AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 9602–9618.
DOI:10.3934/math.2021559
Received: 04 February 2021
Accepted: 23 June 2021
Published: 25 June 2021

Research article

On a Diophantine equation with prime variables

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Abstract: Let $[\alpha]$ denote the integer part of the real number α, N be a sufficiently large integer and (κ, λ) be the exponent pair. In this paper, we prove that for $1 < c < \frac{3+3\kappa-\lambda}{3\kappa+2}$, the Diophantine equation $[p_1^c] + [p_2^c] + [p_3^c] = N$ is solvable in prime variables p_1, p_2, p_3. If we take $(\kappa, \lambda) = \left(\frac{81}{242}, \frac{132}{242}\right)$, we can get the range $1 < c < \frac{837}{727}$, which improves the previous result of Cai.

Keywords: Diophantine equation; prime; exponential sum
Mathematics Subject Classification: 11J25, 11L03, 11P32

1. Introduction

Diophantine equation is a classical problem in number theory. Let $[\alpha]$ denote the integer part of the real number α and N be a sufficiently large integer. In 1933, Segal [27, 28] firstly studied additive problems with non-integer degrees, and proved that there exists a $k_0(c) > 0$ such that the Diophantine equation

$$[x_1^c] + [x_2^c] + \cdots + [x_k^c] = N \tag{1.1}$$

is solvable for $k > k_0(c)$, where $c > 1$ is not an integer. Later, Deshouillers [5] improved Segal's bound of $k_0(c)$ to $6c^3(\log c + 14)$ with $c > 12$. Further Arkhilov and Zhitkov [1] refined Deshouillers's result to $22c^2(\log c + 4)$ with $c > 12$. Afterwards, many results of various Diophantine equations were established (e.g., see [7, 10, 14, 16–19, 21, 25, 41, 42]). In particular, Laporta [17] in 1999 showed that the equation

$$[p_1^c] + [p_2^c] = N \tag{1.2}$$

is solvable in primes p_1, p_2 provided that $1 < c < \frac{17}{16}$ and N is sufficiently large. Recently, the range of c in (1.2) was enlarged to $1 < c < \frac{14}{11}$ by Zhu [40]. Kumchev [15] showed that the equation

$$[m^c] + [p^c] = N \tag{1.3}$$

is solved for almost all N provided that $1 < c < \frac{16}{15}$, where m is an integer and p is a prime. Afterwards, the range of c in (1.3) was enlarged to $1 < c < \frac{17}{11}$ by Balanzario, Garaev and Zuazua [3].

In 1995, Laporta and Tolev [18] considered the equation

$$[p_1^c] + [p_2^c] + [p_3^c] = N \tag{1.4}$$

with prime variables p_1, p_2, p_3. Denote the weighted number of solutions of Eq (1.4) by

$$R(N) = \sum_{[p_1^c]+[p_2^c]+[p_3^c]=N} (\log p_1)(\log p_2)(\log p_3). \tag{1.5}$$

They established the following asymptotic formula

$$R(N) = \frac{\Gamma^3\left(1+\frac{1}{c}\right)}{\Gamma\left(\frac{3}{c}\right)} N^{\frac{3}{c}-1} + O\left(N^{\frac{3}{c}-1}\exp\left(-\log^{\frac{1}{3}-\delta} N\right)\right)$$

for any $0 < \delta < \frac{1}{3}$ and $1 < c < \frac{17}{16}$. Afterwards, the range of c was enlarged to $1 < c < \frac{12}{11}$ by Kumchev and Nedeva [16], to $1 < c < \frac{258}{235}$ by Zhai and Cao [39], and to $1 < c < \frac{137}{119}$ by Cai [4].

In this paper, we first show a more general result related to (1.5) by proving the following theorem.

Theorem 1.1. *Let N be a sufficiently large integer. Then for $1 < c < \frac{3+3\kappa-\lambda}{3\kappa+2}$, we have*

$$R(N) = \frac{\Gamma^3\left(1+\frac{1}{c}\right)}{\Gamma\left(\frac{3}{c}\right)} N^{\frac{3}{c}-1} + O\left(N^{\frac{3}{c}-1}\exp\left(-\log^{\frac{1}{3}-\delta} N\right)\right) \tag{1.6}$$

for any $0 < \delta < \frac{1}{3}$, where (κ, λ) is an exponent pair, and the implied constant in the $O-$symbol depends only on c.

Choosing $(\kappa, \lambda) = BA^2BABABABAB(0, 1) = \left(\frac{81}{242}, \frac{132}{242}\right)$ in Theorem 1.1, we can immediately get the following corollary, which further improves the result of Cai [4].

