Some weighted estimates for the commutators of $p$-adic Hardy operator on two weighted $p$-adic Herz-type spaces

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Abstract: In the present article we discuss the weighted $p$-adic central bounded mean oscillations (CMO) and $p$-adic Lipschtiz estimates for the commutators of $p$-adic Hardy operator on two weighted $p$-adic Herz-type spaces.

Keywords: $p$-adic Hardy operator; commutators; two weighted $p$-adic Morrey-Herz spaces; central bounded mean oscillations; weighted $p$-adic Lipschitz spaces

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1. Introduction

In mathematical analysis Hardy operator is considered an important averaging operator as it plays a vital role in many branches of mathematics, such as complex analysis, partial differential equations and harmonic analysis (for example, see [2,7,8,10,29]). In [6], Hardy introduced the one-dimensional Hardy operator

$$Hf(x) = \frac{1}{x} \int_0^x f(t)dt, \quad x > 0,$$

(1.1)

for a measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$. The operator in (1.1) satisfies the below inequality

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1} \|f\|_{L^q(\mathbb{R}^+)}, \quad 1 < q < \infty,$$

(1.2)

where the constant $q/(q-1)$ is sharp. An extension of the operator $H$ on higher dimensional space $\mathbb{R}^n$ was defined in [3] by Faris as

$$Hf(x) = \frac{1}{|x|^n} \int_{|t| \leq |x|} f(t)dt,$$

(1.3)
where $|x| = \left(\sum_{i=1}^{n} x_i^p\right)^{1/2}$ for $x = (x_1, \ldots, x_n)$. Furthermore, Christ and Grafakos [1] acquired the exact value of the norm of operator $H$ defined by (1.3). Recently, Hardy operator has gained a tremendous amount of consideration, see for example [16, 20, 24, 30, 33, 34] and the references therein.

In the past few decades there has been a relentless attention in $p$-adic models appearing in various branches of science. The applications of $p$-adic analysis are found mainly in the field of mathematical physics (see, for example, [15, 26, 27]). Importantly, many current researchers are paying a valiant effort to harmonic analysis on $p$-adic field [9–11, 13, 17, 21, 22, 25].

Let $\mathbb{Q}$ be a field of rational numbers and $p$ a prime number. We introduce a so called $p$-adic norm $|x|_p$ on $\mathbb{Q}$ by a rule $|x|_p = \{0\} \cup \{p^{-\gamma} : \gamma \in \mathbb{Z}\}$, where $\gamma = \gamma(x)$ is defined from the following representation

$$x = p^\gamma s/t,$$

where $s$ and $t$ are coprime to $p$. $|\cdot|_p$ fulfills all the axioms of a real norm along with the following non-Archimedean property:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}. \quad (1.4)$$

The field of $p$-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. Any nonzero $p$-adic number can be written in canonical form (see [28]) as:

$$x = p^\gamma \sum_{j=0}^{\infty} \alpha_j p^j, \quad (1.5)$$

where $\alpha_j \in \mathbb{Z}, \alpha_j \in \frac{\mathbb{Z}}{p\mathbb{Z}}$, $\alpha_0 \neq 0$. Interestingly, the series in (1.5) is convergent with respect to $|\cdot|_p$ because $|p^\gamma \alpha_1 p^j|_p = p^{-\gamma-j}$.

The higher dimensional $p$-adic vector space $\mathbb{Q}_p^n$ consists of points $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbb{Q}_p$, $i = 1, 2, \ldots, n$, with the following norm

$$|x|_p = \max_{1 \leq i \leq n} |x_i|_p. \quad (1.6)$$

Let

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\}, \quad S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\}$$

be the ball and sphere respectively with center at $a \in \mathbb{Q}_p^n$ and radius $p^\gamma$. If $a = 0$, we may write $B_\gamma(0) = B_\gamma$, $S_\gamma(0) = S_\gamma$.

It is well known that the space $\mathbb{Q}_p^n$ is locally compact commutative group under addition, then there exists a translation invariant Haar measure $dx$ which is normalized such that

$$\int_{B_0} dx = |B_0|_H = 1,$$

where $|A|_H$ represents the Haar measure of a measurable subset $A$ of $\mathbb{Q}_p^n$. Moreover, one can easily show that $|B_\gamma(a)|_H = p^{n\gamma}$ and $|S_\gamma(a)|_H = p^{n\gamma}(1 - p^{-n})$, for any $a \in \mathbb{Q}_p^n$.

In what follows the $p$-adic Hardy operator

$$H_p f(x) = \frac{1}{|x|_p^n} \int_{|t|_p \leq |x|_p} f(t) dt$$
and its commutator
\[ H^p_w f(x) = b H^p(f) - H^p(b f) \]
were defined and studied for \( f, b \in L^{loc}_1(Q^n_p) \) in [4]. In the same paper, Fu et al. acquired the boundedness of \( p \)-adic Hardy operator and its commutator on Lebesgue spaces and Herz spaces. On the Morrey-Herz spaces, the \( p \)-adic Hardy type operators and their commutators are reported in [5]. For complete comprehension of \( p \)-Hardy operator and its commutator, we refer the publications [12, 18, 31, 32].

