Research article

Well posedness for a singular two dimensional fractional initial boundary value problem with Bessel operator involving boundary integral conditions

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Abstract: This paper studies the existence and uniqueness of solutions of a non local initial boundary value problem of a singular two dimensional nonlinear fractional order partial differential equation involving the Caputo fractional derivative by employing the functional analysis. We first establish for the associated linear problem a priori estimate and prove that the range of the operator generated by the considered problem is dense. The technique of obtaining the a priori bound relies on the construction of a suitable multiplicator. From the resulted a priori estimate, we can establish the solvability of the associated linear problem. Then, by applying an iterative process based on the obtained results for the associated linear problem, we establish the existence, uniqueness and continuous dependence of the weak solution of the considered nonlinear problem.

Keywords: fractional differential equation; boundary integral condition; singular initial boundary value problem; well posedness; iterative process; bessel operator

Mathematics Subject Classification: 35D35, 35L20

1. Introduction

Initial boundary value problems with non local and non-classical boundary conditions for integer and fractional order linear and nonlinear evolution partial differential equations, have gained great attention during the last three decades. Especially problems with boundary conditions of integral type (the so called energy specification) which are important from the point of view of their practical application to modelling and investigating various physical phenomena in the context of chemical engineering, thermoelasticity, population dynamics, polymer rheology, aerodynamics, heat conduction processes, plasma physics, underground-water flow, transmission theory, chemical engineering, control theory, fluid flow and many physical and biological processes and systems and so forth, see [19–31]. We should mention that most local phenomena can be examined and modeled in terms of integer
order differential equations, while fractional order differential equations model non local phenomena. Accordingly, fractional order partial differential equations describe real world phenomena that cannot be described by classical mathematics literature. This is due to the fact that many models depend on the present and historical states. For integer order case (see for example [1–11] and references therein. For the fractional order case see for example [15–18, 35–37] and references therein. However, the investigation of initial boundary value problems for nonlinear fractional order partial differential equations still needs too much exploration and investigation.

For the proof of the existence and the uniqueness of the solution of the posed problem, we use the energy inequality method based mainly on some a priori estimates and on the density of the range of the operator generated by the considered problem. In the literature, there are few articles using the method of energy inequalities for the proof of existence and uniqueness of fractional initial-boundary value problems in the fractional case (see [12–14, 32–34]). This work, can be considered as a continuation, improvement and generalization of previous works. Many difficulties are encountered while applying the functional analysis method for the posed problem. These difficulties are mainly due to the fact that the considered equation is nonlinear, singular, with fractional order in a two-dimensional space setting, and supplemented with nonlocal conditions.

The outline of this paper is as follows. In section 2, we set the problem and give some preliminaries. In section 3, we pose the associated linear problem and introduce some function spaces used in the sequel. Section 4 is devoted to the uniqueness results for the associated linear problem. The existence of solution of the associated linear problem is considered in section 5. The main results of this paper are given in section 6, it is consecrated to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the nonlinear problem.

2. Problem setting and some preliminaries

In the bounded domain $Q^T = \Omega \times (0, T)$, where $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$, we consider the two dimensional singular nonlinear fractional partial differential equation in Caputo sense with Bessel operator

$$L(\theta) = \partial^\delta_t \theta - \frac{1}{xy} \text{div}(xy \nabla \theta) = H(x, y, t, \theta, \theta_x, \theta_y), \quad (2.1)$$

where $\partial^\delta_t$ denotes the time fractional Caputo derivative operator of order $\delta \in (1, 2]$, the symbols $\text{div}$ and $\nabla$, denote respectively the divergence and the gradient operators. $\theta_x$ stands for the $x$–derivative of the function $\theta$. Equation (2.1), is supplemented by the initial conditions

$$\ell_1 \theta = \theta(x, y, 0) = F(x, y), \quad \ell_2 \theta = \theta_t(x, y, 0) = G(x, y), \quad (2.2)$$

Neumann boundary conditions

$$\theta_x(a, y, t) = 0, \quad \theta_x(b, y, t) = 0, \quad (2.3)$$

and the non local weighted boundary integral conditions

$$\int_0^a x \theta(x, y, t) dx = 0, \quad \int_0^b y \theta(x, y, t) dx = 0, \quad (2.4)$$
where the functions $F$ and $G$ are given functions which will be specified later on. We shall assume that the function $H$ is a Lipschitzian function, that is there exists a positive constant $\lambda$ such that for all $(x, y, t) \in Q^T$

\[
H(x, y, t, \theta_1, v_1, w_1) - H(x, y, t, \theta_2, v_2, w_2) \leq \lambda (|\theta_1 - \theta_2| + |v_1 - v_2| + |w_1 - w_2|).
\]

In equation (2.1), the fractional derivative $\partial_t^\delta \mathcal{E}$ of order $\delta = \beta + 1$, where $0 < \beta < 1$ (see [17]) for a function $\mathcal{E}$ is defined by

\[
C \partial_t^\delta \mathcal{E}(x,t) = \frac{1}{\Gamma(1-\delta)} \int_0^t \mathcal{E}_{ss}(x,s) \frac{1}{(t-s)^\delta} ds, \quad \forall t \in [0, T],
\]

where $\Gamma(.)$ is the Gamma function.

We begin by giving some important lemmas needed throughout the sequel.

**Lemma 2.1** [38]. Let $R(s)$ be nonnegative and absolutely continuous on $[0, T]$, and suppose that for almost all $s \in [0, T]$, $R$ satisfies the inequality

\[
\frac{dR}{ds} \leq A_1(s)R(t) + B_1(s),
\]

where the functions $A_1(s)$ and $B_1(s)$ are summable and nonnegative on $[0, T]$. Then

\[
R(s) \leq e^{\int_0^s A_1(t)dt} \left( R(0) + \int_0^t B_1(t)dt \right).
\]

**Lemma 2.2.** [14] Let be $M(t)$ a nonnegative absolutely continuous function, such that

\[
C \partial_t^\lambda M(t) \leq b_1 M(t) + b_2(t), \quad 0 < \lambda < 1,
\]

for almost all $t \in [0, T]$, where $b_1$ is a positive constant and $b_2(t)$ is an integrable nonnegative function on $[0, T]$. Then

\[
M(t) \leq M(0)E_{\lambda}(b_1 t^\lambda) + \Gamma(\lambda)E_{\lambda,\lambda}(b_1 t^\lambda)D_{t}^{-\lambda}b_2(t),
\]

where

\[
E_{\lambda}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\lambda n + 1)} \quad \text{and} \quad E_{\lambda,\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\lambda n + \mu)},
\]

are the Mittag-Leffler functions, and $D_{t}^{-\lambda}v(t) = \frac{1}{\Gamma(\lambda)} \int_0^t \frac{v(\tau)}{(t-\tau)^{\lambda}} d\tau$ is the Riemann-Liouville integral of order $0 < \lambda < 1$.

**Lemma 2.3.** [14] For any absolutely continuous function $J(t)$ on the interval $[0, T]$, the following inequality holds

\[
J(t) C \partial_t^\beta J(t) \geq \frac{1}{2} C \partial_t^\beta J^2(t), \quad 0 < \beta < 1.
\]
3. Statement of the associated linear problem

In this section, we set the associated linear problem and introduce different function spaces needed to investigate this problem. We consider the differential equation

$$L(\theta) = \partial_t^\alpha \theta - \frac{1}{xy} \text{div}(xy \nabla \theta) = H(x, y, t),$$  \ (3.1)

supplemented by conditions (2.2)–(2.4). The used method is essentially based on the construction of suitable multipliers for each specific given problem, which provides the a priori estimate from which it is possible to establish the solvability of the posed problem. More precisely, the proofs of uniqueness of the solution is based on an energy inequality and on the density of the range of the operator generated by the abstract formulation of the stated problem.

To investigate the posed problem, we introduce the needed function spaces. We denote by $L^2_\rho(\Omega)$ the Hilbert space of weighted square integrable functions where $\rho = xy$ and with inner product

$$(\mathcal{U}, v)_{L^2_\rho(\Omega)} = (xy \mathcal{U}, v)_{L^2(\Omega)} = \int_\Omega xy \mathcal{U}vdxdy,$$

and with associated norm

$$\|\mathcal{U}\|_{L^2_\rho(\Omega)} = \left( \int_\Omega xy \mathcal{U}^2dxdy \right)^{\frac{1}{2}}.
$$

Let $X$ be Banach space with norm $\|\mathcal{U}\|_X$, and let $\mathcal{U} : (0, T) \to X$ be an abstract function. By $\mathcal{U}(., ., t)\|_X$ we denote the norm of $\mathcal{U}(., ., t) \in X$ for fixed $t$. Let $L^2(0, T; X)$ be the set of all measurable abstract functions $\mathcal{U} : (0, T) \to X$ such that

$$\|\mathcal{U}\|_{L^2(0, T; X)}^2 = \int_0^T \|\mathcal{U}(., ., t)\|^2_X dt < \infty.$$

If $X$ is a Hilbert space, then $L^2(0, T; X)$ is also a Hilbert space. Let $C(0, T; X)$ be the set of all continuous functions $\mathcal{U} : (0, T) \to X$ such that

$$\|\mathcal{U}\|_{C(0, T; X)} = \max_{t \in [0, T]} \|\mathcal{U}(., ., t)\|_X < \infty,$$

and denote by $H^1_\rho(\Omega)$ the weighted Sobolev space with norm

$$\|\mathcal{U}\|_{H^1_\rho(\Omega)}^2 = \|\mathcal{U}\|_{L^2_\rho(\Omega)}^2 + \|\mathcal{U}_x\|_{L^2_\rho(\Omega)}^2 + \|\mathcal{U}_y\|_{L^2_\rho(\Omega)}^2.$$

The given problem (3.1), (2.2)–(2.4) can be viewed as the problem of solving the operator equation $A\theta = W = (H, F, G)$, where $A\theta = (\mathcal{L}\theta, \ell_1\theta, \ell_2\theta), \theta \in D(A)$ where $A$ is the operator given by $A = (\mathcal{L}, \ell_1, \ell_2)$ and $D(A)$ is the set of all functions $\theta \in L^2_\rho(\mathcal{Q}^T) : \partial_t^\alpha \theta, \theta_x, \theta_y, \theta_{xx}, \theta_{yy}, \theta_t \in L^2_\rho(\mathcal{Q}^T)$ and $\theta$ satisfies
conditions (2.2)–(2.4). The operator $A$ acts from $B$ into $Y$, where $B$ is the Banach space obtained by enclosing $D(A)$ with respect to the finite norm

$$
\|\theta\|_{B}^2 = \sup_{0 \leq t \leq T} \|\theta(x, y, \tau)\|_{H^1(\Omega)}^2 = \|\theta(x, y, \tau)\|_{C(0, T; H^1(\Omega))}^2.
$$

