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Research article

On weakly semiprime segments of ordered semihypergroups

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Abstract: In this paper, we introduce the concept of weakly semiprime segments in an ordered semihypergroup and classify weakly semiprime segments of an ordered semihypergroup into four cases which are simple, exceptional, Archimedean and decomposable.

Keywords: ordered semihypergroups; weakly semiprime hyperideals; weakly semiprime segments
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1. Introduction and Preliminaries

The concept of prime segments of a right chain ring was introduced by Brungs and Törner in [1]. Later on, the concept was extended to right chain semigroups [2] and right chain ordered semigroups [3] respectively. In 2006, Mazurek and Törner generalized the concept to a ring and studied the semiprime segments of a ring [4]. Recently, the weakly semiprime segments of ordered semigroups was studied by Luangchaisri and Changphas [5]. The hyperideals of ordered semihypergroups were introduced by Changphas and Davvaz in [6]. In [7], Davvaz et al. introduced the concept of a pseudoorder on an ordered semihypergroup and used it to construct a strongly regular equivalence relation on an ordered semihypergroup for which the corresponding quotient structure is an ordered semigroup. Furthermore, Tang et al. [8] introduced the concept of a weak pseudoorder on an ordered semihypergroup and used it to construct an ordered regular equivalence relation on an ordered semihypergroup for which the corresponding quotient structure is an ordered semihypergroup. Recently, (m,n)-quasi-hyperideal and (m,n)-hyperideal were introduced and studied by A. Mahboob et al. in [9, 10]. Motivated by the previous work on rings and (ordered) semigroups, we introduce the notions of weakly semiprime segments in ordered semihypergroups, and classify them into four subclasses.

We recall first some basic notions of ordered semihypergroups (see [11]).

A *hypergroupoid* $(S,\circ)$ is a nonempty set S together with a *hyperoperation* or *hypercomposition*, that is a mapping $\circ : S\times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S.

If $x \in S$ and A, B are nonempty subsets of S, then we denote $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if $\circ$ is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$ for every $x, y, z \in S$.

An *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ with an order relation $\leq$ which is *compatible* with the hyperoperation $\circ$, meaning that for any $a, b, x \in S$, $a \leq b$ implies that $a \circ x \leq b \circ x$ and $x \circ a \leq x \circ b$. Here, let $A, B \in \mathcal{P}^*(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Let S be an ordered semihypergroup and I be a nonempty subset of S. We say that I is a *hyperideal* of S if (1) $S \circ I \subseteq I, I \circ S \subseteq I$ and (2) $a \in I, b \in S$ and $b \leq a$ imply that $b \in I$. For $\emptyset \neq H \subseteq S$, we use the notation

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For convenience, we write $(a]$ instead of $(\{a\}]$. We denote by $I(a)$ the hyperideal of S generated by a. One can easily verify that

$$I(a) = (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S].$$

2. Main results

Let $(S, \circ, \leq)$ be an ordered semihypergroup. An element e of S is called an *absolute identity* if $e \circ a = a \circ e = \{a\}$ for every $a \in S$. It is easy to see that an ordered semihypergroup has at most an absolute identity. In this section, we introduce the concept of a weakly semiprime segment of an ordered semihypergroup, and classify weakly semiprime segments of an ordered semihypergroup into four cases.

Definition 1. *Let S be an ordered semihypergroup with an absolute identity. A pair $P_2 \subset P_1$ of weakly semiprime hyperideals of S is called a weakly semiprime segment if $\bigcap_{n \in \mathbb{N}} (I^n] \subseteq P_2$ for every hyperideal I of S with $P_2 \subset I \subset P_1$.*

Lemma 1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup with an absolute identity, and $P_2 \subset P_1$ a weakly semiprime segment of S. Then exactly one of the following possibilities occurs.*

(1) There are no further hyperideals of S between P_2 and P_1, and P_2 is comparable with each hyperideal of S contained in P_1;

(2) There exists a weakly semiprime hyperideal Q of S such that $P_2 \subset Q \subset P_1$, and Q is comparable with each hyperideal of S contained in P_1;

(3) $((P_1 \circ a \circ S) \cup (S \circ a \circ P_1)] \subset (S \circ a \circ S]$ for all $a \in P_1 \setminus P_2$;

(4) $((P_1 \circ a \circ S) \cup (S \circ a \circ P_1)] = (S \circ a \circ S]$ for some $a \in P_1 \setminus P_2$ and $P_1 = (S \circ a_1 \circ S] \cup (S \circ a_2 \circ S] \cup P_2$ for some $a_1, a_2 \in P_1$ with $(S \circ a_i \circ S] \subset P_1$.

Proof. Let $a \in P_1 \setminus P_2$. If $(P_1^2 \cup P_2] \subset P_1$, then (3) occurs. Indeed: suppose that $a \in ((P_1 \circ a \circ S) \cup (S \circ a \circ P_1)]$. Then $a \in ((P_1^n \circ a \circ S) \cup (S \circ a \circ P_1^n)] \subseteq (P_1^n]$ for any $n \in \mathbb{N}$. Thus $a \in \bigcap_{n \in \mathbb{N}} (P_1^n] = \bigcap_{n \in \mathbb{N}} (P_1^{2n}] \subseteq \bigcap_{n \in \mathbb{N}} ((P_1^2 \cup P_2)^n] \subseteq P_2$, which contradicts that $a \notin P_2$. Hence, $a \notin ((P_1 \circ a \circ S) \cup (S \circ a \circ P_1)]$ and so $((P_1 \circ a \circ S) \cup (S \circ a \circ P_1)] \subset (S \circ a \circ S]$. To finish the proof, we assume that $(P_1^2 \cup P_2] = P_1$. Let M be the union of all hyperideals I of S with $I \subset P_1$. Then M is a hyperideal of S and $P_2 \subseteq M \subseteq P_1$. Next we consider three cases.