The purpose of the current article is to discuss the weighted central bounded mean oscillations and weighted \( p \)-adic Lipschitz estimates of \( H^p_w \) on two weighted \( p \)-adic Herz spaces and \( p \)-adic Morrey-Herz spaces. Throughout this article a letter \( C \) denotes a constant whose value may change at its different places. It is mandatory to recall the definitions of relevant \( p \)-adic function spaces before moving to our results.

Suppose \( w(x) \) is a nonnegative function on \( Q^n_p \). The weighted measure of \( A \) is denoted and defined as \( w(A) = \int_A w(x) dx \). The weighted \( p \)-adic Lebesgue space \( L^q(w, Q^n_p) \), \((0 < q < \infty)\) is defined to be the space of all measurable functions \( f \) on \( Q^n_p \) such that:

\[ \|f\|_{L^q(w, Q^n_p)} = \left( \int_{Q^n_p} |f(x)|^q w(x) dx \right)^{1/q} < \infty. \]

The theory of \( A_q \) weights on \( \mathbb{R}^n \) was introduced by Benjamin Muckenhoupt in [19]. Let us recall the definition of \( A_q \) weights in \( p \)-adic setting.

**Definition 1.1.** [23] A weight function \( w \in A_q(1 \leq q < \infty) \) if there exists a constant \( C \) free from choice of \( B \subset Q^n_p \) such that

\[ \left( \frac{1}{|B|} \int_B w(x) dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{1}{q-1}} dx \right)^{1/q} \leq C. \]

For the case \( q = 1 \), \( w \in A_1 \), we have

\[ \frac{1}{|B|} \int_B w(x) dx \leq \text{Cess inf}_{x \in B} w(x), \]

for every \( B \subset Q^n_p \).

**Remark 1.2.** A weight function \( w \in A_{\infty} \) if it undergoes the stipulation of \( A_q(1 \leq q < \infty) \) weights.

**Definition 1.3.** Suppose \( w \) is a weight function and \( 1 \leq q < \infty \). The \( p \)-adic space \( CMO^q(w, Q^n_p) \) is defined by

\[ \|f\|_{CMO^q(w, Q^n_p)} = \sup_{\gamma \in \mathbb{Z}} \left( \frac{1}{w(B_{\gamma})} \int_{B_{\gamma}} |f(x) - f_{B_{\gamma}}| w(x)^{1-q} dx \right)^{1/q}, \]

where

\[ f_{B_{\gamma}} = \frac{1}{|B_{\gamma}|} \int_{B_{\gamma}} f(x) dx. \]  

**Definition 1.4.** [22] Suppose \( w_1 \) and \( w_2 \) are weight functions, \( 0 < r, q < \infty \) and \( \alpha \in \mathbb{R} \). Then the two weighted \( p \)-adic Herz space \( K^q_{\alpha,r}(w_1, w_2) \) is defined as

\[ K^q_{\alpha,r}(w_1, w_2) = \{ f \in L^q_{loc}(w_2, Q^n_p \setminus \{0\}) : \|f\|_{K^q_{\alpha,r}(w_1, w_2)} < \infty \}, \]
where
\[ \|f\|_{M_{q}^{\alpha,\lambda}(w, Q_{p})} = \left( \sum_{k=-\infty}^{\infty} w_{1}(B_{k})^{\alpha/n} \|f\chi_{k}\|_{L^{q}(w, Q_{p})}^{\lambda} \right)^{1/r} \]
and \( \chi_{k} \) is the characteristic function of the sphere \( S_{k} = B_{k} \setminus B_{k-1} \).

**Remark 1.7.** Obviously \( K_{q}^{0,0}(w, 1) = L^{q}(w, Q_{p}) \).

**Definition 1.6.** \(^{22}\) Suppose \( w_{1} \) and \( w_{2} \) are weight functions, \( 0 < r, q < \infty, \alpha \in \mathbb{R} \) and \( \lambda \geq 0 \). Then the two weighted \( p \)-adic Morrey-Herz space \( M_{r,q}^{\alpha,\lambda}(w_{1}, w_{2}) \) is defined as follows
\[ M_{r,q}^{\alpha,\lambda}(w_{1}, w_{2}) = \{ f \in L_{\text{loc}}^{q}(w_{2}, Q_{p}) \setminus \{ 0 \}) : \|f\|_{M_{r,q}^{\alpha,\lambda}(w_{1}, w_{2})} < \infty \}, \]
where
\[ \|f\|_{M_{r,q}^{\alpha,\lambda}(w_{1}, w_{2})} = \sup_{k_{0} \in \mathbb{Z}} w_{1}(B_{k_{0}})^{-\lambda/n} \left( \sum_{k=-\infty}^{k_{0}} w_{1}(B_{k})^{\alpha/n} \|f\chi_{k}\|_{L^{q}(w, Q_{p})}^{\lambda} \right)^{1/r}. \]