(3.2)

Functions $\theta \in B$ are continuous on $[0, T]$ with values in $L^2_{\rho}(\Omega)$. Hence the mappings

$$
\ell_1 \theta \in B \rightarrow \ell \theta = \theta(x, y, 0) \in L^2_{\rho}(\Omega),
$$

$$
\ell_2 \theta \in B \rightarrow \ell \theta = \theta(x, y, 0) \in L^2_{\rho}(\Omega),
$$

are defined and continuous on $B$. And $Y$ is the Hilbert space $L^2(0, T, L^2_{\rho}(\Omega)) \times H^1_{\rho}(\Omega) \times L^2_{\rho}(\Omega)$ consisting of vector valued functions $W = (H, F, G)$ for which the norm

$$
\|W\|_Y^2 = \|H\|_{L^2(0, T; L^2_{\rho}(\Omega))}^2 + \|F\|_{H^1_{\rho}(\Omega)}^2 + \|G\|_{L^2_{\rho}(\Omega)}^2,
$$

is finite. Let $\overline{A}$ be the closure of the operator $A$ with domain of definition $D(\overline{A})$.

**Definition 3.** We call a strong solution of problem (3.1), (2.2)-2.4), the solution of the operator equation

$$
\overline{A} \theta = W, \quad \forall \theta \in D(\overline{A}).
$$

4. Main result of uniqueness of solution for the linear problem

We will establish an a priori estimate for the operator $A$ from which we deduce the uniqueness and continuous dependence of the solution upon the initial conditions (2.2).

**Theorem 4.1** For any function $\theta \in D(A)$ we have the a priori estimate

$$
\|\theta(x, y, \tau)\|_{C(0, T; H^1(\Omega))}^2 \leq C \left( \|H\|_{L^2(0, T; L^2_{\rho}(\Omega))}^2 + \|F\|_{H^1_{\rho}(\Omega)}^2 + \|G\|_{L^2_{\rho}(\Omega)}^2 \right),
$$

(4.1)

where $C$ is a positive constant independent of the function $\theta$ given by

$$
C = (C_3 e^{C_3 T} + 1),
$$

with

$$
C_3 = C_2^2 \Gamma(\beta) E_{\beta, \beta} (C_2 T^\beta / \beta \Gamma(\beta)) + C_2,
$$

$$
C_2 = \max \left\{ C_1, 1, \frac{T^{1-\beta}}{(1 - \beta) \Gamma(1 - \beta)} \left( 1 + \frac{a^3 b^3}{16} \right) \right\},
$$

$$
C_1 = \max \left\{ 2, \left( \frac{a^2 b^2}{2} + \frac{a^2}{2} + \frac{b^2}{2} \right) \right\}.
$$

**Proof.**
Let $\beta + 1 = \delta$, where $0 < \beta \leq 1$, then (3.1) takes the form
\[
\partial_t^\beta \theta - \frac{1}{x}(x\theta)_x - \frac{1}{y}(y\theta)_y = H(x, y, t).
\]  
(4.2)

Taking the scalar product in $L^2_p(\Omega)$ of the partial differential equation (4.2) and the integro-differential operator $M\theta = \theta_t + \mathcal{F}_{x_y}(\xi\eta\theta_t)$, where
\[
\mathcal{F}_{x_y}(\xi\eta\theta_t) = \int_0^x \int_0^y \xi\eta\theta_t(\xi, \eta) d\eta d\xi,
\]
then we have
\[
(\partial_t^\beta \theta_t, xy\theta_t)_{L^2_\Omega} = (\partial_t^\beta \theta_t, \theta_t)_{L^2_\Omega} - (\theta_t, x_y \theta_t)_{L^2_\Omega} + (\partial_t^\beta \theta_t, xy\mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega}
\]
\[
- (\theta_t, xy\mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = (\partial_t^\beta \theta_t, \mathcal{F}_{x_y}(\xi\eta\theta_t), \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega}
\]
\[
- (\theta_t, x_y \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = \int_0^a \int_0^b y\theta_t \theta_t dxdy,
\]
\[
- (\theta_t, y \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = \int_0^a \int_0^b x\theta_t \theta_t dxdy,
\]
\[
- (\theta_t, x_y \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = \int_0^a \int_0^b y\theta_t \mathcal{F}_{x_y}(\xi\eta\theta_t) dxdy,
\]
\[
- (\theta_t, y \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = \int_0^a \int_0^b x\theta_t \mathcal{F}_{x_y}(\xi\eta\theta_t) dxdy.
\]
(4.3)

The standard integration by parts of each term in equation (4.3) leads to
\[
(\partial_t^\beta \theta_t, xy\theta_t)_{L^2_\Omega} = (\partial_t^\beta \theta_t, \theta_t)_{L^2_\Omega},
\]  
(4.4)

\[
(\partial_t^\beta \theta_t, xy\mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = (\partial_t^\beta (\mathcal{F}_{x_y}(\xi\eta\theta_t), \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega},
\]
\[
- (\theta_t, x_y \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = - \int_0^a \int_0^b y\theta_t \theta_t dxdy,
\]
\[
- (\theta_t, y \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = - \int_0^a \int_0^b x\theta_t \theta_t dxdy,
\]
\[
- (\theta_t, x_y \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = \int_0^a \int_0^b x\theta_t \mathcal{F}_{x_y}(\xi\eta\theta_t) dxdy,
\]
\[
- (\theta_t, y \mathcal{F}_{x_y}(\xi\eta\theta_t))_{L^2_\Omega} = \int_0^a \int_0^b x\theta_t \mathcal{F}_{x_y}(\xi\eta\theta_t) dxdy.
\]
(4.5)
Substitution of equation (4.4)–(4.13) into equation (4.3) yields

\[-(\theta_{xx}, xy\mathcal{S}_{xy}(\xi\eta\theta))_{L^2(\Omega)} = \int_0^b \int_0^b \theta_x y \mathcal{S}^2_{xy}(\xi\eta\theta) dxdy + (\theta_x, \mathcal{S}_{xy}(\xi\eta\theta))_{L^2(\Omega)}, \quad (4.12)\]

\[-(\theta_{yy}, xy\mathcal{S}^2_{xy}(\xi\eta\theta))_{L^2(\Omega)} = \int_0^b \int_0^b \theta_y x \mathcal{S}^2_{xy}(\xi\eta\theta) dxdy + (\theta_y, \mathcal{S}^2_{xy}(\xi\eta\theta))_{L^2(\Omega)}, \quad (4.13)\]

Substitution of equation (4.4)–(4.13) into equation (4.3) yields

\[
\begin{align*}
(\partial_t^\beta \theta_x, \theta_x)_{L^2(\Omega)} + (\partial_t^\beta (\mathcal{S}_{xy}(\xi\eta\theta)), \mathcal{S}_{xy}(\xi\eta\theta))_{L^2(\Omega)} \\
+ \frac{1}{2}\partial_t \|\theta_x\|^2_{L^2(\Omega)} + \frac{1}{2}\partial_t \|\theta_y\|^2_{L^2(\Omega)} \\
= \int_\Omega xy\theta_t H dxdy + \int_\Omega xy\mathcal{S}^2_{xy}(\xi\eta\theta) dxdy \\
- (\theta_x, \mathcal{S}_{xy}(\xi\eta\theta))_{L^2(\Omega)} - (\theta_y, \mathcal{S}^2_{xy}(\xi\eta\theta))_{L^2(\Omega)}. \quad (4.14)
\end{align*}
\]

Using Cauchy–inequality \(AB \leq \frac{\epsilon_1}{2}A^2 + \frac{1}{2\epsilon_2}B^2\), Poincare’ type inequalities \((\|I_x(\mathcal{E}v)|^2_{L^2(\Omega)} \leq \frac{1}{2}\|v|^2_{L^2(\Omega, \Omega)} \cdot \|I_x(\mathcal{E}v)|^2_{L^2(\Omega, \Omega)} \leq \frac{1}{2}\|I_x(\mathcal{E}v)|^2_{L^2(\Omega, \Omega)} \) \([7]\) and Lemma 2.3, we transform (4.14) to

\[
\begin{align*}
\partial_t^\beta \|\theta_x\|^2_{L^2(\Omega)} + \partial_t^\beta \|\mathcal{S}_{xy}(\xi\eta\theta)\|^2_{L^2(\Omega)} \\
+ \partial_t \|\theta_x\|^2_{L^2(\Omega)} + \partial_t \|\theta_y\|^2_{L^2(\Omega)} \\
\leq \epsilon_1 \|\theta_x\|^2_{L^2(\Omega)} + \left(\frac{1}{\epsilon_1} + \epsilon_2\right)\|H\|^2_{L^2(\Omega)} \\
+ \epsilon_2 \|\theta_x\|^2_{L^2(\Omega)} + \epsilon_1 \|\theta_y\|^2_{L^2(\Omega)} \\
+ \left(\frac{a^2b^2}{2\epsilon_2} + \frac{a^2}{2\epsilon_3} + \frac{b^2}{2\epsilon_3}\right)\|\mathcal{S}_{xy}(\xi\eta\theta)\|^2_{L^2(\Omega)}. \quad (4.15)
\end{align*}
\]

Let \(\epsilon_i = 1, \quad i = 1, 2, 3, 4\), in (4.15), then it follows that

\[
\begin{align*}
\partial_t^\beta \|\theta_x\|^2_{L^2(\Omega)} + \partial_t^\beta \|\mathcal{S}_{xy}(\xi\eta\theta)\|^2_{L^2(\Omega)} \\
+ \partial_t \|\theta_x\|^2_{L^2(\Omega)} + \partial_t \|\theta_y\|^2_{L^2(\Omega)} \\
\leq C_1 \left(\|\theta_x\|^2_{L^2(\Omega)} + \|H\|^2_{L^2(\Omega)} \\
+ \|\theta_x\|^2_{L^2(\Omega)} + \|\theta_y\|^2_{L^2(\Omega)} + \|\mathcal{S}_{xy}(\xi\eta\theta)\|^2_{L^2(\Omega)}\right), \quad (4.16)
\end{align*}
\]

where

\[C_1 = \max \left\{2, \frac{a^2b^2}{2} + \frac{a^2}{2} + \frac{b^2}{2}\right\}.\]

