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Research article

Nonlocal coupled system for ψ-Hilfer fractional order Langevin equations

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Abstract: In the present work a coupled system consisting by ψ-Hilfer fractional order Langevin equations supplemented with nonlocal integral boundary conditions is studied. Existence and uniqueness results are obtained by using standard fixed point theorems. The obtained results are well illustrated by numerical examples.

Keywords: boundary value problems; Langevin equations; ψ-Hilfer fractional derivative; existence; fixed point
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1. Introduction

Fractional calculus deals with the study of fractional order integral and derivative operators over real or complex domains. The subject of fractional differential equations has become a hot topic for the researchers due to its intensive development and applications in the field of physics, mechanics, chemistry, engineering, etc. For a reader interested in the systematic development of the topic, we refer the books [1–8].

Since the beginning of the fractional calculus there are numerous definitions of integrals and fractional derivatives, and over time, new derivatives and fractional integrals arise. Hilfer in [9] generalized both Riemann-Liouville and Caputo fractional derivatives to *Hilfer fractional derivative of order* $\alpha \in (0, 1)$ *and a type* $\beta \in [0, 1]$ which can be reduced to the Riemann-Liouville and Caputo fractional derivatives when $\beta = 0$ and $\beta = 1$, respectively. For more details see [10, 11] and references cited therein.

In [2], a fractional derivative of a function with respect to another function were introduced, by using the fractional derivative in the Riemann-Liouville sense. Almeida [12] using the idea of the fractional derivative in the Caputo sense, proposes a new fractional derivative, called ψ-Caputo derivative with respect to another function ψ, which generalizes a class of fractional derivatives. In [13], the authors, by using the Hilfer fractional derivative idea, proposed a fractional differential operator of a function with respect to another ψ-function, the so-called ψ-Hilfer fractional derivative. The ψ-Hilfer fractional derivative has as advantage the freedom of choice of the classical differential operator, see, for example, [14, 15].

Initial value problems involving Hilfer fractional derivatives were studied by several authors, see for example [16–18] and references therein. In [19], the authors initiated the study of nonlocal boundary value problems for Hilfer fractional derivative. Recently, in [20], the results in [19] was extended to ψ-Hilfer nonlocal implicit fractional boundary value problems. In [21], the existence and uniqueness of solutions were studied, for a new class of boundary value problems of sequential ψ-Hilfer-type fractional differential equations with multi-point boundary conditions.

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [22]. For some new developments on the fractional Langevin equation, see, for example, [23], [24], [25].

In [26], the authors considered a boundary value problem of Langevin fractional differential equations with ψ-Hilfer fractional derivative and nonlocal integral boundary conditions, given by

$$\begin{cases} \mathcal{D}^{\chi_1,\beta_1;\psi}\left(\mathcal{D}^{\chi_2,\beta_2;\psi}+k\right)x(t)=f(t,x(t)), \quad t\in J:=[a,b], \\ x(a)=0, \qquad x(b)=\displaystyle\sum_{i=1}^{m}\lambda_i I^{\delta_i;\psi}x(\tau_i), \end{cases} \tag{1.1}$$

where $\mathcal{D}^{\chi_i,\beta_i;\psi}$, $i = 1, 2$ is the ψ-Hilfer fractional derivative of order χ_i, $0 < \chi_i < 1$ and type β_i, $0 \le \beta_i \le 1$, $i = 1, 2$, $1 < \chi_1 + \chi_2 \le 2$, $k \in \mathbb{R}$, $a \ge 0$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I^{\delta_i;\psi}$ is ψ-Riemann-Liouville fractional integral of order $\delta_i > 0$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ and $0 \le a \le \tau_1 < \tau_2 < \ldots < \tau_m \le b$. Existence and uniqueness results are established by using Krasnosel'skiĭ's fixed point theorem, Leray-Schauder nonlinear alternative and Banach contraction mapping principle.

In the present work, we study a coupled system consisting by ψ-Hilfer fractional order Langevin equations supplemented with nonlocal integral boundary conditions of the form

$$\begin{cases} {}^H\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\psi}\left({}^H\mathfrak{D}_{a^+}^{p_1,q_1;\psi}+\lambda_1\right)x(t)=f(t,x(t),y(t)), \quad t\in J:=[a,b], \\ {}^H\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\psi}\left({}^H\mathfrak{D}_{a^+}^{p_2,q_2;\psi}+\lambda_2\right)y(t)=g(t,x(t),y(t)), \quad t\in J:=[a,b], \\ x(a)=0, \quad x(b)=\displaystyle\sum_{i=1}^{m}\eta_i\mathcal{I}_{a^+}^{\delta_i;\psi}y(\theta_i), \quad y(a)=0, \quad y(b)=\displaystyle\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\kappa_j;\psi}x(\xi_j), \end{cases} \tag{1.2}$$

where ${}^H\mathfrak{D}_{a^+}^{u,v;\psi}$ is ψ-Hilfer fractional derivatives of order $u \in \{\alpha_1, \alpha_2, p_1, p_2\}$ with $0 < u \le 1$ and $v \in \{\beta_1, \beta_2, q_1, q_2\}$ with $0 \le v \le 1$, $\mathcal{I}_{a^+}^{w;\psi}$ is ψ-Riemann-Liouville fractional integral of order $w = \{\delta_i, \kappa_j\}$,

$w > 0$, the points $\theta_i, \xi_j \in [a,b]$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $f, g \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$ and $b > a \geq 0$.

The rest of the paper is organized as follows. In Section 2 we outline the basic concepts from fractional calculus and prove an auxiliary lemma for the linear variant of the problem (1.2). Then, by applying Banach's fixed point theorem, we derive the existence and uniqueness result for the problem (1.2), while the existence result is established via Leray-Schauder alternative. Example illustrating the main results are also presented.

2. Background material and auxiliary results

This section, is assigned to recall some notation in relation to fractional calculus.

Let $\mathcal{S} = C([a,b],\mathbb{R})$ be the space equipped with the norm defined by $\|x\| = \sup\{|x(t)| : t \in J\}$. Obviously $(\mathcal{S}, \|\cdot\|)$ is a Banach space and, consequently, the product space $(\mathcal{S}\times\mathcal{S}, \|\cdot\|)$ is a Banach space with the norm $\|(x,y)\| = \|x\| + \|y\|$ for $(x,y) \in \mathcal{S}\times\mathcal{S}$ and the n-times absolutely continuous functions given by

$$\mathcal{AC}^n([a,b],\mathbb{R}) = \{f : J \to \mathbb{R}; f^{(n-1)} \in \mathcal{AC}([a,b],\mathbb{R})\}.$$

Definition 2.1. *[2] Let (a,b), $(-\infty \leq a < b \leq \infty)$, be a finite or infinite interval of the half-axis $\mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$. Also let $\psi(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). The ψ-Riemann-Liouville fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by*

$$\mathcal{I}_{a^+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1} f(s)ds, \quad t > a > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is represent the Gamma function.

Definition 2.2. *[2] Let $\psi'(t) \neq 0$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann–Liouville derivatives of a function f with respect to another function ψ of order α correspondent to the Riemann–Liouville, is defined by*

$$\mathfrak{D}_{a^+}^{\alpha;\psi} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_{a^+}^{n-\alpha;\psi} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f(s)ds, \tag{2.2}$$

where $n = [\alpha]+1$, $[\alpha]$ is represent the integer part of the real number α.

Definition 2.3. *[13] Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, $[a,b]$ is the interval such that $-\infty \leq a < b \leq \infty$ and $f, \psi \in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a,b]$. The ψ-Hilfer fractional derivative of a function f of order α and type $0 \leq \rho \leq 1$, is defined by*

$${}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} f(t) = \mathcal{I}_{a^+}^{\rho(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(t) = \mathcal{I}_{a^+}^{\gamma-\alpha;\psi}\mathfrak{D}_{a^+}^{\gamma;\psi} f(t), \tag{2.3}$$

where $n = [\alpha]+1$, $[\alpha]$ represents the integer part of the real number α with $\gamma = \alpha + \rho(n-\alpha)$.

Lemma 2.4. [2] *Let $\alpha, \beta > 0$. Then we have the following semigroup property given by,*

$$\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\beta;\psi} f(t) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi} f(t), \quad t > a. \tag{2.4}$$

Next, we present the ψ-fractional integral and derivatives of a power function.

Proposition 2.5. *[2, 13] Let $\alpha \geq 0$, $\upsilon > 0$ and $t > a$. Then, ψ-fractional integral and derivative of a power function are given by*

(i) $\mathcal{I}_{a^+}^{\alpha;\psi} (\psi(s) - \psi(a))^{\upsilon-1} (t) = \dfrac{\Gamma(\upsilon)}{\Gamma(\upsilon+\alpha)} (\psi(t) - \psi(a))^{\upsilon+\alpha-1}$.

(ii) ${}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} (\psi(s) - \psi(a))^{\upsilon-1} (t) = \dfrac{\Gamma(\upsilon)}{\Gamma(\upsilon-\alpha)} (\psi(t) - \psi(a))^{\upsilon-\alpha-1}, \quad n-1 < \alpha < n, \quad \upsilon > n.$

Lemma 2.6. *[27] Let $m-1 < \alpha < m$, $n-1 < \beta < n$, $n, m \in \mathbb{N}$, $n \leq m$, $0 \leq \rho \leq 1$ and $\alpha \geq \beta + \rho(n-\beta)$. If $f \in C^n(J, \mathbb{R})$, then*

$$
{}^H\mathfrak{D}_{a^+}^{\beta,\rho;\psi} \mathcal{I}_{a^+}^{\alpha;\psi} f(t) = \mathcal{I}_{a^+}^{\alpha-\beta;\psi} f(t). \tag{2.5}
$$

Lemma 2.7. *[13] If $f \in C^n(J, \mathbb{R})$, $n-1 < \alpha < n$, $0 \leq \rho \leq 1$ and $\gamma = \alpha + \rho(n-\alpha)$ then*

$$
\mathcal{I}_{a^+}^{\alpha;\psi} {}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} f(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t) - \psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi} f(a), \tag{2.6}
$$

for all $t \in J$, where $f_{\psi}^{[n]} f(t) := \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n f(t)$.

The following lemma deals with a linear variant of the system (1.2).