**Remark 1.7.** It is evident that \( M_{r,q}^{0,0}(w_{1}, w_{2}) = K_{q}^{\alpha,\lambda}(w_{1}, w_{2}) \).

**Definition 1.8.** \(^{23}\) Suppose \( 1 \leq q < \infty, 0 < \beta < 1 \) and \( w \) is a weight function. The \( p \)-adic space \( \text{Lip}_{\beta}(w, Q_{p}) \) is defined as
\[ \|f\|_{\text{Lip}_{\beta}(w, Q_{p})} = \sup_{B \subset Q_{p}} \frac{1}{w(B)^{\beta/n}} \left( \frac{1}{w(B)} \int_{B} |f(x) - f_{B}|^{q} w(x)^{1-q} \, dx \right)^{1/q}, \]
where
\[ f_{B} = \frac{1}{|B|} \int_{B} f(x) \, dx. \]

2. **Weighted CMO estimates of \( H_{p}^{q} \) on two weighted \( p \)-adic Herz-type spaces**

The following section discusses the weighted \( CMO \) estimates of \( H_{p}^{q} \) on two weighted \( p \)-adic Herz-type spaces. We open up the section with few lemmas which are useful in proving key results.

**Lemma 2.1.** \(^{14}\) Suppose \( w \in A_{1} \), then there exist constants \( C_{1}, C_{2} \) and \( 0 < \mu < 1 \) such that
\[ C_{1} \frac{|A|}{|B|} \leq \frac{w(A)}{w(B)} \leq C_{2} \left( \frac{|A|}{|B|} \right)^{\mu}, \]
for any measurable subset \( A \) of a ball \( B \).

**Remark 2.2.** If \( w \in A_{1} \), then it follows from lemma (2.1) that there exist constants \( C_{1} \) and \( \mu \) \( (0 < \mu < 1) \) such that \( \frac{w(B_{i})}{w(B)} \leq C_{1} p^{(k-i)\mu} \) as \( i < k \) and \( \frac{w(B_{i})}{w(B)} \leq C_{1} p^{(k-i)\mu} \) as \( i \geq k \).

**Lemma 2.3.** \(^{23}\) Suppose \( w \in A_{1} \) and \( b \in CMO^{q}(w, Q_{p}) \), then there is a constant \( C \) such that for \( i, k \in \mathbb{Z} \),
\[ |b_{B_{i}} - b_{B_{k}}| \leq C |i - k| \|b\|_{CMO^{q}(w, Q_{p})} \frac{w(B_{i})}{|B_{k}|}. \]
Lemma 2.4. [23] Suppose \( w \in A_1 \), then for \( 1 < q < \infty \),

\[
\int_B w(x)^{1-q'} \, dx \leq C |B|^{q'} w(B)^{1-q'},
\]

where \( 1/q + 1/q' = 1 \).

Now we state the result about the boundedness of \( H_b^p \) on two weighted \( p \)-adic Herz-type spaces.

Theorem 2.5. Let \( 0 < r_1 \leq r_2 < \infty \), \( 1 \leq r, q < \infty \) and let \( w \in A_1 \). If \( \alpha < \frac{nm}{q'} \), then the inequality

\[
\|H_b^p f\|_{\kappa_{r_2}^{\alpha} (w_1, w_2)} \leq C \|b\|_{\text{CMO}^{\alpha}(w, Q_p^n)} \|f\|_{\kappa_{r_1}^{\alpha} (w, w_2)}
\]

holds for all \( b \in \text{CMO}^{\alpha}(w, Q_p^n) \) and \( f \in L_{\text{loc}}(Q_p^n) \).

If \( \alpha = 0 \), \( r_1 = r_2 = q \), then we have the following result.

Corollary 2.6. Let \( 1 \leq r, q < \infty \) and \( w \in A_1 \), then

\[
\|H_b^p f\|_{L^q(w_1, Q_p^n)} \leq C \|b\|_{\text{CMO}^{\alpha}(w, Q_p^n)} \|f\|_{L^q(w, Q_p^n)}
\]

holds for all \( b \in \text{CMO}^{\alpha}(w, Q_p^n) \) and \( f \in L_{\text{loc}}(Q_p^n) \).

Theorem 2.7. Let \( 0 < r_1 \leq r_2 < \infty \), \( 1 \leq r, q < \infty \) and let also \( w \in A_1 \) and \( \lambda > 0 \). If \( \alpha < \frac{nm}{q'} + \lambda \), then

\[
\|H_b^p f\|_{\text{MK}^{\alpha}(w_1, Q_p^n)} \leq C \|b\|_{\text{CMO}^{\alpha}(w, Q_p^n)} \|f\|_{\text{MK}^{\alpha}(w, Q_p^n)}
\]

holds for all \( b \in \text{CMO}^{\alpha}(w, Q_p^n) \) and \( f \in L_{\text{loc}}(Q_p^n) \).

