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Research article

On the fourth-order nonlinear beam equation of a small deflection with nonlocal conditions

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Abstract: The purpose of this paper is to establish an existence and uniqueness theorem for the nonlocal fourth-order nonlinear beam differential equations with a parameter

$$u^{(4)} + A(x)u = \lambda f(x,\ u,\ u''),\ 0 < x < 1$$

subject to the integral boundary conditions:

$$u(0) = u(1) = \int_0^1 p(x)u(x)dx,\ u''(0) = u''(1) = \int_0^1 q(x)u''(x)dx,$$

where $A \in \mathbb{C}[0,1]$, $\lambda > 0$ is a parameter and $p, q \in \mathbb{L}^1[0,1]$.

Keywords: fourth-order beam equation; nonlocal boundary conditions; existence and uniqueness theorem; Schauder's fixed point theorem
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1. Introduction

Fourth-order boundary value problems have important applications in physics and mechanical engineering because they describe deflection (deformation) or bending of elastic beams. The state of the deflection of an elastic beam is modeled by the Euler-Bernoulli equation

$$u^{(4)} = \lambda f(x, u, u''). \tag{1.1}$$

This equation is widely used in mechanics and it is called the beam equation. Here; u represents the deflection of the beam, u'' is the bending moment stiffness, $u^{(4)}$ is the load density stiffness, f is the force per unit length, which represents the distributed load, and λ is a parameter that represents the reciprocal of the flexural rigidity which measures the resistance to bend (see [1], page 199 for further details regarding the mechanics of the beam equation).

In [2], the author considered the linear fourth-order differential equation

$$u^{(4)} + q(x)u = f(x) \tag{1.2}$$

subject to

$$u(0) = a,\ u(1) = b,\ u''(0) = c,\ u''(1) = d, \tag{1.3}$$

where q and f are continuous functions on $[0, 1]$, and established a sufficient condition $\sup\limits_{0\le x\le 1} |q(x)| < \pi^4$ to guarantee a unique solution for this problem. In [3], the author investigated (1.1) under the condition that f is continuous and bounded on $[0, 1] \times \mathbb{R} \times \mathbb{R}$, and subject to (1.3) as well as other types of boundary conditions, and established results on the existence and uniqueness theorems under suitable conditions. In [4] the author considered the same problem of [3] under more general conditions on f, and established the existence of the solution of the equation subject to (1.3) under the condition that f is continuous on $[0, 1] \times \mathbb{R} \times \mathbb{R}$, and replacing the boundedness condition with the growth condition

$$|f(x, u, v)| \le a\,|u| + b\,|v| + c \tag{1.4}$$

for some positive constants a, b, c such that $a + b\pi^2 < \pi^4$. Since then, the problem of investigating existence of solutions of the equation received considerable attention from researchers in the last two decades (see [5–13] and references therein). The authors in [9, 11, 13–15] considered the equation under integral boundary conditions.

In this paper, we establish an existence and uniqueness theorem for the following boundary value problem

$$u^{(4)} + A(x)u = \lambda f(x, u, u''),\ 0 < x < 1 \tag{1.5}$$

subject to the integral boundary conditions

$$u(0) = u(1) = \int_0^1 p(x)u(x)dx,\quad u''(0) = u''(1) = \int_0^1 q(x)u''(x)dx, \tag{1.6}$$

where $A \in \mathbb{C}[0, 1]$, $p, q \in L^1[0, 1]$ and f is continuous on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ and satisfies a growth condition with variable parameters:

$$|f(x, u, v)| \le a(x)\,|u| + b(x)\,|v| + c(x), \tag{1.7}$$

where a, b, c are positive continuous functions on $[0, 1]$. The problem (1.5)-(1.6) generalizes the preceding problems in the following sense:

1. If $A = 0$ then (1.5) reduces to (1.1).
2. If $p = q = 0$ then (1.6) reduces to (1.3).
3. If a, b, c are constants then (1.7) reduces to (1.4).

