Mathematics

Research article

Dynamical significance of generalized fractional integral inequalities via convexity

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Abstract: The main goal of this paper is to develop the significance of generalized fractional integral inequalities via convex functions. We obtain the new version of fractional integral inequalities with the extended Wright generalized Bessel function acting as a kernel for the convex function, which deals with the Hermite-Hadamard type and trapezoid type inequalities. Moreover, we establish new mid-point type and trapezoid type integral inequalities for (η_1, η_2)-convex function related to Hermite-Hadamard type inequality. We establish new version of integral inequality for (η_1, η_2)-convex function related to Fejér type. The results discussed in this paper are a generalized version of many inequalities in literature.

Keywords: fractional inequalities; (η_1, η_2)-convex function; Hadamard inequality; Wright generalized Bessel function; generalized fractional inequalities
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1. Introduction

Fractional calculus is one of the renowned fields in recent research due to its inherent applications in various areas such as mathematical physics, fluid dynamics, mathematical biology etc. [1–6]. On

the other hand, the fractional integral inequalities with the fractional operators are developed by many researchers because these inequalities are used to verify various results of applied problems [7, 8]. In particular, the researchers [30–33] have recently studied many remarkable fractional integral inequalities and their applications. In [39], Mehmood et al. discussed the Hermite-Hadamard-Fejér inequality for fractional integrals involving preinvex functions. Mehreen and Anwar [40] estimated he Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for p-convex functions by utilizing conformable fractional integrals. In [41], Almutairi and Adem Klçman discussed new integral inequalities of Hermite-Hadamard type involving s-convexity and studied their properties. Budak [42] establish Hermite-Hadamard-Fejér type inequalities for convex function involving fractional integrals with respect to another function. The Hermite-Hadamard inequality is defined can be found in [9] for convex function by

$$\Psi\Big(\frac{m+n}{2}\Big) \le \frac{1}{n-m}\int_m^n \Psi(x)dx \le \frac{\Psi(m)+\Psi(n)}{2}$$

$\Psi : I \to \mathbb{R}$, $m, n \in I, m < n, m, n \in \mathbb{R}, I \subseteq \mathbb{R}$ and is playing a significant role in the field of inequalities and are widely used by the researchers [10].

Fejér type integral inequalities can be found in [27–29] by

$$\Psi\Big(\frac{m+n}{2}\Big)\int_m^n \Phi(x)dx \le \int_m^n \Psi(x)\Phi(x)dx \le \frac{\Psi(m)+\Psi(n)}{2}\int_m^n \Phi(x)dx \qquad (1.1)$$

for convex function $\Psi : [m, n] \to \mathbb{R}$ and $\Phi : [m, n] \to \mathbb{R}^+, m, n \in \mathbb{R}$ where the function Φ is integrable and is symmetric about $x = \frac{m+n}{2}$. Note that the Hermite- Hadamard inequality is obtained if $\Phi = 1$ in Fejér inequality (1.1).

The (η_1, η_2)-convex function has been presented [11–13] by obtaining the generalization of η-convex function [14–17] and preinvex function [18–20].

Sarikaya [25] discussed the Hermite and trapezoid inequalities related to the Hermite-Hadamard inequality, and Rostamian Delavar [13] discussed Fejér, midpoint and trapezoid type inequalities related to the Hermite-Hadamard inequalities.

Definition 1.1. *[26] The convex function $\Psi : I \to \mathbb{R}$ is defined for $t \in [0, 1], \forall u, v \in I$ as follows:*

$$\Psi\Big[tu + (1-t)v\Big] \le t\Psi(u) + (1-t)\Psi(v).$$

Definition 1.2. *[13] An invex set $I \subseteq \mathbb{R}$ with respect to a real bifunction $\theta : I \times I \to \mathbb{R}$, is defined for $m, n \in I, \lambda \in [0, 1]$ as follows*

$$n + \lambda\theta(m, n) \in I.$$

Definition 1.3. *[13] The preinvex function $\Psi : I \to \mathbb{R}$ is defined for $m, n \in I$ and $\lambda \in [0, 1]$ as follows*

$$\Psi(n + \lambda\theta(m, n)) \le \lambda\Psi(m) + (1-\lambda)\Psi(n),$$

where I is an invex set with respect to θ.

Definition 1.4. *[13] A function $\Psi : I \to \mathbb{R}$ is said to be convex with respect to η i.e $(\eta - convex)$ if it satisfies*

$$\Psi(\lambda m + (1 - \lambda)n) \le \Psi(n) + \lambda\eta(\Psi(m), \Psi(n))$$

for all $m, n \in I$ and $\lambda \in [0, 1]$ and $I \subseteq \mathbb{R}$ is a convex function and $\eta : \Psi(I) \times \Psi(I) \to \mathbb{R}$ is a bifunction.

Definition 1.5. *[13] Let $\Psi : I \to \mathbb{R}$, $\eta_1 : I \times I \longrightarrow \mathbb{R}$, and $\eta_2 : \Psi(I) \times \Psi(I) \to \mathbb{R}$, then Ψ is called (η_1, η_2)-convex function if*

$$\Psi(x + \lambda\eta_1(y, x)) \le \Psi(x) + \lambda\eta_2(\Psi(y), \Psi(x))$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Example 1.1. *[12] Let Ψ be the function such that $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by*

$$\Psi(x) = \left\{ \begin{array}{ll} x, & for\ 0 \le x \le 1; \\ 1, & for\ x > 1. \end{array} \right\}.$$

Let the two bifunctions $\eta_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\eta_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\eta_1(x, y) = \left\{ \begin{array}{ll} -y, & for\ 0 \le y \le 1; \\ x + y, & for\ y > 1. \end{array} \right\}$$

$$\eta_2(x, y) = \left\{ \begin{array}{ll} x + y, & for\ x \le y; \\ 2(x + y), & for\ x > y. \end{array} \right\}$$

Then Ψ is (η_1, η_2)-convex function.

Definition 1.6. *[43] The Pochhammer's symbol is defined for $t \in \mathbb{N}$ as*

$$(\Im)_t = \left\{ \begin{array}{ll} 1, & for\ t = 0, \Im \ne 0, \\ \Im(\Im + 1) \cdots (\Im + t - 1), & for\ t \ge 1. \end{array} \right\}$$

$$(\Im)_n = \frac{\Gamma(\Im + n)}{\Gamma(\Im)}$$
$$(\Im)_{kn} = \frac{\Gamma(\Im + kn)}{\Gamma(\Im)}$$

for $\Im \in \mathbb{C}$, where Γ being the gamma function.

Definition 1.7. *[43] The integral representation of gamma function is defined as*

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$$

for, $\Re(t) > 0$.

Definition 1.8. *[35–37] The Classical beta function is defined for $\Re(m) > 0$ and $\Re(n) > 0$,*

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt,$$
$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Definition 1.9. *[38] Extended beta functions is defined for $\Re(m) > 0$, $\Re(n) > 0$, $\Re(p) > 0$ is*

$$B_p(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}exp(\frac{-p}{t(1-t)})dt.$$

Definition 1.10. *[34] Ali et al. defined and investigated the generalized Bessel-Maitland function (eight-parameters) with a new fractional integral operator and discussed its properties and relations with Mittag-Leffler functions. The function of generalized Bessel-Maitland as follows:*

$$J_{\mu,\xi,m,\sigma}^{\nu,\eta,\rho,\gamma}(y) = \sum_{p=0}^{\infty} \frac{(\theta)_{\xi p}(\vartheta)_{\sigma p}(-y)^p}{\Gamma(\phi p + \psi + 1)(\delta)_{mp}},$$

where $\phi, \psi, \theta, \delta, \vartheta \in \mathbb{C}$, $\Re(\phi) > 0$, $\Re(\nu) \geq -1$, $\Re(\theta) > 0$, $\Re(\delta) > 0$, $\Re(\vartheta) > 0$; $\xi, m, \sigma \geq 0$ and $m, \xi > \Re(\phi) + \sigma$.