Proof of Theorem 2.5: First, by the definition we have

\[
\|(H_b^p f)\chi_k\|_{L^q(w_1, Q_p^n)}^q = \int_{S_k} |x|^{-qn} \int_{|t|_p \leq |x|_p} f(t)(b(x) - b(t)) \, dt \, w(x)^{1-q} \, dx
\]

\[
\leq C p^{-kn} \int_{S_k} \left( \sum_{i=-\infty}^{k} \int_{S_{i}} |f(t)(b(x) - b(t))| \, dt \right)^q \, w(x)^{1-q} \, dx
\]

\[
= C p^{-kn} \int_{S_k} \left( \sum_{i=-\infty}^{k} \int_{S_{i}} |f(t)(b(x) - b(t))| \, dt \right)^q \, w(x)^{1-q} \, dx
\]

\[
\leq C p^{-kn} \int_{S_k} \left( \sum_{i=-\infty}^{k} \int_{S_{i}} |f(t)(b(x) - b(t))| \, dt \right)^q \, w(x)^{1-q} \, dx
\]

\[
+ C p^{-kn} \int_{S_k} \left( \sum_{i=-\infty}^{k} \int_{S_{i}} |f(t)(b(t) - b(t))| \, dt \right)^q \, w(x)^{1-q} \, dx
\]

\[
= I + II.
\]

(2.1)
Since $w \in A_1 \subset A_q$, making use of Hölder’s inequality along with lemma 2.4, we have

\[
\int_{S_i} f(t)dt \leq \left( \int_{S_i} |f(t)|^q w(t)dt \right)^{1/q} \left( \int_{S_i} w(t)^{-q'q'}dt \right)^{1/q'} \\
\leq C \|fX_i\|_{L^q(w, Q_p^q)} |B_i| w(B_i)^{-1/q}. \tag{2.2}
\]

To estimate $I$, by the application of Hölder’s inequality, Remark 2.2 along with inequality (2.2), we are down to

\[
I \leq C p^{-kq} \|b\|^q_{CMO^*(w, Q_p^q)} w(B_k) \left( \sum_{i=-\infty}^k \|fX_i\|_{L^q(w, Q_p^q)} |B_i| w(B_i)^{-1/q} \right)^q \\
\leq C p^{-kq} \|b\|^q_{CMO^*(w, Q_p^q)} \left( \sum_{i=-\infty}^k \|fX_i\|_{L^q(w, Q_p^q)} |B_i| \left( \frac{w(B_k)}{w(B_i)} \right)^{1/q} \right)^q \\
\leq C \|b\|^q_{CMO^*(w, Q_p^q)} \left( \sum_{i=-\infty}^k p^{i-kq} \|fX_i\|_{L^q(w, Q_p^q)} \right)^q. \tag{2.3}
\]

Now, we estimate $II$ as follows

\[
II \leq C p^{-kq} \int_{S_k} \left( \sum_{i=-\infty}^k \int_{S_i} |f(t)(b(t) - b_{B_i})|dt \right)^q w(x)^{1-q} dx \\
+ C p^{-kq} \int_{S_k} \left( \sum_{i=-\infty}^k \int_{S_i} |f(t)(b_{B_i} - b_{B_i})|dt \right)^q w(x)^{1-q} dx \\
= II_1 + II_2. \tag{2.4}
\]

Next, applying Hölder’s inequality to deduce

\[
\int_{S_i} |f(t)(b(t) - b_{B_i})|dt \\
\leq \left( \int_{S_i} |f(t)|^q w(t)dt \right)^{1/q} \left( \int_{S_i} |b(t) - b_{B_i}|^{q'} w(t)^{-q'q'}dt \right)^{1/q'} \\
\leq w(B_i)^{-1/q} \|fX_i\|_{L^q(w, Q_p^q)} \|b\|_{CMO^*(w, Q_p^q)}. \tag{2.5}
\]

By the application of Hölder’s inequality, inequality (2.5), lemma 2.4 and Remark 2.2, we are in a position to estimate $II_1$.

\[
II_1 \leq C p^{-kq} \int_{S_k} w(x)^{1-q} dx \left( \sum_{i=-\infty}^k \|fX_i\|_{L^q(w, Q_p^q)} w(B_i)^{1/q} \right)^q \\
\leq C p^{-kq} |B_k|^{1-q} \|b\|^q_{CMO^*(w, Q_p^q)} \left( \sum_{i=-\infty}^k \|fX_i\|_{L^q(w, Q_p^q)} w(B_i)^{1/q} \right)^q \\
\leq C \|b\|^q_{CMO^*(w, Q_p^q)} \left( \sum_{i=-\infty}^k \left( \frac{w(B_k)}{w(B_i)} \right)^{1-1/q} \|fX_i\|_{L^q(w, Q_p^q)} \right)^q \\
\leq C \|b\|^q_{CMO^*(w, Q_p^q)} \left( \sum_{i=-\infty}^k p^{i-kq} \|fX_i\|_{L^q(w, Q_p^q)} \right)^q. \tag{2.6}
\]
Next task is to estimate $I_2$. For this, we use Hölder’s inequality, lemmas 2.3 and 2.4, Remark 2.2 and inequality (2.2)