We infer from (4.16) that

\[
D_t^{\beta-1} \|\theta_x\|^2_{L^2(\Omega)} + D_t^{\beta-1} \|\mathcal{S}_{xy}(\xi\eta\theta)\|^2_{L^2(\Omega)} + \|\theta_x\|^2_{L^2(\Omega)} + \|\theta_y\|^2_{L^2(\Omega)}
\]
By dropping the last two terms from the left side of (4.18), and applying Lemma 2.2 by taking
\[ \mathcal{A} \mathcal{M} \mathcal{I} \mathcal{S} \mathcal{M} \mathcal{A} \mathcal{T} \mathcal{M} \mathcal{E} \mathcal{S} \mathcal{H} \quad \text{V o l u m e} \quad 6, \quad I s s u e \quad 9, \quad 9786–9812. \]
\[ \text{Since} \quad h \quad \text{and} \quad \vartheta = (0), \quad \text{we have} \]
\[ \leq C_1 \left( \int_0^t \| \theta_t \|^2_{l^2_\beta(\Omega)} d\tau + \int_0^t \| \mathcal{H} \|^2_{l^2_\beta(\Omega)} d\tau + \int_0^t \| \mathcal{I}_x(y \vartheta) \|^2_{l^2_\beta(\Omega)} d\tau \right) \\
+ \int_0^t \| \vartheta_t \|^2_{l^2_\beta(\Omega)} d\tau + \int_0^t \| F \|^2_{l^2_\beta(\Omega)} d\tau \]
\[ + \int_0^t \| F \|^2_{l^2_\beta(\Omega)} d\tau + \int_0^t \| G \|^2_{l^2_\beta(\Omega)} d\tau \] \quad (4.17)
By using a Poincaré-type inequality, (4.17) becomes
\[ D^{\beta-1}_t \| \theta \|^2_{l^2_\beta(\Omega)} + D^{\beta-1}_t \| \mathcal{I}_x(y \vartheta) \|^2_{l^2_\beta(\Omega)} + \| \theta_t \|^2_{l^2_\beta(\Omega)} + \| \vartheta \|^2_{l^2_\beta(\Omega)} \]
\[ \leq C_2 \left( \int_0^t \| \theta_t \|^2_{l^2_\beta(\Omega)} d\tau + \int_0^t \| \mathcal{I}_x(y \vartheta) \|^2_{l^2_\beta(\Omega)} d\tau \right) \\
+ \int_0^t \| \theta_t \|^2_{l^2_\beta(\Omega)} d\tau + \int_0^t \| \vartheta_t \|^2_{l^2_\beta(\Omega)} d\tau \]
\[ + \int_0^t \| H \|^2_{l^2_\beta(\Omega)} d\tau + \| F \|^2_{l^2_\beta(\Omega)} + \| G \|^2_{l^2_\beta(\Omega)} \right), \quad (4.18) \]
where
\[ C_2 = \max \left\{ C_1, \frac{T^{1-\beta}}{(1-\beta) \Gamma(1-\beta)} (1 + \frac{a^3 b^3}{16}) \right\}. \]
By dropping the last two terms from the left side of (4.18), and applying Lemma 2.2 by taking
\[ h(t) = \int_0^t \| \theta_t \|^2_{l^2_\beta(\Omega)} d\tau + \int_0^t \| \mathcal{I}_x(y \vartheta) \|^2_{l^2_\beta(\Omega)} d\tau, \]
\[ \vartheta \right) = D^{\beta-1}_t \| \theta \|^2_{l^2_\beta(\Omega)} + D^{\beta-1}_t \| \mathcal{I}_x(y \vartheta) \|^2_{l^2_\beta(\Omega)}, \]
and \( h(0) = 0 \), we have
\[ h(t) \leq C_2 \Gamma(\beta) E_{\beta, \beta}(C_1 T^\beta) D_t^{\beta} \left( \int_0^t \| \theta_t \|^2_{l^2_\beta(\Omega)} d\tau \right) \\
+ \int_0^t \| \vartheta_t \|^2_{l^2_\beta(\Omega)} d\tau + \int_0^t \| H \|^2_{l^2_\beta(\Omega)} d\tau \]
\[ + \| F \|^2_{l^2_\beta(\Omega)} + \| G \|^2_{l^2_\beta(\Omega)} \right), \quad (4.19) \]
Since
\[ D_t^{\beta} \int_0^t \| J(x, y, \tau) \|^2_{l^2_\beta(\Omega)} d\tau \leq \frac{T^\beta}{\beta \Gamma(\beta)} \int_0^t \| J(x, y, \tau) \|^2_{l^2_\beta(\Omega)} d\tau, \]
then

\[
\begin{align*}
    h(t) & \leq C_2 \Gamma(\beta) E_{\beta, \beta} (C_2 T^\beta) \left( \frac{T^\beta}{\beta \Gamma(\beta)} \right) \left( \int_0^t \| \theta_t \|_{L_2^2(\Omega)}^2 \, d\tau \right) \\
    & \quad + \int_0^t \| \theta_y \|_{L_2^2(\Omega)}^2 \, d\tau + \int_0^t \| H \|_{L_2^2(\Omega)}^2 \, d\tau \\
    & \quad + \| F \|_{H^1_p(\Omega)}^2 + \| G \|_{L_2^2(\Omega)}^2, \\
\end{align*}
\]

(4.20)

Hence (4.18) becomes

\[
\begin{align*}
    D_t^\beta \| \theta \|_{L_2^2(\Omega)}^2 + D_t^\beta \| \Im x (\xi \eta \theta_t) \|_{L_2^2(\Omega)}^2 \\
    + ||\theta_t\|_{L_2^2(\Omega)}^2 + ||\theta_x\|_{L_2^2(\Omega)}^2 \\
    \leq C_3 \left( \int_0^t \| \theta_t \|_{L_2^2(\Omega)}^2 \, d\tau + \int_0^t \| \theta_y \|_{L_2^2(\Omega)}^2 \, d\tau + \int_0^t \| H \|_{L_2^2(\Omega)}^2 \, d\tau \\
    + \| F \|_{H^1_p(\Omega)}^2 + \| G \|_{L_2^2(\Omega)}^2 \right), \\
\end{align*}
\]

(4.21)

where

\[
C_3 = C_2^2 \Gamma(\beta) E_{\beta, \beta} (C_2 T^\beta) \left( \frac{T^\beta}{\beta \Gamma(\beta)} \right) + C_2.
\]

Now discard the first two terms from left hand side in (4.21) and use lemma 2.1 with

\[
S(t) = \int_0^t \| \theta_t \|_{L_2^2(\Omega)}^2 \, d\tau + \int_0^t \| \theta_x \|_{L_2^2(\Omega)}^2 \, d\tau,
\]

\[
dS(t)/dt = \| \theta_t \|_{L_2^2(\Omega)}^2 + \| \theta_x \|_{L_2^2(\Omega)}^2,
\]

\[
S(0) = 0,
\]

we see that

\[
\begin{align*}
    D_t^\beta \| \theta \|_{L_2^2(\Omega)}^2 + D_t^\beta \| \Im x (\xi \eta \theta_t) \|_{L_2^2(\Omega)}^2 \\
    + ||\theta_t\|_{L_2^2(\Omega)}^2 + ||\theta_x\|_{L_2^2(\Omega)}^2 \\
    \leq C \left( \int_0^t \| H \|_{L_2^2(\Omega)}^2 \, d\tau + \| F \|_{H^1_p(\Omega)}^2 + \| G \|_{L_2^2(\Omega)}^2 \right), \\
\end{align*}
\]

(4.22)

where

\[
C = (C_3 e^{C_3 T} + 1).
\]

Now if we omit the first two terms from the left-hand side of (4.22) and use the fact that (see Lemma 6.3)

\[
||\nabla \theta||_{L_2^2(\Omega)}^2 \sim ||\theta||_{H^1_p(\Omega)}^2,
\]

\[
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\]

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then we get after passing to the supremum over \((0, T)\)

\[
\sup_{0 \leq t \leq T} \left\{ \|\theta\|_{H^l_1(\Omega)}^2 \right\} \leq C \left( \|H\|_{L^2(0,T;L^2_\rho(\Omega))}^2 + \|F\|_{H^l_1(\Omega)}^2 + \|G\|_{L^2_\rho(\Omega)}^2 \right).
\]  

(4.23)

Since the only information we have about the range of the operator \(A\), is that \(R(A) \subset Y\), we must extend \(A\) so that the estimate (4.23) holds for the extension and its range is the hole space \(Y\). To this end, we establish the following proposition.

**Proposition 4.2** The operator \(A : B \to Y\) admits a closure.

**Proof.** The proof can be done as in [7].

Let \(A\) be the closure of the operator \(A\), and \(D(A)\) be its domain. The inequality (4.1) can be extended to strong solutions after passing to limit, that is we have

\[
\|\theta\|_B \leq C\|A\theta\|_Y, \quad \forall \theta \in D(A),
\]

from which we deduce that \(R(A)\) is closed in \(Y\) and that \(R(A) = \overline{R(A)}\).

**Definition 4.3.** A solution of the equation

\[
A\theta = (L\theta, \ell_1\theta, \ell_2\theta) = (H, F, G)
\]

is called a strong solution of problem (3.1), (2.2)-2.4).

5. Solvability of the posed associated linear problem

**Theorem 5.1.** Problem (3.1), (2.2)-2.4), has a unique strong solution \(\theta = L^{-1}(H,F,G) = \overline{L^{-1}}(H,F,G)\), that depends continuously on the data, for all \(H \in L^2(0,T;L^2_\rho(\Omega))\), \(G \in L^2_\rho(\Omega)\) and \(F \in H^l_1(\Omega)\).

**Proof.** To prove that problem (3.1), (2.2)-2.4) has unique strong solution for all \(W = (H, F, G) \in Y\), it suffices to prove that the range \(R(A)\) of the operator \(A\) is dense in \(Y\). For this we need to prove the following proposition.

