. Then $2a^2 + 1 = 3^2, 4a^2 + 1 = 17$ and $4a^2 + 3 = 19$. Thus

$$(a, p, q, r, x, y, z) = (2, 2, 17, 19, 682, 305, 12)$$

is the only solution of the simultaneous Pell equations (1.7) satisfying the condition β) of Theorem 1.3.

For $a = a_n = \frac{(3+2\sqrt{2})^n-(3-2\sqrt{2})^n}{2\sqrt{2}}$, by calculation, one can find that when $1 \le n \le 30$, there is only one $a = a_{19} = 124145519261542$ such that $8a^4 + 8a^2 + 1$ is a product of a prime

$$q = 38780919705251594795650690075845230810352482561374918769$$

and a square 49. But both $4a^2 + 1$ and $16a^4 + 12a^2 + 1$ are not a product of a prime and a square. Moreover, $a_{30} > 3 \cdot 10^{22}$, hence we guess that the simultaneous Pell equations (1.7) have no positive integer solution satisfying the condition γ) or δ) of Theorem 1.3.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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