$$I_2 \leq Cp^{-knq}$$

$$\times \int_{S_k} \left( \sum_{l=-\infty}^{\infty} \int_{S_l} |f(y)(i-k)||\|b\|_{CMO^r(w,Q^p)} w(B_l)\|B_l\|^{-1} \right) \frac{w(y)}{w(x)} \lesssim q$$

$$\leq C p^{-knq} \|b\|_{CMO^r(w,Q^p)}^q w(B_k)^{1-q}$$

$$\times \left( \sum_{l=-\infty}^{\infty} \frac{w(B_l)^{1-1/q}}{|B_l|} \|fX_l\|_{L^q(w,Q^p)} \right)^q$$

$$\leq C \|b\|_{CMO^r(w,Q^p)}^q$$

$$\times \left( \sum_{l=-\infty}^{\infty} \frac{(k-i)w(B_l)^{1-1/q}}{w(B_k)} \|fX_l\|_{L^q(w,Q^p)} \right)^q$$

$$\leq C \|b\|_{CMO^r(w,Q^p)}^q$$

$$\times \left( \sum_{l=-\infty}^{\infty} \frac{(k-i)p^{(i-k)\mu/q} \|fX_l\|_{L^q(w,Q^p)}^q} \right)^q.$$  \tag{2.7}

From (2.3), (2.6) and (2.7) together with Jensen’s Inequality, we have

$$\|H^2 f\|_{K_{q^2}^{\alpha^{r_2}}(w, w^{1-r})}$$

$$= \left( \sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_2/n} \|H^2 f\|_{L^2(w^{1-r}, Q^p)} \right)^{1/r_2}$$

$$\leq \left( \sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n} \|H^2 f\|_{L^1(w^{1-r}, Q^p)} \right)^{1/r_1}$$

$$\leq C \|b\|_{CMO^r(w,Q^p)} \left( \sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n} \left( \sum_{l=-\infty}^{\infty} \frac{p^{(i-k)\mu/q} \|fX_l\|_{L^q(w,Q^p)}}{r_1} \right) \right)^{1/r_1}$$

$$+ C \|b\|_{CMO^r(w,Q^p)} \left( \sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n} \left( \sum_{l=-\infty}^{\infty} p^{(i-k)\mu/q} \|fX_l\|_{L^q(w,Q^p)} \right)^{r_1} \right)^{1/r_1}$$

$$+ C \|b\|_{CMO^r(w,Q^p)} \left( \sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n} \left( \sum_{l=-\infty}^{\infty} (k-i)p^{(i-k)\mu/q} \|fX_l\|_{L^q(w,Q^p)} \right) \right)^{1/r_1}$$

$$= J.$$ 

Therefore,

$$J^{r_1} \leq C \|b\|_{CMO^{\mu/q_1}(w,Q^p)}^r$$

$$\times \sum_{k=-\infty}^{\infty} w(B_k)^{\alpha r_1/n} \left( \sum_{l=-\infty}^{\infty} (k-i)p^{(i-k)\mu/q} \|fX_l\|_{L^q(w,Q^p)} \right)^{r_1}$$

$$\leq C \|b\|_{CMO^{\mu/q_1}(w,Q^p)}^r.$$
\[
\times \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{k} (k - i)p^{(i-k)\mu/q' - \alpha} ||f_X||_{L^p(w,Q_p^\alpha)} \right)^{r_1}.
\]

In what follows we consider two cases, \(0 < r_1 \leq 1\) and \(r_1 > 1\).

**Case 1:** When \(0 < r_1 \leq 1\) and \(\alpha < n\mu/q'\), we have

\[
J^{r_1} \leq C||b||_{CMO_{max,q'}^\alpha(w,Q_p^\alpha)}^{r_1} \times \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{k} (k - i)^n (B_i)^{\alpha q}/n \right)^{r_1} p^{[(i-k)(n\mu/q' - \alpha)]^{r_1}} ||f_X||_{L^p(w,Q_p^\alpha)}^{r_1} = C||b||_{CMO_{max,q'}^\alpha(w,Q_p^\alpha)}^{r_1} \times \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{k} (k - i)^n (B_i)^{\alpha q}/n \right)^{r_1} p^{[(i-k)(n\mu/q' - \alpha)]^{r_1}} ||f_X||_{L^p(w,Q_p^\alpha)}^{r_1} = C||b||_{CMO_{max,q'}^\alpha(w,Q_p^\alpha)}^{r_1} ||f||_{K_q^{r_1}(w,w)}^{r_1}.
\]

**Case 2:** Whenever \(r_1 > 1\), an application of Hölder’s inequality with \(\alpha < n\mu/q'\), we get

\[
J^{r_1} \leq C||b||_{CMO_{max,q'}^\alpha(w,Q_p^\alpha)}^{r_1} \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{k} (k - i)^{n} (B_i)^{\alpha q}/n \right)^{r_1} p^{[(i-k)(n\mu/q' - \alpha)]^{r_1}} ||f_X||_{L^p(w,Q_p^\alpha)}^{r_1} = C||b||_{CMO_{max,q'}^\alpha(w,Q_p^\alpha)}^{r_1} \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{k} (k - i)^{n} (B_i)^{\alpha q}/n \right)^{r_1} p^{[(i-k)(n\mu/q' - \alpha)]^{r_1}} ||f_X||_{L^p(w,Q_p^\alpha)}^{r_1} = C||b||_{CMO_{max,q'}^\alpha(w,Q_p^\alpha)}^{r_1} ||f||_{K_q^{r_1}(w,w)}^{r_1}.
\]

Hence, the proof of theorem is completed.