We are particularly interested in the case of small deflection. Deflections are small as long as they are below the elastic limit, and in this case the deflection curve might be almost flat, so the bending

won't be noticed by eye. When the deflection is small, Hook's law is applicable, and this will give Euler-Bernoulli Equation. Here, shear distortion and effects of rotatory inertia are negligible due to the absence of the axial forces. If the axial force exists and becomes a function of transverse displacement, large deflection occurs and we need to seek help from nonlinear beam theory. It should be noted that most of the beams used in industry and constructions (Towers, bridges, aircrafts,...) possess enery small deflections, since large deflections can cause cracks in the beams, and this may eventually lead to catastrophic damages. For more details about the beam theory we refer the reader to ([16], pp 758–760). Small deflections usually occur when either the loaded force f is small (so a, b, c are small), or the material of the beam has high flexural rigidity which implies that λ is small.
We propose the following hypothesis

$$\sup_{0\le x\le 1} A(x) = A_1 < 16\alpha\beta,\ 1 - \int_0^1 p^2(x)dx = \alpha > 0,\ 1 - \int_0^1 q^2(x)dx = \beta > 0,\ \lambda < \frac{16\alpha\beta - A_1}{5d}, \tag{1.8}$$

where $d = \max\{a, b\}$. Note that if $A_1 = 0$ then $\lambda < \dfrac{16\alpha\beta}{5d}$. This seems natural and reflects the fact that small amount of force and large amount of force loaded on a beam of high flexural rigidity will always produce small deflections.

2. Existence and uniqueness theorems

2.1. Existence theorem

The Pr.(1.5)-(1.6) can be converted into the following system:

$$\begin{cases} u'' &= v,\ u(0) = u(1) = \int_0^1 p(x)u(x)dx, \\ v'' &= -A(x)u + \lambda f(x,\ u,\ v),\ v(0) = v(1) = \int_0^1 q(x)v(x)dx. \end{cases} \tag{2.1}$$

Thus, we shall prove the following statement

Proposition 2.1. *If (1.7) and (1.8) hold, then there exists a constant $M > 0$ such that for any $x \in [0, 1]$ and any solution u to Pr.(1.5)-(1.6), we have*

$$\| u \|_{\rho,0} + \| u \|_{\rho,1} \le M, \tag{2.2}$$

where $\| u \|_{\rho,0} = \max\limits_{0\le x\le 1} | \rho(x)u(x) |$, $\| u \|_{\rho,1} = \| u'' \|_{\rho,0}$ *and* $\rho(x) = x(1 - x),\ x \in [0, 1]$.

Proof. Multiplying both sides of the first equation of (2.1) by $\rho(x)u$ and integrating the resulting equation from 0 to 1, then employing integration by parts, we obtain

$$2\int_0^1 u^2(x)dx + 2\int_0^1 \rho(x)u'^2(x)dx = \left[u^2(1) + u^2(0)\right] - 2\int_0^1 \rho(x)u(x)v(x)dx. \tag{2.3}$$

Taking into account $u(0) = u(1) = \int_0^1 p(x)u(x)dx$, we have

$$\int_0^1 u^2(x)dx + \int_0^1 \rho(x)(u'(x))^2 dx = \left[\int_0^1 p(x)u(x)dx\right]^2 - \int_0^1 \rho(x)u(x)v(x)dx. \tag{2.4}$$

The integrals $\int_0^1 p(x)u(x)dx$ and $\int_0^1 \rho(x)u(x)v(x)dx$ can be estimated by means of the Cauchy-Schwarz inequality

$$\left[\int_0^1 p(x)u(x)dx\right]^2 \leq \left(\int_0^1 p^2(x)dx\right)\left(\int_0^1 u^2(x)dx\right) \tag{2.5}$$

and

$$\mid \int_0^1 \rho(x)u(x)v(x)dx \mid \ \leq \ \left(\int_0^1 \rho(x)u^2(x)dx\right)^{\frac{1}{2}} \left(\int_0^1 \rho(x)v^2(x)dx\right)^{\frac{1}{2}}. \tag{2.6}$$

Since $\sup\limits_{0\leq x\leq 1} \rho(x) = \frac{1}{4}$, we have

$$\mid \int_0^1 \rho(x)u(x)v(x)dx \mid \ \leq \ \frac{1}{4}\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}} \left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}. \tag{2.7}$$