Definition 1.11. *The extended generalized Bessel-Maitland function is defined for $\mu, \nu, \eta, \rho, \gamma, c \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$; $\xi, m, \sigma \geq 0$ and $m, \xi > \Re(\mu) + \sigma$ by*

$$J_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + \nu + 1)(\rho)_{mn}}(-\omega)^n.$$

In the recent era of research, the field of fractional calculus has gained more recognition due to its wide range of applications in different sciences [44,45]. Such new developments in fractional calculus motivate the researchers to establish some new innovative ideas to unify the fractional operators and propose new inequalities involving new fractional operators.

The Hermite-Hadamard integral inequalities and their extensions have been widely studied for a different type of convexities [46–50]. Here, we defined the following generalized fractional integral operators containing generalized Bessel-Maitland function as its kernel defined, which are the generalization of many well-known fractional integrals:

Definition 1.12. *The generalized fractional integral operators with extended generalized Bessel-Maitland function acting as kernel, is defined for $\mu, \nu, \eta, \rho, \gamma, c \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$; $\xi, m, \sigma \geq 0$ and $m, \xi > \Re(\mu) + \sigma$ as follows*

$$\left(\mathfrak{T}_{\nu,\eta,\rho,\gamma;p^+}^{\mu,\xi,m,\sigma,c}f\right)(x,r) = \int_p^x (x-t)^\nu J_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(x-t)^\mu; r)f(t)dt$$

and

$$\left(\mathfrak{T}_{\nu,\eta,\rho,\gamma;q^-}^{\mu,\xi,m,\sigma,c}f\right)(x,r) = \int_x^q (t-x)^\nu J_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t-x)^\mu; r)f(t)dt.$$

Remark 1.1.   *1. If we put r = 0, w = 0 and replacing v by v − 1 in definition* 1.12*, we get the Riemann-Liouville fractional integral operators* [21]*.*

2. *If we put* σ = 0 *and replace v by v − 1 in definition* 1.12*,* ω *by* −ω*, we get generalized fractional integral operator containing extended generalized Mittag-Leffler function as their kernels defined by Andric et al.* [22]*.*

3. *If we put r = 0,* ξ = 0 *and replacing v by v − 1,* ω *by* −ω *in definition* 1.12*, we get generalized fractional integral operator containing generalized Mittag-Leffler function as their kernels defined in* [23]*.*

4. *If we put r = 0,* ξ = 0*,* σ = 0*,* ρ = m = 1 *in definition* 1.12*, we get the Srivastava fractional integral operator* [24]*.*

This paper aims to obtain Hermite Hadamard and Fejér inequalities using generalized fractional integral having extended generalized Bessel-Maitland function as its kernel.

The structure of the paper follows: In section 2, we present Hermite-Hadamard inequalities for convex function using generalized fractional operator. Section 3 is devoted to Trapezoid type inequalities related to Hermite-Hadamard inequalities. Fejér type inequalities for (η_1, η_2)- convex function using the generalized fractional operator are presented in section 4.

2. Hermite-Hadamard inequalities

In this section, we obtain the Hermite-Hadamard inequalities for convex function using generalized fractional operator as follows:

Theorem 2.1. *Let* $\Psi : [u, v] \to \mathbb{R}$ *be a convex function where* $0 < u < v$ *and* $\Psi \in L_1[u, v]$*. If* Ψ *is an increasing function on* $[u, v]$*, then for the generalized fractional integrals defined in definition* 1.12*, we have*

$$\Psi\left(\frac{u+v}{2}\right)(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;v^-} 1)(u, p) \leq \frac{1}{2}\left[\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(v)^-} \Psi\right)(u, p) + \left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+} \Psi\right)(v, p)\right]$$

$$\leq \frac{\Psi(u) + \Psi(v)}{2}(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;u^+} 1)(v, p).$$

Proof. By the convexity of Ψ on the interval $[u, v]$, let $x, y \in [u, v]$ with $t = \frac{1}{2}$, we have

$$\Psi(\frac{x+y}{2}) \leq \frac{\Psi(x) + \Psi(y)}{2},$$

where if we takes

$$x = tu + (1 − t)v, y = (1 − t)u + tv$$

leads to

$$2\Psi(\frac{u+v}{2}) \leq \Psi(tu + (1 − t)v) + \Psi((1 − t)u + tv).$$

Multiplying both sides by $(1-t)^{v'} \mathfrak{S}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-t)^{\mu}; p)$ and integrating the resulting inequality on $[0, 1]$ with respect to t, we have

$$2\Psi(\frac{u+v}{2}) \int_0^1 (1-t)^{v'} \mathfrak{S}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(1-t)^{\mu}; p)dt$$

$$\leq \int_0^1 (1-t)^{v'} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)\Psi(tu+(1-t)v)dt$$

$$+ \int_0^1 (1-t)^{v'} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)\Psi((1-t)u+tv)dt$$

$$2\Psi(\frac{u+v}{2})\sum_{n=0}^\infty \frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n \int_0^1 (1-t)^{v'+\mu n}dt$$

$$\leq \sum_{n=0}^\infty \frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n \times$$

$$\Big[\int_0^1 (1-t)^{v'+\mu n}\Psi(tu+(1-t)v)dt + \int_0^1 (1-t)^{v'+\mu n}\Psi((1-t)u+tv)dt\Big]. \tag{2.1}$$

By making suitable substitutions in inequality (2.1), we obtain

$$2\Psi(\frac{u+v}{2})(\mathfrak{T}_{v',\eta,\rho,\gamma;v^-}^{\mu,\xi,m,\sigma,c}1)(u,p) \leq \Big[\int_u^v (\frac{z-u}{v-u})^{v'} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{z-u}{v-u})^\mu; p)\Psi(z)dz$$

$$+ \int_u^v (\frac{v-z}{v-u})^{v'} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{v-z}{v-u})^\mu; p)\Psi(z)dz\Big]$$

or

$$\Psi(\frac{u+v}{2})(\mathfrak{T}_{v',\eta,\rho,\gamma;v^-}^{\mu,\xi,m,\sigma,c}1)(u,p) \leq \frac{1}{2}\Big[(\mathfrak{T}_{v',\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(v;p) + (\mathfrak{T}_{v',\eta,\rho,\gamma;v^-}^{\mu,\xi,m,\sigma,c}\Psi)(u;p)\Big]. \tag{2.2}$$

For second part of inequality, again using the convexity of Ψ,

$$\Psi(tu+(1-t)v) \leq t\Psi(u)+(1-t)\psi(v)$$

and

$$\Psi((1-t)u+tv) \leq (1-t)\Psi(u)+t\psi(v).$$

Addition of these inequalities, gives

$$\Psi(tu+(1-t)v)+\Psi((1-t)u+tv) \leq (\Psi(u)+\psi(v)).$$

Multiplying both sides by $(1-t)^{v'}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)$ and integrating the resulting inequality on $[0,1]$ with respect to t, we get

$$\int_0^1 (1-t)^{v'}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)\Psi(tu+(1-t)v)dt$$

$$+ \int_0^1 (1-t)^{v'}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)\Psi((1-t)u+tv)dt$$

$$\leq (\Psi(u)+\Psi(v))\int_0^1 (1-t)^{v'}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)dt.$$

Making substitution in the integrals involved leads to

$$\frac{1}{2}\Big[(\mathfrak{T}_{v',\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(v;p) + (\mathfrak{T}_{v',\eta,\rho,\gamma;v^-}^{\mu,\xi,m,\sigma,c}\Psi)(u;p)\Big] \leq \frac{\Psi(u)+\Psi(v)}{2}(\mathfrak{T}_{v',\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}1)(v,p), \tag{2.3}$$

combining (2.2) and (2.3), we get the desired result. □

3. Trapezoid inequalities related to the Hermite-Hadamard inequalities

The Trapezoid type inequalities related to the Hermite-Hadamard inequalities are presented in this section.