**Proof of Theorem 2.7:** From theorem 2.5, we have

\[
||H_{k}^{\alpha}f_X||_{L^p(w^{1-q},Q_p)} \leq C||b||_{CMO_{max,q'}^\alpha(w,Q_p^\alpha)}^{r_1} \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{k} (k - i)^{(i-k)(n\mu/q')} \right)^{r_1} ||f_X||_{L^p(w,Q_p^\alpha)}^{r_1}.
\]

By definition of weighted \(p\)-adic Morrey-Herz spaces and Jensen’s Inequality along with \(\alpha < n\mu/q' + \lambda\),
\( \lambda > 0 \) and \( 1 < r_1 < \infty \), we reach at
\[
\| H^p_{b} f \|_{W^p_{q_1} \cap (w, w^{1/r_1})} = \sup_{k_0 \in \mathbb{Z}} w(B_{k_0})^{-\lambda/n} \left( \sum_{k = -\infty}^{k_0} w(B_k) \| (H^p_{b} f) \chi_k \|_{L^q(w^{1/q} \cap Q_p)}^{r_1} \right)^{1/r_1} \\
\leq \sup_{k_0 \in \mathbb{Z}} w(B_{k_0})^{-\lambda/n} \left( \sum_{k = -\infty}^{k_0} w(B_k) \| (H^p_{b} f) \chi_k \|_{L^q(w^{1/q} \cap Q_p)}^{r_1} \right)^{1/r_1} \\
\leq C \| b \|_{CMO^{\max\mu/q_1}(w, Q_p)} \sup_{k_0 \in \mathbb{Z}} w(B_{k_0})^{-\lambda/n} \left( \sum_{k = -\infty}^{k_0} w(B_k) \| (H^p_{b} f) \chi_k \|_{L^q(w^{1/q} \cap Q_p)}^{r_1} \right)^{1/r_1} \\
\leq C \| b \|_{CMO^{\max\mu/q_1}(w, Q_p)} \sup_{k_0 \in \mathbb{Z}} w(B_{k_0})^{-\lambda/n} \left( \sum_{k = -\infty}^{k_0} w(B_k) \| (H^p_{b} f) \chi_k \|_{L^q(w^{1/q} \cap Q_p)}^{r_1} \right)^{1/r_1} \\
\leq C \| b \|_{CMO^{\max\mu/q_1}(w, Q_p)} \sup_{k_0 \in \mathbb{Z}} w(B_{k_0})^{-\lambda/n} \left( \sum_{k = -\infty}^{k_0} w(B_k) \| (H^p_{b} f) \chi_k \|_{L^q(w^{1/q} \cap Q_p)}^{r_1} \right)^{1/r_1} \\
\leq C \| b \|_{CMO^{\max\mu/q_1}(w, Q_p)} \| f \|_{W^p_{q_1} \cap (w, w^{1/r_1})}.
\]

3. Weighted \( p \)-adic Lipschitz estimates of \( H^p_{b} \) on two weighted \( p \)-adic Herz-type Spaces

The current section deals the weighted \( p \)-adic Lipschitz estimates of \( H^p_{b} \) on two weighted \( p \)-adic Herz-type spaces. The outset of a section is with a lemma which is helpful in proving main results.

\textbf{Lemma 3.1.} [23] Suppose \( w \in A_1 \) and \( b \in Lip_\beta(w, \mathbb{Q}_p^n) \), then there is a constant \( C \) such that for \( i, k \in \mathbb{Z} \),
\[
| b_{B_i} - b_{B_k} | \leq C(i - k) \| b \|_{Lip_\beta(w, \mathbb{Q}_p^n)} w(B_i)^{\beta/n} w(B_k)^{1/n}.
\]

Now, we state the result about the boundedness of commutator of \( p \)-adic Hardy operator on two weighted \( p \)-adic Herz-type spaces.

\textbf{Theorem 3.2.} Let \( 0 < r_1 \leq r_2 < \infty, 1 \leq q_1, q_2 < \infty, 1/q_1 - 1/q_2 = \beta/n \) and let \( w \in A_1 \). If \( \alpha \leq \frac{\mu}{q_1} \), then the inequality
\[
\| H^p_{b} f \|_{K^p_{q_2} \cap (w, w^{1/q_2})} \leq C \| b \|_{Lip_\beta(w, \mathbb{Q}_p^n)} \| f \|_{K^p_{q_1} \cap (w, w)}
\]
holds for all \( b \in Lip_\beta(w, \mathbb{Q}_p^n) \) and \( f \in L_{loc}(\mathbb{Q}_p^n) \).