Corollary 1.2. *Under the notations of Theorem 1.1, for $1 < c < \frac{837}{727}$ the asymptotic formula (1.6) follows.*

It is easy to verify that the range of c in Corollary 1.2 is larger than one of Cai's result. Our improvement mainly derives from more accurate estimates of exponential sums by combining Van der Corput's method, exponent pairs and some elementary methods. Also the estimates of exponential sums has lots of applications in problems including automorphic forms (e.g., see [8, 11, 12, 20, 22, 24, 29–38]).

Notation. Throughout the paper, N always denotes a sufficiently large integer. The letter p, with or without subscripts, is always reserved for primes. Let $\varepsilon \in \left(0, 10^{-10}(\frac{3+3\kappa-\lambda}{3\kappa+2} - c)\right)$. We denote by $\{x\}$ and $||x||$ the fraction part of x and the distance from x to the nearest integer, respectively. Let $1 < c < \frac{3+3\kappa-\lambda}{3\kappa+2}$ and

$$P = N^{\frac{1}{c}}, \quad \tau = P^{1-c-\varepsilon}, \quad e(x) = e^{2\pi i x}, \quad S(\alpha) = \sum_{p\le P} (\log p) e(\alpha[p^c]).$$

2. Auxiliary lemmas

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([9, Lemma 5]). *Suppose that z_n is a sequence of complex numbers, then we have*

$$\left|\sum_{N\le n\le 2N} z_n\right|^2 \le \left(1+\frac{N}{Q}\right)\sum_{q=0}^{Q}\left(1-\frac{q}{Q}\right)\mathrm{Re}\left(\sum_{N\le n\le 2N-q}\overline{z_n}z_{n+q}\right),$$

where $\mathrm{Re}(t)$ *and* $\bar{t}$ *denote the real part and the conjugate of the complex number t, respectively.*

Lemma 2.2. *Suppose that* $|x|>0$ *and* $c>1$. *Then for any exponent pair* (κ,λ) *and* $M\le a<b\le 2M$*, we have*

$$\sum_{a\le n\le b} e(xn^c) \ll (|x|M^c)^{\kappa}M^{\lambda-\kappa}+\frac{M^{1-c}}{|x|}.$$

Proof. We can get this lemma from [6, (3.3.4)]. □

Lemma 2.3 ([2, Lemma 12]). *Suppose that t is not an integer and* $H\ge 3$. *Then for any* $\alpha\in(0,1)$*, we have*

$$e(-\alpha\{t\})=\sum_{|h|\le H}c_h(\alpha)e(ht)+O\left(\min\left(1,\frac{1}{H\|t\|}\right)\right),$$

where

$$c_h(\alpha)=\frac{1-e(-\alpha)}{2\pi i(h+\alpha)}.$$

Lemma 2.4 ([9, Lemma 3]). *Suppose that* $3<U<V<Z<X$*, and* $\{Z\}=\frac{1}{2}$, $X\ge 64Z^2U$, $Z\ge 4U^2$, $V^3\ge 32X$. *Further suppose that* $F(n)$ *is a complex valued function such that* $|F(n)|\le 1$. *Then the sum*

$$\sum_{X\le n\le 2X}\Lambda(n)F(n)$$

can be decomposed into $O(\log^{10}X)$ *sums, each of which either of type I:*

$$\sum_{M\le m\le 2M}a(m)\sum_{N\le n\le 2N}F(mn)$$

with $N>Z$*, where* $a(m)\ll m^{\varepsilon}$ *and* $X\ll MN\ll X$*, or of type II:*

$$\sum_{M\le m\le 2M}a(m)\sum_{N\le n\le 2N}b(n)F(mn)$$

with $U\ll M\ll V$*, where* $a(m)\ll m^{\varepsilon}, b(n)\ll n^{\varepsilon}$ *and* $X\ll MN\ll X$.