**Proposition 5.2.** If for some function \(g(x,y,t) \in L^2_\rho(Q^T)\) and for all \(k(x,y,t) \in D(A)\) satisfies homogeneous initial conditions we have

\[
(Lk, g)_{L^2_\rho(Q^T)} = 0,
\]

then \(g \equiv 0\) a.e in \(Q^T\).

**Proof.** Equation (5.1) implies

\[
(\partial_\beta^l k - \frac{1}{x}k_x - k_{xx} - \frac{1}{y}k_y - k_{yy}, g)_{L^2_\rho(Q^T)} = 0,
\]

(5.2)

Let \(P(x,y,t)\) be a function satisfying conditions (2.2)-2.4) and such that \(P, P_x, P_y, \mathcal{H}, \mathcal{H}^2 P, \mathcal{H} P_x, \mathcal{H} P_y, \partial_\beta^l P\) are all in \(L^2_\rho(Q^T)\), then we set

\[
k(x,y,t) = \mathcal{H}^2 P = \int_0^t \int_0^z P(x,y,z)dzds,
\]
and let
\[ g(x, y, t) = \mathfrak{F}_t P + \mathfrak{F}^2(x \xi \mathfrak{F}_t P), \]
then equation (5.2) becomes
\[
(\partial_t^{\beta+1}(\mathfrak{F}^2_t P) - \frac{1}{x}(\mathfrak{F}^2_t P_x) - (\mathfrak{F}^2_t P_{xx} ) - \frac{1}{y}(\mathfrak{F}^2_t P_y))
\]
\[ - (\mathfrak{F}^2_t P_{xy}), \mathfrak{F}_t P + \mathfrak{F}^2(x \xi \mathfrak{F}_t P) )_{L^2(\Omega^r)} = 0, \]
that is
\[
(\partial_t^{\beta+1}(\mathfrak{F}^2_t P), \mathfrak{F}_t P)_{L^2(\Omega)} + (\partial_t^{\beta+1}(\mathfrak{F}^2_t P), \mathfrak{F}^2(x \xi \mathfrak{F}_t P))_{L^2(\Omega)}
\]
\[ - (\mathfrak{F}^2_t P_x, \mathfrak{F}_t P)_{L^2(\Omega)} - (\mathfrak{F}^2_t P_{xx}), \mathfrak{F}^2(x \xi \mathfrak{F}_t P))_{L^2(\Omega)}
\]
\[ - ((\mathfrak{F}^2_t P_{yy}), \mathfrak{F}_t P)_{L^2(\Omega)} - ((\mathfrak{F}^2_t P_{yy}), \mathfrak{F}^2(x \xi \mathfrak{F}_t P))_{L^2(\Omega)} \]
\[ - (\mathfrak{F}^2_t P_{xy}), \mathfrak{F}_t P)_{L^2(\Omega)} - ((\mathfrak{F}^2_t P_{xy}), \mathfrak{F}^2(x \xi \mathfrak{F}_t P))_{L^2(\Omega)} = 0. \]  
(5.3)

Put in mind that the function \( P \) verifies the given boundary and initial conditions (2.2)–(2.4), then all terms in (5.3) can be computed as
\[
(\partial_t^{\beta+1}(\mathfrak{F}^2_t P), \mathfrak{F}_t P)_{L^2(\Omega)} = (\partial_t^{\beta}(\mathfrak{F}_t P), \mathfrak{F}_t P)_{L^2(\Omega)}, \]  
(5.4)

\[
(\partial_t^{\beta+1}(\mathfrak{F}^2_t P), \mathfrak{F}^2(x \xi \mathfrak{F}_t P))_{L^2(\Omega)} = (\partial_t^{\beta}(\mathfrak{F}_t P), \mathfrak{F}^2(x \xi \mathfrak{F}_t P))_{L^2(\Omega)}, \]  
(5.5)

\[
- (\frac{1}{x}(\mathfrak{F}^2_t P_x), \mathfrak{F}_t P)_{L^2(\Omega)} = -(y \mathfrak{F}^2_t P_x, \mathfrak{F}_t P)_{L^2(\Omega)}, \]  
(5.6)

\[
- (\frac{1}{y}(\mathfrak{F}^2_t P_y), \mathfrak{F}_t P)_{L^2(\Omega)} = -(x \mathfrak{F}^2_t P_y, \mathfrak{F}_t P)_{L^2(\Omega)}, \]  
(5.7)

\[
- (\mathfrak{F}^2_t P_{xx}), \mathfrak{F}_t P)_{L^2(\Omega)} = - \int_{\Omega} xy(\mathfrak{F}^2_t P_{xx})(\mathfrak{F}_t P)dxdy
\]
\[ = (y \mathfrak{F}^2_t P_x, \mathfrak{F}_t P)_{L^2(\Omega)} + \frac{1}{2} \frac{\partial}{\partial t} \left\| \mathfrak{F}^2_t P_x \right\|_{L^2(\Omega)}, \]  
(5.8)

\[
- (\mathfrak{F}^2_t P_{yy}), \mathfrak{F}_t P)_{L^2(\Omega)} = - \int_{\Omega} xy(\mathfrak{F}^2_t P_{yy})(\mathfrak{F}_t P)dxdy
\]
\[ = (x \mathfrak{F}^2_t P_y, \mathfrak{F}_t P)_{L^2(\Omega)} + \frac{1}{2} \frac{\partial}{\partial t} \left\| \mathfrak{F}^2_t P_y \right\|_{L^2(\Omega)}, \]  
(5.9)
Insertion of equations (5.4)–(5.13) into (5.3), yields

\[ -\left(\frac{1}{x}(\mathcal{I}^2_x P_x, \mathcal{I}^2_y(\xi\eta\mathcal{I}, P))\right)_{L^2(\Omega)} = -\left(\mathcal{I}^2_x P_x, \mathcal{I}^2_y(\xi\eta\mathcal{I}, P)\right)_{L^2(\Omega)}, \]  

\[ -\left(\frac{1}{y}(\mathcal{I}^2_y P_y, \mathcal{I}^2_x(\xi\eta\mathcal{I}, P))\right)_{L^2(\Omega)} = -\left(\mathcal{I}^2_y P_y, \mathcal{I}^2_x(\xi\eta\mathcal{I}, P)\right)_{L^2(\Omega)}, \]  

\[-(\mathcal{I}^2_x P_{xx}, \mathcal{I}^2_y(\xi\eta\mathcal{I}, P))_{L^2(\Omega)} = -\int_\Omega xy(\mathcal{I}^2_x P_{xx})\mathcal{I}^2_y(\xi\eta\mathcal{I}, P)dxdy = \left(\mathcal{I}^2_y P_x, \mathcal{I}^2_x(\xi\eta\mathcal{I}, P)\right)_{L^2(\Omega)}, \]  

\[-(\mathcal{I}^2_y P_{yy}, \mathcal{I}^2_x(\xi\eta\mathcal{I}, P))_{L^2(\Omega)} = \int_\Omega xy\mathcal{I}^2_y P_{yy}\mathcal{I}^2_x(\xi\eta\mathcal{I}, P)dxdy = \left(\mathcal{I}^2_y P_y, \mathcal{I}^2_x(\xi\eta\mathcal{I}, P)\right)_{L^2(\Omega)}. \]

Insertion of equations (5.4)–(5.13) into (5.3), yields

\[
2(\partial_t^0(\mathcal{I}, P, \mathcal{I}, P)_{L^2(\Omega)} + 2(\partial_t^0(\mathcal{I}_{xy}(\xi\eta\mathcal{I}, P))\mathcal{I}_{xy}(\xi\eta\mathcal{I}, P))_{L^2(\Omega)}
+ \frac{\partial}{\partial t} \|\mathcal{I}^2_x P_x\|_{L^2(\Omega)} + \frac{\partial}{\partial t} \|\mathcal{I}^2_y P_y\|_{L^2(\Omega)}
= -2\left(\mathcal{I}^2_x P_x, \mathcal{I}_{xy}(\xi\eta\mathcal{I}, P)\right)_{L^2(\Omega)}
-2\left(\mathcal{I}^2_y P_y, \mathcal{I}_{xy}(\xi\eta\mathcal{I}, P)\right)_{L^2(\Omega)}. \]  

Applying Lemma 2.3 and Poincaré type inequality to (5.14), we obtain

\[
\partial_t^0\|\mathcal{I}, P\|_{L^2(\Omega)}^2 + \partial_t^0\|\mathcal{I}_{xy}(\xi\eta\mathcal{I}, P)\|_{L^2(\Omega)}^2
+ \frac{\partial}{\partial t} \|\mathcal{I}^2_x P_x\|_{L^2(\Omega)} + \frac{\partial}{\partial t} \|\mathcal{I}^2_y P_y\|_{L^2(\Omega)}
\leq \|\mathcal{I}^2_x P_x\|_{L^2(\Omega)}^2 + \|\mathcal{I}^2_y P_y\|_{L^2(\Omega)}^2 + \frac{(ab^3 + a^3b)}{4} \|\mathcal{I}_{xy}(\xi\eta\mathcal{I}, P)\|_{L^2(\Omega)}^2, \]  

We infer from (5.15) that

\[
D_t^{0,1}\|\mathcal{I}, P\|_{L^2(\Omega)}^2 + D_t^{0,1}\|\mathcal{I}_{xy}(\xi\eta\mathcal{I}, P)\|_{L^2(\Omega)}^2
+ \|\mathcal{I}^2_x P_x\|_{L^2(\Omega)}^2 + \|\mathcal{I}^2_y P_y\|_{L^2(\Omega)}^2
\leq C\int_0^T \|\mathcal{I}_{xy}(\xi\eta\mathcal{I}, P)\|_{L^2(\Omega)}^2 d\tau + \int_0^T \|\mathcal{I}^2_x P_x\|_{L^2(\Omega)}^2 d\tau + \int_0^T \|\mathcal{I}^2_y P_y\|_{L^2(\Omega)}^2 d\tau. \]  

\[9797\]
where
\[
C = \max \left\{ 1, \frac{ab^3 + a^3b}{4} \right\}.
\]