Thus

$$\left(1 - \int_0^1 p^2(x)dx\right)\int_0^1 u^2(x)dx + \int_0^1 \rho(x)u'^2(x)dx \leq \frac{1}{4}\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}} \left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}. \tag{2.8}$$

Hence

$$\int_0^1 u^2(x)dx + \int_0^1 \rho(x)u'^2(x)dx \leq \frac{1}{4\alpha}\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}} \left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}. \tag{2.9}$$

Since

$$\int_0^1 u^2(x)dx \leq \int_0^1 u^2(x)dx + \int_0^1 \rho(x)u'^2(x)dx. \tag{2.10}$$

Thus

$$\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}} \leq \frac{1}{4\alpha}\left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}. \tag{2.11}$$

It follows that

$$\int_0^1 u^2(x)dx + \int_0^1 \rho(x)u'^2(x)dx \leq C_1 \int_0^1 v^2(x)dx, \tag{2.12}$$

where $C_1 = \dfrac{1}{16\alpha^2}$ and $\alpha = 1 - \int_0^1 p^2(x)dx$.

Proceeding as before, multiplying both sides of the second equation of (2.1) by $\rho(x)v$ and integrating the resulting equation from 0 to 1, then employing integration by parts, taking into account the nonlocal boundary conditions $v(0) = v(1) = \int_0^1 q(x)v(x)dx$, we obtain

$$\int_0^1 v^2(x)dx + \int_0^1 \rho(x)(v'(x))^2dx = \left[\int_0^1 q(x)v(x)dx\right]^2 + \int_0^1 A(x)\rho(x)u(x)v(x)dx$$

$$-\lambda\int_0^1 f(x,u,v)\rho(x)v(x)dx. \tag{2.13}$$

Note that

$$\mid\int_0^1 A(x)\rho(x)u(x)v(x)dx\mid\leq\frac{A_1}{4}\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}} \tag{2.14}$$

and

$$\left[\int_0^1 q(x)v(x)dx\right]^2\leq\left(\int_0^1 q^2(x)dx\right)\left(\int_0^1 v^2(x)dx\right). \tag{2.15}$$

Applying (1.7) to $f(x,\ u,\ v)$ by assuming that $a(x)\leq a,\ b(x)\leq b,\ c(x)\leq c,\ \forall x\in[0,1]$ with $a,b,c>0$, to obtain

$$\mid\int_0^1 f(x,u,v)\rho(x)v(x)dx\mid\leq\frac{a}{4}\int_0^1\mid u(x)v(x)\mid dx+\frac{b}{4}\int_0^1 v^2(x)dx+\frac{1}{4}\int_0^1\mid c(x)v(x)\mid dx. \tag{2.16}$$

The integral $\int_0^1\mid c(x)v(x)\mid dx$ can be estimated by means of the $\epsilon-$inequality

$$\int_0^1\mid c(x)v(x)\mid dx\leq\frac{1}{2\epsilon}c^2+\frac{\epsilon}{2}\int_0^1 v^2(x)dx,\ \epsilon>0. \tag{2.17}$$

Thus

$$\begin{aligned}\mid\int_0^1 f(x,u,v)\rho(x)v(x)dx\mid\leq{}&\frac{a}{4}\left(\int_0^1 u^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 v^2(x)dx\right)^{\frac{1}{2}}+\frac{b}{4}\int_0^1 v^2(x)dx\\&+\left(\frac{c^2}{8\epsilon}+\frac{\epsilon}{8}\int_0^1 v^2(x)dx\right),\ \epsilon>0.\end{aligned} \tag{2.18}$$

Since

$$\int_0^1 u^2(x)dx\leq\int_0^1 u^2(x)dx+\int_0^1\rho(x)u'^2(x)dx\leq C_1\int_0^1 v^2(x)dx. \tag{2.19}$$

Substituting (2.19) into (2.14) and (2.18), we obtain

$$\mid\int_0^1 A(x)\rho(x)u(x)v(x)dx\mid\leq\frac{A_1}{4}\sqrt{C_1}\int_0^1 v^2(x)dx \tag{2.20}$$

and

$$\mid\int_0^1 f(x,u,v)\rho(x)v(x)dx\mid\leq\left[\frac{a}{4}\sqrt{C_1}+\frac{b}{4}+\frac{\epsilon}{8}\right]\int_0^1 v^2(x)dx+\frac{c^2}{8\epsilon},\ \epsilon>0. \tag{2.21}$$