Lemma 3.1. *Let a function* $\Psi : I \to \mathbb{R}$ *with* $I = [u, v] \subseteq \mathbb{R}$, $\Psi' \in L_1[u, v]$ *be a differentiable function on* (u, v). *Then for the generalized fractional integrals defined in definition* (1.12), *we have*

$$\frac{\Psi(u) + \Psi(v)}{2} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2(v-u)}\left[(\mathfrak{I}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(v; p) + (\mathfrak{I}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p)\right]$$

$$= \frac{v-u}{2}I$$

where

$$I = \int_0^1 (1-t)^{v'} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)\Psi'(ut + (1-t)v)dt +$$

$$\int_0^1 -t^{v'} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^\mu; p)\Psi'(ut + (1-t)v)dt.$$

Proof. If we consider the integral

$$I = \int_0^1 (1-t)^{v'} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)\Psi'(ut + (1-t)v)dt$$

$$+ \int_0^1 -t^{v'} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^\mu; p)\Psi'(ut + (1-t)v)dt.$$

Let

$$I = I_1 + I_2.$$

Firstly, we consider I_1

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \int_0^1 (1-t)^{v'+\mu n}\Psi'(ut + (1-t)v)dt.$$

Integrating by parts, we have

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\left[(1-t)^{v'+\mu n}\frac{\Psi(ut + (1-t)v}{u-v}\Big|_0^1\right.$$

$$+ \frac{v' + \mu n}{u - v} \int_0^1 (1-t)^{v'+\mu n-1}\Psi(ut + (1-t)v)dt\Big]$$

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\left[\frac{\Psi(v)}{v - u} - \frac{v' + \mu n}{(v - u)^2} \int_u^v (\frac{x - u}{v - u})^{v'+\mu n-1}\Psi(x)dx\right]$$

$$I_1 = \frac{\Psi(v)}{v - u}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{(v - u)^2}(\mathfrak{I}_{v'-1,\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Psi)(v; p).$$

On the same lines, we get

$$I_2 = \frac{\Psi(u)}{v-u}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{(v-u)^2}(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p)$$

implies

$$I = \frac{\Psi(u)+\Psi(v)}{v-u}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{(v-u)^2}\times$$
$$\Big[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p) + \mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Psi](v; p)\Big].$$

Multiplying by $\frac{v-u}{2}$, we have the required result. $\qquad\qquad\qquad\square$

By using Lemma 3.1, we present the following theorem.

Theorem 3.1. *Let a function* $\Psi : I = [u, v] \to \mathbb{R}$ *with* $I \in \mathbb{R}$ *be a differentiable function on* (u, v). *Also, suppose that* $|\Psi'|$ *is a convex function on I, then for the generalized fractional integrals in definition 1.12, we have*

$$\Big|\frac{\Psi(u)+\Psi(v)}{2}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2(v-u)}\times$$
$$\Big[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(v; p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p)\Big]\Big|$$
$$\le \frac{v-u}{2}\Big|\mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2^{v'}}\mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\tfrac{1}{2})^\mu; p)\Big|[|\Psi'(u)| + |\Psi'(v)|],$$

where $v' \ge 0$.

Proof. If we consider the following integral expression

$$\Big|\frac{\Psi(u)+\Psi(v)}{2}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2(v-u)}\times$$
$$\Big[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(v; p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p)\Big]\Big|$$
$$= \Big|\frac{v-u}{2}I\Big|$$
$$\le \frac{v-u}{2}\sum_{n=0}^\infty \Big|\frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\Big|\times$$
$$\int_0^1 \Big|(1-t)^{v'+\mu n} - t^{v'+\mu n}\Big|\Big|\Psi'(ut + (1-t)v)\Big|dt$$
$$\le \frac{v-u}{2}\sum_{n=0}^\infty \Big|\frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\Big|\times$$
$$\int_0^1 \Big|(1-t)^{v'+\mu n} - t^{v'+\mu n}\Big|\Big[|\Psi'(u)|t + (1-t)|\Psi'(v)|\Big]dt\Big]$$
$$= \frac{v-u}{2}\sum_{n=0}^\infty \Big|\frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\Big|\times$$
$$\Big[\int_0^{\frac{1}{2}}((1-t)^{v'+\mu n} - t^{v'+\mu n})\Big[|\Psi'(u)|t + (1-t)|\Psi'(v)|\Big]dt$$

$$+ \int_{\frac{1}{2}}^{1} (t^{v'+\mu n} - (1-t)^{v'+\mu n})[|\Psi'(u)|t + (1-t)|\Psi'(v)|]dt\Big].$$

Solving the integrals involved by using integrating by parts method, we obtain

$$\Big|\frac{\Psi(u) + \Psi(v)}{2} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2(v-u)} \times$$
$$\Big[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(v; p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p)\Big]\Big|$$
$$\le \frac{v-u}{2}\Big|\mathfrak{J}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2^{v'}}\mathfrak{J}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{1}{2})^\mu; p)\Big|\Big[|\Psi'(u)| + |\Psi'(v)|\Big].$$

$\square$

4. Fejér type inequalities for (η_1, η_2)-convex function

Here, we present Fejér type inequalities for (η_1, η_2)-convex function by using the generalized fractional operator in definition 1.12.

Theorem 4.1. *Let $\Psi : I \to \mathbb{R}$, be an (η_1, η_2)-convex functions such that η_2 is an integrable bi-function on $\Psi(I) \times \Psi(I)$ and for any $u, v \in I, \eta_1(v, u) > 0$ with $\Psi \in L_1[u, u + \eta_1(v, u)]$ and the function $\Phi : [u, u + \eta_1(v, u)] \to \mathbb{R}^+$ is integrable and symmetric to $u + \frac{1}{2}\eta_1(v, u)$ i.e., $\Phi(2u + \eta_1(v, u) - x) = \Phi(x)$, where $I \subseteq \mathbb{R}$ be an invex set with respect to η_1 such that*

$$\eta_1(v + t_2\eta_1(u, v), v + t_1\eta_1(u, v)) = (t_2 - t_1)\eta_1(u, v), \tag{4.1}$$

for all $t_1, t_2 \in [0, 1]$. Then for the generalized fractional integrals defined in definition (1.12), the following Fejér type inequality holds:

$$\Psi\Big(\frac{2u + \eta_1(v, u)}{2}\Big)\Big[\Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Phi\Big)(u, p) + \Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Phi\Big)(u + \eta_1(v, u), p)\Big]$$
$$-\frac{1}{2}\int_u^{u+\eta_1(v,u)}\Big[(u + \eta_1(v, u) - x)^{v'}\mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(u + \eta_1(v, u) - x)^\mu; p)$$
$$+ (x - u)^{v'}\mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(x - u)^\mu; p)\Big] \times \eta_2(\Psi(x), \Psi(2u + \eta_1(v, u) - x)).\Phi(x)dx$$
$$\le \Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi\Phi\Big)(u, p) + \Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Psi\Phi\Big)(u + \eta_1(v, u), p).$$

Proof. By the (η_1, η_2)-convexity of the function Ψ and using (4.1),we get

$$\Psi\Big(\frac{2u + \eta_1(v, u)}{2}\Big) = \Psi\Big(\frac{2u + (1+t)\eta_1(v, u)}{2} - \frac{t\eta_1(v, u)}{2}\Big)$$
$$= \Psi\Big(\frac{2u + (1+t)\eta_1(v, u)}{2} + \frac{1}{2}\eta_1\Big(u + \frac{(1-t)}{2}\eta_1(v, u), u + \frac{(1+t)}{2}\eta_1(v, u)\Big)\Big),$$

or

$$\Psi\left(\frac{2u + \eta_1(v, u)}{2}\right)$$
$$\leq \Psi\left(\frac{2u + (1 + t)\eta_1(v, u)}{2} + \frac{1}{2}\eta_2\left(\Psi\left(\frac{2u + (1 - t)\eta_1(v, u)}{2}\right), \Psi\left(\frac{2u + (1 + t)\eta_1(v, u)}{2}\right)\right)\right). \tag{4.2}$$

Now by adapting the same procedure as above, we obtain

$$\Psi\left(\frac{2u + \eta_1(v, u)}{2}\right)$$
$$\leq \Psi\left(\frac{2u + (1 - t)\eta_1(v, u)}{2} + \frac{1}{2}\eta_2\left(\Psi\left(\frac{2u + (1 + t)\eta_1(v, u)}{2}\right), \Psi\left(\frac{2u + (1 - t)\eta_1(v, u)}{2}\right)\right)\right). \tag{4.3}$$