By omitting the first two terms from the left-hand side of (5.16) and applying Lemma 2.1 by letting

\[
S(t) = \int_0^t \| \mathcal{I}_x^2 P_x \|_{L^2(\Omega)}^2 d\tau + \int_0^t \| \mathcal{I}_y^2 P_y \|_{L^2(\Omega)}^2 d\tau,
\]

\[
\frac{dS(t)}{dt} = \| \mathcal{I}_x^2 P_x \|_{L^2(\Omega)}^2 + \| \mathcal{I}_y^2 P_y \|_{L^2(\Omega)}^2,
\]

\[S(0) = 0\],

then

\[
S(t) = \int_0^t \| \mathcal{I}_x^2 P_x \|_{L^2(\Omega)}^2 d\tau + \int_0^t \| \mathcal{I}_y^2 P_y \|_{L^2(\Omega)}^2 d\tau
\]

\[\leq Te^{CT} \int_0^t \| \mathcal{I}_x^2(\xi \eta P) \|_{L^2(\Omega)}^2 d\tau,\]  

(5.17)

By inserting (5.17) into (5.16), we obtain

\[
D^\beta_{\xi}(\xi \eta P) + D^\beta_{\eta}(\xi \eta P)
\]

\[\leq C(1 + Te^{CT}) \int_0^t \| \mathcal{I}_x^2(\xi \eta P) \|_{L^2(\Omega)}^2 d\tau,\]  

(5.18)

if we drop the last three terms on the left hand side of (5.18) and apply Lemma 2.2 by taking

\[
z(t) = \int_0^t \| \mathcal{I}_x^2(\xi \eta P) \|_{L^2(Q_T)}^2 d\tau,
\]

\[
D^\beta_{\xi} z(t) = D^\beta_{\xi}(\xi \eta P)
\]

\[\leq 0,\]  

(5.19)

Consequently, inequality (5.19) implies that \( g \) is zero a.e in \( Q_T \).
To complete the proof of Theorem 5.1, we suppose that for some element $W = (H, F, G) \in R(A)^ot$, we have
\[
(L\theta, H)_{L^2(0,T;L^2(\Omega))} + (\ell_1\theta, F)_{H^1(\Omega)} + (\ell_2\theta, G)_{L^2(\Omega)} = 0.
\]
We must prove that $W = 0$. If we put $\theta \in D(A)$ satisfying homogeneous conditions into (5.20), we have
\[
(L\theta, H)_{L^2(0,T;L^2(\Omega))} = 0.
\]
Applying Proposition 5.2 to (5.21), it follows from that $H = 0$. Thus (5.20) takes the form
\[
(\ell_1\theta, F)_{H^1(\Omega)} + (\ell_2\theta, G)_{L^2(\Omega)} = 0.
\]
But since the range of the operators $\ell_1, \ell_2$ are dense in the spaces $H^1(\Omega), L^2(\Omega)$ respectively then relation (5.22) implies $G = F = 0$. Consequently $W = 0$ and Theorem 5.1 follows.

6. The nonlinear problem

This section is consecrated to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the problem (2.1)–(2.4). Let us consider the following auxiliary problem with homogeneous equation
\[
L(U) = \partial_t^{\alpha+1} U - \frac{1}{xy} \text{div}(xy\nabla U) = 0,
\]
\[
(\ell_1 U = U(x, y, 0) = F(x, y), \quad (\ell_2 U = U_t(x, y, 0) = G(x, y),
\]
\[
U_x(a, y, t) = 0, \quad U_y(x, b, t) = 0,
\]
\[
\int_0^a xU(x, y, t)dx = 0, \quad \int_0^b yU(x, y, t)dx = 0.
\]
If $\theta$ is a solution of problem (2.1)-(2.4) and $U$ is a solution of problem (6.1)-(6.4), then $w = \theta - U$ satisfies
\[
\partial_t^{\alpha+1} w - \frac{1}{xy} \text{div}(xy\nabla w) = \tilde{H}(x, y, t, w, w_x, w_y),
\]
\[
w(x, y, 0) = 0, \quad w_t(x, y, 0) = 0,
\]
\[
w_x(a, y, t) = 0, \quad w_y(x, b, t) = 0,
\]
\[
\int_0^a xw(x, y, t)dx = 0, \quad \int_0^b yw(x, y, t)dx = 0.
\]
where

\[ \tilde{H}(x, y, t, w, w_x, w_y) = H(x, y, t, \theta - U, (\theta - U)_x, (\theta - U)_y). \]

The function \( \tilde{H} \) satisfies the Lipschitz condition

\[
\begin{align*}
\tilde{H}(x, y, t, w_1, w_2, w_3) - \tilde{H}(x, y, t, v_1, v_2, v_3) & \leq \lambda (|w_1 - v_1| + |w_2 - v_2| + |w_3 - v_3|), \\
& \quad (6.9)
\end{align*}
\]

for all \((x, y, t) \in Q^T = (0, a) \times (0, b) \times (0, T)\). According to Theorem (5.1) problem (6.1)–(6.4) has a unique solution depending continuously on \( F \in H^1_p(\Omega) \) and \( G \in L^2_2(\Omega) \). It remains to solve problem (6.5)–(6.8). We shall prove that problem (6.5)-(6.8) has a unique weak solution. Suppose that \( \nu \) and \( w \) belong to \( C^1(Q^T) \) such that \( \nu(x, T) = 0, w(x, y, 0) = 0, w_t(x, y, 0) = 0, \int_0^a xw dx = 0, \int_0^b yw dx = 0, \) For \( \nu \in C^1(Q^T) \), we have

\[
(\mathcal{L} \nu, \mathcal{G}_{xy}(\xi \eta \nu))_{L^2(0,T,L^2_2(\Omega))} = \left( \partial^2_t w_t, \mathcal{G}_{xy}(\xi \eta \nu) \right)_{L^2(0,T,L^2_2(\Omega))} - \int_0^1 \left( -w_x, \mathcal{G}_{xy}(\xi \eta \nu) \right)_{L^2(0,T,L^2_2(\Omega))} - \int_0^1 \left( -w_y, \mathcal{G}_{xy}(\xi \eta \nu) \right)_{L^2(0,T,L^2_2(\Omega))} = (\tilde{H}, \mathcal{G}_{xy}(\xi \eta \nu))_{L^2(0,T,L^2_2(\Omega))},
\]

(6.10)

By using conditions on \( w \) and \( \nu \), a quick computation of each term on the right and left-hand side of (6.10), gives

\[
\left( \partial^2_t w_t, \mathcal{G}_{xy}(\xi \eta \nu) \right)_{L^2(0,T,L^2_2(\Omega))} = (\nu, \partial^2_t (\mathcal{G}_{xy}(\xi \eta \nu)))_{L^2(0,T,L^2_2(\Omega))},
\]

(6.11)

\[
-1 \left( -w_x, \mathcal{G}_{xy}(\xi \eta \nu) \right)_{L^2(0,T,L^2_2(\Omega))} = (w, \mathcal{G}_y(\eta \nu))_{L^2(0,T,L^2_2(\Omega))},
\]

(6.12)

\[
-(w_y, \mathcal{G}_{xy}(\xi \eta \nu))_{L^2(0,T,L^2_2(\Omega))} = - (w, \mathcal{G}_y(\eta \nu))_{L^2(0,T,L^2_2(\Omega))} + (xw_x, \mathcal{G}_y(\eta \nu))_{L^2(0,T,L^2_2(\Omega))},
\]

(6.13)

\[
-(w_y, \mathcal{G}_{xy}(\xi \eta \nu))_{L^2(0,T,L^2_2(\Omega))} = (yw_y, \mathcal{G}_x(\xi \nu))_{L^2(0,T,L^2_2(\Omega))} - (w, \mathcal{G}_x(\xi \nu))_{L^2(0,T,L^2_2(\Omega))},
\]

(6.14)

\[
- \left( -w_y, \mathcal{G}_{xy}(\xi \eta \nu) \right)_{L^2(0,T,L^2_2(\Omega))} = (w, \mathcal{G}_x(\xi \nu))_{L^2(0,T,L^2_2(\Omega))},
\]

(6.15)

\[
(\tilde{H}, \mathcal{G}_{xy}(\xi \eta \nu))_{L^2(0,T,L^2_2(\Omega))} = (\mathcal{G}_{xy}(\xi \eta \tilde{H}), \nu)_{L^2(0,T,L^2_2(\Omega))},
\]

(6.16)

Insertion of (5.11)-(5.16) into (5.10) yields

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\[ M(w, v) = (\mathcal{S}_{xy}(\xi \eta \tilde{H}), v)_{L^2(0,T;L^2_2(\Omega))}, \]  
\[ \text{where} \]
\[ M(w, v) = (v, \partial_t^2 \mathcal{S}_{xy}(\xi \eta w))_{L^2(0,T;L^2_2(\Omega))} + (xw_x, \mathcal{S}_y(\eta v))_{L^2(0,T;L^2_2(\Omega))} + (yw_y, \mathcal{S}_x(\xi v))_{L^2(0,T;L^2_2(\Omega))}. \]

**Definition 6.1.** A function \( w \in L^2(0, T; \mathcal{H}_0^1(\Omega)) \) is called weak solution of problem (6.5)-(6.8) if (6.7) and (6.17) hold.