Lemma 2.5. *Let $f(t)$ be a real value function and continuous differentiable at least three times on $[a, b]$ $(1 \le a < b \le 2a)$, $|f'''(x)| \sim \Delta > 0$, then we have*

$$\sum_{a<n\le b} e(f(n)) \ll a\Delta^{\frac{1}{6}} + \Delta^{-\frac{1}{3}}.$$

Moreover, if $0 < c_1\lambda_1 \le c_2\lambda_1$, $|f''(x)| \sim \lambda_1 a^{-1}$, *then we have*

$$\sum_{a<n\le b} e(f(n)) \ll a^{\frac{1}{2}}\lambda_1^{\frac{1}{2}} + \lambda_1^{-1};$$

if $c_2\lambda_1 \le \frac{1}{2}$, *then we have*

$$\sum_{a<n\le b} e(f(n)) \ll \lambda_1^{-1}.$$

Proof. The first result was proved by Sargos [26]. And the remaining two results were due to Jia [13]. □

Lemma 2.6 ([23, Lemma 2]). *Let $M > 0$, $N > 0$, $u_m > 0$, $\upsilon_n > 0$, $A_m > 0$, $B_n > 0$ $(1 \le m \le M, 1 \le n \le N)$. Let also Q_1 and Q_2 be given non-negative numbers, $Q_1 \le Q_2$. Then there is one q such that $Q_1 \le q \le Q_2$ and*

$$\sum_{m=1}^{M} A_m q^{u_m} + \sum_{n=1}^{N} B_n q^{-\upsilon_n} \ll \sum_{m=1}^{M}\sum_{n=1}^{N} (A_m^{\upsilon_n} B_n^{u_m})^{\frac{1}{u_m+\upsilon_n}} + \sum_{m=1}^{M} A_m Q_1^{u_m} + \sum_{n=1}^{N} B_n Q_2^{-\upsilon_n}.$$

Lemma 2.7 ([36, Lemma 5]). *Let $f(x)$, $g(x)$ be algebraic functions in $[a, b]$, $|f''(x)| \sim \frac{1}{R}$, $f'''(x) \ll \frac{1}{RU}$, $U \ge 1$, $|g(x)| \ll G$, $|g'(x)| \ll GU^{-1}$. $[\alpha, \beta]$ is the image of $[a, b]$ under the mapping $y = f'(x)$. n_u is the solution of $f'(n) = u$.*

$$b_u = \begin{cases} 1, & \alpha < u < \beta, \\ \frac{1}{2}, & u = \alpha \in \mathbb{N} \text{ or } u = \beta \in \mathbb{N}. \end{cases}$$

Then we have

$$\begin{aligned} \sum_{a<n\le b} g(n)e(f(n)) = {} & \sum_{\alpha<u\le\beta} b_u \frac{g(n_u)}{\sqrt{|f''(n_u)|}} e\left(f(n_u) - un_u + \frac{1}{8}\right) \\ & + O\left(G\log(\beta - \alpha + 2) + G(b - a + R)u^{-1}\right) \\ & + O\left(G\min\left(\sqrt{R}, \frac{1}{\|\alpha\|}\right) + G\min\left(\sqrt{R}, \frac{1}{\|\beta\|}\right)\right). \end{aligned}$$

Lemma 2.8 ([13, Lemma 3]). *Suppose that $x \sim N$, $f(x) \ll P$, and $f'(x) \gg \Delta$. Then we have*

$$\sum_{n\sim N} \min\left(D, \frac{1}{\|f(n)\|}\right) \ll (P + 1)\left(D + \Delta^{-1}\right)\log\left(2 + \Delta^{-1}\right).$$

Lemma 2.9. *For $0 < \alpha < 1$ and any exponent pair (κ, λ), we have*

$$T(\alpha, X) = \sum_{X<n\le 2X} e(\alpha[n^c]) \ll X^{\frac{\kappa c+\lambda}{1+\kappa}} \log X + \frac{X}{\alpha X^c}.$$

Proof. Throughout the proof of this lemma, we write $H = X^{\frac{-\kappa c+1-\lambda+\kappa}{1+\kappa}}$ for convenience. Using Lemma 2.3 we can get

$$T(\alpha, X) = \sum_{|h|\le H} c_h(\alpha) \sum_{X<n\le 2X} e((h+\alpha)n^c) + O\left((\log X) \sum_{X<n\le 2X} \min\left(1, \frac{1}{H\|n^c\|}\right)\right).$$

Then by the expansion

$$\min\left(1, \frac{1}{H\|\theta\|}\right) = \sum_{h=-\infty}^{\infty} a_h e(h\theta),$$

where

$$|a_h| = \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{h^2}\right),$$

we have

$$\begin{aligned}
\sum_{X<n\le 2X} \min\left(1, \frac{1}{H\|n^c\|}\right) &\le \sum_{h=-\infty}^{\infty} |a_h| \left|\sum_{X<n\le 2X} e(hn^c)\right| \\
&\ll \frac{X\log 2H}{H} + \sum_{1\le h\le H} \frac{1}{h}\left((hX^c)^\kappa X^{\lambda-\kappa} + \frac{X}{hX^c}\right) \\
&\quad + \sum_{h\ge H} \frac{H}{h^2}\left((hX^c)^\kappa X^{\lambda-\kappa} + \frac{X}{hX^c}\right) \\
&\ll X^{\frac{\kappa c+\lambda}{1+\kappa}} \log X,
\end{aligned}$$

where we estimated the sum over n by Lemma 2.2 .