If \( \alpha = 0, r_1 = q_1 = p \) and \( r_2 = q_2 = q \), then we have the following corollary.
Corollary 3.3. Let $1 \leq q < \infty$, $1/q_1 - 1/q_2 = \beta/n$ and $w \in A_1$, then
\[
\|H_b^p f\|_{L^q(w, Q_{p,B})} \leq C \|b\|_{L^{p,q}(w, Q_{p,B})} \|f\|_{L^q(w, Q_{p,B})}
\]
holds for all $b \in \text{Lip}(w, Q_{p,B})$ and $f \in L^q_{\text{loc}}(Q_{p,B})$.

Theorem 3.4. Let $0 < r_1 \leq r_2 < \infty$, $1 \leq q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = \beta/n$ and let $w \in A_1$. If $\alpha < \frac{\mu}{q_1} + \lambda$, then
\[
\|H_b^p f\|_{M^{\alpha,\lambda}_{r_1, r_2}(w, Q_{p,B})} \leq C \|b\|_{L^{p,q}(w, Q_{p,B})} \|f\|_{M^{\alpha,\lambda}_{r_1, r_2}(w, Q_{p,B})}
\]
holds for all $b \in \text{Lip}(w, Q_{p,B})$ and $f \in L^q_{\text{loc}}(Q_{p,B})$.

Proof of Theorem 3.2: In a similar fashion as that of theorem 2.5, we get
\[
\|(H_b^p f)_x\|_{L^{q_2}(w, Q_{p,B})} \leq C P^{-kq_2/n} \int_{S_k} \left( \sum_{i=-\infty}^k \int_{S_i} |f(t)(b(x) - b_{B_x})| dt \right)^{q_2} w(x)^{1-q_2} dx
\]
\[
+ C P^{-kq_2/n} \int_{S_k} \left( \sum_{i=-\infty}^k \int_{S_i} |f(t)(b(t) - b_{B_t})| dt \right)^{q_2} w(x)^{1-q_2} dx
\]
\[
= L + LL. \quad (3.1)
\]
For the evaluation of $L$, we apply Hölder’s inequality, Remark 2.2, $\beta/n = 1/q_1 - 1/q_2$, $w \in A_1 \subset A_{q_1}$, and inequality (2.2) to get
\[
L \leq C P^{-kq_2/n} \|b\|_{L^{p,q}(w, Q_{p,B})} w(B_t)^{1+\beta/n} \left\{ \sum_{i=-\infty}^k \|f(x)\|_{L^{q_1}(w, Q_{p,B})} |B_i| \right\}^{q_2}
\]
\[
\leq C P^{-kq_2/n} \|b\|_{L^{p,q}(w, Q_{p,B})} \left\{ \sum_{i=-\infty}^k \|f(x)\|_{L^{q_1}(w, Q_{p,B})} |B_i| \left( \frac{w(B_i)}{w(B_t)} \right)^{1/q_1} \right\}^{q_2}
\]
\[
\leq C \|b\|_{L^{p,q}(w, Q_{p,B})}^q \left( p^{k-\beta/n q_2} \|f(x)\|_{L^{q_1}(w, Q_{p,B})} \right)^{q_2} \quad (3.2)
\]
In order to evaluate $LL$, we proceed as follows
\[
LL \leq C P^{-kq_2/n} \int_{S_k} \left( \sum_{i=-\infty}^k \int_{S_i} |f(t)(b(t) - b_{B_t})| dt \right)^{q_2} w(x)^{1-q_2} dx
\]
\[
+ C P^{-kq_2/n} \int_{S_k} \left( \sum_{i=-\infty}^k \int_{S_i} |f(t)(b_{B_t} - b_{B_t})| dt \right)^{q_2} w(x)^{1-q_2} dx
\]
\[
= LL_1 + LL_2. \quad (3.3)
\]
The following preparation will do world of good to estimate $LL_1$. Using Hölder’s inequality, we have
\[
\int_{S_i} |f(t)(b(t) - b_{B_t})| dt
\]
\[
\leq \left( \int_{S_i} |f(t)|^{q_1} w(t) |dt| \right)^{1/q_1} \left( \int_{S_i} |b(t) - b_{B_t}|^{\mu} w(t)^{-q_1/q_2} |dt| \right)^{1/q_2}
\]
\[
\leq w(B_t)^{-1/q_2 + \beta/n} \|f(x)\|_{L^{q_1}(w, Q_{p,B})} |b| \|L^{p,q}(w, Q_{p,B})|. \quad (3.4)
\]
To evaluate $LL_1$, we apply Hölder’s inequality, inequality (3.4), lemma 2.4 and Remark 2.2.