Our main purpose is to construct an iteration sequence \((w^n)_{n \in \mathbb{N}}\) which converges to a certain function \( w \in L^2(0, T; \mathcal{H}_0^1(\Omega)) \) which solves problem (6.5)–(6.8). Starting with \( w^{(0)} = 0 \), the sequence \((w^n)_{n \in \mathbb{N}}\) is defined as follows: given the element \( w^{(n-1)} \), then for \( n = 1, 2, \ldots \) we solve the iterated problems:

\[ \partial_t^2 w_{i}^{(n)} - \frac{1}{x}w_{x}^{(n)} - \frac{1}{y}w_{y}^{(n)} - w_{yy}^{(n)} = \tilde{H}(x, y, t, w^{(n)}_{x}, w_{x}^{(n)}, w_{y}^{(n)}), \]  
\[ w^{(n)}(x, y, 0) = 0, \quad w_{y}^{(n)}(x, y, 0) = 0, \]  
\[ w_{x}^{(n)}(a, y, t) = 0, \quad w_{y}^{(n)}(x, b, t) = 0, \]  
\[ \int_{0}^{a} x w^{(n)}(x, y, t) dx = 0, \quad \int_{0}^{b} y w^{(n)}(x, y, t) dx = 0, \]  
(6.19)  
(6.20)  
(6.21)  
(6.22)

Theorem (5.1) asserts that for fixed \( n \), each problem (6.19)-(6.22) has unique solution \( w^{(n)}(x, y, t) \). If we set \( \mathcal{Z}^{(n)}(x, y, t) = w_{i}^{(n+1)}(x, y, t) - w_{i}^{(n)}(x, y, t) \), then we have the new problem

\[ \partial_t^2 \mathcal{Z}^{(n)} - \frac{1}{x} \mathcal{Z}_{x}^{(n)} - \mathcal{Z}_{xx}^{(n)} - \frac{1}{y} \mathcal{Z}_{y}^{(n)} - \mathcal{Z}_{yy}^{(n)} = \sigma^{(n-1)}(x, y, t), \]  
\[ \mathcal{Z}^{(n)}(x, y, 0) = 0, \quad \mathcal{Z}_{y}^{(n)}(x, y, 0) = 0, \]  
\[ \mathcal{Z}_{x}^{(n)}(a, y, t) = 0, \quad \mathcal{Z}_{y}^{(n)}(x, b, t) = 0, \]  
\[ \int_{0}^{a} x \mathcal{Z}^{(n)}(x, y, t) dx = 0, \quad \int_{0}^{b} y \mathcal{Z}^{(n)}(x, y, t) dx = 0, \]  
(6.23)  
(6.24)  
(6.25)  
(6.26)

where

\[ \sigma^{(n-1)}(x, y, t) = \tilde{H}(x, y, t, w^{(n)}_{x}, w_{x}^{(n)}, w_{y}^{(n)}) - \tilde{H}(x, y, t, w^{(n-1)}_{x}, w_{x}^{(n-1)}, w_{y}^{(n-1)}). \]

**Theorem 6.2.** Assume that condition (6.9) holds, then for the linearized problem (6.23)–(6.26), we have the a priori estimate

\[ \| \mathcal{Z}^{(n)} \|_{L^2(0,T;\mathcal{H}_0^1(\Omega))} \leq K \| \mathcal{Z}^{(n-1)} \|_{L^2(0,T;\mathcal{H}_0^1(\Omega))}, \]  
(6.27)
where $K$ is positive constant given by

$$K = TC_4 \left( \frac{a^2 + b^2}{8} + 1 \right).$$

**Proof.** Taking the inner product in $L^2(0, \tau; L^2_\rho(\Omega))$, with $0 \leq \tau \leq T$ of equation (6.23) and integro-differential operator

$$\mathcal{M}Z = Z_t^{(n)} - \mathfrak{S}_{xy}^2(\xi \eta Z_t^{(n)}),$$

we have

$$\left( \frac{\partial^2}{\partial t^2} Z_t^{(n)}, Z_t^{(n)} \right)_{L^2(0, \tau; L^2_\rho(\Omega))} - \left( \frac{1}{x} \mathcal{Z}_x^{(n)}, Z_t^{(n)} \right)_{L^2(0, \tau; L^2_\rho(\Omega))} - \left( \mathcal{Z}_x^{(n)}, \frac{\partial}{\partial t} Z_t^{(n)} \right)_{L^2(0, \tau; L^2_\rho(\Omega))} - \left( \mathcal{Z}_y^{(n)}, \frac{\partial}{\partial t} Z_t^{(n)} \right)_{L^2(0, \tau; L^2_\rho(\Omega))}$$

$$+ \frac{1}{x} \left( \mathcal{Z}_x^{(n)}, \mathfrak{S}_{xy}^2(\xi \eta Z_t^{(n)}) \right)_{L^2(0, \tau; L^2_\rho(\Omega))} + \left( \mathcal{Z}_x^{(n)}, \mathfrak{S}_{xy}^2(\xi \eta Z_t^{(n)}) \right)_{L^2(0, \tau; L^2_\rho(\Omega))}$$

$$+ \left( \mathcal{Z}_y^{(n)}, \mathfrak{S}_{xy}^2(\xi \eta Z_t^{(n)}) \right)_{L^2(0, \tau; L^2_\rho(\Omega))} + \left( \mathcal{Z}_y^{(n)}, \mathfrak{S}_{xy}^2(\xi \eta Z_t^{(n)}) \right)_{L^2(0, \tau; L^2_\rho(\Omega))}
$$

$$= (\sigma^{(n-1)}, Z_t^{(n)})_{L^2(0, \tau; L^2_\rho(\Omega))} - (\sigma^{(n-1)}, \mathfrak{S}_{xy}^2(\xi \eta Z_t^{(n)}))_{L^2(0, \tau; L^2_\rho(\Omega))}.
$$

(6.28)

In the light of conditions (6.25) and (6.26), Cauchy’s inequality, Lemma 2.2, Lemma 2.3 and successive integrations by parts of each term of (6.28) leads to

$$\left( \frac{\partial^2}{\partial t^2} Z_t^{(n)}, Z_t^{(n)} \right)_{L^2(0, \tau; L^2_\rho(\Omega))} \geq \frac{1}{2} \int_0^\tau \frac{\partial}{\partial t} \left\| Z_t^{(n)} \right\|^2_{L^2_\rho(\Omega)} dt,$$

(6.29)

$$= \frac{1}{2} D_{\tau}^{-1} \left\| Z_t^{(n)}(x, y, t) \right\|^2_{L^2_\rho(\Omega)},$$

(6.30)

$$- \left( \frac{1}{x} \mathcal{Z}_x^{(n)}, Z_t^{(n)} \right)_{L^2(0, \tau; L^2_\rho(\Omega))} = - \left( y \mathcal{Z}_x^{(n)}, Z_t^{(n)} \right)_{L^2(0, \tau; L^2_\rho(\Omega))},$$

(6.31)

$$- \left( \mathcal{Z}_y^{(n)}, \frac{\partial}{\partial t} Z_t^{(n)} \right)_{L^2(0, \tau; L^2_\rho(\Omega))} = - \left( \frac{\partial}{\partial t} \frac{\partial}{\partial t} Z_t^{(n)}, \mathfrak{S}_{xy}^2(\xi \eta Z_t^{(n)}) \right)_{L^2(0, \tau; L^2_\rho(\Omega))},$$

(6.32)

$$+ \frac{1}{2} \left\| Z_t^{(n)}(x, y, t) \right\|^2_{L^2_\rho(\Omega)}.$$

(6.33)
\[ - \left( \partial_t^q \mathbf{Z}_t^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} \]
\[ \geq \frac{1}{2} \int_0^r \partial_t^q \| \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \|^2_{L^2(\Omega)} \, dt \]
\[ = \frac{1}{2} D_r^{q-1} \| \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}(x, y, t)) \|^2_{L^2(\Omega)}, \tag{6.34} \]
\[ \left( \frac{1}{x} \mathbf{Z}_x^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} = \left( y \mathbf{Z}_x^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))}, \tag{6.35} \]
\[ \left( \frac{1}{y} \mathbf{Z}_y^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} = \left( x \mathbf{Z}_y^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))}, \tag{6.36} \]
\[ \left( \mathbf{Z}_{xx}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} = - \left( y \mathbf{Z}_x^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} \]
\[ - \left( \mathbf{Z}_x^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))}, \tag{6.37} \]
\[ \left( \mathbf{Z}_{yy}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} = - \left( x \mathbf{Z}_y^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} \]
\[ - \left( \mathbf{Z}_y^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))}, \tag{6.38} \]

Combination of (6.28)–(6.38) yields
\[ D_r^{q-1} \| \mathbf{Z}_t^{(n)}(x, y, t) \|^2_{L^2(\Omega)} + D_r^{q-1} \| \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}(x, y, t)) \|^2_{L^2(\Omega)} \]
\[ + \| \mathbf{Z}_x^{(n)}(x, y, t) \|^2_{L^2(\Omega)} + \| \mathbf{Z}_y^{(n)}(x, y, t) \|^2_{L^2(\Omega)} = 2 \left( \mathbf{Z}_x^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} + 2 \left( \mathbf{Z}_y^{(n)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \right)_{L^2(0,r \mathbf{L}_d^2(\Omega))} \]
\[ + 2(\sigma^{(n-1)}, \mathbf{Z}_t^{(n)})_{L^2(0,r \mathbf{L}_d^2(\Omega))} + 2(\sigma^{(n-1)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)})_{L^2(0,r \mathbf{L}_d^2(\Omega))} \tag{6.39} \]

Estimation of the right-hand side of (6.39) gives
\[ 2(\sigma^{(n-1)}, \mathbf{Z}_t^{(n)})_{L^2(0,r \mathbf{L}_d^2(\Omega))} \]
\[ \leq \| \mathbf{Z}_t^{(n)} \|^2_{L^2(0,r \mathbf{L}_d^2(\Omega))} + \| \sigma^{(n-1)} \|^2_{L^2(0,r \mathbf{L}_d^2(\Omega))} \]
\[ \leq \| \mathbf{Z}_t^{(n)} \|^2_{L^2(0,r \mathbf{L}_d^2(\Omega))} + \frac{3}{2} \lambda^2 \left( \| \mathbf{Z}^{(n-1)} \|^2_{L^2(0,r \mathbf{L}_d^2(\Omega))} \right) \]
\[ + \| \mathbf{Z}_x^{(n-1)} \|^2_{L^2(0,r \mathbf{L}_d^2(\Omega))} + \| \mathbf{Z}_y^{(n-1)} \|^2_{L^2(0,r \mathbf{L}_d^2(\Omega))} \tag{6.40} \]

\[ 2(\sigma^{(n-1)}, \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)})_{L^2(0,r \mathbf{L}_d^2(\Omega))} \]
\[ \leq \frac{a^2 b^2}{4} \| \mathbf{G}_{xy}(\xi \eta \mathbf{Z}_t^{(n)}) \|^2_{L^2(0,r \mathbf{L}_d^2(\Omega))} + \frac{3}{2} \lambda^2 \left( \| \mathbf{Z}^{(n-1)} \|^2_{L^2(0,r \mathbf{L}_d^2(\Omega))} \right) \]
Now by discarding the first two terms from left hand side of (6.44) and using Lemma 2.1 with

\[ \|Z^{(n)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n)}_y\|_{L^2(0,r,\xi^2)} \],

(6.41)

\[ 2(\mathcal{L}_t^{(n)}, \mathcal{S}_x(\xi \eta Z_t^{(n)}))_{L^2(0,r,\xi^2)} \]

\[ \leq \|Z^{(n)}_x\|_{L^2(0,r,\xi^2)} + b^2 \|\mathcal{S}_x(\xi \eta Z_t^{(n)})\|_{L^2(0,r,\xi^2)}, \]