Lemma 2.8. *Let $0 < \alpha_1, \alpha_2, p_1, p_2 \leq 1$, $0 \leq \beta_1, \beta_2, q_1, q_2 \leq 1$, $\gamma_1 = \alpha_1 + \beta_1(1-\alpha_1)$, $\gamma_2 = p_1 + q_1(1-p_1)$, $\varphi_1 = \alpha_2 + \beta_2(1-\alpha_2)$, $\varphi_2 = p_2 + q_2(1-p_2)$, h_1, $h_2 \in \mathcal{S}$ and $\Omega \neq 0$. Then the solution for the linear system of ψ-Hilfer fractional Langevin differential equations of the form:*

$$
\begin{cases}
{}^H\mathfrak{D}_{a^+}^{\alpha_1,\beta_1;\psi} \left({}^H\mathfrak{D}_{a^+}^{p_1,q_1;\psi} + \lambda_1\right) x(t) = h_1(t), \quad t \in J, \\
{}^H\mathfrak{D}_{a^+}^{\alpha_2,\beta_2;\psi} \left({}^H\mathfrak{D}_{a^+}^{p_2,q_2;\psi} + \lambda_2\right) y(t) = h_2(t), \quad t \in J, \\
x(a) = 0, \quad x(b) = \displaystyle\sum_{i=1}^{m} \eta_i \mathcal{I}_{a^+}^{\delta_i;\psi} y(\theta_i), \quad y(a) = 0, \quad y(b) = \displaystyle\sum_{j=1}^{n} \mu_j \mathcal{I}_{a^+}^{\kappa_j;\psi} x(\xi_j),
\end{cases} \tag{2.7}
$$

is equivalent to the integral equations

$$
\begin{aligned}
x(t) \;=\; & \mathcal{I}_{a^+}^{\alpha_1+p_1;\psi} h_1(t) - \lambda_1 \mathcal{I}_{a^+}^{p_1;\psi} x(t) \\
& + \frac{(\psi(t) - \psi(a))^{\gamma_1+p_1-1}}{\Omega\Gamma(\gamma_1+p_1)} \Bigg[\Omega_4 \Bigg(- \mathcal{I}_{a^+}^{\alpha_1+p_1;\psi} h_1(b) + \sum_{i=1}^{m} \eta_i \mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi} h_2(\theta_i) \\
& + \lambda_1 \mathcal{I}_{a^+}^{p_1;\psi} x(b) - \lambda_2 \sum_{i=1}^{m} \eta_i \mathcal{I}_{a^+}^{p_2+\delta_i;\psi} y(\theta_i) \Bigg) + \Omega_2 \Bigg(\sum_{j=1}^{n} \mu_j \mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi} h_1(\xi_j) \\
& - \mathcal{I}_{a^+}^{\alpha_2+p_2;\psi} h_2(b) - \lambda_1 \sum_{j=1}^{n} \mu_j \mathcal{I}_{a^+}^{p_1+\kappa_j;\psi} x(\xi_j) + \lambda_2 \mathcal{I}_{a^+}^{p_2;\psi} y(b) \Bigg) \Bigg],
\end{aligned} \tag{2.8}
$$

and

$$
y(t) \;=\; \mathcal{I}_{a^+}^{\alpha_2+p_2;\psi} h_2(t) - \lambda_2 \mathcal{I}_{a^+}^{p_2;\psi} y(t)
$$

$$+\frac{(\psi(t)-\psi(a))^{\varphi_1+p_2-1}}{\Omega\Gamma(\varphi_1+p_2)}\Bigg[\Omega_1\Bigg(\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi}h_1(\xi_j)-\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}h_2(b)$$

$$-\lambda_1\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{p_1+\kappa_j;\psi}x(\xi_j)+\lambda_2\mathcal{I}_{a^+}^{p_2;\psi}y(b)\Bigg)+\Omega_3\Bigg(-\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}h_1(b)$$

$$+\sum_{i=1}^{m}\eta_i\mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi}h_2(\theta_i)+\lambda_1\mathcal{I}_{a^+}^{p_1;\psi}x(b)-\lambda_2\sum_{i=1}^{m}\eta_i\mathcal{I}_{a^+}^{p_2+\delta_i;\psi}y(\theta_i)\Bigg)\Bigg], \tag{2.9}$$

where

$$\Omega_1 = \frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}, \tag{2.10}$$

$$\Omega_2 = \sum_{i=1}^{m}\frac{\eta_i\,(\psi(\theta_i)-\psi(a))^{\varphi_1+p_2+\delta_i-1}}{\Gamma(\varphi_1+p_2+\delta_i)}, \tag{2.11}$$

$$\Omega_3 = \sum_{j=1}^{n}\frac{\mu_j\left(\psi(\xi_j)-\psi(a)\right)^{\gamma_1+p_1+\kappa_j-1}}{\Gamma(\gamma_1+p_1+\kappa_j)}, \tag{2.12}$$

$$\Omega_4 = \frac{(\psi(b)-\psi(a))^{\varphi_1+p_2-1}}{\Gamma(\varphi_1+p_2)}, \tag{2.13}$$

$$\Omega = \Omega_1\Omega_4-\Omega_2\Omega_3. \tag{2.14}$$

Proof. Let $x \in \mathcal{S}$ be a solution of the problem (2.7). Taking the operator $\mathcal{I}_{a^+}^{\alpha_1;\psi}$ on both sides of (2.7) and using Lemma 2.7, we have

$$^{H}\mathfrak{D}_{a^+}^{p_1,q_1;\psi}x(t)+\lambda_1 x(t)=\mathcal{I}_{a^+}^{\alpha_1;\psi}h_1(t)+\frac{(\psi(t)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)}c_1, \tag{2.15}$$

where $c_1 \in \mathbb{R}$. Applying the operator $\mathcal{I}_{a^+}^{p_1;\psi}$ on both sides of (2.15) and using Lemma 2.7 again, we get

$$x(t)=\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}h_1(t)-\lambda_1\mathcal{I}_{a^+}^{p_1;\psi}x(t)+\frac{(\psi(t)-\psi(a))^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}c_1+\frac{(\psi(t)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)}c_2, \tag{2.16}$$

where c_1, c_2 are arbitrary constants.

In the same process, let $y \in \mathcal{S}$ be a solution of the problem (2.7). Applying the operators $\mathcal{I}_{a^+}^{\alpha_2;\psi}$ and $\mathcal{I}_{a^+}^{p_2;\psi}$, respectively, on both sides of the second equation in (2.7) and using Lemma 2.7, we obtain

$$y(t)=\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}h_2(t)-\lambda_2\mathcal{I}_{a^+}^{p_2;\psi}y(t)+\frac{(\psi(t)-\psi(a))^{\varphi_1+p_2-1}}{\Gamma(\varphi_1+p_2)}d_1+\frac{(\psi(t)-\psi(a))^{\varphi_2-1}}{\Gamma(\varphi_2)}d_2, \tag{2.17}$$

where d_1, d_2 are arbitrary constants. Using the boundary conditions $x(a)=0$ and $y(a)=0$, respectively in (2.16) and (2.17), we find that $c_2=0$ and $d_2=0$. Hence we have

$$x(t) = \mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}h_1(t)-\lambda_1\mathcal{I}_{a^+}^{p_1;\psi}x(t)+\frac{(\psi(t)-\psi(a))^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}c_1, \tag{2.18}$$

$$y(t) = \mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}h_2(t)-\lambda_2\mathcal{I}_{a^+}^{p_2;\psi}y(t)+\frac{(\psi(t)-\psi(a))^{\varphi_1+p_2-1}}{\Gamma(\varphi_1+p_2)}d_1. \tag{2.19}$$

Using (2.18) and (2.19) in the conditions $x(b) = \sum_{i=1}^{m} \eta_i \mathcal{I}_{a^+}^{\delta_i;\psi} y(\theta_i)$ and $y(b) = \sum_{j=1}^{n} \mu_j \mathcal{I}_{a^+}^{\kappa_j;\psi} x(\xi_j)$, we obtain a system of equations in the unknown constants c_1 and d_1 given by

$$\Omega_1 c_1 - \Omega_2 d_1 = A_1, \tag{2.20}$$

$$-\Omega_3 c_1 + \Omega_4 d_1 = A_2, \tag{2.21}$$

where $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ are given by (2.10), (2.11), (2.12), (2.13), respectively, and

$$\begin{aligned} A_1 &= -\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi} h_1(b) + \sum_{i=1}^{m} \eta_i \mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi} h_2(\theta_i) + \lambda_1 \mathcal{I}_{a^+}^{p_1;\psi} x(b) - \lambda_2 \sum_{i=1}^{m} \eta_i \mathcal{I}_{a^+}^{p_2+\delta_i;\psi} y(\theta_i), \\ A_2 &= \sum_{j=1}^{n} \mu_j \mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi} h_1(\xi_j) - \mathcal{I}_{a^+}^{\alpha_2+p_2;\psi} h_2(b) - \lambda_1 \sum_{j=1}^{n} \mu_j \mathcal{I}_{a^+}^{p_1+\kappa_j;\psi} x(\xi_j) + \lambda_2 \mathcal{I}_{a^+}^{p_2;\psi} y(b). \end{aligned}$$

Solving the system (2.20)-(2.21) for c_1 and d_1, it follows that

$$c_1 = \frac{\Omega_4 A_1 + \Omega_2 A_2}{\Omega_1 \Omega_4 - \Omega_2 \Omega_3} \qquad \text{and} \qquad d_1 = \frac{\Omega_1 A_2 + \Omega_3 A_1}{\Omega_1 \Omega_4 - \Omega_2 \Omega_3}.$$

Inserting values c_1 and d_1 in (2.18) and (2.19) respectively together with notations $\Omega_1, \Omega_2, \Omega_3, \Omega_4, A_1$, and A_2 lead to solutions (2.8) and (2.9).

Conversely, it is easily to shown, by a direct computation, that the solution $x(t)$ and $y(t)$ are given by (2.8) and (2.9) satisfies the problem (2.7) under the nonlocal integral boundary conditions. The proof of Lemma 2.8 is completed. □

Fixed point theorems play a major role in establishing the existence theory for the system (1.2). We collect here the well-known fixed point theorems used in this paper.

Lemma 2.9. (Banach fixed point theorem) *[28] Let X be a Banach space, $D \subset X$ closed and $F : D \to D$ a strict contraction, i.e. $|Fx - Fy| \le k|x - y|$ for some $k \in (0, 1)$ and all $x, y \in D$. Then F has a unique fixed point in D.*

Lemma 2.10. *(Leray-Schauder alternative [29]). Let $\mathcal{K} : D \to D$ be a complete continuous operator (*i.e., a map that restricted to any bounded set in D is compact*). Let*

$$M(\mathcal{K}) = \{x \in D : x = \sigma \mathcal{K}(x) \text{ for some } 0 < \sigma < 1\}.$$

Then either the set $M(\mathcal{K})$ is unbounded, or $\mathcal{K}$ has at least one fixed point.

3. Existence and uniqueness results

In this section, we discuss existence and uniqueness results to the purposed system (1.2).

