Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 9911–9918.
DOI: 10.3934/math.2021576
Received: 27 April 2021
Accepted: 30 June 2021
Published: 05 July 2021

Research article

Prime decomposition of quadratic matrix polynomials

Yunbo Tian[1,*] **and Sheng Chen**[2]

[1] School of Mathematics and Statistics, Linyi University, Linyi 276000, China
[2] Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

* **Correspondence:** Email: tianyunbo@lyu.edu.cn.

Abstract: We study the prime decomposition of a quadratic monic matrix polynomial. From the prime decomposition of a quadratic matrix polynomial, we obtain a formula of the general solution to the corresponding second-order differential equation. For a quadratic matrix polynomial with pairwise commuting coefficients, we get a sufficient condition for the existence of a prime decomposition.

Keywords: differential equation; matrix polynomials; factorization; matrix equation; matrix pencil; explicit solution
Mathematics Subject Classification: 15A23, 15A24, 34A05

1. Introduction

The linear second-order differential equation

$$\ddot{q}(t) + A\dot{q}(t) + Bq(t) = f(t), \tag{1.1}$$

where $A, B \in \mathbb{C}^{n\times n}$ and $q(t)$ is an nth-order vector, frequently arise in the fields of mechanical and electrical oscillation [16]. The study of the solutions of the Eq (1.1) lead to the research of a quadratic matrix polynomial

$$L(\lambda) = \lambda^2 + A\lambda + B, \tag{1.2}$$

where $A, B \in M_n(\mathbb{C})$ and I is the identity matrix of order n [7]. In this work we investigate the prime decomposition of quadratic matrix polynomial (1.2). By the prime decomposition of (1.2), we represent the general solution of Eq (1.1).

Consider the solution of the homogeneous equation (1.1) with $n = 1$ and $f = 0$, rewritten in the form

$$\ddot{q}(t) + a\dot{q}(t) + bq(t) = 0. \tag{1.3}$$

We knows that the general solution of Eq (1.3) with $a^2 - 4b \neq 0$ is

$$q(t) = c_1 e^{\lambda_1} + c_2 e^{\lambda_2}, \tag{1.4}$$

where c_1, c_2 are constants and λ_1, λ_2 are the roots of $\lambda^2 + a\lambda + b = 0$. In analogy with Formula (1.4) we ask whether Eq (1.1) has the formula of the general solution.

The quadratic matrix polynomial (1.2) is called factorizable if it can be factorized into a product of two linear matrix polynomial, i.e.

$$L(\lambda) = (I\lambda + C)(I\lambda + D), \tag{1.5}$$

where $C, D \in \mathbb{C}^{n\times n}$, and $I\lambda + D$ is called a right divisor of $L(\lambda)$. Right divisors $I\lambda + D_1$, $I\lambda + D_2$ for $L(\lambda)$ are said to form a complete pair if $D_1 - D_2$ is invertible.

There have been extensive study and application of the matrix polynomial factorization (see [4, 9–11, 14, 15]). The quadratic eigenvalue problems (see [13, 16]) received much attention because of its applications in the dynamic analysis of mechanical systems in acoustics and linear stability of flows in fluid mechanics. A solution of a quadratic matrix equation can be obtained by the fraction of a quadratic matrix polynomial [7]. A system of second-order differential equation with self-adjoint coefficients may describe the ubiquitous problem of damped oscillatory systems with a finite number of degrees of freedom. This leads to the study of Hermitian quadratic matrix polynomials [1, 12].

Gohberg, Lancaster and Rodman analyzed some properties of linear second-order differential equation (1.1) and quadratic matrix polynomial (1.2) in [7]. It was shown that Eq (1.1) has a formula of the general solution similar to Formula (1.4) if the quadratic matrix polynomial (1.2) has a complete pair [7]. Motivated by results in [7], we propose a concept of prime decomposition to generalized this result.

This paper is organized as follows. In section 2, we give the definition of prime decomposition of a quadratic monic matrix polynomial $L(\lambda)$ and some properties. By the prime decomposition, we get an integral formula for the corresponding second order matrix differential equation. In section 3, we investigate the prime decomposition of a class of quadratic matrix polynomial. A sufficient condition for $L(\lambda)$ having a prime decomposition is presented in Theorem 3.1.

2. Prime decomposition and its applications

First we give the definition of prime decomposability of a quadratic matrix polynomial $L(\lambda)$. Some equivalent conditions of prime decomposability are given in Remark 2.2.