\[
LL_1 \leq C p^{-kq_2n} \int_{S_k} w(x)^{1-q_2} dx \left( \sum_{i=-\infty}^{\infty} \|f \chi_i\|_{L^{q_1}(w^{1/q_1})} \|w(B_i)^{1/q_1+\beta/n} \|_{L^{p}(w^{2/q_1})} \right)^{q_2}
\]

\[
\leq C p^{-kq_2n} |B_k|^{q_2} w(B_k)^{1-q_2} \|b\|_{L^{p}(w^{2/q_1})} \left( \sum_{i=-\infty}^{\infty} \|f \chi_i\|_{L^{q_1}(w^{1/q_1})} \|w(B_i)^{1/q_1+\beta/n} \|_{L^{p}(w^{2/q_1})} \right)^{q_2}
\]

\[
\leq C |\|b\|_{L^{p}(w^{2/q_1})} \left( \sum_{i=-\infty}^{\infty} \|w(B_i)^{1/q_1} \|_{L^{q_1}(w^{1/q_1})} \right)^{q_2}
\]

\[
\leq C |\|b\|_{L^{p}(w^{2/q_1})} \left( \sum_{i=-\infty}^{\infty} p^{1-k\eta_i/q_1} \|f \chi_i\|_{L^{q_1}(w^{1/q_1})} \right)^{q_2}.
\]

Next step is to evaluate $LL_2$. For this we use Hölder’s inequality, lemmas 3.1 and 2.4, inequality (2.2), and Remark 2.2 to get

\[
LL_2 \leq C p^{-kq_1n} \|b\|_{L^{p}(w^{2/q_1})} \|B_k\|^{q_2} w(B_k)^{1-q_2}
\]

\[
\times \left( \sum_{i=-\infty}^{\infty} \|k-i\| w(B_k)^{\beta/n} \|w(B_k)^{1/q_1} \|_{L^{q_1}(w^{1/q_1})} \right)^{q_2}
\]

\[
= C |\|b\|_{L^{p}(w^{2/q_1})} \left( \sum_{i=-\infty}^{\infty} \|w(B_k)^{1/q_1} \|_{L^{q_1}(w^{1/q_1})} \right)^{q_2}
\]

\[
\times \left( \sum_{i=-\infty}^{\infty} \|k-i\| p^{1-k\eta_i/q_1} \|f \chi_i\|_{L^{q_1}(w^{1/q_1})} \right)^{q_2}.
\]

Remaining proof is more or less same to the proof of theorem 2.5. Thus, we conclude the theorem.

**Proof of Theorem 3.4:** Let $\alpha < \eta_i/q_1 + \lambda$. By the definition of weighted $p$-adic Morrey-Herz spaces together with inequalities (3.2), (3.5), and (3.6), we have

\[
\|H^p w \|_{MK}^{\alpha} \leq C |\|b\|_{L^{p}(w^{2/q_1})} \sup_{k_i \in \mathbb{Z}} \|w(B_k)^{1/q_1} \|_{L^{q_1}(w^{1/q_1})} \left( \sum_{i=-\infty}^{\infty} p^{1-k\eta_i/q_1} \|f \chi_i\|_{L^{q_1}(w^{1/q_1})} \right)^{1/r_2}
\]

\[
+ C |\|b\|_{L^{p}(w^{2/q_1})} \sup_{k_i \in \mathbb{Z}} \|w(B_k)^{1/q_1} \|_{L^{q_1}(w^{1/q_1})} \left( \sum_{i=-\infty}^{\infty} p^{1-k\eta_i/q_1} \|f \chi_i\|_{L^{q_1}(w^{1/q_1})} \right)^{1/2/r_2}
\]

\[
+ C |\|b\|_{L^{p}(w^{2/q_1})} \sup_{k_i \in \mathbb{Z}} \|w(B_k)^{1/q_1} \|_{L^{q_1}(w^{1/q_1})} \left( \sum_{i=-\infty}^{\infty} p^{1-k\eta_i/q_1} \|f \chi_i\|_{L^{q_1}(w^{1/q_1})} \right)^{1/r_2}
\]

\[
= S_1 + S_2 + S_3.
\]
Next by applying the similar arguments as in theorem 2.7, we get

\[ S_1 \leq C \|b\|_{\text{Lip}(w, \mathbb{Q}_p^n)} \|f\|_{M_{K^{\alpha,\lambda}_{q_1,q_2}}(w, w)}, \quad \alpha < n/q'_1 + \lambda, \]

\[ S_2 \leq C \|b\|_{\text{Lip}(w, \mathbb{Q}_p^n)} \|f\|_{M_{K^{\alpha,\lambda}_{q_1,q_2}}(w, w)}, \quad \alpha < n\mu/q'_2 + \lambda, \]

\[ S_3 \leq C \|b\|_{\text{Lip}(w, \mathbb{Q}_p^n)} \|f\|_{M_{K^{\alpha,\lambda}_{q_1,q_2}}(w, w)}, \quad \alpha < n\mu/q'_1 + \lambda. \]

So, the proof of the theorem is finished.

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Conflict of interest

The authors declare that they have no conflict of interest.

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