Throughout this paper, the expression $\mathcal{I}_{0^+}^{q,\rho} f(s, x(s), y(s))(c)$ means that

$$\mathcal{I}_{a^+}^{u;\psi} f(s, x(s), y(s))(c) = \frac{1}{\Gamma(u)} \int_a^c \psi'(s)(\psi(c) - \psi(s))^{u-1} f(s, x(s), y(s)) ds,$$

where $u \in \{p_1, p_2, \alpha_1 + p_1, \alpha_2 + p_2, p_2 + \delta_i, \alpha_2 + p_2 + \delta_i, p_1 + \kappa_j, \alpha_1 + p_1 + \kappa_j\}$ and $c \in \{t, a, b, \theta_i, \xi_j\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

In view of Lemma 2.8, we define an operator $Q : \mathcal{S}\times\mathcal{S} \to \mathcal{S}\times\mathcal{S}$ by

$$Q(x,y)(t) := (Q_1(x,y)(t), Q_2(x,y)(t)), \tag{3.1}$$

where

$$\begin{aligned} Q_1(x,y)(t) &= \mathcal{I}_{a^+}^{\alpha_1+p_1;\psi} f(s,x(s),y(s))(t) - \lambda_1 \mathcal{I}_{a^+}^{p_1;\psi} x(t) \\ &+ \frac{(\psi(t)-\psi(a))^{\gamma_1+p_1-1}}{\Omega\Gamma(\gamma_1+p_1)}\Bigg[\Omega_4\Bigg(-\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi} f(s,x(s),y(s))(b) \\ &+ \sum_{i=1}^{m}\eta_i \mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi} g(s,x(s),y(s))(\theta_i) + \lambda_1 \mathcal{I}_{a^+}^{p_1;\psi} x(b) - \lambda_2 \sum_{i=1}^{m}\eta_i \mathcal{I}_{a^+}^{p_2+\delta_i;\psi} y(\theta_i)\Bigg) \\ &+ \Omega_2\Bigg(\sum_{j=1}^{n}\mu_j \mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi} f(s,x(s),y(s))(\xi_j) - \mathcal{I}_{a^+}^{\alpha_2+p_2;\psi} g(s,x(s),y(s))(b) \\ &- \lambda_1 \sum_{j=1}^{n}\mu_j \mathcal{I}_{a^+}^{p_1+\kappa_j;\psi} x(\xi_j) + \lambda_2 \mathcal{I}_{a^+}^{p_2;\psi} y(b)\Bigg)\Bigg], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} Q_2(x,y)(t) &= \mathcal{I}_{a^+}^{\alpha_2+p_2;\psi} g(s,x(s),y(s))(t) - \lambda_2 \mathcal{I}_{a^+}^{p_2;\psi} y(t) \\ &+ \frac{(\psi(t)-\psi(a))^{\varphi_1+p_2-1}}{\Omega\Gamma(\varphi_1+p_2)}\Bigg[\Omega_1\Bigg(\sum_{j=1}^{n}\mu_j \mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi} f(s,x(s),y(s))(\xi_j) \\ &- \mathcal{I}_{a^+}^{\alpha_2+p_2;\psi} g(s,x(s),y(s))(b) - \lambda_1 \sum_{j=1}^{n}\mu_j \mathcal{I}_{a^+}^{p_1+\kappa_j;\psi} x(\xi_j) + \lambda_2 \mathcal{I}_{a^+}^{p_2;\psi} y(b)\Bigg) \\ &+ \Omega_3\Bigg(-\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi} f(s,x(s),y(s))(b) + \sum_{i=1}^{m}\eta_i \mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi} g(s,x(s),y(s))(\theta_i) \\ &+ \lambda_1 \mathcal{I}_{a^+}^{p_1;\psi} x(b) - \lambda_2 \sum_{i=1}^{m}\eta_i \mathcal{I}_{a^+}^{p_2+\delta_i;\psi} y(\theta_i)\Bigg)\Bigg]. \end{aligned} \tag{3.3}$$

It should be noticed that the system (1.2) has solutions if and only if Q has fixed points.

For the sake of convenience, we use the following notations:

$$\Psi_1(B,u) = \frac{(\psi(B)-\psi(a))^u}{\Gamma(u+1)}, \tag{3.4}$$

$$\Lambda_1(U) = \Psi_1(b,U) + \frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Bigg(|\Omega_4|\Psi_1(b,U) + |\Omega_2|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j, U+\kappa_j)\Bigg), \tag{3.5}$$

$$\Lambda_2(U) = \frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Bigg(|\Omega_2|\Psi_1(b,U) + |\Omega_4|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i, U+\delta_i)\Bigg), \tag{3.6}$$

$$\Lambda_3(U) = \frac{\Psi_1(b,\varphi_1+p_2-1)}{|\Omega|}\Bigg(|\Omega_3|\Psi_1(b,U) + |\Omega_1|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j, U+\kappa_j)\Bigg), \tag{3.7}$$

$$\Lambda_4(U) = \Psi_1(b,U) + \frac{\Psi_1(b,\varphi_1+p_2-1)}{|\Omega|}\left(|\Omega_1|\Psi_1(b,U) + |\Omega_3|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,U+\delta_i)\right). \quad (3.8)$$

3.1. Uniqueness result via Banach's fixed point theorem

In the first result, we establish the existence and uniqueness of solutions for system (1.2), by applying Banach's fixed point theorem.

Theorem 3.1. *Let $\Omega \neq 0$, $f,\ g : [a,b]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous functions. In addition, we assume that:*

(H_1) There exist constants $L_1,\ L_2 > 0$ such that, $\forall t \in [a,b]$ and $x_1,\ x_2,\ y_1\ y_2 \in \mathbb{R}$,

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &\leq L_1(|x_1-y_1| + |x_2-y_2|), \\ |g(t,x_1,y_1) - g(t,x_2,y_2)| &\leq L_2(|x_1-y_1| + |x_2-y_2|). \end{aligned}$$

Then system (1.2) has a unique solution on $[a,b]$, provided that $\Phi_1 < 1$, where

$$\begin{aligned} \Phi_1 := {} & [\Lambda_1(\alpha_1+p_1) + \Lambda_3(\alpha_1+p_1)]L_1 + [\Lambda_2(\alpha_2+p_2) + \Lambda_4(\alpha_2+p_2)]L_2 \\ & +[\Lambda_1(p_1) + \Lambda_3(p_1)]|\lambda_1| + [\Lambda_2(p_2) + \Lambda_4(p_2)]|\lambda_2|. \end{aligned} \quad (3.9)$$

Proof. Firstly, we transform the system (1.2) into a fixed point problem, $(x,y)(t) = Q(x,y)(t)$, where the operator Q is defined as in (3.1). Applying the Banach contraction mapping principle, we shall show that the operator Q has a unique fixed point, which is the unique solution of system (1.2).

Let $\sup_{t\in[a,b]}|f(t,0,0)| := M_1 < \infty$ and $\sup_{t\in[a,b]}|g(t,0,0)| := N_1 < \infty$. Next, we set $B_{r_1} := \{(x,y) \in \mathcal{S}\times\mathcal{S} : \|(x,y)\| \leq r_1\}$ with

$$r_1 \geq \frac{[\Lambda_1(\alpha_1+p_1) + \Lambda_3(\alpha_1+p_1)]M_1 + [\Lambda_2(\alpha_2+p_2) + \Lambda_4(\alpha_2+p_2)]N_1}{1-\Phi_1}. \quad (3.10)$$

Observe that B_{r_1} is a bounded, closed, and convex subset of $\mathcal{S}$. The proof is divided into two steps:

Step I. We show that $QB_{r_1} \subset B_{r_1}$.

For any $(x,y) \in B_{r_1}$, $t \in [a,b]$, using the condition (H_1), we have

$$|f(t,x,y)| \leq |f(t,x,y) - f(t,0,0)| + |f(t,0,0)| \leq L_1(\|x\| + \|y\|) + M_1 \leq L_1 r_1 + M_1,$$

and

$$|g(t,x,y)| \leq |g(t,x,y) - g(t,0,0)| + |g(t,0,0)| \leq L_2(\|x\| + \|y\|) + N_1 \leq L_2 r_1 + N_1.$$

Then, we get

$$\begin{aligned} \|Q_1(x,y)\| \leq {} & \sup_{t\in[a,b]}\Bigg\{\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}|f(s,x(s),y(s))|(t) + |\lambda_1|\mathcal{I}_{a^+}^{p_1;\psi}|x(s)|(t) \\ & + \frac{(\psi(t)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\Bigg[|\Omega_4|\Bigg(\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}|f(s,x(s),y(s))|(b) \\ & + \sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi}|g(s,x(s),y(s))|(\theta_i) + |\lambda_1|\mathcal{I}_{a^+}^{p_1;\psi}|x(s)|(b) \end{aligned}$$

$$
\begin{aligned}
&+|\lambda_2|\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{p_2+\delta_i;\psi}|y(s)|(\theta_i)\Bigg)+|\Omega_2|\Bigg(\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi}|f(s,x(s),y(s))|(\xi_j)\\
&+\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}|g(s,x(s),y(s))|(b)+|\lambda_1|\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{p_1+\kappa_j;\psi}|x(s)|(\xi_j)+|\lambda_2|\mathcal{I}_{a^+}^{p_2;\psi}|y(s)|(b)\Bigg)\Bigg]\Bigg\}\\
\le\;&\Bigg[\frac{(\psi(b)-\psi(a))^{\alpha_1+p_1}}{\Gamma(\alpha_1+p_1+1)}+\frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\Bigg(\frac{|\Omega_4|\,(\psi(b)-\psi(a))^{\alpha_1+p_1}}{\Gamma(\alpha_1+p_1+1)}\\
&+\sum_{j=1}^{n}\frac{|\Omega_2||\mu_j|\left(\psi(\xi_j)-\psi(a)\right)^{\alpha_1+p_1+\kappa_j}}{\Gamma(\alpha_1+p_1+\kappa_j+1)}\Bigg)\Bigg]L_1r_1+\Bigg[\frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\\
&\times\Bigg(\frac{|\Omega_2|\,(\psi(b)-\psi(a))^{\alpha_2+p_2}}{\Gamma(\alpha_2+p_2+1)}+\sum_{i=1}^{m}\frac{|\Omega_4||\eta_i|\,(\psi(\theta_i)-\psi(a))^{\alpha_2+p_2+\delta_i}}{\Gamma(\alpha_2+p_2+\delta_i+1)}\Bigg)\Bigg]L_2r_1\\
&+\Bigg[|\lambda_1|\Bigg(\frac{(\psi(b)-\psi(a))^{p_1}}{\Gamma(p_1+1)}+\frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\Bigg(\frac{|\Omega_4|\,(\psi(b)-\psi(a))^{p_1}}{\Gamma(p_1+1)}\\
&+\sum_{j=1}^{n}\frac{|\Omega_2||\mu_j|\left(\psi(\xi_j)-\psi(a)\right)^{p_1+\kappa_j}}{\Gamma(p_1+\kappa_j+1)}\Bigg)\Bigg)+|\lambda_2|\frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\Bigg(\frac{|\Omega_2|\,(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)}\\
&+\sum_{i=1}^{m}\frac{|\Omega_4||\eta_i|\,(\psi(\theta_i)-\psi(a))^{p_2+\delta_i}}{\Gamma(p_2+\delta_i+1)}\Bigg)\Bigg]r_1+\Bigg[\frac{(\psi(b)-\psi(a))^{\alpha_1+p_1}}{\Gamma(\alpha_1+p_1+1)}+\frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\\
&\times\Bigg(\frac{|\Omega_4|\,(\psi(b)-\psi(a))^{\alpha_1+p_1}}{\Gamma(\alpha_1+p_1+1)}+\sum_{j=1}^{n}\frac{|\Omega_2||\mu_j|\left(\psi(\xi_j)-\psi(a)\right)^{\alpha_1+p_1+\kappa_j}}{\Gamma(\alpha_1+p_1+\kappa_j+1)}\Bigg)\Bigg]M_1\\
&+\Bigg[\frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\Bigg(\frac{|\Omega_2|\,(\psi(b)-\psi(a))^{\alpha_2+p_2}}{\Gamma(\alpha_2+p_2+1)}\\
&+\sum_{i=1}^{m}\frac{|\Omega_4||\eta_i|\,(\psi(\theta_i)-\psi(a))^{\alpha_2+p_2+\delta_i}}{\Gamma(\alpha_2+p_2+\delta_i+1)}\Bigg)\Bigg]N_1\\
=\;&\Bigg[\Psi_1(b,\alpha_1+p_1)+\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Bigg(|\Omega_4|\Psi_1(b,\alpha_1+p_1)\\
&+|\Omega_2|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,\alpha_1+p_1+\kappa_j)\Bigg)\Bigg]L_1r_1+\Bigg[\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Bigg(|\Omega_2|\Psi_1(b,\alpha_2+p_2)\\
&+|\Omega_4|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,\alpha_2+p_2+\delta_i)\Bigg)\Bigg]L_2r_1+\Bigg[|\lambda_1|\Bigg(\Psi_1(b,p_1)+\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\\
&\times\Bigg(|\Omega_4|\Psi_1(b,p_1)+|\Omega_2|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,p_1+\kappa_j)\Bigg)\Bigg)+|\lambda_2|\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\\
&\times\Bigg(|\Omega_2|\Psi_1(b,p_2)+|\Omega_4|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,p_2+\delta_i)\Bigg)\Bigg]r_1+\Bigg[\Psi_1(b,\alpha_1+p_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Big(|\Omega_4|\Psi_1(b,\alpha_1+p_1)+|\Omega_2|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,\alpha_1+p_1+\kappa_j)\Big)\Big]M_1\\
&+\Big[\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Big(|\Omega_2|\Psi_1(b,\alpha_2+p_2)+\sum_{i=1}^{m}|\Omega_4||\eta_i|\Psi_1(\theta_i,\alpha_2+p_2+\delta_i)\Big)\Big]N_1\\
= \;& \Big[\Lambda_1(\alpha_1+p_1)L_1+\Lambda_2(\alpha_2+p_2)L_2+|\lambda_1|\Lambda_1(p_1)+|\lambda_2|\Lambda_2(p_2)\Big]r_1\\
&+\Lambda_1(\alpha_1+p_1)M_1+\Lambda_2(\alpha_2+p_2)N_1.
\end{aligned}
$$