Definition 2.1. *The matrix polynomial $L(\lambda)$ has a prime decomposition if there exist $C, D \in M_n(\mathbb{C})$ such that*

$$L(\lambda) = (\lambda I + C)(\lambda I + D) \tag{2.1}$$

and there are $U, V \in M_n(\mathbb{C})$ such that

$$U(\lambda I + C) + (\lambda I + D)V = I. \tag{2.2}$$

Remark 2.2. *Our definition of prime decomposition of matrix polynomials is different from the definition of coprime factorization of matrix polynomials which was studied in many papers (e.g.*

Definition 3.1 in [6]). By Lemma 3.1 in [2], it is easy to verify that condition (2.2) is equivalent to each of the following statements:

(1) There is $H \in M_n(\mathbb{C})$ such that

$$DH - HC = I. \tag{2.3}$$

(2) There are matrix polynomials $U(\lambda), V(\lambda)$ such that

$$U(\lambda)(\lambda I + C) + (\lambda I + D)V(\lambda) = I.$$

Motivated by Lemma 3.1 in [8], we have the following theorem. It shows the significance of prime decomposability in solving equations.

Theorem 2.3. *Let R be a ring with identity 1. Suppose that $f, g, a, b \in R$ satisfy the condition*

$$af + gb = 1.$$

Let m be in a left R module M. If $y, z \in M$ and

$$\begin{cases} f(y) = m, \\ g(z) = a(m), \end{cases}$$

then $x = by + z$ is a solution of the equation $fg(x) = m$. Conversely, any solution $x \in M$ of the equation $fg(x) = m$ can be written as the following form, $x = by + z$, where $y, z \in M$ and

$$\begin{cases} f(y) = m, \\ g(z) = a(m). \end{cases}$$

In particular, by taking $m = 0$ we get $\ker(fg) = b\ker(f) + \ker(g)$.

Proof. Suppose that

$$\begin{cases} f(y) = m, \\ g(z) = a(m). \end{cases}$$

Then we have

$$fg(x) = fgb(y) + fg(z) = f(1 - af)(y) + fa(m) = f(y) - faf(y) + fa(m) = m.$$

So $x = by + z$ is a solution to $fg(x) = m$.

Conversely, if $fg(x) = m$, then we take

$$\begin{cases} y = g(x), \\ z = x - by. \end{cases}$$

We will get

$$\begin{cases} f(y) = m, \\ g(z) = a(m). \end{cases}$$

The proof is complete. □

The following result, whose proof is obvious, presents an immediate application of the the above theorem to differential equations.

Corollary 2.4. *Suppose that $L(\lambda) = \lambda^2 I + \lambda A + B$ is prime decomposable, i.e., there are $C, D, H \in M_n(\mathbb{C})$ such that $L(\lambda) = (\lambda I + C)(\lambda I + D)$ and $DH - HC = I$. Then the solution of $L(d/dt)u = f$ has the form*

$$u(t) = He^{-Ct}\alpha_1 + e^{-Dt}\alpha_2 + \int_0^t He^{(s-t)C} f(s)ds - \int_0^t e^{(s-t)D} Hf(s)ds$$

for some $\alpha_1, \alpha_2 \in \mathbb{C}^n$. In particular, the solution of $L(d/dt)u = 0$ has the form

$$u(t) = He^{-Ct}\alpha_1 + e^{-Dt}\alpha_2$$

for some $\alpha_1, \alpha_2 \in \mathbb{C}^n$.

Remark 2.5. *Suppose that*

$$I\lambda - S_1, I\lambda - S_2$$

is a complete pair for $L(\lambda)$ (see Section 2.5 in [7]). Then $S_2 - S_1$ is some invertible matrix, say P. Let $C = PS_2P^{-1}, D = S_1$. Then $P^{-1}C - DP^{-1} = I$. So

$$L(\lambda) = (\lambda I - PS_2P^{-1})(\lambda I - S_1)$$

is a prime decomposition. Thus we can recover Theorem 2.16 in [7] from Corollary 2.4.