In a similar way, we have

$$\begin{aligned}
&\sum_{|h|\le H} c_h(\alpha) \sum_{X<n\le 2X} e((h+\alpha)n^c) \\
= \;& c_0(\alpha) \sum_{X<n\le 2X} e(\alpha n^c) + \sum_{1\le h\le H} c_h(\alpha) \sum_{X<n\le 2X} e((h+\alpha)n^c) \\
\ll \;& X^{\frac{\kappa c+\lambda}{1+\kappa}} \log X + \frac{X}{\alpha X^c}.
\end{aligned}$$

Then this lemma follows. □

Lemma 2.10 ([42, Lemma 2.1]). *Suppose that $f(n)$ is a real-valued function in the interval $[N, N_1]$, where $2 \le N < N_1 \le 2N$. If $0 < c_1\lambda_1 \le |f'(n)| \le c_2\lambda_1 \le \frac{1}{2}$, then we have*

$$\sum_{N<n\le N_1} e(f(n)) \ll \lambda_1^{-1}.$$

If $|f^{(j)}(n)| \sim \lambda_1 N^{-j+1}$ $(j = 1, 2)$, then we have

$$\sum_{N<n\le N_1} e(f(n)) \ll \lambda_1^{-1} + N^{\frac{1}{2}} \lambda_1^{\frac{1}{2}}.$$

If $|f^{(j)}(n)| \sim \lambda_1 N^{-j+1}$ $(j = 1,2,3,4,5,6)$*, then we have*

$$\sum_{N<n\le N_1} e(f(n)) \ll \lambda_1^{-1} + N^{\lambda}\lambda_1^{\kappa},$$

where (κ,λ) *is any exponent pair.*

Lemma 2.11 ([9, Lemma 6])**.** *Suppose that* $0 < a < b \le 2a$ *and* R *is an open convex set in* $\mathbb{C}$ *containing the real segment* $[a,b]$*. Suppose further that* $f(z)$ *is analytic on* R*.* $f(x)$ *is real for real* $x \in R$*.* $f''(z) \le M$ *for* $z \in R$*. There is a constant* $k > 0$ *such that* $f''(x) \le -kM$ *for all real* $x \in R$*. Let* $f'(b) = \alpha$ *and* $f'(a) = \beta$*, and define* x_υ *for each integer* υ *in the range* $\alpha < \upsilon < \beta$ *by* $f'(x_\upsilon) = \upsilon$*. Then we have*

$$\sum_{a<n\le b} e(f(n)) = e\left(-\frac{1}{8}\right)\sum_{\alpha<\upsilon\le\beta} |f''(x_\upsilon)|^{-\frac{1}{2}} e\big(f(x_\upsilon) - \upsilon x_\upsilon\big) + O\Big(M^{-\frac{1}{2}} + \log\left(2 + M(b-a)\right)\Big).$$

3. The estimate of $S(\alpha)$

Lemma 3.1. *Let* $P^{\frac{5}{6}} \ll X \ll P$*,* $H = X^{1-\frac{(1+2\kappa)c+\lambda}{2+2\kappa}}$ *and* $c_h(\alpha)$ *denote complex numbers such that* $c_h(\alpha) \ll (1+|h|)^{-1}$*. Then uniformly for* $\alpha \in (\tau, 1-\tau)$*, we have*

$$S_I = \sum_{|h|\sim H} c_h(\alpha) \sum_{M\le m\le 2M} a(m) \sum_{N\le n\le 2N} e\left((h+\alpha)(mn)^c\right) \ll X^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}+2\varepsilon} \tag{3.1}$$

for any $a(m) \ll m^{\varepsilon}$*, where* (κ,λ) *is any exponent pair,* $X \ll MN \ll X$ *and* $M \ll Y$ *with* $Y = \min\{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8\}$*,*