(i) If $M = P_2$, then (1) occurs obviously.

(ii) If $P_2 \subset M \subset P_1$, then (2) occurs. In fact: Let A be a hyperideal of S with $A^2 \subseteq M$. Then $A^2 \subseteq P_1$. Since P_1 is weakly semiprime, we have $A \subseteq P_1$. Suppose that $A = P_1$. Then $P_1 = (P_1^2 \cup P_2] = (A^2 \cup P_2] \subseteq M$ which contradicts that $M \subset P_1$. Thus $A \subset P_1$ and so $A \subseteq M$. Hence, M is weakly semiprime.

(iii) Let $M = P_1$. Then $(S \circ x \circ S] \subset P_1$ for any $x \in P_1$. Assume that (3) does not occur. Then there exists $a \in P_1 \setminus P_2$ such that $((P_1 \circ a \circ S) \cup (S \circ a \circ P_1)] = (S \circ a \circ S]$. Thus $a \in ((s \circ a \circ S) \cup (S \circ a \circ t)]$ for some $s, t \in P_1$. Set $I = (S \circ s \circ S] \cup (S \circ t \circ S]$. Then I is a hyperideal of S and $a \in ((s \circ a \circ S) \cup (S \circ a \circ t)] \subseteq ((S \circ s \circ S \circ a \circ S) \cup (S \circ a \circ S \circ t \circ S)] \subseteq ((I \circ a \circ S) \cup (S \circ a \circ I)]$. If $I \cup P_2 \subset P_1$, then by the beginning of the proof, we have $a \in \bigcap_{n \in \mathbb{N}} (I^n] \subseteq \bigcap_{n \in \mathbb{N}} ((I \cup P_2)^n] \subseteq P_2$ which contradicts that $a \notin P_2$. Hence, $P_1 = I \cup P_2$ and so (4) occurs.

It is easy to verify that the possibilities (1),(2),(3),(4) are mutually exclusive. □

Theorem 1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup with an absolute identity, and $P_2 \subset P_1$ a weakly semiprime segment of S. Then exactly one of the following possibilities occurs.*

(a) The semiprime segment $P_2 \subset P_1$ is simple; that is, there are no further hyperideals of S between P_2 and P_1, and P_2 is comparable with each hyperideal of S contained in P_1;

(b) The semiprime segment $P_2 \subset P_1$ is exceptional; that is, there exists a weakly semiprime hyperideal Q of S such that $P_2 \subset Q \subset P_1$, and Q is comparable with each hyperideal of S contained in P_1;

(c) The semiprime segment $P_2 \subset P_1$ is Archimedean; that is, for every $a \in P_1 \setminus P_2$ there exists a hyperideal $I \subseteq P_1$ of S such that $a \in I$ and $\bigcap_{n \in \mathbb{N}} (I^n] \subseteq P_2$;

(d) The semiprime segment $P_2 \subset P_1$ is decomposable; that is, the semiprime segment $P_2 \subset P_1$ is not Archimedean and $P_1 = A \cup B$ for some hyperideals A, B of S properly contained in P_1.

Proof. From Lemma 1, we know that there are four cases to consider. Clearly, in the case (1) the semiprime segment $P_2 \subset P_1$ is simple; in the case (2) the semiprime segment $P_2 \subset P_1$ is exceptional; and in the case (4) the segment $P_2 \subset P_1$ is either Archimedean or decomposable.

Assume that the case (3) occurs and the segment $P_2 \subset P_1$ is not Archimedean. Then there exists $a \in P_1 \setminus P_2$ such that $\bigcap_{n \in \mathbb{N}} (I^n] \nsubseteq P_2$ for all hyperideals I with $a \in I \subseteq P_1$. If $((S \circ a \circ S]^2] \cup P_2 \subset P_1$, then $\bigcap_{n \in \mathbb{N}} ((S \circ a \circ S]^n] = \bigcap_{n \in \mathbb{N}} ((S \circ a \circ S]^{2n}] \subseteq \bigcap_{n \in \mathbb{N}} (((S \circ a \circ S]^2 \cup P_2)^n] \subseteq P_2$, which is a contradiction. Thus $((S \circ a \circ S]^2] \cup P_2 = P_1$. Moreover, $((S \circ a \circ S]^2] \subset (P_1 \circ a \circ S] \subset (S \circ a \circ S] \subseteq P_1$. Hence, the segment $P_2 \subset P_1$ is decomposable.

It is easy to see that the possibilities (a),(b),(c),(d) are mutually exclusive. □

3. Conclusions

Let S be an ordered semihypergroup with an absolute identity. A pair $P_2 \subset P_1$ of weakly semiprime hyperideals of S is called a weakly semiprime segment if $\bigcap_{n \in \mathbb{N}} (I^n] \subseteq P_2$ for every hyperideal I of S with $P_2 \subset I \subset P_1$. In this paper, we classify weakly semiprime segments of an ordered semihypergroup into four cases which are simple, exceptional, Archimedean and decomposable.

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Conflict of interest

The authors declare no conflict of interest.

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