Using the generalized fractional integral operators defined in 1.12, we have

$$I_1 = \left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\Phi\right)(u; p)$$
$$= \int_u^{u+\eta_1(v,u)} (x - u)^{v'} J^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(x - u)^\mu; p)\Psi(x)\Phi(x)dx$$
$$= \sum_{n=0}^\infty \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \int_u^{u+\eta_1(v,u)} (x - u)^{v'+\mu n}\Psi(x)\Phi(x)dx$$

$$I_1 = \sum_{n=0}^\infty \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v + 1)(\rho)_{mn}}(-\omega)^n\left[\int_u^{u+\frac{1}{2}\eta_1(v,u)} (x - u)^{v'+\mu n}\Psi(x)\Phi(x)dx\right.$$
$$\left. + \int_{u+\frac{1}{2}\eta_1(v,u)}^{u+\eta_1(v,u)} (x - u)^{v'+\mu n}\Psi(x)\Phi(x)dx\right]. \tag{4.4}$$

By making the substitution $x = \frac{2u+(1-t)\eta_1(v,u)}{2}$ and $x = \frac{2u+(1+t)\eta_1(v,u)}{2}$ respectively in the integrals appearing in (4.4), we have

$$I_1 = \sum_{n=0}^\infty \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\left(\frac{\eta_1(v, u)}{2}\right)^{v'+\mu n+1}$$
$$\left[\int_0^1 (1 - t)^{v'+\mu n}\Psi\left(\frac{2u + (1 - t)\eta_1(v, u)}{2}\right)\Phi\left(\frac{2u + (1 - t)\eta_1(v, u)}{2}\right)dt\right.$$
$$\left. + \int_0^1 (1 + t)^{v'+\mu n}\Psi\left(\frac{2u + (1 + t)\eta_1(v, u)}{2}\right)\Phi\left(\frac{2u + (1 + t)\eta_1(v, u)}{2}dt\right)\right].$$

By using the inequalities (4.2) and (4.3), we proceed

$$I_1 \geq \sum_{n=0}^\infty \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\left(\frac{\eta_1(v, u)}{2}\right)^{v'+\mu n+1}$$
$$\left[\int_0^1 (1 - t)^{v'+\mu n}\Psi\left(\frac{2u + \eta_1(v, u)}{2}\right)\Phi\left(\frac{2u + (1 - t)\eta_1(v, u)}{2}\right)dt\right.$$

$$- \frac{1}{2} \int_0^1 (1-t)^{v'+\mu n} \eta_2 \left(\Psi\left(\frac{2u+(1+t)\eta_1(v,u)}{2}\right), \Psi\left(\frac{2u+(1-t)\eta_1(v,u)}{2}\right) \right)$$

$$\Phi\left(\frac{2u+(1-t)\eta_1(v,u)}{2}\right) dt$$

$$+ \int_0^1 (1+t)^{v'+\mu n} \Psi\left(\frac{2u+\eta_1(v,u)}{2}\right) \Phi\left(\frac{2u+(1+t)\eta_1(v,u)}{2}\right) dt$$

$$- \frac{1}{2} \int_0^1 (1+t)^{v'+\mu n} \eta_2 \left(\Psi\left(\frac{2u+(1-t)\eta_1(v,u)}{2}\right), \Psi\left(\frac{2u+(1+t)\eta_1(v,u)}{2}\right) \right)$$

$$\Phi\left(\frac{2u+(1+t)\eta_1(v,u)}{2}\right) dt \bigg],$$

or

$$I_1 \geq \Psi\left(\frac{2u+\eta_1(v,u)}{2}\right) \left(\frac{\eta_1(v,u)}{2}\right)^{v'+\mu n+1} \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}} (-\omega)^n$$

$$\left[\int_0^1 (1-t)^{v'+\mu n} \Phi\left(\frac{2u+(1-t)\eta_1(v,u)}{2}\right) dt - \frac{1}{2} \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}} (-\omega)^n \times \right.$$

$$\left(\frac{\eta_1(v,u)}{2}\right)^{v'+\mu n+1} \int_0^1 (1-t)^{v'+\mu n} \eta_2 \left(\Psi\left(\frac{2u+(1+t)\eta_1(v,u)}{2}\right), \Psi\left(\frac{2u+(1-t)\eta_1(v,u)}{2}\right) \right) \times$$

$$\Phi\left(\frac{2u+(1-t)\eta_1(v,u)}{2}\right) dt$$

$$+ \Psi\left(\frac{2u+\eta_1(v,u)}{2}\right) \left(\frac{\eta_1(v,u)}{2}\right)^{v'+\mu n+1} \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}} (-\omega)^n \times$$

$$\int_0^1 (1+t)^{v'+\mu n} \Phi\left(\frac{2u+(1+t)\eta_1(v,u)}{2}\right) dt$$

$$- \frac{1}{2} \left(\frac{\eta_1(v,u)}{2}\right)^{v'+\mu n+1} \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}} (-\omega)^n$$

$$\int_0^1 (1+t)^{v'+\mu n} \eta_2 \left(\Psi\left(\frac{2u+(1-t)\eta_1(v,u)}{2}\right), \Psi\left(\frac{2u+(1+t)\eta_1(v,u)}{2}\right) \right)$$

$$\Phi\left(\frac{2u+(1+t)\eta_1(v,u)}{2}\right) dt \bigg].$$

Again by simplification and using the above mentioned substitution as well as the symmetry of Φ to $u + \frac{1}{2}\eta_1(v,u)$ leads to the following

$$I_1 \geq \Psi\left(\frac{2u+\eta_1(v,u)}{2}\right) \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}} (-\omega)^n$$

$$\left[\int_u^{u+\eta_1(v,u)} (x-u)^{v'+\mu n} \Phi(x) dx - \frac{1}{2} \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}} (-\omega)^n \times \right.$$

$$\int_u^{u+\eta_1(v,u)} (u+\eta_1(v,u)-x)^{v'+\mu n} \eta_2 \left(\Psi(x), \Psi(2u+\eta_1(v,u)-x) \right) \Phi(x) dx.$$

It follows that

$$I_1 \geq \Psi\left(\frac{2u + \eta_1(v,u)}{2}\right)\left[\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Phi\right)(u,p) - \frac{1}{2}\int_u^{u+\eta_1(v,u)}(u+\eta_1(v,u)-x)^{v'}\times\right.$$
$$\mathrm{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(u+\eta_1(v,u)-x)^\mu;p)\times\eta_2(\Psi(x),\Psi(2u+\eta_1(v,u)-x)).\Phi(x)dx.$$

Now, consider

$$I_2 = \left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\Phi\right)(u+\eta_1(v,u),p)$$
$$= \int_u^{u+\eta_1(v,u)}(u+\eta_1(v,u)-x)^{v'}\mathrm{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(u+\eta_1(v,u)-x)^\mu;p)\Psi(x)\Phi(x)dx.$$

Solving on the same pattern as used above, we get

$$I_2 \geq \Psi\left(\frac{2u + \eta_1(v,u)}{2}\right)\left[\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;u^+}\Phi\right)(u+\eta_1(v,u),p)\times\right.$$
$$-\frac{1}{2}\int_u^{u+\eta_1(v,u)}(x-u)^{v'}\mathrm{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(x-u)^\mu;p)\times\eta_2(\Psi(x),\Psi(2u+\eta_1(v,u)-x)).\Phi(x)dx.$$

By adding I_1 and I_2, we get the required inequality. $\qquad\square$

Lemma 4.1. *Let I be an invex subset of $\mathbb{R}$ with respect to $\eta_1 : I \times I \longrightarrow \mathbb{R}$. Let $u, v \in I$ satisfying $\eta_1(v,u) > 0$ and $\Phi : [u, u + \eta_1(v,u)] \longrightarrow \mathbb{R}$ be integrable and symmetric about $u + \frac{1}{2}\eta_1(v,u)$. Then for the integrals defined in definition (1.12), the following holds;*

$$\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\right)(u+\eta_1(v,u);p) = \left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u)^-}\Psi\right)(u;p)$$
$$= \frac{1}{2}\left[\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\right)(u+\eta_1(v,u);p) + \left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\right)(u;p)\right].$$