$$X_1 = X^{\frac{15}{2}\frac{(1+2\kappa)c+\lambda}{2+2\kappa}-\frac{c}{2}-\frac{11}{2}},\ X_2 = X^{\frac{52}{11}\frac{(1+2\kappa)c+\lambda}{2+2\kappa}-\frac{4c}{11}-\frac{8}{11}},$$
$$X_3 = X^{\frac{31}{8}\frac{(1+2\kappa)c+\lambda}{2+2\kappa}-\frac{c}{8}-\frac{23}{8}},\ X_4 = X^{\frac{2(1+2\kappa)c+2\lambda}{1+\kappa}-3},$$
$$X_5 = X^{\frac{4(1+2\kappa)c+4\lambda}{1+\kappa}-\frac{46}{7}},\quad X_6 = X^{\frac{16}{7}\frac{(1+2\kappa)c+\lambda}{1+\kappa}-\frac{25}{7}},$$
$$X_7 = X^{\frac{20}{3}\frac{(1+2\kappa)c+\lambda}{1+\kappa}-\frac{34}{3}},\quad X_8 = X^{\frac{7}{3}\frac{(1+2\kappa)c+\lambda}{1+\kappa}-\frac{11}{3}}.$$

Proof. It is easy to deduce that

$$S_I \ll M^{\varepsilon} \sum_{h\sim H} K_h,$$

where $K_h = \sum_{m\sim M} \left|\sum_{n\sim N} e\left((\alpha+h)(mn)^c\right)\right|$. According to Hölder's inequality, we have

$$K_h^4 \ll M^3 \sum_{m\sim M} \left|\sum_{n\sim N} e\left((\alpha+h)(mn)^c\right)\right|^4. \tag{3.2}$$

Let $z_n = z_n(m,\alpha) = (\alpha+h)(mn)^c$. Suppose that Q, J are two positive integers such that $1 \le Q \le N\log^{-1} X$, $1 \le J \le N\log^{-1} X$. For the inner sum in (3.2), applying Lemma 2.1 twice, we can get

$$K_h^4 \ll \frac{X^4}{Q^2} + \frac{X^4}{J} + \frac{X^3}{JQ}\sum_{j=1}^{J}\sum_{q=1}^{Q} |E_{q,j}|, \tag{3.3}$$

where

$$E_{q,j} = \sum_{m\sim M} \sum_{N<n\le 2N-q-j} e(z_n - z_{n+q} + z_{n+q+j} - z_{n+j}). \tag{3.4}$$

Let $\Delta(n^c; q, j) = (n+q+j)^c - (n+q)^c - (n+j)^c + n^c$, $G(m,n) = (\alpha+h)m^c\Delta(n^c; q, j)$. Then $z_n - z_{n+q} - z_{n+j} + z_{n+q+j} = G(m,n)$. Thus we have

$$E_{q,j} = \sum_m \sum_n e(G(m,n)). \tag{3.5}$$

For any $t \ne 1, 0$, we have

$$\Delta(n^t; q, j) = t(t-1)qjn^{t-2} + O(N^{t-3}qj(q+j)), \tag{3.6}$$

then

$$\frac{\partial G}{\partial n} = c(c-1)(c-2)(\alpha+h)qjm^c n^{c-3}\left(1 + O\left(\frac{q+j}{N}\right)\right)$$

and

$$\frac{\partial^2 G}{\partial n^2} = c(c-1)(c-2)(c-3)(\alpha+h)qjm^c n^{c-4}\left(1 + O\left(\frac{q+j}{N}\right)\right). \tag{3.7}$$

If $c(c-1)(c-2)(\alpha+h)qjM^cN^{c-3} \le \frac{1}{100}$, by Lemma 2.5 we have

$$\sum_m \sum_n e(G(m,n)) \ll MN^3((\alpha+h)qjM^cN^c)^{-1}.$$

From now we always suppose that $c(c-1)(c-2)(\alpha+h)qjM^cN^{c-3} \ge \frac{1}{100}$. By Lemma 2.7 we have

$$\sum_{N<n\le 2N-j-q} e(G(m,n)) = e\left(\frac{1}{8}\right) \sum_{\alpha<\upsilon<\beta} \left|\frac{\partial^2 G}{\partial n^2}(m, n_\upsilon)\right|^{-\frac{1}{2}} e(G(m,n_\upsilon) - \upsilon n_\upsilon) + R(m,q,j),$$

where

$$\begin{aligned} &\frac{\partial G}{\partial n}(m, n_\upsilon) = \upsilon, \beta = \frac{\partial G}{\partial n}(m, N), \alpha = \frac{\partial G}{\partial n}(m, 2N-q-j),\\ &R = N^4\left[(h+\alpha)qjX^c\right]^{-1}, \upsilon \sim (h+\alpha)qjM^cN^{c-3},\\ &R(m,q,j) = O\left(\log X + RN^{-1} + \min\left(\sqrt{R}, \max\left(\frac{1}{||\alpha||}, \frac{1}{||\beta||}\right)\right)\right). \end{aligned} \tag{3.8}$$