3. The prime decomposability of a quadratic matrix polynomial

We first propose some notations used in this section. Let R_1,R_2 be $n \times n$ matrices with $R_1R_2 = R_2R_1$. The joint spectrum, denoted by $\sigma(R_1, R_2)$, is a subset of $\mathbb{C}^2$ defined by

$$\sigma(R_1, R_2) = \{\mathbf{k} = (k_1, k_2) \in \mathbb{C}^2 | \exists x \in \mathbb{C}^n \text{ s.t. } x \neq 0, \text{and } R_i x = k_i x, i = 1, 2\}.$$

Since $R_1R_2 = R_2R_1$, there exists an invertible matrix T such that

$$TR_iT^{-1} = \begin{bmatrix} k_1^{(i)} & * & \cdots & \cdots \\ 0 & k_2^{(i)} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_n^{(i)} \end{bmatrix}, \ i = 1, 2, \tag{3.1}$$

and the joint spectrum $\sigma(R_1, R_2)$ can be read off from the diagonal elements in (3.1), namely,

$$\sigma(R_1, R_2) = \{\mathbf{k}_j = (k_j^{(1)}, k_j^{(2)}) | j = 1, \ldots, n\}.$$

The multiplicity of $\mathbf{k} \in \sigma(R_1, R_2)$ is the number of $\mathbf{k}_i, i = 1, \ldots, n$ which are same as $\mathbf{k}$. The matrix M is called upper Toeplitz matrix if

$$M = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & \cdots & 0 & a_1 \end{bmatrix}.$$

The polynomial $F(\lambda)$ is called upper Toeplitz matrix polynomial if

$$F(\lambda)=\begin{bmatrix} f_1(\lambda) & f_2(\lambda) & \cdots & f_{n-1} & f_n(\lambda) \\ 0 & f_1(\lambda) & \cdots & f_{n-2} & f_{n-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_1(\lambda) & f_2(\lambda) \\ 0 & 0 & \cdots & 0 & f_1(\lambda) \end{bmatrix},$$

where $f_1(\lambda)=\lambda^2+a\lambda+b$, and $\deg(f_i)\le 1, i=2,\ldots,n$.

It is shown that a matrix polynomial with pairwise commuting coefficients of the simple structure can be represented in the form of a product of linear factors [15]. Motivated by [15], we investigate the prime decomposability of $L(\lambda)$ with pairwise commuting coefficients. The following theorem gives a sufficient condition for $L(\lambda)$ having a prime decomposition.

Theorem 3.1. *If polynomial matrix $L(\lambda)=\lambda^2 I+A\lambda+B$ satisfies the following conditions:*
(i) $AB=BA$,
(ii) each nonlinear elementary divisor of A and B is coprime with the other elementary divisors of A and B, respectively,
(iii) the degrees of elementary divisors of $L(\lambda)$ are not great than 2,
(iv) the multiplicity of eigenvalue $\mathbf{k}=(a,b)\in\sigma(A_1,A_0)$ satisfying $a^2-4b=0$ is even.
Then $L(\lambda)$ has a prime decomposition.

Prior to the proof of this theorem, we formulate several auxiliary statements. By the following two lemmas, the problem of prime decomposability of a quadratic monic matrix polynomials can be reduced in some sense.

Lemma 3.2. *Suppose that $A,B,T\in M_n(\mathbb{C})$, where T is invertible. Then $L(\lambda)=\lambda^2 I+A\lambda+B$ is prime decomposable if and only if $TL(\lambda)T^{-1}$ is prime decomposable.*

Proof. Note that $TL(\lambda)T^{-1}=(\lambda I+TCT^{-1})(\lambda I+TDT^{-1})$, where T is invertible. Furthermore, Eq (2.3) is equivalent to equation $(TDT^{-1})(THT^{-1})-(THT^{-1})(TCT^{-1})=I$. By Definition 2.1, the result follows. The proof is complete. □

Lemma 3.3. *Suppose $a\in\mathbb{C}$. Then $L(\lambda)=\lambda^2 I+A\lambda+B$ is prime decomposable if and only if $\lambda^2 I+\lambda\tilde{A}+\tilde{B}$ is prime decomposable, where $\tilde{A}=2aI+A$, $\tilde{B}=a^2I+aA+B$.*

Proof. If $L(\lambda)$ is decomposable, then there exist $C,D\in M_n(\mathbb{C})$ such that

$$\lambda^2 I+\lambda A+B=(\lambda I+C)(\lambda I+D).$$

Let $\tilde{C}=aI+C$, and $\tilde{D}=aI+D$. We have

$$\lambda^2 I+\lambda(2aI+A)+a^2I+aA+B=(\lambda I+\tilde{C})(\lambda I+\tilde{D}).$$

Furthermore, the condition that there exists H such that $DH-HC=I$ is equivalent to the condition $\tilde{C}H-H\tilde{D}=I$. By Definition 2.1, the result follows. □

The following lemma is a well known result about Sylvester equation which was studied in many papers [3, 5].