Now using (2.15), (2.20) and (2.21), we obtain

$$\left[1-\left(\int_0^1 q^2(x)dx+(\frac{A_1}{4}+\frac{\lambda a}{4})\sqrt{C_1}+\frac{\lambda b}{4}+\frac{\lambda\epsilon}{8}\right)\right]\int_0^1 v^2(x)dx$$

$$+\int_0^1 \rho(x)v'^2(x)dx \leq \frac{\lambda c^2}{8\epsilon}. \tag{2.22}$$

Let

$$\gamma = 1-\left(\int_0^1 q^2(x)dx + (\frac{A_1}{4}+\frac{\lambda a}{4})\sqrt{C_1}+\frac{\lambda b}{4}+\frac{\lambda\epsilon}{8}\right). \tag{2.23}$$

Writing $\sqrt{C_1} = \dfrac{1}{4\alpha}$ and using the fact that $\alpha < 1$ and $\beta = 1-\int_0^1 q^2(x)dx$, we see that

$$(\frac{A_1}{4}+\frac{\lambda a}{4})\frac{1}{4\alpha}+\frac{\lambda b}{4} = \frac{1}{16\alpha}\left[A_1+\lambda a+4\lambda\alpha b\right]$$

$$< \frac{1}{16\alpha}\left[A_1+5\lambda d\right]$$

$$< 1-\int\limits_0^1 q^2(x)dx.$$

This implies that $1-\left(\int_0^1 q^2(x)dx + (\frac{A_1}{4}+\frac{\lambda a}{4})\sqrt{C_1}+\frac{\lambda b}{4}\right) > 0$.

Now, we can choose ϵ small such that $\gamma > 0$. It follows that

$$\int_0^1 q^2(x)dx + (\frac{A_1}{4}+\frac{\lambda a}{4})\sqrt{C_1}+\frac{\lambda b}{4}+\frac{\lambda\epsilon}{8} < 1. \tag{2.24}$$

Hence

$$\int_0^1 v^2(x)dx + \int_0^1 \rho(x)v'^2(x)dx \leq M_1, \tag{2.25}$$

where $M_1 = \dfrac{\frac{\lambda c^2}{8\epsilon}}{\gamma}$. Combining (2.25) with (2.12), we have

$$\int_0^1 u^2(x)dx + \int_0^1 \rho(x)u'^2(x)dx \leq M_2, \tag{2.26}$$

where $M_2 = C_1 M_1$.
On the other hand, we have

$$\rho(x)u(x) = \int_0^x (\rho(x)u(x))'\,dx + \rho(0)u(0) = \int_0^x (\rho(x)u(x))'\,dx. \tag{2.27}$$

Thus

$$|\,\rho(x)u(x)\,| \leq \int_0^1 |\,(\rho(x)u(x))'\,|\,dx = \int_0^1 \left[|\,\rho'(x)u(x)+\rho(x)u'(x)\,|\right]dx. \tag{2.28}$$

Using $\sup\limits_{0\leq x\leq 1} \mid \rho'(x) \mid= 1$, applying Hölder's inequality and using $\rho^2(x) \leq \rho(x),\ x \in [0, 1]$, we obtain

$$\mid \rho(x)u(x) \mid\leq \left[\int_0^1 (\mid u(x) + \rho(x)u'(x) \mid)^2\right]^{\frac{1}{2}} dx \leq \sqrt{2}\left[\int_0^1 \left(u^2(x) + (\rho(x)u'(x))^2\right) dx\right]^{\frac{1}{2}} \leq \sqrt{2M_2}. \quad (2.29)$$

Similarly,

$$\mid \rho(x)v(x) \mid\leq \sqrt{2}\left[\int_0^1 \left(v^2(x) + \rho(x)v'^2(x)\right) dx\right]^{\frac{1}{2}} \leq \sqrt{2M_1}. \quad (2.30)$$

These two inequalities imply the required result, and complete the proof of the proposition. □

The fundamental theorem used in proving the existence of the solution is Schauder's theorem. In order to make use of this theorem, it is sufficient to present the following lemmas.