By Lemma 2.8, the contribution of $R(m,q,j)$ to $E(q,j)$ is

$$\begin{aligned} &\ll M\log X + MRN^{-1} + \sum_{m\sim M} \min\left(\sqrt{R}, \frac{1}{||\alpha||}\right) + \sum_{m\sim M} \min\left(\sqrt{R}, \frac{1}{||\beta||}\right)\\ &\ll M\log X + X^{3-c}[(h+\alpha)qjM^2]^{-1} + [(h+\alpha)qj]^{\frac{1}{2}} MX^{\frac{c}{2}-1}\log X. \end{aligned} \tag{3.9}$$

Then we only need to deal with the following exponential sum

$$\begin{aligned}&\sum_{m\sim M}\sum_{\alpha<\upsilon<\beta}\left|\frac{\partial^2 G}{\partial n^2}(m,n_\upsilon)\right|^{-\frac{1}{2}} e(G(m,n_\upsilon)-\upsilon n_\upsilon)\\ =&\sum_{\upsilon}\sum_{m\in I_\upsilon}\left|\frac{\partial^2 G}{\partial n^2}(m,n_\upsilon)\right|^{-\frac{1}{2}} e(G(m,n_\upsilon)-\upsilon n_\upsilon),\end{aligned}$$

where I_υ is a subinterval of $[M,2M]$. For a fixed υ, we define $\Delta_{\lambda'}=\Delta\left(n_\upsilon^{\lambda'};q,j\right)$, where λ' is an arbitrary real number. We take the derivative of m in (3.8) and get

$$n'_\upsilon=-\frac{c\Delta_{c-1}}{(c-1)m\Delta_{c-2}}. \tag{3.10}$$

It follows from (3.7) that

$$\frac{d}{dm}\left(\frac{\partial^2 G}{\partial n^2}(m,n_\upsilon)\right)=\frac{c^2(h+\alpha)m^{c-1}}{\Delta_{c-2}}\left((c-1)\Delta_{c-2}^2-(c-2)\Delta_{c-1}\Delta_{c-3}\right).$$

Recalling (3.6), we can get

$$\frac{d}{dm}\left(\frac{\partial^2 G}{\partial n^2}(m,n_\upsilon)\right)=c^2(c-1)(c-2)(c-3)(h+\alpha)qjm^{c-1}n_\upsilon^{c-4}\left(1+O\left(\frac{q+j}{N}\right)\right).$$

Thus for m, $\left|\frac{\partial^2 G}{\partial n^2}(m,n_\upsilon)\right|^{-\frac{1}{2}}$ is monotonic. Let $g(m)=G(m,n_\upsilon(m))-\upsilon n_\upsilon(m)$. Then we have

$$\begin{aligned}g'(m)=&c(\alpha+h)m^{c-1}\Delta_c,\\ g''(m)=&\frac{c(\alpha+h)}{c-1}\frac{(c-1)^2\Delta_c\Delta_{c-2}-c^2\Delta_{c-1}^2}{m^{2-c}\Delta_{c-2}}=\frac{c(\alpha+h)}{(c-1)}\frac{g_1(m)-g_2(m)}{g_0(m)},\\ g'''(m)=&\frac{c(\alpha+h)}{(c-1)}\frac{(g_1'-g_2')g_0-g_0'(g_1-g_2)}{g_0^2},\end{aligned}$$

where

$$\begin{aligned}g_1'=&\left((c-1)^2c\Delta_{c-1}\Delta_{c-2}+(c-1)^2(c-2)\Delta_c\Delta_{c-3}\right)n'_\upsilon(m),\\ g_2'=&2c^2(c-1)\Delta_{c-1}\Delta_{c-2}n'_\upsilon(m),\\ g_0'=&\frac{(2-c)m^{1-c}}{(c-1)\Delta_{c-2}}\left((c-1)\Delta_{c-2}^2+c\Delta_{c-1}\Delta_{c-3}\right).\end{aligned}$$

From the above formulas we can obtain

$$g'''(m)\sim(h+\alpha)qjM^{-1}X^{c-2}.$$

Using Lemma 2.5 and partial summation we can get

$$\sum_{m\sim M}\sum_{\upsilon}\left|\frac{\partial^2 G}{\partial n^2}(m,n_\upsilon)\right|^{-\frac{1}{2}} e(G(m,n_\upsilon)-\upsilon n_\upsilon)$$

$$\ll\left(M\left((h+\alpha)qjM^{-1}X^{c-2}\right)^{\frac{1}{6}}+\left((h+\alpha)qjM^{-1}X^{c-2}\right)^{-\frac{1}{3}}\right)$$
$$\times(h+\alpha)qjM^{c}N^{c-3}\left((h+\alpha)qjM^{c}N^{c-4}\right)^{-\frac{1}{2}}$$
$$\ll((h+\alpha)qj)^{\frac{2}{3}}M^{\frac{11}{6}}X^{\frac{2c}{3}-\frac{4}{3}}+((h+\alpha)qj)^{\frac{1}{6}}M^{\frac{4}{3}}X^{\frac{c}{6}-\frac{1}{3}}. \tag{3.11}$$