(6.42)

\[ 2(\mathcal{L}_t^{(n)}, \mathcal{S}_y(\xi \eta Z_t^{(n)}))_{L^2(0,r,\xi^2)} \]

\[ \leq \|Z^{(n)}_y\|_{L^2(0,r,\xi^2)} + b^2 \|\mathcal{S}_y(\xi \eta Z_t^{(n)})\|_{L^2(0,r,\xi^2)}. \]

(6.43)

Upon substitution of (6.40)–(6.43) into (6.39), we obtain

\[ D^\alpha \|Z^{(n)}(x,y,t)\|_{L^2(\xi^2)}^2 + D^\alpha \|\mathcal{S}_x(\xi \eta Z_t^{(n)})(x,y,t)\|_{L^2(\xi^2)}^2 \]

\[ + \|Z^{(n)}_y(x,y,\tau)\|_{L^2(\xi^2)}^2 + \|Z^{(n)}_x(x,y,\tau)\|_{L^2(\xi^2)}^2, \]

\[ \leq C_1 \left( \|Z^{(n)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n)}_y\|_{L^2(0,r,\xi^2)} \right) \]

\[ + \|Z^{(n-1)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n-1)}_y\|_{L^2(0,r,\xi^2)} \]

\[ + \|Z^{(n)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n)}_y\|_{L^2(0,r,\xi^2)} \]

\[ + \|\mathcal{S}_x(\xi \eta Z_t^{(n)})\|_{L^2(0,r,\xi^2)} \],

(6.44)

where

\[ C_1 = \max\{1, \frac{3}{2} \alpha^2, \frac{3}{4}, \alpha^2, b^2\}. \]

Now by discarding the first two terms from left hand side of (6.44) and using Lemma 2.1 with

\[ \mathcal{P}(\tau) = \|Z^{(n)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n)}_y\|_{L^2(0,r,\xi^2)}, \quad \mathcal{P}(0) = 0, \]

we obtain

\[ \|Z^{(n)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n)}_y\|_{L^2(0,r,\xi^2)} \]

\[ \leq C_2 \left( \|Z^{(n-1)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n-1)}_y\|_{L^2(0,r,\xi^2)} \right) \]

\[ + \|Z^{(n-1)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n)}_x\|_{L^2(0,r,\xi^2)} \]

\[ + \|\mathcal{S}_x(\xi \eta Z_t^{(n)})\|_{L^2(0,r,\xi^2)} \],

(6.45)

where \( C_2 = C_1 T e^{C_1 T} \). Hence, inequality (6.44) becomes

\[ D^\alpha \|Z^{(n)}_x(x,y,t)\|_{L^2(\xi^2)}^2 + D^\alpha \|\mathcal{S}_x(\xi \eta Z_t^{(n)})(x,y,t)\|_{L^2(\xi^2)}^2 \]

\[ + \|Z^{(n)}_x(x,y,\tau)\|_{L^2(\xi^2)}^2 + \|Z^{(n)}_y(x,y,\tau)\|_{L^2(\xi^2)}^2, \]

\[ \leq C_4 \left( \|Z^{(n-1)}_x\|_{L^2(0,r,\xi^2)} + \|Z^{(n-1)}_y\|_{L^2(0,r,\xi^2)} \right) \]
where \( C_3 = C_1(1 + C_2) \).

We now need to eliminate the last two terms on the right-hand side of (6.46) by using Lemma 2.2 and setting

\[
\mathcal{K}(\tau) = \|Z^{(n)}_y\|_{L^2(0, r; L^2_\beta(\Omega))}^2 + \|\mathcal{S}_{xy}(\xi \eta Z^{(n)}_x)\|_{L^2(0, r; L^2_\beta(\Omega))}^2,
\]

\[
\partial^\beta_\tau \mathcal{K}(\tau) = D^\beta_\tau \|Z^{(n)}_x(x, y, \tau)\|_{L^2_\beta(\Omega)}^2 + D^\beta_\tau \|\mathcal{S}_{xy}(\xi \eta Z^{(n)}_x)(x, y, \tau)\|_{L^2_\beta(\Omega)}^2
\]

\[
\mathcal{K}(0) = 0,
\]

then we have

\[
\|Z^{(n)}_y\|_{L^2(0, r; L^2_\beta(\Omega))}^2 + \|\mathcal{S}_{xy}(\xi \eta Z^{(n)}_x)\|_{L^2(0, r; L^2_\beta(\Omega))}^2 \\
\leq \mathcal{K}(0) E_\beta(C_3 r^\delta) + \Gamma(\beta) E_\beta(C_3 r^\delta) D^\beta_\tau \left( \|Z^{(n-1)}_y\|_{L^2_\beta(\Omega)}^2 \\
+ \|Z^{(n-1)}_x\|_{L^2(0, r; L^2_\beta(\Omega))}^2 + \|Z^{(n-1)}_y\|_{L^2(0, r; L^2_\beta(\Omega))}^2 \right). \tag{6.47}
\]

Inequality (6.47) implies

\[
\|Z^{(n)}_y\|_{L^2(0, r; L^2_\beta(\Omega))}^2 + \|\mathcal{S}_{xy}(\xi \eta Z^{(n)}_x)\|_{L^2(0, r; L^2_\beta(\Omega))}^2 \\
\leq \Gamma(\beta) E_\beta(C_3 r^\delta) \left( D^\beta_\tau \|Z^{(n-1)}_y\|_{L^2_\beta(\Omega)}^2 \\
+ D^\beta_\tau \|Z^{(n-1)}_x\|_{L^2(0, r; L^2_\beta(\Omega))}^2 + D^\beta_\tau \|Z^{(n-1)}_y\|_{L^2(0, r; L^2_\beta(\Omega))}^2 \right). \tag{6.48}
\]

It is obvious that

\[
D^\beta_\tau \|Z^{(n-1)}_y\|_{L^2_\beta(\Omega)}^2 \leq \frac{T^\frac{\delta}{\Gamma(\beta + 1)}}{\Gamma(\beta + 1)} \int_0^\tau \|Z^{(n-1)}_y\|_{L^2_\beta(\Omega)}^2 d\tau, \tag{6.49}
\]

If we discard the first two terms on the left-hand side of (6.46) and combine with (6.48) and (6.49), we obtain

\[
\|\nabla Z^{(n)}(x, y, \tau)\|_{L^2_\beta(\Omega)}^2 \leq C_4 \|Z^{(n-1)}_y\|_{L^2(0, r; L^2_\beta(\Omega))}^2 \tag{6.50}
\]

where

\[
C_4 = C_1 \left( 1 + \Gamma(\beta) E_\beta(C_3 T^\delta) \frac{T^\frac{\delta}{\Gamma(\beta + 1)}}{\Gamma(\beta + 1)} \right). \tag{6.51}
\]

**Lemma 6.3** For \( \theta \in H^1_\beta(\Omega) \) satisfying \( \int_0^a x \theta dx = 0, \int_0^b y \theta dy = 0 \), we have

\[
\|\theta\|_{L^2_\beta(\Omega)}^2 \leq \frac{a^2 + b^2}{8} \|\nabla \theta\|_{L^2_\beta(\Omega)}^2 \tag{6.52}
\]

Now by using the above lemma and equivalence of norms

\[
\|\nabla \theta\|_{L^2_\beta(\Omega)} \sim \|\theta\|_{H^1_\beta(\Omega)} \tag{6.53}
\]
which comes from

\[ \| \nabla \theta \|_{L^2_\mu(\Omega)}^2 \leq \| \theta \|_{H^1_\mu(\Omega)}^2 \leq \left( \frac{a^2 + b^2}{8} + 1 \right) \| \nabla \theta \|_{L^2_\mu(\Omega)}^2, \]

we infer from (6.50) and (6.53) that

\[ \| \mathcal{Z}^{(m)} \|_{L^2(0,T;H^1_\mu(\Omega))} \leq K \| \mathcal{Z}^{(m-1)} \|_{L^2(0,T;H^1_\mu(\Omega))}, \quad (6.54) \]

where

\[ K = TC_4 \left( \frac{a^2 + b^2}{8} + 1 \right). \quad (6.55) \]

From the criteria of convergence of series, we see from (6.54) that the series \( \sum_{n=0}^\infty \mathcal{Z}^{(n)} \) converges if \( K < 1 \). Since \( \mathcal{Z}^{(n)}(x,y,t) = w^{(n+1)}(x,y,t) - w^{(n)}(x,y,t) \), then it follows that the sequence \( (w^{(n)})_{n \in \mathbb{N}} \) defined by

\[
\begin{align*}
  w^{(n)}(x,y,t) &= \sum_{k=0}^{n-1} (w^{(n+1)}(x,y,t) - w^{(k)}(x,y,t)) + w^{(0)}(x,y,t), \\
  &= \sum_{k=0}^{n-1} \mathcal{Z}^{(k)}(x,y,t), \quad n = 1, 2, 3, \ldots \quad (6.56)
\end{align*}
\]

converges to \( w \in L^2(0,T;H^1_\mu(\Omega)) \). Now to prove that this limit function \( w \) is a solution of problem under consideration (6.5)–(6.8), we should show that \( w \) satisfies (6.7) and (6.17) as mentioned in Definition 6.1. For problem (6.19)–(6.22), we have

\[ M(w^{(n)}, v) = \left( v, \mathcal{J}_{xy}(\xi, \eta \bar{\mathcal{H}}(x,y,t, w^{(n-1)}), \frac{\partial w^{(n-1)}}{\partial \xi}, \frac{\partial w^{(n-1)}}{\partial \eta} \right)_{L^2(0,T;L^2_\mu(\Omega))}. \quad (6.57) \]