Proof. If we consider

$$\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\right)(u+\eta_1(v,u);r)$$
$$= \int_u^{u+\eta_1(v,u)}(u+\eta_1(v,u)-x)^{v'}\mathrm{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(u+\eta_1(v,u)-x)^\mu;p)\Psi(x)dx. \qquad (4.5)$$

By substituting $x = 2u + \eta_1(v,u) - t$ in (4.5), we get

$$\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\right)(u+\eta_1(v,u);r) = \int_u^{u+\eta_1(v,u)}(t-u)^{v'}\mathrm{J}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(t-u)^\mu;p)\Psi(2u+\eta_1(v,u)-t)dt$$

$$\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\right)(u+\eta_1(v,u);p) = \left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\right)(u;p). \qquad (4.6)$$

Addition of $\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\right)(u;p)$ in Eq (4.6) on both sides, leads to the required result. $\qquad\square$

Theorem 4.2. *Let* $\Psi : I \to \mathbb{R}$, *be an* (η_1, η_2)-*convex functions such that* η_2 *is an integrable bifunction on* $\Psi(I) \times \Psi(I)$ *and for any* $u, v \in I$, $\eta_1(v, u) > 0$ *with* $\Psi \in L_1[u, u + \eta_1(v, u)]$ *and the function* $\Phi : [u, u + \eta_1(v, u)] \to \mathbb{R}^+$ *is integrable and symmetric to* $u + \frac{1}{2}\eta_1(v, u)$ *i.e* $\Phi(2u + \eta_1(v, u) - x) = \Phi(x)$, *where* $I \subseteq \mathbb{R}$ *be an invex set with respect to* η_1. *Then for the generalized fractional integrals defined in definition* (1.12), *the following Fejér type inequality holds:*

$$
\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\Phi\right)(u, p) + \left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\Phi\right)(u + \eta_1(v, u), p)
$$

$$
\leq \left(\frac{2\Psi(u) + \eta_2(\Psi(v), \Psi(u))}{2}\right)\times
$$

$$
\left[\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\Phi\right)(u, p) + \left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\Phi\right)(u + \eta_1(v, u), p)\right].
$$

Proof. If we consider the integral

$$
\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\Phi\right)(u; p) = \int_u^{u+\eta_1(v,u)} (x - u)^{v'} J^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(x - u)^\mu; p)\Psi(x)\Phi(x)dx
$$

$$
= \sum_{n=0}^\infty \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \int_u^{u+\eta_1(v,u)} (x - u)^{v'+\mu n}\Psi(x)\Phi(x)dx.
$$

Making substitution $x = u + t\eta_1(v, u)$ leads to following integral,

$$
\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\Phi\right)(u; p) = \sum_{n=0}^\infty \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\times
$$

$$
(\eta_1(v, u))^{v'+\mu n+1} \int_0^1 (t)^{v'+\mu n}\Psi(u + t\eta_1(v, u))\Phi(u + t\eta_1(v, u))dt.
$$

Using the (η_1, η_2)-convexity of Ψ, we get

$$
\leq \sum_{n=0}^\infty \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n\times
$$

$$
(\eta_1(v, u))^{v'+\mu n+1} \int_0^1 (t)^{v'+\mu n}\Big(\Psi(u) + t\eta_2(\Psi(v), \Psi(u))\Big)\Phi(u + t\eta_1(v, u))dt
$$

$$
= \sum_{n=0}^\infty \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n(\eta_1(v, u))^{v'+\mu n+1}
$$

$$
\left[\Psi(u) \int_0^1 (t)^{v'+\mu n}\Phi(u + t\eta_1(v, u))dt + \eta_2(\Psi(v), \Psi(u)) \int_0^1 (t)^{v'+\mu n+1}\Phi(u + t\eta_1(v, u))dt\right]. \quad (4.7)
$$

Now, consider

$$
\left(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\Phi\right)(u + \eta_1(v, u); p) = \int_u^{u+\eta_1(v,u)} (u + \eta_1(v, u) - x)^{v'}\times
$$

$$\mathrm{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(u+\eta_1(v,u)-x)^\mu;p)\Psi(x)\Phi(x)dx$$

$$= \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \int_u^{u+\eta_1(v,u)} (u+\eta_1(v,u)-x)^{v'+\mu n}\Psi(x)\Phi(x)dx$$

$$\leq \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \times$$

$$(\eta_1(v,u))^{v'+\mu n+1} \int_0^1 (t)^{v'+\mu n}\Big(\Psi(u)+(1-t)\eta_2(\Psi(v),\Psi(u))\Big)\Phi(u+(1-t)\eta_1(v,u))dt$$

$$= \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n(\eta_1(v,u))^{v'+\mu n+1}$$

$$\Big[\Psi(u) \int_0^1 (t)^{v'+\mu n}\Phi(u+(1-t)\eta_1(v,u))dt + \eta_2(\Psi(v),\Psi(u))$$

$$\times \int_0^1 (t)^{v'+\mu n}(1-t)\Phi(u+(1-t)\eta_1(v,u))dt\Big]. \tag{4.8}$$

Adding Eqs (4.7) and (4.8) and using the symmetry of Φ about $u+\frac{1}{2}\eta_1(v,u)$, we have

$$\Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Psi\Phi\Big)(u+\eta_1(v,u);p) + \Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi\Phi(u;p)\Big)$$

$$\leq \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n(\eta_1(v,u))^{v'+\mu n+1} \times$$

$$\Big[2\Psi(u) \int_0^1 (t)^{v'+\mu n}\Phi(u+t\eta_1(v,u))dt + \eta_2(\Psi(v),\Psi(u)) \int_0^1 (t)^{v'+\mu n}\Phi(u+t\eta_1(v,u))dt\Big]$$

$$= \Big(2\Psi(u)+\eta_2(\Psi(v),\Psi(u))\Big) \sum_{n=0}^{\infty} \frac{\beta_p(\eta+\xi n, c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c-\eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n(\eta_1(v,u))^{v'+\mu n+1} \times$$

$$\int_0^1 (t)^{v'+\mu n}\Phi(u+(t)\eta_1(v,u))dt.$$

$$= \Big(\frac{2\Psi(u)+\eta_2(\Psi(v),\Psi(u))}{2}\Big)\Big(2\mathfrak{T}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Phi\Big)(u;p).$$

By using lemma 4.1, we have

$$\Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi\Phi\Big)(u;p) + \Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Psi\Phi\Big)(u+\eta_1(v,u);p)$$

$$\leq \Big(\frac{2\Psi(u)+\eta_2(\Psi(v),\Psi(u))}{2}\Big)\Big[\Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Phi\Big)(u+\eta_1(v,u);p) + \Big(\mathfrak{T}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Phi\Big)(u;p)\Big].$$

$$\square$$

Corollary 4.1. *In Fejér type inequalities defined in Theorems 4.1 and 4.2, if we take $\eta_1(u,v) = u - v$, $\forall u, v \in I$,*

$$\Psi(\frac{u+v}{2})\Big[\big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Phi\big)(v;p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(v)^-}\Phi\big)(u;p)\Big]$$

$$- \frac{1}{2}\int_u^v \Big[(v-x)^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(v-x)^\mu;p)$$

$$+ (x-u)^{v'}\mathfrak{I}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma}(\omega(x-u)^\mu;p)\Big] \times \eta_2(\Psi(x),\Psi(u+v-x)).\Phi(x)dx$$

$$\leq \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(v)^-}\Psi\Phi\big)(u;p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\Phi\big)(v;p)$$

$$\leq \Big(\frac{2\Psi(u) + \eta_2(\Psi(v),\Psi(u))}{2}\Big)\Big[\big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Phi\big)(v;p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(v)^-}\Phi\big)(u;p)\Big],$$

which is Fejér inequality for generalized fractional integral can be obtained by considering the function Ψ to be η-convex.