By (3.5), (3.9) and (3.11), we have

$$E_{q,j}\log^{-1}X\ll M+\left((h+\alpha)qjM^{2}\right)^{-1}X^{3-c}+((h+\alpha)qj)^{\frac{1}{2}}MX^{\frac{c}{2}-1}$$
$$+((h+\alpha)qj)^{\frac{2}{3}}M^{\frac{11}{6}}X^{\frac{2c}{3}-\frac{4}{3}}+((h+\alpha)qj)^{\frac{1}{6}}M^{\frac{4}{3}}X^{\frac{c}{6}-\frac{1}{3}}. \tag{3.12}$$

Inserting (3.12) into (3.3), we obtain

$$K_h^4\log^{-1}X\ll\frac{X^4}{Q^2}+\frac{X^4}{J}+MX^3+\left((h+\alpha)QJM^{2}\right)^{-1}X^{6-c}+((h+\alpha)QJ)^{\frac{1}{2}}MX^{\frac{c}{2}+2}$$
$$+((h+\alpha)QJ)^{\frac{2}{3}}M^{\frac{11}{6}}X^{\frac{2c}{3}+\frac{5}{3}}+((h+\alpha)QJ)^{\frac{1}{6}}M^{\frac{4}{3}}X^{\frac{c}{6}+\frac{8}{3}}.$$

Then choosing optimal $J\in[0,N\log^{-1}X]$ and $Q\in[0,N\log^{-1}X]$ and using Lemma 2.6 twice we can get

$$K_h\log^{-3}X\ll B(h),$$

where

$$B(h)=X^{\frac{5}{6}}+(\alpha+h)^{\frac{1}{14}}M^{\frac{1}{7}}X^{\frac{c}{14}+\frac{5}{7}}+(\alpha+h)^{\frac{1}{12}}M^{\frac{11}{48}}X^{\frac{c}{12}+\frac{17}{24}}+(\alpha+h)^{\frac{1}{30}}M^{\frac{4}{15}}X^{\frac{c}{30}+\frac{11}{15}}$$
$$+X^{\frac{3}{4}}M^{\frac{1}{4}}+(\alpha+h)^{-\frac{1}{4}}X^{1-\frac{c}{4}}+X^{\frac{23}{28}}M^{\frac{1}{8}}+X^{\frac{25}{32}}M^{\frac{7}{32}}+X^{\frac{17}{20}}M^{\frac{3}{40}}+X^{\frac{11}{14}}M^{\frac{3}{14}}.$$

Recalling the definitions of H and Y, we have

$$S_I\log^{-3}X\ll M^{\varepsilon}HB(H)\ll X^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}+2\varepsilon},$$

and Lemma 3.1 is proved. □

Lemma 3.2. *Let* $P^{\frac{5}{6}}\ll X\ll P$, $H=X^{1-\frac{(1+2\kappa)c+\lambda}{2+2\kappa}}$, $F=(h+\alpha)X^{c}$*and* $c_h(\alpha)$ *denote complex numbers such that* $c_h(\alpha)\ll(1+|h|)^{-1}$. *Then uniformly for* $\alpha\in(\tau,1-\tau)$, *we have*

$$S_{II}=\sum_{|h|\sim H}c_h(\alpha)\sum_{M\le m\le 2M}a(m)\sum_{N\le n\le 2N}b(n)e\left((h+\alpha)(mn)^{c}\right)\ll X^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}+2\varepsilon}, \tag{3.13}$$

for any $a(m)\ll m^{\varepsilon}$, $b(n)\ll n^{\varepsilon}$, *where* (κ,λ) *is any exponent pair,* $X\ll MN\ll X$ *and*

$$\max\{X^{3-\frac{(1+2\kappa)c+\lambda}{1+\kappa}}F^{-1},X^{4-\frac{2(1+2\kappa)c+2\lambda}{1+\kappa}},X^{26-\frac{53}{3}\frac{(1+2\kappa)c+\lambda}{1+\kappa}}F^{\frac{53}{9}}\}\ll M\ll X^{\frac{(1+2\kappa)c+\lambda}{1+\kappa}-1}.$$