From (6.57), we have

\[
\begin{align*}
  M(w^{(n)} - w, v) + M(w, v) &= (v, \mathcal{J}_{xy}(\xi, \eta \bar{\mathcal{H}}(x,y,t, w^{(n-1)}), \frac{\partial w^{(n-1)}}{\partial \xi}, \frac{\partial w^{(n-1)}}{\partial \eta}))_{L^2(0,T;L^2_\mu(\Omega))} \\
  &= (v, \mathcal{J}_{xy}(\xi, \eta \bar{\mathcal{H}}(x,y,t, w^{(n-1)}), \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}))_{L^2(0,T;L^2_\mu(\Omega))} \\
  &\quad - \mathcal{J}_{xy}(\xi, \eta \bar{\mathcal{H}}(x,y,t, w^{(n)}), \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta})_{L^2(0,T;L^2_\mu(\Omega))} \\
  &\quad + (v, \mathcal{J}_{xy}(\xi, \eta \bar{\mathcal{H}}(x,y,t, w^{(n)}), \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}))_{L^2(0,T;L^2_\mu(\Omega))}, \quad (6.58)
\end{align*}
\]

From the partial differential equation (6.19), we have

\[
\begin{align*}
  \left( v, \partial_{t}^{n+1} \mathcal{J}_{xy}(\xi, \eta \bar{\mathcal{H}}(w^{(n)} - w)) \right)_{L^2(0,T;L^2_\mu(\Omega))} \\
  - (v, \mathcal{J}_{xy}(w^{(n)}_\xi - w_\xi))_{L^2(0,T;L^2_\mu(\Omega))} \\
  - (v, \mathcal{J}_{xy}(w^{(n)}_\eta - w_\eta))_{L^2(0,T;L^2_\mu(\Omega))} \\
  - (v, \mathcal{J}_{xy}(w^{(n)}_{\xi\xi} - w_{\xi\xi}))_{L^2(0,T;L^2_\mu(\Omega))}
\end{align*}
\]
Inner products in (6.59) can be evaluated by using conditions on functions \( v \) and \( w \), and this leads to

\[
- \left( v, \mathcal{J}_y \xi \eta \left( w^{(n)}_{xy} - w_{x \eta} \right) \right)_{L^2(0,T;L^2_p(\Omega))} = M(w^{(n)} - w, v). \tag{6.59}
\]

We now estimate terms on the left-hand side of (6.60) to see that

\[
\begin{align*}
(\mathcal{J}_x (\xi \eta \partial_T^{\frac{1}{2}} v), (w^{(n)} - w))_{L^2(0,T;L^2_p(\Omega))} & \leq ||w^{(n)} - w||_{L^2(0,T;L^2_p(\Omega))} \times ||\mathcal{J}_x (\xi \eta \partial_T^{\frac{1}{2}} v)||_{L^2(0,T;L^2_p(\Omega))} \\
& \leq \frac{a^2 b^2}{4} ||w^{(n)} - w||_{L^2(0,T;L^2_p(\Omega))} \times ||\partial_T^{\frac{1}{2}} v||_{L^2(0,T;L^2_p(\Omega))}, \tag{6.61}
\end{align*}
\]

\[
\begin{align*}
(\mathcal{J}_y \eta v, \mathcal{J}_x (w^{(n)}_{xy} - w_{x \eta}))_{L^2(0,T;L^2_p(\Omega))} & \leq ||\mathcal{J}_x (w^{(n)}_{xy} - w_{x \eta})||_{L^2(0,T;L^2_p(\Omega))} \times ||\mathcal{J}_y \eta v||_{L^2(0,T;L^2_p(\Omega))} \\
& \leq \frac{a^2 b^2}{4} ||v||_{L^2(0,T;L^2_p(\Omega))} \times ||w^{(n)} - w||_{L^2(0,T;H^1_0(\Omega))}, \tag{6.62}
\end{align*}
\]

\[
\begin{align*}
(\mathcal{J}_x \xi v, \mathcal{J}_y (w^{(n)}_{x \eta} - w_{x \eta}))_{L^2(0,T;L^2_p(\Omega))} & \leq ||\mathcal{J}_y (w^{(n)}_{x \eta} - w_{x \eta})||_{L^2(0,T;L^2_p(\Omega))} \times ||\mathcal{J}_x \xi v||_{L^2(0,T;L^2_p(\Omega))} \\
& \leq \frac{a^2 b^2}{4} ||v||_{L^2(0,T;L^2_p(\Omega))} \times ||w^{(n)} - w||_{L^2(0,T;H^1_0(\Omega))}, \tag{6.63}
\end{align*}
\]

\[
\begin{align*}
(\mathcal{J}_y (\eta v), \mathcal{x}(w^{(n)} - w_x))_{L^2(0,T;L^2_p(\Omega))} & \leq a ||\mathcal{J}_y (\eta v)||_{L^2(0,T;L^2_p(\Omega))} \times ||w^{(n)} - w_x||_{L^2(0,T;L^2_p(\Omega))}
\end{align*}
\]
If we combine equality (6.60) and inequalities (6.61)–(6.67), we obtain
\[
\begin{align*}
    M(w^{(n)} - w, v) & \leq C_s \|v\|_{L^2(0,T;L^2(\Omega))} \left( \|w^{(n)}\|_{L^2(0,T;H^1(\Omega))} + \|\delta^{\mu+1}_t v\|_{L^2(0,T;L^2(\Omega))} \|w^{(n)}\|_{L^2(0,T;L^2(\Omega))} \right) \\
    & \leq \frac{a^2 b^2}{2} \|v\|_{L^2(0,T;L^2(\Omega))} \left( \|w^{(n)}\|_{L^2(0,T;H^1(\Omega))} + \|\partial_{xx} \|_{L^2(0,T;L^2(\Omega))} \|w^{(n)}\|_{L^2(0,T;L^2(\Omega))} \right) \\
    & \leq \frac{a^2 b^2}{2} \|v\|_{L^2(0,T;L^2(\Omega))} \left( \|w^{(n)}\|_{L^2(0,T;H^1(\Omega))} + \|\partial_{xx} \|_{L^2(0,T;L^2(\Omega))} \|w^{(n)}\|_{L^2(0,T;L^2(\Omega))} \right),
\end{align*}
\]

where
\[
C_s = \max \left\{ \frac{a^2 b^2}{4}, \frac{a^2 b}{2}, \frac{a^2}{2}, \frac{b^2}{2} \right\}.
\]

On the other side we have
\[
\begin{align*}
    & \left( v, \mathcal{J}_y \xi \eta \tilde{H} (\xi, \eta, t, w^{(n-1)}, \frac{\partial w^{(n-1)}}{\partial \xi}, \frac{\partial w^{(n-1)}}{\partial \eta}) \right) - \left( \mathcal{J}_y \xi \eta \tilde{H} (\xi, \eta, t, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}) \right)_{L^2(0,T;L^2(\Omega))} \\
    & \leq \frac{(ab)^2 \lambda}{8} \|v\|_{L^2(0,T;L^2(\Omega))} \left( \|w^{(n-1)}\|_{L^2(0,T;H^1(\Omega))} \right).
\end{align*}
\]

Taking into account (6.68) and (6.69) and passing to the limit in (6.58) as \( n \to \infty \) to obtain
\[
M(w, v) = \left( v, \mathcal{J}_y \xi \eta \tilde{H} (\xi, \eta, t, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}) \right)_{L^2(0,T;L^2(\Omega))}.
\]
Hence (6.17) holds. Now to conclude that problem (6.5)-(6.8) has weak solution, we show that (6.7) holds. Since \( w \in L^2(0, T; H^1_\rho(\Omega)) \), then \( \int_0^t \frac{\partial w}{\partial x}(x, y, s)ds \), \( \int_0^t \frac{\partial w}{\partial y}(x, y, s)ds \in C(Q^T) \) and we conclude that \( \frac{\partial w}{\partial x}(a, y, t) = 0, \frac{\partial w}{\partial y}(x, b, t) = 0 \), a.e.

Thus we have proved the following

**Theorem 6.4.** Suppose that condition (6.9) holds and that \( K < 1 \), then problem (6.5)–(6.8) has a weak solution belonging to \( L^2(0, T; H^1_\rho(\Omega)) \).

It remains to prove that problem (6.5)-(6.8) admits a unique solution.

**Theorem 6.5** Assume that condition (6.9) holds, then problem (6.5)-(6.8) admits a unique solution.

**Proof.** Suppose that \( w_1, w_2 \in L^2(0, T; H^1_\rho(\Omega)) \) are two solutions of (6.5)-(6.8), then \( V = w_1 - w_2 \in L^2(0, T; H^1_\rho(\Omega)) \) and satisfies

\[
\sigma(x, y, t) = \tilde{H} \left( x, y, t, w_1, \frac{\partial w_1}{\partial x}, \frac{\partial w_1}{\partial y} \right) - \tilde{H} \left( x, y, t, w_2, \frac{\partial w_2}{\partial x}, \frac{\partial w_2}{\partial y} \right). \tag{6.75}
\]

Taking the inner product in \( L^2(0, T; L^2_\rho(\Omega)) \) of (6.71) and the integro-differential operator

\[
MV = V_t - \mathcal{M} \xi \eta V_t, \tag{6.76}
\]

and following the same procedure done in establishing the proof of Theorem 6.2, we have

\[
\|V\|_{L^2(0,T;H^1_\rho(\Omega))} \leq K \|V\|_{L^2(0,T;H^1_\rho(\Omega))}, \tag{6.77}
\]

where

\[
K = TC_4 \left( \frac{a^2 + b^2}{8} + 1 \right). \tag{6.78}
\]

Since \( K < 1 \), it follows from (6.77) that

\[
(1 - K)\|V\|_{L^2(0,T;H^1_\rho(\Omega))} = 0. \tag{6.79}
\]

Consequently (6.79) implies that \( V = w_1 - w_2 = 0 \) and hence \( w_1 = w_2 \in L^2(0, T; H^1_\rho(\Omega)) \).
7. Conclusions

Here we studied a non local mixed problem for a two dimensional singular nonlinear fractional order equation in the Caputo sense. We prove the existence, uniqueness and continuous dependence of a strong solution of the posed problem. We first establish for the associated linear problem a priori estimate and prove that the range of the operator generated by the considered problem is dense. The technique of deriving the a priori estimate is based on constructing a suitable multiplier. From the resulted energy estimate, it is possible to establish the solvability of the linear problem. Then, by applying an iterative process based on the obtained results for the linear problem, we establish the existence, uniqueness and continuous dependence of the weak solution of the nonlinear problem. The main contribution is that we applied and developed the a priori estimate method for a two dimensional singular nonlinear fractional order partial differential equation with Bessel operator that have never been treated in the literature of integer and fractional differential equations.

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Conflict of interest

The authors declare no conflict of interest.

References


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