Lemma 2.2. *[17] Let $g : [0, 1] \to \mathbb{R}$ be a continuous function. If $1 - \int_0^1 p^2(x)dx = \alpha > 0$, then the unique solution u of the following boundary value problem*

$$u'' = g(x) \quad (2.31)$$

subject to the nonlocal boundary conditions $u(0) = u(1) = \int_0^1 p(x)u(x)dx$ is given by

$$u(x) = \int_0^1 G_1(x, y)g(y)dy, \quad (2.32)$$

where $G_1(x; y)$ is the Green function of this BVP and given by (Eqs.(7)-(10) see [17]).

Thus from (2.1), we obtain an equivalent integral system

$$\begin{cases} u &= \int_0^1 G_1(x, s)v(s)ds,\ x \in [0, 1], \\ v &= -\int_0^1 G_1(x, s)A(s)u(s)ds + \lambda \int_0^1 G_1(x, s)f(s, u(s), v(s))ds,\ x \in [0, 1]. \end{cases} \quad (2.33)$$

Define the Banach space

$$\mathbb{Y}_\rho = \left\{u \in \mathbb{C}^2[0, 1] :\ u(0) = u(1) = \int_0^1 p(x)u(x)dx, u''(0) = u''(1) = \int_0^1 q(x)u(x)dx\right\} \quad (2.34)$$

with norm $\| u \|_{\rho,2}=\| u \|_{\rho,0} + \| u \|_{\rho,1}$, where $\| u \|_{\rho,1}=\| u'' \|_{\rho,0}$. Also, define the operator $T : \mathbb{X} \longrightarrow \mathbb{X}$ by $T(u, v) = (T_1(u, v), T_2(u, v))$, where $\mathbb{X} = \mathbb{Y}_\rho \times \mathbb{Y}_\rho$ with norm $\| (u, v) \|_{\rho,2}=\| u \|_{\rho,0} + \| v \|_{\rho,0}$ and

$$T_1(u, v) = \int_0^1 G_1(x, s)v(s)ds \quad (2.35)$$

and

$$T_2(u, v) = -\int_0^1 G_1(x, s)A(s)u(s)ds + \lambda \int_0^1 G_1(x, s)f(s, u(s), v(s))ds. \quad (2.36)$$

Consider the closed and convex set

$$\mathbb{S} = \left\{(u, v) \in \mathbb{X} :\ \|(u, v)\|_{\rho,2} \leq M\right\}. \quad (2.37)$$

Lemma 2.3. *For any $(u, v) \in \mathbb{S}$, $T(u, v)$ is contained in $\mathbb{S}$.*

Proof. From the definition of $T(u, v)$ we have

$$|\rho(x)T_1(u, v)| \leq \rho(x) \int_0^1 |G_1(x, s)||v(s)|\,ds \leq \frac{1}{4} \int_0^1 |G_1(x, s)||v(s)|\,ds, \tag{2.38}$$

Thus

$$|\rho(x)T_1(u, v)| \leq \frac{1}{4} \left(\int_0^1 |G_1(x, s)|^2 \right)^{\frac{1}{2}} \left(\int_0^1 |v(s)|^2 \right)^{\frac{1}{2}}. \tag{2.39}$$

Assume that $\sup_{0 \leq x \leq 1} |G_1(x, s)| \leq L$ and from (2.25), in particular, we have $\int_0^1 v^2(x)dx \leq M_1$, thus

$$|\rho(x)T_1(u, v)| \leq \frac{1}{4} L M_1^{\frac{1}{2}} = M_1^*. \tag{2.40}$$

Similarly

$$|\rho(x)T_2(u, v)| \leq \rho(x) \int_0^1 |G_1(x, s)||A(s)||u(s)|\,ds + \lambda\rho(x) \int_0^1 |G_1(x, s)||f(s, u(s), v(s))|\,ds. \tag{2.41}$$