Corollary 4.2. *In Theorems 4.1 and 4.2, if we put $\eta_2(u,v) = u - v$, for all $u, v \in \Psi(I)$, then*

$$\Psi(\frac{2u + \eta_1(v,u)}{2})\Big[\big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Phi\big)(u + \eta_1(v,u);p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Phi\big)(u;p)\Big]$$

$$\leq \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Psi\Phi\big)(u;p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\Phi\big)(u + \eta_1(v,u);p)$$

$$\leq \Big(\frac{\Psi(u) + \Psi(v)}{2}\Big)\Big[\big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Phi\big)(u + \eta_1(v,u);p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}\Phi\big)(u;p)\Big],$$

which is Fejér inequality for generalized fractional integral can be obtained by considering the function Ψ to be preinvex convex.

Corollary 4.3. *In Fejér type inequalities 4.1 and 4.2, if we put $\eta_1(u,v) = u - v$, $\forall u, v \in I$ and $\eta_2(x,y) = x - y$, $\forall x, y \in \Psi(I)$,*

$$\Psi(\frac{u+v}{2})\Big[\big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Phi\big)(v;p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(v)^-}\Phi\big)(u;p)\Big]$$

$$\leq \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(v)^-}\Psi\Phi\big)(u;p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Psi\Phi\big)(v;p)$$

$$\leq \Big(\frac{\Psi(u) + \Psi(v)}{2}\Big)\Big[\big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u)^+}\Phi\big)(v;p) + \big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(v)^-}\Phi\big)(u;p)\Big],$$

which is Fejér type inequality for generalized fractional integral can be obtained by considering the function Ψ to be convex.

Corollary 4.4. *In corollary 4.3, if we take $\Phi = 1$, we get Hermite-Hadamard type inequality for convex function discussed in 2.1.*

Corollary 4.5. *In Fejér type inequalities defined in 4.1 and 4.2, if we take $\Phi = 1$ then can obtain the Hermite-Hadamard type inequality for (η_1, η_2)-convex function as*

$$\Psi\Big(\frac{2u + \eta_1(v,u)}{2}\Big)\big(\mathfrak{T}^{\mu,\xi,m,\sigma,c}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}1\big)(u,p)$$

$$- \frac{1}{4} \int_u^{u+\eta_1(v,u)} \left[(u + \eta_1(v,u) - x)^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(u + \eta_1(v,u) - x)^{\mu}; p) \right.$$

$$\left. + (x - u)^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(x - u)^{\mu}; p) \right] \times \eta_2(\Psi(x), \Psi(2u + \eta_1(v,u) - x)) dx$$

$$\leq \frac{1}{2} \left[\left(\mathfrak{T}_{v',\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c} \Psi \right)(u + \eta_1(v,u), p) + \left(\mathfrak{T}_{v',\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c} \Psi \right)(u, p) \right]$$

$$\leq \left(\frac{2\Psi(u) + \eta_2(\Psi(v), \Psi(u))}{2} \right) \left(\mathfrak{T}_{v',\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c} 1 \right)(u + \eta_1(v,u), p).$$

5. Mid point and trapezoid type inequalities related to Hermite-Hadamard type inequalities

In the section, we discuss the midpoint and trapezoid type inequalities connected to Hermite-Hadamard inequalities for the function whose absolute value of the derivative is (η_1, η_2)-convex function. The following lemma will help us in the next result.

Lemma 5.1. *Let a function* $\Psi : I \to \mathbb{R}$ *with* $I \in \mathbb{R}$ *,* $\Psi \in L_1[u, u + \eta_1(v,u)]$ *be a differentiable function where* I *is taken to be an open invex set with respect to* $\eta_1 : I \times I \to \mathbb{R}$ *with* $\eta_1(v,u) > 0$, *for* $u, v \in I$. *Then for the generalized fractional integrals defined in definition 1.12, we have*

$$\frac{\eta_1(v,u)}{2} \sum_{k=1}^4 I_k = \Psi\left(\frac{2u + \eta_1(v,u)}{2}\right) \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^{\mu}; p) - \frac{1}{2\eta_1(v,u)}$$

$$\left[\left(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c} \Psi \right)(u + \eta_1(v,u); p) + \left(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c} \Psi \right)(u; p) \right]$$

where

$$I_1 = \int_0^{\frac{1}{2}} t^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^{\mu}; p) \Psi'(u + t\eta_1(v,u)) dt$$

$$I_2 = \int_0^{\frac{1}{2}} -(t)^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^{\mu}; p) \Psi'(u + (1-t)\eta_1(v,u)) dt$$

$$I_3 = \int_{\frac{1}{2}}^1 t^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^{\mu}; p) \Psi'(u + t\eta_1(v,u)) dt - \int_{\frac{1}{2}}^1 \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^{\mu}; p) \Psi'(u + t\eta_1(v,u)) dt$$

$$I_4 = \int_{\frac{1}{2}}^1 \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^{\mu}; p) \Psi'(u + (1-t)\eta_1(v,u)) dt$$

$$- \int_{\frac{1}{2}}^1 t^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^{\mu}; p) \Psi'(u + (1-t)\eta_1(v,u)) dt.$$

Proof. If we consider

$$I_1 = \int_0^{\frac{1}{2}} t^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^{\mu}; p) \Psi'(u + t\eta_1(v,u)) dt$$

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \int_0^{\frac{1}{2}} (t)^{v'+\mu n} \Psi'(u + t\eta_1(v,u)) dt.$$

Solving the integrals by using integrating by parts method leads to

$$
\begin{aligned}
I_1 = & \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \Big[t^{v' + \mu n} \frac{\Psi(u + t\eta_1(v, u))}{\eta_1(v, u)} \Big|_0^{\frac{1}{2}} \\
& - \frac{v' + \mu n}{\eta_1(v, u)} \int_0^{\frac{1}{2}} (t)^{v' + \mu n - 1} \Psi(u + t\eta_1(v, u)) dt \Big] \\
= & \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \Big[\frac{(2)^{-(v' + \mu n)}}{\eta_1(v, u)} \Psi \frac{(2u + \eta_1(v, u))}{2} \\
& - \frac{v' + \mu n}{\eta_1(v, u)} \int_0^{\frac{1}{2}} (t)^{v' + \mu n - 1} \Psi(u + t\eta_1(v, u)) dt \Big].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 = & \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \Big[\frac{(2)^{-(v' + \mu n)}}{\eta_1(v, u)} \Psi \frac{(2u + \eta_1(v, u))}{2} \\
& - \frac{v' + \mu n}{\eta_1(v, u)} \int_0^{\frac{1}{2}} (t)^{v' + \mu n - 1} \Psi(u + (1 - t)\eta_1(v, u)) dt \Big].
\end{aligned}
$$

Now

$$
\begin{aligned}
I_3 = & \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \int_{\frac{1}{2}}^{1} (t^{v' + \mu n} - 1)\Psi'(u + t\eta_1(v, u)) dt \\
I_3 = & \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \Big[(t^{v' + \mu n} - 1) \frac{\Psi(u + t\eta_1(v, u))}{\eta_1(v, u)} \Big|_{\frac{1}{2}}^{1} \\
& - \frac{v' + \mu n}{\eta_1(v, u)} \int_{\frac{1}{2}}^{1} (t)^{v' + \mu n - 1} \Psi(u + t\eta_1(v, u)) dt \Big] \\
= & \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \Big[\frac{1 - 2^{-(v' + \mu n)}}{\eta_1(v, u)} \Psi(\frac{(2u + \eta_1(v, u))}{2}) \\
& - \frac{v' + \mu n}{\eta_1(v, u)} \int_{\frac{1}{2}}^{1} (t)^{v' + \mu n - 1} \Psi(u + t\eta_1(v, u)) dt \Big].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_4 = & \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \Big[\frac{1 - 2^{-(v' + \mu n)}}{\eta_1(v, u)} \Psi(\frac{(2u + \eta_1(v, u))}{2}) \\
& - \frac{v' + \mu n}{\eta_1(v, u)} \int_{\frac{1}{2}}^{1} (t)^{v' + \mu n - 1} \Psi(u + (1 - t)\eta_1(v, u)) dt \Big].
\end{aligned}
$$