Similarly, we find that

$$
\begin{aligned}
\|Q_2(x,y)\| \;\le\;& \Big[\Lambda_3(\alpha_1+p_1)L_1+\Lambda_4(\alpha_2+p_2)L_2+|\lambda_1|\Lambda_3(p_1)+|\lambda_2|\Lambda_4(p_2)\Big]r_1\\
&+\Lambda_3(\alpha_1+p_1)M_1+\Lambda_4(\alpha_2+p_2)N_1.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\|Q(x,y)\| \;\le\;& \Big[(\Lambda_1(\alpha_1+p_1)+\Lambda_3(\alpha_1+p_1))L_1+(\Lambda_2(\alpha_2+p_2)+\Lambda_4(\alpha_2+p_2))L_2\\
&+(\Lambda_1(p_1)+\Lambda_3(p_1))|\lambda_1|+(\Lambda_2(p_2)+\Lambda_4(p_2))|\lambda_2|\Big]r_1\\
&+[\Lambda_1(\alpha_1+p_1)+\Lambda_3(\alpha_1+p_1)]M_1+[\Lambda_2(\alpha_2+p_2)+\Lambda_4(\alpha_2+p_2)]N_1\\
\le\;& r_1,
\end{aligned}
$$

which implies that $QB_{r_1}\subset B_{r_1}$.

Step II. We show that $Q:\mathcal{S}\to\mathcal{S}$ is a contraction.

Using condition (H_1), for any $(x_1,y_1),(x_2,y_2)\in\mathcal{S}\times\mathcal{S}$ and for each $t\in[a,b]$, we have

$$
\begin{aligned}
&\|Q_1(x_1,y_1)-Q_1(x_2,y_2)\|\\
\le\;& \sup_{t\in[a,b]}\Big\{\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}|f(s,x_1(s),y_1(s))-f(s,x_2(s),y_2(s))|(t)+|\lambda_1|\mathcal{I}_{a^+}^{p_1;\psi}|x_1(s)-x_2(s)|(t)\\
&+\frac{(\psi(t)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\Big[|\Omega_4|\Big(\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}|f(s,x_1(s),y_1(s))-f(s,x_2(s),y_2(s))|(b)\\
&+\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi}|g(s,x_1(s),y_1(s))-g(s,x_2(s),y_2(s))|(\theta_i)\\
&+|\lambda_1|\mathcal{I}_{a^+}^{p_1;\psi}|x_1(s)-x_2(s)|(b)+|\lambda_2|\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{p_2+\delta_i;\psi}|y_1(s)-y_2(s)|(\theta_i)\Big)\\
&+|\Omega_2|\Big(\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi}|f(s,x_1(s),y_1(s))-f(s,x_2(s),y_2(s))|(\xi_j)\\
&+\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}|g(s,x_1(s),y_1(s))-g(s,x_2(s),y_2(s))|(b)\\
&+|\lambda_1|\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{p_1+\kappa_j;\psi}|x_1(s)-x_2(s)|(\xi_j)+|\lambda_2|\mathcal{I}_{a^+}^{p_2;\psi}|y_1(s)-y_2(s)|(b)\Big)\Big]\Big\}\\
\le\;& (\|x_1-x_2\|+\|y_1-y_2\|)\Big[\Psi_1(b,\alpha_1+p_1)+\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Big(|\Omega_4|\Psi_1(b,\alpha_1+p_1)
\end{aligned}
$$

$$
\begin{aligned}
& +|\Omega_2|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,\alpha_1+p_1+\kappa_j)\Big)\Big]L_1+(\|x_1-x_2\|+\|y_1-y_2\|)\\
& \times\Big[\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Big(|\Omega_2|\Psi_1(b,\alpha_2+p_2)+|\Omega_4|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,\alpha_2+p_2+\delta_i)\Big)\Big]L_2\\
& +\Big[|\lambda_1|\Big(\Psi_1(b,p_1)+\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Big(|\Omega_4|\Psi_1(b,p_1)+|\Omega_2|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,p_1+\kappa_j)\Big)\Big)\\
& +\frac{|\lambda_2|\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Big(|\Omega_2|\Psi_1(b,p_2)+|\Omega_4|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,p_2+\delta_i)\Big)\Big]\\
& \times(\|x_1-x_2\|+\|y_1-y_2\|)\\
\le\ & \Big\{[\Lambda_1(\alpha_1+p_1)+\Lambda_3(\alpha_1+p_1)]L_1+[\Lambda_2(\alpha_2+p_2)+\Lambda_4(\alpha_2+p_2)]L_2\\
& +[\Lambda_1(p_1)+\Lambda_3(p_1)]|\lambda_1|+[\Lambda_2(p_2)+\Lambda_4(p_2)]|\lambda_2|\Big\}(\|x_1-x_2\|+\|y_1-y_2\|).
\end{aligned}
$$

Similarly, we get that

$$
\begin{aligned}
& \|\mathcal{Q}_2(x_1,y_1)-\mathcal{Q}_2(x_2,y_2)\|\\
\le\ & \sup_{t\in[a,b]}\Big\{\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}|g(s,x_1(s),y_1(s))-g(s,x_2(s),y_2(s))|(t)+|\lambda_2|\mathcal{I}_{a^+}^{p_2;\psi}|y_1(s)-y_2(s)|(t)\\
& +\frac{(\psi(t)-\psi(a))^{\varphi_1+p_2-1}}{|\Omega|\Gamma(\varphi_1+p_2)}\Big[|\Omega_1|\Big(\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi}|f(s,x_1(s),y_1(s))-f(s,x_2(s),y_2(s))|(\xi_j)\\
& +\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}|g(s,x_1(s),y_1(s))-g(s,x_2(s),y_2(s))|(b)+|\lambda_1|\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{p_1+\kappa_j;\psi}|x_1(s)-x_2(s)|(\xi_j)\\
& +|\lambda_2|\mathcal{I}_{a^+}^{p_2;\psi}|y_1(s)-y_2(s)|(b)\Big)+\Omega_3\Big(\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}|f(s,x_1(s),y_1(s))-f(s,x_2(s),y_2(s))|(b)\\
& +\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi}|g(s,x_1(s),y_1(s))-g(s,x_2(s),y_2(s))|(\theta_i)+|\lambda_1|\mathcal{I}_{a^+}^{p_1;\psi}|x_1(s)-x_2(s)|(b)\\
& +|\lambda_2|\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{p_2+\delta_i;\psi}|y_1(s)-y_2(s)|(\theta_i)\Big)\Big]\Big\}\\
\le\ & \Big[\Lambda_3(\alpha_1+p_1)L_1+\Lambda_4(\alpha_2+p_2)L_2+|\lambda_1|\Lambda_3(p_1)+|\lambda_2|\Lambda_4(p_2)\Big](\|x_1-x_2\|+\|y_1-y_2\|).
\end{aligned}
$$

Consequently, we get

$$\|\mathcal{Q}(x_1,y_1)-\mathcal{Q}(x_2,y_2)\|\le\Phi(\|x_1-x_2\|+\|y_1-y_2\|).$$

which, by condition (3.9), implies that the operator $\mathcal{Q}$ is a contraction. Therefore, by the conclusion of Banach contraction mapping principle (Lemma 2.9), the operator $\mathcal{Q}$ has a unique fixed point. Hence, system (1.2) has a unique solution on $[a.b]$. The proof is completed. □

3.2. Existence result via Leray-Schauder alternative

In the second result, we apply the Leray-Schauder alternative to investigate the existence of solutions for the system (1.2).

For convenience, we set:

$$E_1(U) = \Lambda_1(U) + \Lambda_3(U), \tag{3.11}$$
$$E_2(U) = \Lambda_2(U) + \Lambda_4(U), \tag{3.12}$$

where $\Lambda_1(\cdot)$, $\Lambda_2(\cdot)$, $\Lambda_3(\cdot)$ and $\Lambda_4(\cdot)$ are given by (3.5), (3.6), (3.7) and (3.8) respectively.

Theorem 3.2. *Let $f,\ g : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions such that the following condition holds:*

(H_2) *There exist real constants K_i, $\overline{K}_i \geq 0$ for $i = 1, 2, 3$, such that, for $x, y \in \mathbb{R}$,*

$$|f(t,x,y)| \leq K_1 + K_2|x| + K_3|y|, \quad \text{and} \quad |g(t,x,y)| \leq \overline{K}_1 + \overline{K}_2|x| + \overline{K}_3|y|.$$

Then, the system (1.2) *has at least one solution on* $[a,b]$ *provided that*

$$\Phi_2 = E_1(\alpha_1 + p_1)K_2 + E_2(\alpha_2 + p_2)\overline{K}_2 + E_1(p_1)|\lambda_1| < 1, \tag{3.13}$$
$$\Phi_3 = E_1(\alpha_1 + p_1)K_3 + E_2(\alpha_2 + p_2)\overline{K}_3 + E_2(p_2)|\lambda_2| < 1, \tag{3.14}$$

where $E_1(\cdot)$ and $E_2(\cdot)$ are given by (3.11) *and* (3.12) *respectively.*

Proof. The process of the proof is divided into several steps:

Firstly, we show that the operator $\mathcal{Q} : \mathcal{S} \times \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is completely continuous. Note that the operator $\mathcal{Q}$, defined by (3.1), is continuous in view of the continuity of f and g.