Thus

$$|\rho(x)T_2(u, v)| \leq \frac{1}{16\alpha} L A_1 M_2^{\frac{1}{2}} + \frac{\sqrt{3}L\lambda}{4} \left(a^2 M_2 + b^2 M_1 + c^2 \right)^{\frac{1}{2}} = M_2^*. \tag{2.42}$$

It follows that $\| T(u, v) \|_{\rho,2} \leq M$, where $M = M_1^* + M_2^*$. On account of the continuity of $f(x, u, v)$, u and v, it follows that $T(u, v)$ is continuous. This shows that $T(u, v)$ is also contained in $\mathbb{S}$. □

To prove that $T(u, v)$ is compact, we use the Arzela-Ascoli lemma, that is $T(\mathbb{S})$ must be closed, bounded and equicontinuous.

In order to prove that $T(\mathbb{S})$ is equicontinuous, it is sufficient to prove that the inequality

$$|T(u, v)| \leq K |x - y| \tag{2.43}$$

is satisfied for any x and y in the interval $[0, 1]$. It follows by the definition of $T(u, v)$ that

$$|T_1(u(x), v(x)) - T_1(u(y), v(y))| \leq \left| \int_y^x G_1(x, s)v(s)ds \right| \leq L M_1^{\frac{1}{2}} |x - y| = 4M_1^* |x - y|. \tag{2.44}$$

Similarly, for $T_2(u(x), v(x))$, we have

$$|T_2(u(x), v(x)) - T_2(u(y), v(y))| \leq 4M_2^* |x - y| \quad \text{for any} \quad x, y \in [0, 1], \tag{2.45}$$

which proves the equicontinuous of $T(u, v)$.
Consequently, $T(u, v)$ has a fixed point by the Schauder's fixed point theorem.
Thus, we have

Theorem 2.4. *Under the hypothesis of Proposition 2.1, there exists a continuous solution (u, v) which satisfies system (2.1).*

2.2. A uniqueness theorem

A uniqueness theorem can also be obtained if we assume that $f(x,\ u,\ u'')$ satisfies a Lipschitz condition in u and u'' with constants $k_i > 0,\ i = 0, 1$ such that

$$\int_0^1 q^2(x)dx + (\frac{A_1\sqrt{C_1}}{4} + \frac{\lambda k_0\sqrt{C_1}}{4}) + \frac{\lambda k_1}{4} < 1 \tag{2.46}$$

and

$$| f(x,u,u'') - f(x,v,v'') | \leq k_0 \mid u - v \mid + k_1 \mid u'' - v'' \mid . \tag{2.47}$$

Thus, we have

Theorem 2.5. *If f is Lipschitz in u and u'', where the constants $k_i > 0,\ i = 0, 1$ satisfy (2.46) and if (1.7) and (1.8) hold, then the system (2.1) has a unique solution $(u,\ v)$.*

Proof. Suppose there are two solutions u and v such that $u \neq v$. Then from Pr.(1.5)-(1.6), we have

$$w^{(4)} + A(x)w = \lambda\left[f(x,\ u,\ u'') - f(x,\ v,\ v'')\right],\ 0 < x < 1, \tag{2.48}$$

subject to

$$w(0) = w(1) = \int_0^1 p(x)w(x)dx,\ w''(0) = w''(1) = \int_0^1 q(x)w''(x)dx, \tag{2.49}$$

where $w = u - v$.
Thus

$$\begin{cases} w'' &= z,\ w(0) = w(1) = \int_0^1 p(x)w(x)dx, \\ z'' &= -A(x)w + \lambda\left[f(x,\ u,\ u'') - f(x,\ v,\ v'')\right],\ z(0) = z(1) = \int_0^1 q(x)z(x)dx. \end{cases} \tag{2.50}$$