Adding I_1, I_2, I_3 and I_4, we proceed to the desired result as,

$$
\sum_{k=1}^{4} I_k = \frac{2}{\eta_1(v, u)} \Psi(\frac{2u + \eta_1(v, u)}{2}) \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n
$$

$$-\sum_{n=0}^{\infty}\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\frac{v'+\mu n}{\eta_1(v,u)}\times$$

$$\left[\int_0^1 t^{v'+\mu n-1}\Psi(u+t\eta_1(v,u))dt+\int_0^1 t^{v'+\mu n-1}\Psi(u+(1-t)\eta_1(v,u))dt\right]$$

or

$$\frac{\eta_1(v,u)}{2}\sum_{k=1}^{4}I_k=\Psi(\frac{2u+\eta_1(v,u)}{2})\mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^{\mu};p)-\frac{1}{2\eta_1(v,u)}$$

$$\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(u+\eta_1(v,u);p)+(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u;p)\right].$$

$\square$

Next, we present mid-point type inequalities related to Hermite-Hadamard inequalities:

Theorem 5.1. *Consider a function* $\Psi:I\to\mathbb{R}$ *with* $I\in\mathbb{R}$ *,$\Psi'\in L_1[u,u+\eta_1(v,u)]$ be a differentiable function where I is taken to be an open invex set with respect to* $\eta_1:I\times I\to\mathbb{R}$ *with* $\eta_1(v,u)>0$ *for* $u,v\in I$. *Then for the generalized fractional integrals defined in definition 1.12, we have*

$$\left|\Psi(\frac{2u+\eta_1(v,u)}{2})\mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^{\mu};p)-\frac{1}{2\eta_1(v,u)}\right.$$

$$\left.\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(u+\eta_1(v,u);p)+(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u;p)\right]\right|$$

$$\le\frac{\eta_1(v,u)}{2}[|\Psi'(u)|+|\Psi'(v)|+\frac{1}{2}\eta_2(|\Psi'(u)|,|\Psi'(v)|)+\frac{1}{2}.\eta_2(|\Psi'(v)|,|\Psi'(u)|)],$$

for $0<v'+\mu n\le 1$.

Proof. By using lemma 5.1 and using the property of absolute function for addition,leads to

$$\left|\Psi(\frac{2u+\eta_1(v,u)}{2})\mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^{\mu};p)-\frac{1}{2\eta_1(v,u)}\right. \tag{5.1}$$

$$\left.\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(u+\eta_1(v,u);p)+(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u;p)\right]\right| \tag{5.2}$$

$$\le\frac{\eta_1(v,u)}{2}\sum_{k=1}^{4}|I_k|. \tag{5.3}$$

To solve $|I_k|,k=1,2,3,4$, we further move by using (η_1,η_2)-convexity of $|\Psi'|$

$$|I_1|\le\int_0^{\frac{1}{2}}t^{v'}|\mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^{\mu};p)||\Psi'(u+t\eta_1(v,u))|dt$$

$$\le\int_0^{\frac{1}{2}}t^{v'}|\mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^{\mu};p)||\Psi'(u)|dt+\int_0^{\frac{1}{2}}(t)^{v'+1}|\mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^{\mu};p)|\eta_2(|\Psi'(v)|,|\Psi'(u)|)dt$$

$$\le\sum_{n=0}^{\infty}\left|\frac{\beta_p(\eta+\xi n,c-\eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta,c-\eta)\Gamma(\mu n+v'+1)(\rho)_{mn}}(-\omega)^n\right|$$

$$\left[|\Psi'(u)|\frac{1}{2^{v'+\mu n+1}(v'+\mu n+1)}+\eta_2(|\Psi'(v)|,|\Psi'(u)|)\frac{1}{2^{v'+\mu n+2}(v'+\mu n+2)}\right]$$

Analogously

$$|I_2| \le \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \right|$$

$$\left[|\Psi'(v)| \frac{1}{2^{v' + \mu n + 1}(v' + \mu n + 1)} + \eta_2(|\Psi'(u)|, |\Psi'(v)|) \frac{1}{2^{v' + \mu n + 2}(v' + \mu n + 2)} \right].$$

For $|I_k|, k = 3, 4$. We will use the fact that for all $j \in (0, 1]$ and $u_1, u_2 \in [0, 1]$. Therefore, we have

$$|u_1^j - u_2^j| \le |u_1 - u_2|^j$$

$$|I_3| \le \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \right| \times$$

$$\left[|\Psi'(u)| \frac{1}{2^{v' + \mu n + 1}(v' + \mu n + 1)} + \eta_2(|\Psi'(v)|, |\Psi'(u)|) \frac{v' + \mu n + 3}{2^{v' + \mu n + 2}(v' + \mu n + 1)(v' + \mu n + 2)} \right]$$

and

$$|I_4| \le \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \right| \times$$

$$\left[|\Psi'(v)| \frac{1}{2^{v' + \mu n + 1}(v' + \mu n + 1)} + \eta_2(|\Psi'(u)|, |\Psi'(v)|) \frac{v' + \mu n + 3}{2^{v' + \mu n + 2}(v' + \mu n + 1)(v' + \mu n + 2)} \right].$$

Using the above evaluated absolute values in (5.1), we have

$$\left| \Psi\left(\frac{2u + \eta_1(v, u)}{2}\right) \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2\eta_1(v, u)} \times \right.$$

$$\left. \left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(u + \eta_1(v, u); p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p) \right] \right|$$

$$\le \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \right| \frac{\eta_1(v, u)}{2^{v' + \mu n + 1}(v' + \mu n + 1)}$$

$$\left[|\Psi'(u)| + |\Psi'(v)| + \frac{1}{2}.\eta_2(|\Psi'(v)|, |\Psi'(u)|) + \frac{1}{2}\eta_2(|\Psi'(u)|, |\Psi'(v)|) \right]$$

$$= \frac{\eta_1(v, u)}{2^{v+1}} \left| \mathfrak{J}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{1}{2})^\mu; p) \right|$$

$$\left[|\Psi'(u)| + |\Psi'(v)| + \frac{1}{2}.\eta_2(|\Psi'(v)|, |\Psi'(u)|) + \frac{1}{2}\eta_2(|\Psi'(u)|, |\Psi'(v)|) \right].$$

$\square$

Corollary 5.1. *In Theorem 5.1, if we take $\eta_1(u, v) = u - v$, $\forall u, v \in I$, then*

$$\left| \Psi\left(\frac{u + v}{2}\right) \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2(v - u)} \left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(v; p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p) \right] \right|$$

$$\le \frac{v - u}{2^{v+1}} |\mathfrak{J}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{1}{2})^\mu; p)| \left[|\Psi'(u)| + |\Psi'(v)| + \frac{1}{2}.\eta_2(|\Psi'(v)|, |\Psi'(u)|) + \frac{1}{2}\eta_2(|\Psi'(u)|, |\Psi'(v)|) \right].$$

Corollary 5.2. *In Theorem 5.1, if we take $\eta_1(u,v) = u - v$, $\forall u, v \in I$ and $\eta_2(x, y) = x - y$, $\forall x, y \in \Psi(I)$, then*

$$\left| \Psi(\frac{u+v}{2}) \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2(v-u)} \left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(v; p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p) \right] \right|$$

$$\leq \frac{v-u}{2^{v'+1}} |\mathfrak{J}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{1}{2})^\mu; p)| \left[|\Psi'(u)| + |\Psi'(v)| \right].$$

Lemma 5.2. *If we consider a function $\Psi : I \to \mathbb{R}$ with $I \in \mathbb{R}$, $\Psi \in L_1[u, u + \eta_1(v, u)]$ be a differentiable function where I is taken to be an open invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$ with $\eta_1(v, u) > 0$ for $u, v \in I$. Then for the generalized fractional integrals defined in definition (1.12), we have*

$$\frac{\Psi(u) + \Psi(u + \eta_1(v, u))}{2} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2\eta_1(v, u)}$$

$$\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(u + \eta_1(v, u); p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p) \right]$$

$$= \frac{\eta_1(v, u)}{2} I$$

where

$$I = \int_0^1 t^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^\mu; p)\Psi'(u + t\eta_1(v, u))dt$$

$$+ \int_0^1 -(1-t)^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)\Psi'(u + t\eta_1(v, u))dt.$$