Let $B_{r_2} \subset \mathcal{S} \times \mathcal{S}$ be a bounded set, where $B_{r_2} = \{(x,y) \in \mathcal{S} \times \mathcal{S} : \|(x,y)\| \leq r_2\}$. Then, for any $(x,y) \in B_{r_2}$, there exist positive real numbers $\overline{f}$ and $\overline{g}$ such that $|f(t,x(t),y(t))| \leq \overline{f}$ and $|g(t,x(t),y(t))| \leq \overline{g}$.

Thus, for each $(x,y) \in B_{r_2}$ we have

$$\begin{aligned}
\|\mathcal{Q}_1(x,y)\| \leq & \sup_{t\in[a,b]} \Bigg\{ \mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}|f(s,x(s),y(s))|(t) + |\lambda_1|\mathcal{I}_{a^+}^{p_1;\psi}|x(s)|(t) \\
& + \frac{(\psi(t)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\Bigg[|\Omega_4|\Bigg(\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}|f(s,x(s),y(s))|(b) \\
& + \sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi}|g(s,x(s),y(s))|(\theta_i) + |\lambda_1|\mathcal{I}_{a^+}^{p_1;\psi}|x(s)|(b) \\
& + |\lambda_2|\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{p_2+\delta_i;\psi}|y(s)|(\theta_i)\Bigg) + |\Omega_2|\Bigg(\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi}|f(s,x(s),y(s))|(\xi_j) \\
& + \mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}|g(s,x(s),y(s))|(b) + |\lambda_1|\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{p_1+\kappa_j;\psi}|x(s)|(\xi_j) + |\lambda_2|\mathcal{I}_{a^+}^{p_2;\psi}|y(s)|(b)\Bigg)\Bigg]\Bigg\} \\
\leq & \Bigg[\Psi_1(b,\alpha_1+p_1) + \frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Bigg(|\Omega_4|\Psi_1(b,\alpha_1+p_1) \\
& + |\Omega_2|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,\alpha_1+p_1+\kappa_j)\Bigg)\Bigg]\overline{f}
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg[\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Bigg(|\Omega_2|\Psi_1(b,\alpha_2+p_2)+|\Omega_4|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,\alpha_2+p_2+\delta_i)\Bigg)\Bigg]\overline{g}\\
&+\Bigg[|\lambda_1|\Bigg(\Psi_1(b,p_1)+\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Bigg(|\Omega_4|\Psi_1(b,p_1)+|\Omega_2|\sum_{j=1}^{n}\mu_j\Psi_1(\xi_j,p_1+\kappa_j)\Bigg)\Bigg)\\
&+|\lambda_2|\Bigg(\frac{\Psi_1(b,\gamma_1+p_1-1)}{|\Omega|}\Bigg(|\Omega_2|\Psi_1(b,p_2)+|\Omega_4|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,p_2+\delta_i)\Bigg)\Bigg)\Bigg]r_2\\
= \ &\Lambda_1(\alpha_1+p_1)\overline{f}+\Lambda_2(\alpha_2+p_2)\overline{g}+\Big[|\lambda_1|\Lambda_1(p_1)+|\lambda_2|\Lambda_2(p_2)\Big]r_2.
\end{aligned}
$$

In similar process, we have that

$$\|Q_2(x,y)\|\leq\Lambda_3(\alpha_1+p_1)\overline{f}+\Lambda_4(\alpha_2+p_2)\overline{g}+\Big[|\lambda_1|\Lambda_3(p_1)+|\lambda_2|\Lambda_4(p_2)\Big]r_2.$$

Therefore, it follows that

$$\|Q(x,y)\|\leq E_1(\alpha_1+p_1)\overline{f}+E_2(\alpha_2+p_2)\overline{g}+\Big[E_1(p_1)|\lambda_1|+E_2(p_2)|\lambda_2|\Big]r_2,$$

which implies that the operator Q is uniformly bounded.

In the next step, we show that the operator Q is equicontinuous. Let $\tau_1,\tau_2\in[a,b]$ with $\tau_1<\tau_2$. Then, we can compute that

$$
\begin{aligned}
&|Q_1(x,y)(\tau_2)-Q_1(x,y)(\tau_1)|\\
\leq\ &\left|\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}f(s,x(s),y(s))(\tau_2)-\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}f(s,x(s),y(s))(\tau_1)\right|\\
&+|\lambda_1|\left|\mathcal{I}_{a^+}^{p_1;\psi}x(\tau_2)-\mathcal{I}_{a^+}^{p_1;\psi}x(\tau_1)\right|+\frac{\left|(\psi(\tau_2)-\psi(a))^{\gamma_1+p_1-1}-(\psi(\tau_1)-\psi(a))^{\gamma_1+p_1-1}\right|}{|\Omega|\Gamma(\gamma_1+p_1)}\\
&\times\Bigg[|\Omega_4|\Bigg(\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}|f(s,x(s),y(s))|(b)+\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi}|g(s,x(s),y(s))|(\theta_i)\\
&+|\lambda_1|\mathcal{I}_{a^+}^{p_1;\psi}|x(s)|(b)+|\lambda_2|\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{p_2+\delta_i;\psi}|y(s)|(\theta_i)\Bigg)\\
&+|\Omega_2|\Bigg(\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi}|f(s,x(s),y(s))|(\xi_j)+\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}|g(s,x(s),y(s))|(b)\\
&+|\lambda_1|\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{p_1+\kappa_j;\psi}|x(s)|(\xi_j)+|\lambda_2|\mathcal{I}_{a^+}^{p_2;\psi}|y(s)|(b)\Bigg)\Bigg]\\
\leq\ &\frac{\overline{f}}{\Gamma(\alpha_1+p_1+1)}\Big[2\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\alpha_1+p_1}+\left|(\psi(\tau_2)-\psi(a))^{\alpha_1+p_1}-(\psi(\tau_1)-\psi(a))^{\alpha_1+p_1}\right|\Big]\\
&+\frac{|\lambda_1|r_2}{\Gamma(p_1+1)}\Big[2\left(\psi(\tau_2)-\psi(\tau_1)\right)^{p_1}+|(\psi(\tau_2)-\psi(a))^{p_1}-(\psi(\tau_1)-\psi(a))^{p_1}|\Big]\\
&+\frac{\left|(\psi(\tau_2)-\psi(a))^{\gamma_1+p_1-1}-(\psi(\tau_1)-\psi(a))^{\gamma_1+p_1-1}\right|}{|\Omega|\Gamma(\gamma_1+p_1)}\Bigg[\Bigg(|\Omega_4|\Psi_1(b,\alpha_1+p_1)\\
&+|\Omega_2|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,\alpha_1+p_1+\kappa_j)\Bigg)\overline{f}+\Bigg(|\Omega_4|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,\alpha_2+p_2+\delta_i)
\end{aligned}
$$

$$+|\Omega_2|\Psi_1(b,\alpha_2+p_2)\Big)\overline{g}+\Big(|\Omega_4||\lambda_1|\Psi_1(b,p_1)+|\Omega_2||\lambda_1|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,p_1+\kappa_j)$$
$$+|\Omega_2||\lambda_2|\Psi_1(b,p_2)+|\Omega_4||\lambda_2|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,p_2+\delta_i)\Big)r_2\Big].$$

In the same process, we deduce that

$$\begin{aligned}
&|Q_2(x,y)(\tau_2)-Q_2(x,y)(\tau_1)|\\
\leq\;& \frac{\overline{g}}{\Gamma(\alpha_2+p_2+1)}\Big[2\left(\psi(\tau_2)-\psi(\tau_1)\right)^{\alpha_2+p_2}+\left|(\psi(\tau_2)-\psi(a))^{\alpha_2+p_2}-(\psi(\tau_1)-\psi(a))^{\alpha_2+p_2}\right|\Big]\\
&+\frac{|\lambda_2|r_2}{\Gamma(p_2+1)}\Big[2\left(\psi(\tau_2)-\psi(\tau_1)\right)^{p_2}+|(\psi(\tau_2)-\psi(a))^{p_2}-(\psi(\tau_1)-\psi(a))^{p_2}|\Big]\\
&+\frac{\left|(\psi(\tau_2)-\psi(a))^{\varphi_1+p_2-1}-(\psi(\tau_1)-\psi(a))^{\varphi_1+p_2-1}\right|}{|\Omega|\Gamma(\varphi_1+p_2)}\Big[\Big(|\Omega_3|\Psi_1(b,\alpha_1+p_1)\\
&+|\Omega_1|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,\alpha_1+p_1+\kappa_j)\Big)\overline{f}+\Big(|\Omega_3|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,\alpha_2+p_2+\delta_i)\\
&+|\Omega_1|\Psi_1(b,\alpha_2+p_2)\Big)\overline{g}+\Big(|\Omega_3||\lambda_1|\Psi_1(b,p_1)+|\Omega_1||\lambda_1|\sum_{j=1}^{n}|\mu_j|\Psi_1(\xi_j,p_1+\kappa_j)\\
&+|\Omega_1||\lambda_2|\Psi_1(b,p_2)+|\Omega_3||\lambda_2|\sum_{i=1}^{m}|\eta_i|\Psi_1(\theta_i,p_2+\delta_i)\Big)r_2\Big].
\end{aligned}$$

Consequently, the operator $Q(x,y)$ is equicontinuous. By Arzelá-Ascoli theorem, the operator $Q(x,y)$ is completely continuous.

At last process, we show that the set $\mathcal{U}=\{(x,y)\in\mathcal{S}\times\mathcal{S}|(x,y)=\sigma Q(x,y),\ 0\leq\sigma\leq 1\}$ is a bounded. Let $(x,y)\in\mathcal{U}$, with $(x,y)=\sigma Q(x,y)$. For any $t\in[a,b]$, we obtain

$$x(t)=\sigma Q(x,y)(t),\quad \text{and}\quad y(t)=\sigma Q(x,y)(t).$$

By using condition (H_3), we can calculate that

$$\begin{aligned}
|x(t)| &= |\sigma Q_1(x,y)(t)|\\
&\leq (K_1+K_2\|x\|+K_3\|y\|)\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}(1)(b)+|\lambda_1|\|x\|\mathcal{I}_{a^+}^{p_1;\psi}(1)(b)+\frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{|\Omega|\Gamma(\gamma_1+p_1)}\\
&\quad\times\Big[|\Omega_4|\Big((K_1+K_2\|x\|+K_3\|y\|)\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}(1)(b)+|\lambda_1|\|x\|\mathcal{I}_{a^+}^{p_1;\psi}(1)(b)\\
&\quad+(\overline{K}_1+\overline{K}_2\|x\|+\overline{K}_3\|y\|)\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi}(1)(\theta_i)+|\lambda_2|\|y\|\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{p_2+\delta_i;\psi}(1)(\theta_i)\Big)\\
&\quad+|\Omega_2|\Big((K_1+K_2\|x\|+K_3\|y\|)\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi}(1)(\xi_j)+|\lambda_2|\|y\|\mathcal{I}_{a^+}^{p_2;\psi}(1)(b)\\
&\quad+(\overline{K}_1+\overline{K}_2\|x\|+\overline{K}_3\|y\|)\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}(1)(b)+|\lambda_1|\|x\|\sum_{j=1}^{n}\mu_j\mathcal{I}_{a^+}^{p_1+\kappa_j;\psi}(1)(\xi_j)\Big)\Big]
\end{aligned}$$