Lemma 3.4. *[7] Let $C, D \in M_n(\mathbb{C})$. If $\sigma(C) \cap \sigma(D) = \emptyset$, where $\sigma(X)$ is the set of eigenvalues of the matrix X, then for any $U \in M_n(\mathbb{C})$, there exists a unique $H \in M_n(\mathbb{C})$ such that $DH - HC = U$.*

Lemma 3.5. *Suppose $F(\lambda)$ is upper Toeplitz matrix polynomial with $f_1(\lambda) = \lambda^2 + a\lambda + b$ on the diagonal. If $a^2 \neq 4b$, then $F(\lambda)$ has a prime decomposition.*

Proof. As shown in [15], $F(\lambda)$ can be decomposed into a product of linear factors which has only one (without regard for multiplicity) characteristic root. By Lemma 3.4, we know that the decomposition is a prime decomposition. □

Lemma 3.6. *Suppose $F(\lambda)$ is upper Toeplitz matrix polynomial with $f_1(\lambda) = \lambda^2 + a\lambda + b$ on the diagonal, and n is the order of $F(\lambda)$. If $a^2 = 4b$ and the degrees of elementary divisors of $F(\lambda)$ are not great than 2, then $F(\lambda)$ has a prime decomposition if and only if n is even.*

Proof. Since the degrees of elementary divisors of $F(\lambda)$ are not great than 2, we have

$$F(\lambda) = \lambda^2 I_n + 2c_1 I_n + c_1^2 I,$$

where $c_1 = \frac{a}{2}$. By Lemma 3.3, it suffices to prove the prime decomposability of $\lambda^2 I_n$, i.e., there exists C such that

$$\lambda^2 I_n = (\lambda I_n + C)(\lambda I_n - C), \tag{3.2}$$

where $C^2 = 0$ and there is H satisfying

$$-CH - HC = I. \tag{3.3}$$

We prove the result in two cases.

Case 1. $n = 2k$, k is an nonnegative integer.

Let

$$C_i = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, H_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$C = diag(C_1, \dots, C_k), H = diag(H_1, \dots, H_k),$$

satisfy Eqs (3.2) and (3.3).

Case 2. $n = 2k + 1$, k is an nonnegative integer.

By Lemma 3.2, we suppose $C = diag(J_1, \dots, J_s)$, where J_i is a Jordan block with zero on the diagonal for $i = 1, \dots, s$. Since $C^2 = 0$, the order of J_i is not great than 2 for $i = 1, \dots, s$. Thus there exists r such that $J_r = 0$. Without loss of generality, we assume $J_1 = 0$. Then the $(1, 1)$ entry of $-CH - HC$ is 0. Hence there does not exist H satisfying Eq (3.3). □

Now we are ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1. Suppose that $L(\lambda) = \lambda^2 I + \lambda A + B$ satisfies the conditions of Theorem 3.1. Hence there exists an invertible matrix T such that

$$TBT^{-1} = diag(b_1 I_{r_1}, \dots, b_l I_{r_l}, J_{l+1}, \dots, J_s),$$

where J_i is a Jordan block with b_i on the diagonal for $i = l+1, \ldots, s$, and $b_i \neq b_j$ for $i \neq j (i, j = 1, \ldots, s)$. Since $AB = BA$, we have

$$TAT^{-1} = diag(M_1, \ldots, M_l, M_{l+1}, \ldots, M_s),$$

where $M_i (i > l)$ is upper Toeplitz matrix(see Theorem S2.2 in [7]). Note that there exist invertible T_i such that

$$T_i M_i T_i^{-1} = diag(J_{i_1}, \ldots, J_{i_{m_i}}),\ i = 1, \ldots, l,$$

where J_{i_j} is a Jordan block with a_{i_j} on the diagonal for $j = l, \ldots, m_i$.