Proceeding as before, we obtain

$$\int_0^1 w^2(x)dx + \int_0^1 \rho(x)w'^2(x)dx \leq C_1 \int_0^1 z^2(x)dx \tag{2.51}$$

and

$$\begin{aligned} \int_0^1 z^2(x)dx + \int_0^1 \rho(x)(z'(x))^2dx = \left[\int_0^1 q(x)z(x)dx\right]^2 + \int_0^1 A(x)\rho(x)w(x)z(x)dx \\ -\lambda\int_0^1 \left[f(x,\ u,\ u'') - f(x,\ v,\ v'')\right]\rho(x)z(x)dx, \end{aligned} \tag{2.52}$$

where

$$\left| \int_0^1 A(x)\rho(x)w(x)z(x)dx \right| \leq \frac{A_1}{4}\left(\int_0^1 w^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 z^2(x)dx\right)^{\frac{1}{2}} \tag{2.53}$$

and

$$\left[\int_0^1 q(x)z(x)dx\right]^2 \leq \left(\int_0^1 q^2(x)dx\right)\left(\int_0^1 z^2(x)dx\right). \tag{2.54}$$

Applying the Lipschitz condition to $f(x,\ u,\ u'') - f(x,\ v,\ v'')$ to obtain

$$\mid \int_0^1 f(x,\ u,\ u'') - f(x,\ v,\ v'')\rho(x)z(x)dx \mid\leq \frac{k_0}{4}\int_0^1 \mid w(x)z(x)\mid dx + \frac{k_1}{4}\int_0^1 z^2(x)dx. \tag{2.55}$$

Combining these inequalities, we obtain

$$\gamma_1 \int_0^1 z^2(x)dx + \int_0^1 \rho(x)z'^2(x)dx \leq 0, \tag{2.56}$$

where $\gamma_1 = 1 - \left(\int_0^1 q^2(x)dx + (\frac{A_1\sqrt{C_1}}{4} + \frac{\lambda k_0\sqrt{C_1}}{4}) + \frac{\lambda k_1}{4}\right) > 0$. This is a contradiction.
This completes the proof. □

Example 1. *Consider*

$$u^{(4)} + \pi^2 xu = \frac{1}{2}\sqrt{r^2(x) + u^2 + u''^2},\ 0 < x < 1 \tag{2.57}$$

under the boundary conditions (1.6) with $p(x) = q(x) = \frac{1}{2}x^2$. *Here* $r : [0,1] \to \mathbb{R}$ *is a continuous function. Let*

$$f(x,\ u,\ u'') = \sqrt{r^2(x) + u^2 + u''^2},\ \lambda = \frac{1}{2}. \tag{2.58}$$

It is clear that

$$f(x,\ u,\ u'') \leq\mid r(x)\mid + \mid u(x)\mid + \mid u''(x)\mid. \tag{2.59}$$

So that $a(x) = b(x) = 1$ *and* $c(x) = r(x)$. *However,* $\lambda = \frac{1}{2} < \frac{16\alpha\beta - A_1}{5d}$ *since* $\sup_{0\leq x\leq 1} A(x) = A_1 = \pi^2$ *and* $d = \max\{a, b\} = 1$.
Hence by Theorem 2.4, the solution exists.
To prove the uniqueness, note that $f(x,\ u,\ u'')$ *is Lipschitz:*

$$\mid f(x,\ u,\ u'') - f(x,\ v,\ v'')\mid\leq\mid u - v\mid + \mid u'' - v''\mid. \tag{2.60}$$

Indeed,

$$\begin{aligned} &\mid \sqrt{r^2 + u^2 + u''^2} - \sqrt{r^2 + v^2 + (v'')^2}\mid \\ &\leq\mid u - v\mid \frac{\mid u\mid + \mid v\mid}{\sqrt{r^2 + u^2 + u''^2} + \sqrt{r^2 + v^2 + v''^2}} + \mid u'' - v''\mid \frac{\mid u''\mid + \mid v''\mid}{\sqrt{r^2 + u^2 + u''^2} + \sqrt{r^2 + v^2 + v''^2}} \\ &\leq\mid u - v\mid + \mid u'' - v''\mid. \end{aligned} \tag{2.61}$$

So $k_0 = 1$ *and* $k_1 = 1$. *But the condition (2.46) implies that* $\lambda < \frac{16\alpha\beta - A_1}{1+4\alpha}$. *A simple substitution gives* $\frac{16\alpha\beta - A_1}{1+4\alpha} \approx 0.95 > \frac{1}{2}$, *this means that the condition (2.46) is satisfied. So by Theorem 2.5 the solution is unique.*

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Conflict of interest

The authors declare no conflict of interest.

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