$$\leq \Lambda_1(\alpha_1+p_1)(K_1+K_2\|x\|+K_3\|y\|)+\Lambda_2(\alpha_2+p_2)(\overline{K}_1+\overline{K}_2\|x\|+\overline{K}_3\|y\|)$$
$$+|\lambda_1|\Lambda_1(p_1)\|x\|+|\lambda_2|\Lambda_2(p_2)\|y\|,$$

and

$$\begin{aligned}
|y(t)| &= |\sigma \mathcal{Q}_2(x,y)(t)| \\
&\leq (\overline{K}_1+\overline{K}_2\|x\|+\overline{K}_3\|y\|)\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}(1)(b)+|\lambda_2|\|y\|\mathcal{I}_{a^+}^{p_2;\psi}(1)(b)+\frac{(\psi(b)-\psi(a))^{\varphi_1+p_2-1}}{|\Omega|\Gamma(\varphi_1+p_2)} \\
&\quad \times\Bigg[|\Omega_1|\Bigg((K_1+K_2\|x\|+K_3\|y\|)\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{\alpha_1+p_1+\kappa_j;\psi}(1)(\xi_j)+|\lambda_2|\|y\|\mathcal{I}_{a^+}^{p_2;\psi}(1)(b) \\
&\quad +(\overline{K}_1+\overline{K}_2\|x\|+\overline{K}_3\|y\|)\mathcal{I}_{a^+}^{\alpha_2+p_2;\psi}(1)(b)+|\lambda_1|\|x\|\sum_{j=1}^{n}|\mu_j|\mathcal{I}_{a^+}^{p_1+\kappa_j;\psi}(1)(\xi_j)\Bigg) \\
&\quad +|\Omega_3|\Bigg((\overline{K}_1+\overline{K}_2\|x\|+\overline{K}_3\|y\|)\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{\alpha_2+p_2+\delta_i;\psi}(1)(\theta_i)+|\lambda_1|\|x\|\mathcal{I}_{a^+}^{p_1;\psi}(1)(b) \\
&\quad +(K_1+K_2\|x\|+K_3\|y\|)\mathcal{I}_{a^+}^{\alpha_1+p_1;\psi}(1)(b)+|\lambda_2|\|y\|\sum_{i=1}^{m}|\eta_i|\mathcal{I}_{a^+}^{p_2+\delta_i;\psi}(1)(\theta_i)\Bigg)\Bigg] \\
&\leq \Lambda_3(\alpha_1+p_1)(K_1+K_2\|x\|+K_3\|y\|)+\Lambda_4(\alpha_2+p_2)(\overline{K}_1+\overline{K}_2\|x\|+\overline{K}_3\|y\|) \\
&\quad +|\lambda_1|\Lambda_3(p_1)\|x\|+|\lambda_2|\Lambda_4(p_2)\|y\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x\| &\leq \Big[\Lambda_1(\alpha_1+p_1)K_1+\Lambda_2(\alpha_2+p_2)\overline{K}_1\Big] \\
&\quad +\Big[\Lambda_1(\alpha_1+p_1)K_2+\Lambda_2(\alpha_2+p_2)\overline{K}_2+|\lambda_1|\Lambda_1(p_1)\Big]\|x\| \\
&\quad +\Big[\Lambda_1(\alpha_1+p_1)K_3+\Lambda_2(\alpha_2+p_2)\overline{K}_3+|\lambda_2|\Lambda_2(p_2)\Big]\|y\|,
\end{aligned}$$

and

$$\begin{aligned}
\|y\| &\leq \Big[\Lambda_3(\alpha_1+p_1)K_1+\Lambda_4(\alpha_2+p_2)\overline{K}_1\Big] \\
&\quad +\Big[\Lambda_3(\alpha_1+p_1)K_2+\Lambda_4(\alpha_2+p_2)\overline{K}_2+|\lambda_1|\Lambda_3(p_1)\Big]\|x\| \\
&\quad +\Big[\Lambda_3(\alpha_1+p_1)K_3+\Lambda_4(\alpha_2+p_2)\overline{K}_3+|\lambda_2|\Lambda_4(p_2)\Big]\|y\|.
\end{aligned}$$

From the above inequalities, we get

$$\begin{aligned}
\|x\|+\|y\| &\leq E_1(\alpha_1+p_1)K_1+E_2(\alpha_2+p_2)\overline{K}_1 \\
&\quad +\Big(E_1(\alpha_1+p_1)K_2+E_2(\alpha_2+p_2)\overline{K}_2+E_1(p_1)|\lambda_1|\Big)\|x\| \\
&\quad +\Big(E_1(\alpha_1+p_1)K_3+E_2(\alpha_2+p_2)\overline{K}_3+E_2(p_2)|\lambda_2|\Big)\|y\|,
\end{aligned}$$

which implies that

$$\|(x,y)\| \leq \frac{E_1(\alpha_1+p_1)K_1+E_2(\alpha_2+p_2)\overline{K}_1}{E_0},$$

where $E_0 = \min\{1-\Phi_2,\ 1-\Phi_3\}$. Hence the set $\mathcal{U}$ is bounded. Thus, by Lemma 2.10, the operator Q has at least one fixed point, which implies that system (1.2) has at least one solution on $[a,b]$. This completes the proof. □

4. Examples

This section is devoted to the illustration of the results derived in previous sections.

Example 4.1. *Consider the following nonlocal boundary problem of the form:*

$$\begin{cases} {}^H\mathfrak{D}_{0^+}^{\frac{\sqrt{e}}{2},0.5;\frac{e^t}{6}}\left({}^H\mathfrak{D}_{0^+}^{\frac{\sqrt[3]{e}}{2},0.4;\frac{e^t}{6}}+\dfrac{1}{10}\right)x(t)=f(t,x(t),y(t)), \quad t\in[0,1.2], \\ {}^H\mathfrak{D}_{0^+}^{\frac{\sqrt{e}}{3},0.6;\frac{e^t}{6}}\left({}^H\mathfrak{D}_{0^+}^{\frac{\sqrt[3]{e}}{3},0.7;\frac{e^t}{6}}+\dfrac{1}{20}\right)y(t)=g(t,x(t),y(t)), \quad t\in[0,1.2], \\ x(0)=0, \qquad x(1.2)=\displaystyle\sum_{i=1}^{3}\left(\frac{\ln 2}{i+1}\right)\mathcal{I}_{0^+}^{\ln\left(\frac{6-i}{2}\right);\frac{e^t}{6}}y\left(\frac{2i-1}{10}\right), \\ y(0)=0, \qquad y(1.2)=\displaystyle\sum_{j=1}^{2}\left(\frac{\ln(j+1)}{j}\right)\mathcal{I}_{0^+}^{\ln\left(\frac{j+3}{3}\right);\frac{e^t}{6}}x\left(\frac{2j+3}{10}\right). \end{cases} \tag{4.1}$$

Here $\alpha_k=\sqrt{e}/(k+1)$, $\beta_k=(k+4)/10$, $p_k=\sqrt[3]{e}/(k+1)$, $q_k=(3k+1)/10$, $\lambda_k=(15-5k)/100$, $k=1,2$, $\psi(t)=e^t/6$, $\eta_i=\ln 2/(i+1)$, $\delta_i=\ln((6-i)/2)$, $\theta_i=(2i-1)/10$, $i=1,2,3$, $\mu_j=\ln(j+1)/j$, $\kappa_j=\ln((j+3)/3)$, $\xi_j=(2j+3)/10$, $j=1,2$, $a=0$ *and* $b=1.2$. *From* (2.14) *with the given data, we can compute that of*

$$\begin{aligned} \Omega_1 &= \frac{(\psi(b)-\psi(a))^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}=\frac{\left(\frac{e^{1.2}}{6}-\frac{1}{6}\right)^{0.61}}{\Gamma(1.61)}, \\ \Omega_2 &= \sum_{i=1}^{m}\frac{\eta_i\,(\psi(\theta_i)-\psi(a))^{\varphi_1+p_2+\delta_i-1}}{\Gamma(\varphi_1+p_2+\delta_i)}=\sum_{i=1}^{3}\left(\frac{\ln 2}{i+1}\right)\frac{\left(\frac{e^{\frac{2i-1}{10}}}{6}-\frac{1}{6}\right)^{0.285+\ln\left(\frac{6-i}{2}\right)}}{\Gamma\left(1.285+\ln\left(\frac{6-i}{2}\right)\right)}, \\ \Omega_3 &= \sum_{j=1}^{n}\frac{\mu_j\left(\psi(\xi_j)-\psi(a)\right)^{\gamma_1+p_1+\kappa_j-1}}{\Gamma(\gamma_1+p_1+\kappa_j)}=\sum_{j=1}^{n}\left(\frac{\ln(j+1)}{j}\right)\frac{\left(\frac{e^{\frac{2j+3}{10}}}{6}-\frac{1}{6}\right)^{0.61+\ln\left(\frac{j+3}{3}\right)}}{\Gamma\left(1.61+\ln\left(\frac{j+3}{3}\right)\right)}, \\ \Omega_4 &= \frac{(\psi(b)-\psi(a))^{\varphi_1+p_2-1}}{\Gamma(\varphi_1+p_2)}=\frac{\left(\frac{e^{1.2}}{6}-\frac{1}{6}\right)^{0.285}}{\Gamma(1.285)}. \end{aligned}$$

Then, $\Omega_1\approx 0.6261$, $\Omega_2\approx 0.0580$, $\Omega_3\approx 0.1687$, $\Omega_4\approx 0.8476$, *and* $\Omega=\Omega_1\Omega_4-\Omega_2\Omega_3\approx 0.5209\neq 0$. *In addition, Table* 1 *shows the numerical results of* Ω_i *for* $i=1,2,3,4$, *and* Ω *for a variety of* $t\in(0,1.2)$. *These results are shown in Figure* 1.