Let $H = T diag(T_1, \ldots, T_l, I, \ldots, I)$. We have

$$HL(\lambda)H^{-1} = diag(L_1(\lambda), \ldots, L_h(\lambda)),$$

where $h = \sum_{i=0}^{l} m_i + s - l$, and $L_i(\lambda)$ is upper Toeplitz matrix polynomial for $i = 1, \ldots, h$. Because $L_i(\lambda)(i = 1, \ldots, h)$ satisfy the conditions of Theorem 3.1, from Lemmas 3.5 and 3.6, we know that $L_i(\lambda)(i = 1, \ldots, h)$ has a prime decomposition, say

$$L_i(\lambda) = (\lambda I + C_i)(\lambda I + D_i).$$

Hence $HL(\lambda)H^{-1}$ has a prime decomposition

$$HL(\lambda)H^{-1} = diag(\lambda I + C_1, \ldots, \lambda I + C_h) diag(\lambda I + D_1, \ldots, \lambda I + D_h).$$

By Lemma 3.2, we know that $L(\lambda)$ has a prime decomposition. $\square$

4. Conclusions

In this paper, we have presented the prime decomposition of a quadratic monic matrix polynomial and the application in solving corresponding second-order differential equation. We have got a sufficient condition for the existence of a prime decomposition for a quadratic matrix polynomial with pairwise commuting coefficients. As has been said, a complete pair can be used to form a prime decomposition of the quadratic matrix polynomial. Thus, we expect that the relation between a prime decomposition and a complete pair can be studied more thoroughly. The prime decomposition of a matrix polynomial with degree n will be investigated in a future paper.

Acknowledgments

The first author is supported by the Natural Science Foundation of Shandong Province, China (Grant No. ZR2018PA002).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. C. Campos, J. E. Roman, Inertia-based spectrum slicing for symmetric quadratic eigenvalue problems, *Numer. Linear Algebra Appl.*, **27** (2020), e2293.
2. S. Chen, Y. Tian, On solutions of generalized Sylvester equation in polynomial matrices, *J. Franklin Inst.*, **351** (2014), 5376–5385.
3. P. M. Cohn, The range of derivations on a skew field and the equation $ax - xb = c$, *J. Indian Math. Soc.*, **37** (1973), 61–69.
4. P. V. Dooren, Rational and polynomial matrix factorizations via recursive pole-zero cancellation, *Linear Algebra Appl.*, **137** (1990), 663–697.
5. H. Flanders, H. K. Wimmer, On the matrix equations $AX - XB = C$ and $AX - YB = C$, *SIAM J. Appl. Math.*, **32** (1977), 707–710.
6. S. D. Garveya, P. Lancaster, A. A. Popov, U. Prells, I. Zaballa, Filters connecting isospectral quadratic systems, *Linear Algebra Appl.*, **438** (2013), 1497–1516.
7. I. Gobherg, P. Lancaster, L. Rodman, Matrix polynomials, In: *Indefinite linear algebra and applications*, Birkhäuser Basel, 2009.
8. S. K. Jain, T. Y. Lam, A. Leroy, *Ore extensions and V-domains*, Rings, Modules and Representations: International Conference on Rings and Things in Honor of Carl Faith and Barbara Osofsky, Ohio University-Zanesville, **480** (2009), 249–262.
9. P. S. Kazimirski, Separation of a regular linear factor of a simple structure from a matrix polynomial, *Theor. Appl. Probl. Algebra Differ. Equations*, **119** (1976), 29–40.
10. P. S. Kazimirski, V. M. Petrichkovich, Decomposability of polynomial matrices into a product of linear factors, *Mat. Met. Fiz.-Mekh. Polya*, **8** (1978), 3–9.
11. I. N. Krupnik, Decomposition of a monic polynomial into a product of linear factors, *Linear Algebra Appl.*, **167** (1992), 239–242.
12. P. Lancaster, F. Tisseur, Hermitian quadratic matrix polynomials: Solvents and inverse problems, *Linear Algebra Appl.*, **436** (2012), 4017–4026.
13. A. Malyshev, M. Sadkane, Computing the distance to continuous-time instability of quadratic matrix polynomials, *Numer. Math.*, **145** (2020), 149–165.
14. B. Z. Shavarovskii, Factorable polynomial matrices, *Math. Notes*, **68** (2000), 507–518.
15. B. Z. Shavarovs'kyi, Decomposability of matrix polynomials with commuting coefficients into a product of linear factors, *Ukr. Math. J.*, **62** (2011), 1295–1306.
16. F. Tisseur, K. Meerbergen, The quadratic eigenvalue problem, *SIAM Rev.*, **43** (2001), 235–286.