Proof. We consider the fractional integral

$$I = \int_0^1 t^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(t)^\mu; p)\Psi'(u + t\eta_1(v, u))dt$$

$$+ \int_0^1 -(1-t)^{v'} \mathfrak{J}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1-t)^\mu; p)\Psi'(u + t\eta_1(v, u))dt.$$

Let

$$I = I_1 + I_2.$$

First, we consider I_1

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \int_0^1 t^{v'+\mu n}\Psi'(u + t\eta_1(v, u))dt.$$

Integrating by parts, we have

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \Big[t^{v'+\mu n} \frac{\Psi(u + t\eta_1(v, u))}{\eta_1(v, u)} \Big|_0^1$$

$$- \frac{v' + \mu n}{\eta_1(v, u)} \int_0^1 t^{v'+\mu n-1}\Psi(u + t\eta_1(v, u))dt \Big]$$

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \times$$

$$\left[\frac{\Psi(u + \eta_1(v, u))}{\eta_1(v, u)} - \frac{v + \mu n}{\eta_1(v, u)}\int_0^1 t^{v'+\mu n-1}\Psi(u + t\eta_1(v, u))dt\right]$$

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}}(-\omega)^n \times$$

$$\left[\frac{\Psi(u + \eta_1(v, u))}{\eta_1(v, u)} - \frac{v' + \mu n}{(\eta_1(v, u))^2}\int_u^{u+\eta_1(v,u)}(\frac{x - u}{u + \eta_1(v, u) - u})^{v'+\mu n-1}\Psi(x)dx\right]$$

$$I_1 = \frac{\Psi(u + \eta_1(v, u))}{\eta_1(v, u)}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{(\eta_1(v, u))^2}(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p).$$

On the same lines, we get

$$I_2 = \frac{\Psi(u)}{\eta_1(v, u)}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{(\eta_1(v, u))^2}(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Psi)(u + \eta_1(v, u); p)$$

$$I = \frac{\Psi(u) + \Psi(u + \eta_1(v, u))}{\eta_1(v, u)}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{(\eta_1(v, u))^2}\times$$

$$\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p) + \mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u)^+}^{\mu,\xi,m,\sigma,c}\Psi)(u + \eta_1(v, u); p)\right].$$

Multiplying by $\frac{\eta_1(v,u)}{2}$, we get the required result. $\qquad\square$

Here, we are able to give trapezoid-type inequalities related to Hermite-Hadamard inequalities:

Theorem 5.2. *If we consider a function $\Psi : I \to \mathbb{R}$ with $I \in \mathbb{R}$, $\Psi' \in L_1[u, u + \eta_1(v, u)]$ be a differentiable function where I is taken to be an open invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$ with $\eta_1(v, u) > 0$ for $u, v \in I$. Suppose also that $|\Psi'|$ is an (η_1, η_2)-convex function on I. Then for the generalized fractional integrals defined in definition 1.12, we have*

$$\left|\frac{\Psi(u) + \Psi(u + \eta_1(v, u))}{2}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2\eta_1(v, u)}\times\right.$$

$$\left.\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(u + \eta_1(v, u); p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p)\right]\right|$$

$$\leq \frac{\eta_1(v, u)}{2}\left|\mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2^{v'}}\mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{1}{2})^\mu; p)\right|\left[2|\Psi'(u)| + \eta_2(|\Psi'(v)|, |\Psi'(u)|)\right],$$

where $v' \geq 0$.

Proof.

$$\left|\frac{\Psi(u) + \Psi(u + \eta_1(v, u))}{2}\mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2\eta_1(v, u)}\times\right.$$

$$\left.\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c}\Psi)(u + \eta_1(v, u); p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c}\Psi)(u; p)\right]\right|$$

$$= |\frac{\eta_1(v, u)}{2}I|$$

$$\leq \frac{\eta_1(v,u)}{2} \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \right|$$

$$\times \int_0^1 |t^{v'+\mu n} - (1-t)^{v'+\mu n}| |\Psi'(u + t\eta_1(v,u))| dt$$

$$\leq \frac{\eta_1(v,u)}{2} \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \right| \times$$

$$\int_0^1 |t^{v'+\mu n} - (1-t)^{v'+\mu n}| \left[|\Psi'(u)| + t\eta_2(|\Psi'(v)|, |\Psi'(u)|) \right] dt$$

$$= \frac{\eta_1(v,u)}{2} \sum_{n=0}^{\infty} \left| \frac{\beta_p(\eta + \xi n, c - \eta)(c)_{\xi n}(\gamma)_{\sigma n}}{\beta(\eta, c - \eta)\Gamma(\mu n + v' + 1)(\rho)_{mn}} (-\omega)^n \right| \times$$

$$\left[\int_0^{\frac{1}{2}} ((1-t)^{v'+\mu n} - t^{v'+\mu n}) \left[|\Psi'(u)| + t\eta_2(|\Psi'(v)|, |\Psi'(u)|) dt \right] \right.$$

$$\left. + \int_{\frac{1}{2}}^1 (t^{v'+\mu n} - (1-t)^{v'+\mu n}) \left[|\Psi'(u)| + t\eta_2(|\Psi'(v)|, |\Psi'(u)|) dt \right] \right.$$

Solving the integrals involved by using integrating by parts method, we obtain the desired result

$$\left| \frac{\Psi(u) + \Psi(u + \eta_1(v,u))}{2} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2\eta_1(v,u)} \times \right.$$

$$\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c} \Psi)(u + \eta_1(v,u); p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(u+\eta_1(v,u))^-}^{\mu,\xi,m,\sigma,c} \Psi)(u; p) \right] \right|$$

$$\leq \frac{\eta_1(v,u)}{2} \left| \mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2^{v'}} \mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{1}{2})^\mu; p) \right| \left[2|\Psi'(u)| + \eta_2(|\Psi'(v)|, |\Psi'(u)|) \right].$$

$\square$

Corollary 5.3. *In theorem 5.2, $\eta_1(u,v) = u - v, \forall u, v \in I$ gives*

$$\left| \frac{\Psi(u) + \Psi(v)}{2} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2(v-u)} \times \right.$$

$$\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c} \Psi)(v; p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c} \Psi)(u; p) \right]$$

$$\leq \frac{v-u}{2} \left| \mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2^{v'}} \mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{1}{2})^\mu; p) \right| \left[2|\Psi'(u)| + \eta_2(|\Psi'(v)|, |\Psi'(u)|) \right].$$

Corollary 5.4. *In theorem (5.2), $\eta_1(u,v) = u - v, \forall u, v \in I$, and $\eta_2(u,v) = u - v, \forall u, v \in \Psi(I)$ gives*

$$\left| \frac{\Psi(u) + \Psi(v)}{2} \mathfrak{I}_{v',\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2(v-u)} \times \right.$$

$$\left[(\mathfrak{T}_{v'-1,\eta,\rho,\gamma;u^+}^{\mu,\xi,m,\sigma,c} \Psi)(v; p) + (\mathfrak{T}_{v'-1,\eta,\rho,\gamma;(v)^-}^{\mu,\xi,m,\sigma,c} \Psi)(u; p) \right] \right|$$

$$\leq \frac{v-u}{2} \left| \mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(1)^\mu; p) - \frac{1}{2^v} \mathfrak{I}_{v'+1,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma,c}(\omega(\frac{1}{2})^\mu; p) \right| \left[|\Psi'(u)| + |\Psi'(v)| \right].$$

6. Concluding remarks

Various researchers have studied integral inequalities due to their wide applications in both pure and applied mathematics. This paper discussed the new version of integral inequalities such as Hermite-Hadamard type and trapezoid type inequalities for the convex function by utilizing generalized fractional integrals concerning the extended Wright generalized Bessel function as a kernel. Also, we established new mid-point type and trapezoidal type integral inequalities for (η_1, η_2)-convex function related to Hermite-Hadamard and Fejér type inequalities. All the inequalities presented in this paper are more general than the inequalities available in the literature, which can easily observe from the corollaries.

Conflict of interest

The authors declare that they have no competing interest.

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