Table 1. Numerical results of Ω_i for $i = 1, 2, 3, 4$ and Ω in Ex 4.1.

n	t	Ω_1	Ω_2	Ω_3	Ω_4	Ω
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.05	0.0612	0.0104	0.0125	0.2860	0.0174
3	0.10	0.0949	0.0186	0.0247	0.3509	0.0328
4	0.15	0.1234	0.0267	0.0372	0.3968	0.0480
5	0.20	0.1494	0.0348	0.0502	0.4339	0.0631
6	0.25	0.1739	0.0431	0.0637	0.4658	0.0782
7	0.30	0.1974	0.0517	0.0777	0.4943	0.0936
8	0.35	0.2204	0.0605	0.0925	0.5204	0.1091
9	0.40	0.2430	0.0697	0.1079	0.5447	0.1249
10	0.45	0.2654	0.0793	0.1240	0.5676	0.1408
11	0.50	0.2878	0.0893	0.1409	0.5895	0.1570
12	0.55	0.3101	0.0997	0.1586	0.6104	0.1735
13	0.60	0.3325	0.1107	0.1771	0.6306	0.1901
14	0.65	0.3551	0.1221	0.1966	0.6503	0.2069
15	0.70	0.3778	0.1341	0.2170	0.6694	0.2238
16	0.75	0.4009	0.1466	0.2384	0.6882	0.2409
17	0.80	0.4242	0.1597	0.2609	0.7066	0.2581
18	0.85	0.4479	0.1735	0.2845	0.7248	0.2752
19	0.90	0.4719	0.1880	0.3093	0.7427	0.2923
20	0.95	0.4964	0.2032	0.3354	0.7605	0.3093
21	1.00	0.5213	0.2192	0.3627	0.7781	0.3261
22	1.05	0.5467	0.2360	0.3914	0.7956	0.3426
23	1.10	0.5726	0.2536	0.4216	0.8130	0.3586
24	1.15	0.5990	0.2721	0.4532	0.8303	0.3741
25	1.20	0.6261	0.2916	0.4865	0.8476	0.3888

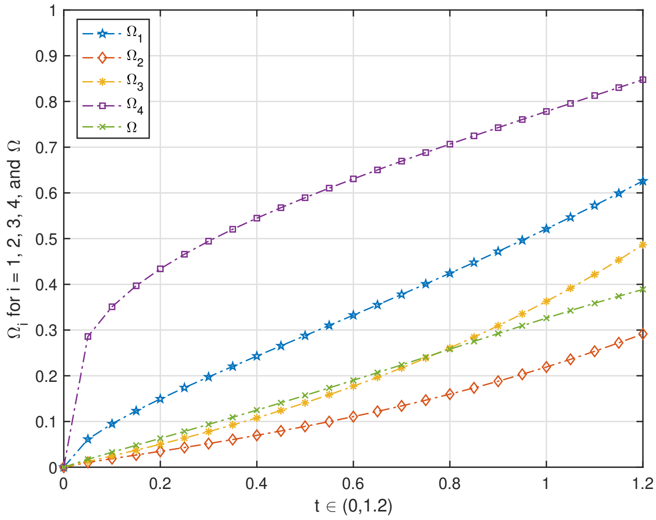

Figure 1. Graphical representation of Ω_i for $i = 1, 2, 3, 4$, and Ω in Example 4.1.

(i) To demonstrate the application of Theorem 3.1, let us take

$$\begin{aligned} f(t,x,y) &= \frac{1}{2} + \frac{6e^{-t}}{(\sqrt{5}+2\sin^2\pi t)^2}\cdot\frac{|x|}{10+|x|} + \frac{3e^{-t}}{5}\cdot\frac{|y|}{5+|y|}, \\ g(t,x,y) &= 1 + \frac{2^{5t-4}}{(10t+8)(1+|x|+|y|)}. \end{aligned} \tag{4.2}$$

For $x_i,\ y_i \in \mathbb{R}$, $i = 1,2$ *and* $t \in [0, 1.2]$, *we get the inequalities*

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &\le \frac{3}{25}\left(|x_1-x_2| + |y_1-y_2|\right), \\ |g(t,x_1,y_1) - g(t,x_2,y_2)| &\le \frac{1}{5}\left(|x_1-x_2| + |y_1-y_2|\right). \end{aligned}$$

The assumption (H_1) *is fulfilled with* $\Lambda_1(\alpha_1+p_1) \approx 0.3530$, $\Lambda_2(\alpha_2+p_2) \approx 0.0336$, $\Lambda_3(\alpha_1+p_1) \approx 0.0626$, $\Lambda_4(\alpha_2+p_2) \approx 0.7669$, $\Lambda_1(p_1) \approx 1.1549$, $\Lambda_2(p_2) \approx 0.0866$, $\Lambda_3(p_1) \approx 1.2941$, $\Lambda_4(p_2) \approx 1.4749$, $L_1 = 3/25$ *and* $L_2 = 1/5$. *Also* $\Phi_1 \approx 0.4329 < 1$, *and thus by Theorem 3.1, the system* (4.1), *with f and g given by (4.2), has a unique solution on* $[0, 1.2]$. *In addition, Tables* 2 *and* 3 *show the numerical results of* $\Lambda_i(U)$ *for* $i = 1,2,3,4$, *where* $U = \{\alpha_1+p_1, \alpha_2+p_2, p_1, p_2\}$ *and* Φ_1 *for a variety of* $t \in (0, 1.2)$. *These results are shown in Figures* 2-4.

Table 2. Numerical results of $\Lambda_i(U)$ for $i = 1, 2, 3, 4$ and $U = \{\alpha_1 + p_1, \alpha_2 + p_2\}$ in Ex. 4.1 (i).

n	t	$\Lambda_1(\alpha_1 + p_1)$	$\Lambda_2(\alpha_2 + p_2)$	$\Lambda_3(\alpha_1 + p_1)$	$\Lambda_4(\alpha_2 + p_2)$
1	0.00	0.0000	0.0000	0.0000	0.0000
2	0.05	0.0008	0.0002	0.0000	0.0092
3	0.10	0.0025	0.0009	0.0002	0.0206
4	0.15	0.0050	0.0020	0.0006	0.0339
5	0.20	0.0085	0.0035	0.0012	0.0491
6	0.25	0.0128	0.0056	0.0022	0.0664
7	0.30	0.0181	0.0083	0.0777	0.0857
8	0.35	0.0244	0.0117	0.0037	0.1074
9	0.40	0.0320	0.0158	0.0058	0.1316
10	0.45	0.0408	0.0208	0.0087	0.1584
11	0.50	0.0510	0.0267	0.0124	0.1882
12	0.55	0.0628	0.0338	0.0173	0.2211
13	0.60	0.0763	0.0421	0.0234	0.2576
14	0.65	0.0918	0.0517	0.0311	0.2979
15	0.70	0.1094	0.0629	0.0407	0.3423
16	0.75	0.1294	0.0759	0.0524	0.3914
17	0.80	0.1520	0.0908	0.0667	0.4455
18	0.85	0.1776	0.1079	0.0841	0.5052
19	0.90	0.2064	0.1274	0.1049	0.5709
20	0.95	0.2389	0.1497	0.1300	0.6434
21	1.00	0.2755	0.1751	0.1953	0.7234
22	1.05	0.3166	0.2039	0.2373	0.8115
23	1.10	0.3628	0.2365	0.2868	0.9087
24	1.15	0.4146	0.2734	0.3451	1.0160
25	1.20	0.4728	0.3152	0.4134	1.1344

Table 3. Numerical results of $\Lambda_i(U)$ for $i = 1, 2, 3, 4$, $U = \{p_1, p_2\}$ and Φ_1 in Ex. 4.1 (i).

n	t	$\Lambda_1(p_1)$	$\Lambda_2(p_2)$	$\Lambda_3(p_1)$	$\Lambda_4(p_2)$	Φ_1
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.05	0.0586	0.0041	0.0026	0.1434	0.0155
3	0.10	0.1040	0.0102	0.0085	0.2162	0.0272
4	0.15	0.1477	0.0178	0.0174	0.2808	0.0393
5	0.20	0.1912	0.0269	0.0292	0.3426	0.0522
6	0.25	0.2353	0.0374	0.0440	0.4038	0.0662
7	0.30	0.2804	0.0493	0.0622	0.4655	0.0814
8	0.35	0.3269	0.0628	0.0839	0.5285	0.0981
9	0.40	0.3750	0.0779	0.1094	0.5933	0.1163
10	0.45	0.4249	0.0948	0.1391	0.6603	0.1364
11	0.50	0.4769	0.1135	0.1733	0.7300	0.1584
12	0.55	0.5313	0.1341	0.2125	0.8029	0.1826
13	0.60	0.5882	0.1569	0.2570	0.8791	0.2091
14	0.65	0.6478	0.1819	0.3075	0.9593	0.2384
15	0.70	0.7105	0.2094	0.3645	1.0437	0.2706
16	0.75	0.7764	0.2395	0.4285	1.1328	0.3061
17	0.80	0.8459	0.2724	0.5003	1.2270	0.3452
18	0.85	0.9192	0.3084	0.5805	1.3269	0.3882
19	0.90	0.9967	0.3476	0.6700	1.4328	0.4357
20	0.95	1.0787	0.3904	0.7696	1.5455	0.4881
21	1.00	1.1656	0.4370	0.8803	1.6654	0.5459
22	1.05	1.2577	0.4877	1.0030	1.7934	0.6097
23	1.10	1.3555	0.5429	1.1391	1.9300	0.6801
24	1.15	1.4596	0.6029	1.2896	2.0760	0.7579
25	1.20	1.5704	0.6680	1.4559	2.2325	0.8439

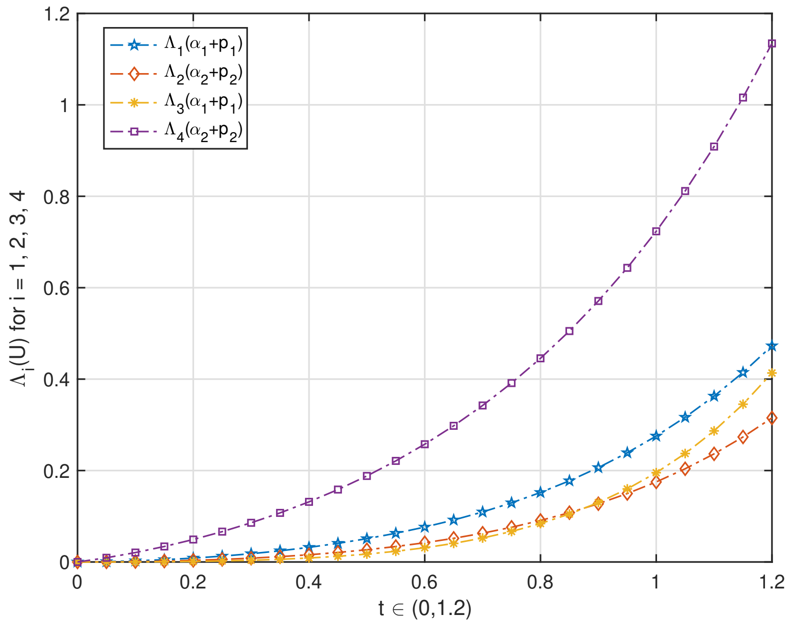

Figure 2. Graphical representation of $\Lambda_i(U)$ for $i = 1, 2, 3, 4$, and $U = \{\alpha_1 + p_1, \alpha_2 + p_2\}$ in Ex. 4.1 (i).

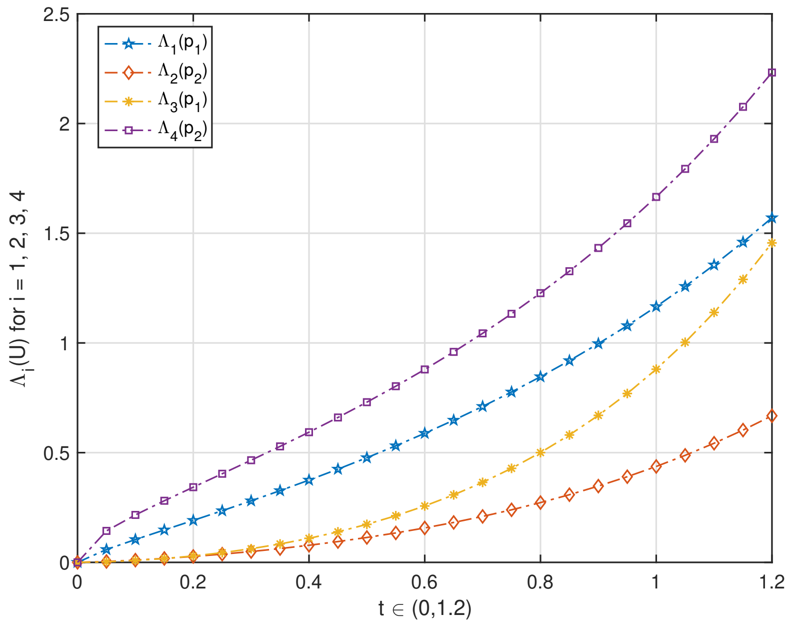

Figure 3. Graphical representation of $\Lambda_i(U)$ for $i = 1, 2, 3, 4$, and $U = \{p_1, p_2\}$ in Ex. 4.1 (i).

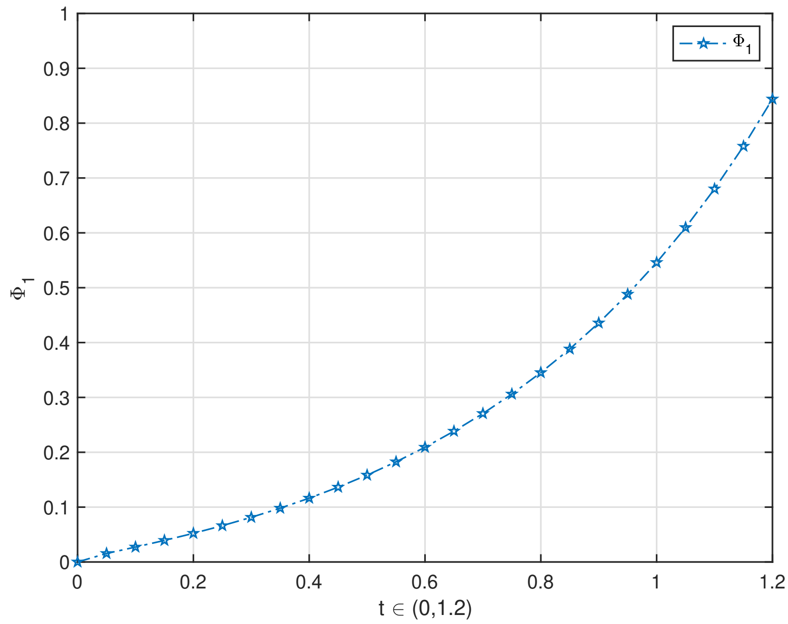

Figure 4. Graphical representation of Φ_1 in Ex. 4.1 (i).

(ii) To illustrate Theorem 3.2 let

$$\begin{aligned} f(t,x,y) &= \frac{1}{10}\sqrt{3+\sin^2(\pi t)} + \frac{6\sin|x|}{2^{2+t}(5+t^2)} + \frac{3\cos^2(\pi t)}{(3t+5)^2}\cdot\frac{|y|}{1+|y|}, \\ g(t,x,y) &= \frac{1+2\cos^2(5\pi t)}{(t+2)^3+2} + \frac{4\sqrt{8+e^{-t^2}}}{(t+5)^2}\cdot\frac{|x|}{2+|x|} + \frac{5t-2}{5}\cdot\frac{\sin|y|}{5+\sin|y|}. \end{aligned} \tag{4.3}$$

For $x, y \in \mathbb{R}$ and $t \in [0, 1.2]$, we have

$$|f(t,x,y)| \le \frac{1}{5} + \frac{3}{10}|x| + \frac{3}{25}|y| \quad \text{and} \quad |g(t,x,y)| \le \frac{3}{10} + \frac{6}{25}|x| + \frac{4}{25}|y|.$$

From (3.11)-(3.12) *with the given datas, we have $E_1(\alpha_1 + p_1) \approx 0.4156$, $E_2(\alpha_2 + p_2) \approx 0.8005$, $E_1(p_1) \approx 1.4489$, and $E_2(p_2) \approx 1.5614$. The assumption (H_2) is satisfied with $K_1 = 1/5$, $K_2 = 3/10$, $K_3 = 3/25$, $\overline{K}_1 = 3/10$, $\overline{K}_2 = 6/25$ and $\overline{K}_3 = 4/25$. Hence, we get $\Phi_2 \approx 0.4617 < 1$ and $\Phi_3 \approx 0.2560 < 1$. Since, all the assumptions of Theorem 3.2 are fulfilled, the system* (4.1)*, with f and g given by (4.3), has at least one solution on $[0, 1.2]$. In addition, Table* 4 *show the numerical results of $E_i(U)$ for $i = 1, 2$, where $U = \{\alpha_1 + p_1, \alpha_2 + p_2, p_1, p_2\}$ and Φ_i for $i = 2, 3$ for a variety of $t \in (0, 1.2)$. These results are shown in Figures* 5-6.

Table 4. Numerical results of $E_i(U)$ for $i = 1, 2$, $U = \{\alpha_1 + p_1, \alpha_2 + p_2, p_1, p_2\}$, Φ_2, and Φ_3 in Ex. 4.1 (ii).

n	t	$E_1(\alpha_1 + p_1)$	$E_2(\alpha_2 + p_2)$	$E_1(p_1)$	$E_2(p_2)$	Φ_2	Φ_3
1	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.000
2	0.05	0.0008	0.0095	0.0612	0.1475	0.0086	0.0090
3	0.10	0.0027	0.0215	0.1126	0.2264	0.0172	0.0151
4	0.15	0.0056	0.0359	0.1651	0.2986	0.0268	0.0213
5	0.20	0.0097	0.0526	0.2204	0.3695	0.0376	0.0281
6	0.25	0.0150	0.0720	0.2793	0.4412	0.0497	0.0354
7	0.30	0.0218	0.0940	0.3426	0.5149	0.0634	0.0434
8	0.35	0.0303	0.1191	0.4107	0.5913	0.0787	0.0522
9	0.40	0.0406	0.1473	0.4843	0.6712	0.0960	0.0620
10	0.45	0.0532	0.1792	0.5640	0.7551	0.1154	0.0728
11	0.50	0.0683	0.2149	0.6502	0.8435	0.1371	0.0848
12	0.55	0.0862	0.2549	0.7438	0.9370	0.1614	0.0980
13	0.60	0.1074	0.2997	0.8452	1.0360	0.1887	0.1126
14	0.65	0.1325	0.3496	0.9554	1.1412	0.2192	0.1289
15	0.70	0.1618	0.4053	1.0750	1.2531	0.2533	0.1469
16	0.75	0.1961	0.4673	1.2050	1.3723	0.2915	0.1669
17	0.80	0.2361	0.5363	1.3462	1.4994	0.3342	0.1891
18	0.85	0.2825	0.6130	1.4997	1.6352	0.3819	0.2138
19	0.90	0.3364	0.6984	1.6667	1.7804	0.4352	0.2411
20	0.95	0.3987	0.7932	1.8483	1.9359	0.4948	0.2715
21	1.00	0.4708	0.8985	2.0458	2.1024	0.5614	0.3054
22	1.05	0.5539	1.0154	2.2607	2.2811	0.6359	0.3430
23	1.10	0.6496	1.1453	2.4946	2.4729	0.7192	0.3848
24	1.15	0.7597	1.2894	2.7492	2.6789	0.8123	0.4314
25	1.20	0.8862	1.4495	3.0263	2.9005	0.9164	0.4833

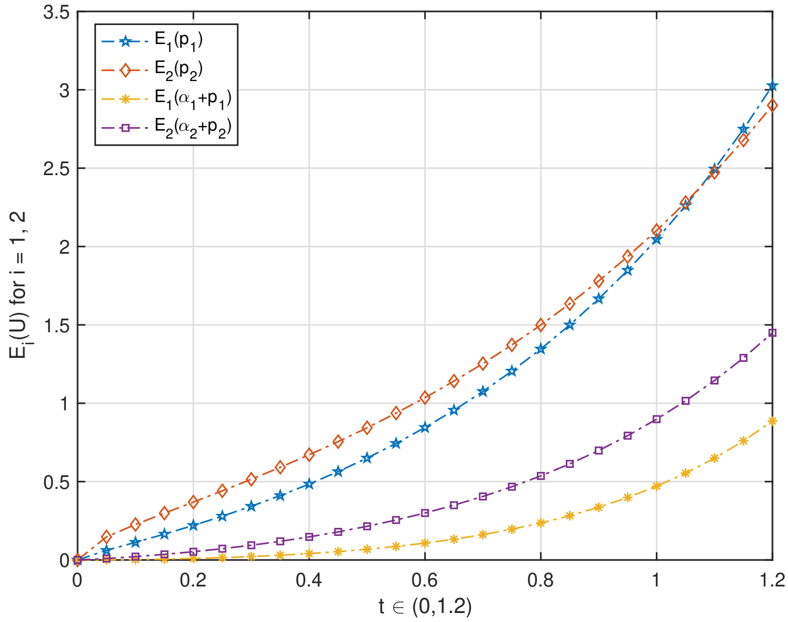

Figure 5. Graphical representation of $E_i(U)$ for $i = 1, 2$, $U = \{\alpha_1 + p_1, \alpha_2 + p_2, p_1, p_2\}$ in Ex. 4.1 (ii).

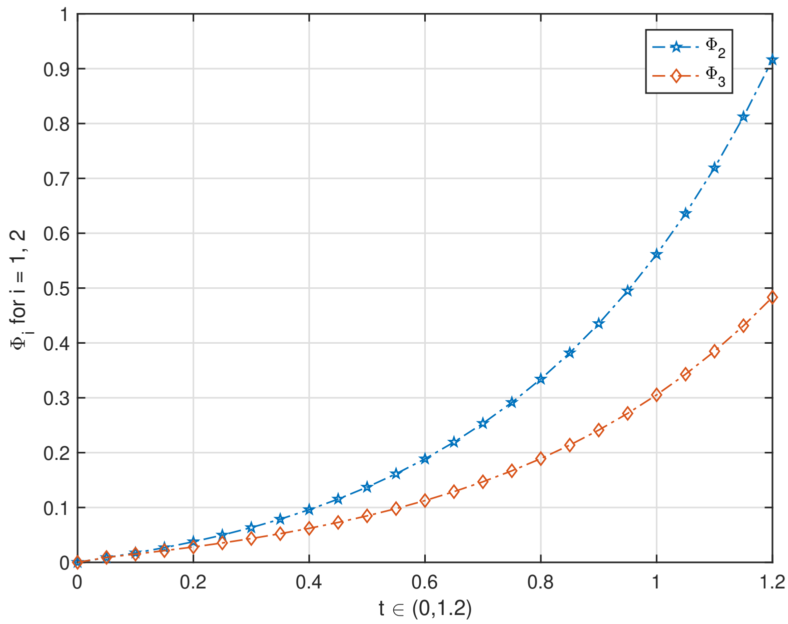

Figure 6. Graphical representation of Φ_2 and Φ_3 in Ex. 4.1 (ii).

5. Conclusions

We have discussed the existence and uniqueness of solutions for a coupled system consisting by ψ-Hilfer fractional order Langevin equations supplemented with nonlocal integral boundary conditions.

We proved the uniqueness of the solutions in the first case using the Banach contraction mapping principle, and we established the existence of the findings in the second case using the Leray-Schauder alternative. The results of the present paper are new and significantly contribute to the existing literature on the topic. Moreover, several new results follow as special cases of the present one.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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