Proof. Taking $Q=[X^{2-\frac{(1+2\kappa)c+\lambda}{1+\kappa}}\log^{-1}X]$, then we have $Q=o(N)$. By Cauchy's inequality and Lemma 2.1, we have

$$|S_{II}|^2\ll\sum_{m\sim M}|a(m)|^2\sum_{m\sim M}\left|\sum_{n\sim N}b(n)e(f(mn))\right|^2$$

$$\ll \frac{M^2N^2\log^{2A+2B}X}{Q} + \frac{MN\log^{2A}X}{Q}\sum_{q=1}^{Q}E_q, \tag{3.14}$$

where $E_q = \sum_{n\sim N}|b(n+q)b(n)|\,|\sum_{m\sim M}e(G(mn))|$ and $G(m,n) = G(m,n,q) = (h+\alpha)m^c\Delta(n,q;c)$, $\Delta(n,q;c) = (n+q)^c - n^c$.

If $|\frac{\partial G}{\partial m}| \le 10^3 Mq^{-2}$, by Lemma 2.10 we have

$$\begin{aligned} E_q &\ll \sum_{n\sim N}|b(n+q)b(n)|\left(\frac{MN}{Fq} + \left(\frac{Fq}{MN}\right)^{\frac{1}{2}}M^{\frac{1}{2}}\right) \\ &\ll \sum_{n\sim N}(|b(n+q)|^2 + |b(n)|^2)\left(\frac{MN}{Fq} + \frac{M}{q}\right) \\ &\ll \frac{MN}{q}\log^{2B}X \end{aligned} \tag{3.15}$$

noting that $M \gg \frac{X}{F}$.

Now we suppose $|\partial G/\partial m| > 10^3 Mq^{-2}$. By Lemma 2.11 we get

$$\sum_{m\sim M}e\left(G(m,n)\right) \ll \frac{MN^{1/2}}{(Fq)^{1/2}}\left|\sum_{r_1(n)\le r\le r_2(n)}\varphi(n,r)e(s(r,n))\right| + \log X + MN^{1/2}(Fq)^{-1/2},$$

where $s(r,n) = G(M(r,n),n) - rm(r,n)$, $\varphi(r,n) = \frac{(Fq)^{\frac{1}{2}}}{MN^{\frac{1}{2}}}\left|\frac{\partial^2 G(m(r,n),n)}{\partial m^2}\right|^{-\frac{1}{2}}$ and

$$r_1(n) = \frac{\partial G}{\partial m}(M,n), \quad r_2(n) = \frac{\partial G}{\partial m}(2M,n).$$

Thus we have

$$\begin{aligned} E_q \ll\ & \frac{MN^{1/2}}{(Fq)^{1/2}}\sum_{n\sim N}|b(n+q)b(n)|\left|\sum_{r_1(n)<r\le r_2(n)}\varphi(n,r)e(s(r,n))\right| \\ & + N\log^{2B+1}X + MN^{3/2}(Fq)^{-1/2}\log^{2B}X. \end{aligned} \tag{3.16}$$

So it suffices to bound the sum

$$\Sigma_1 = \sum_{n\sim N}|b(n+q)b(n)|\left|\sum_{r_1(n)<r\le r_2(n)}\varphi(n,r)e(s(r,n))\right|.$$

Let $T = [Fq^3/M^2N]$ and $R = Fq/MN$. By Cauchy's inequality and Lemma 2.1 again we get

$$\begin{aligned} {\Sigma_1}^2 &\ll \sum_{n\sim N}|b(n+q)b(n)|^2\sum_{n\sim N}\left|\sum_{r_1(n)<r\le r_2(n)}\varphi(n,r)e(s(r,n))\right|^2 \\ &\ll \frac{N^2R^2\log^{4B}X}{T} + \frac{NR\log^{4B}X}{T}\Sigma_2, \end{aligned} \tag{3.17}$$

where

$$\Sigma_2 = \sum_{t=1}^{T} \left| \sum_{n\sim N} \sum_{r_1(n)<r\le r_2(n)-t} \varphi(n, r+t)\varphi(n, r)e(s(r+t) - s(r, n)) \right|$$

and where we used the estimate

$$\sum_{n\sim N} |b(n+q)b(n)|^2 \ll \sum_{n\sim N} (|b(n+q)|^4 + |b(n)|^4) \ll N \log^{4B} X.$$

It is easy to check that $10 < T = o(R)$.

Recall that $s(r, n) = G(m(r, n), n) - rm(r, n)$, where $m(r, n)$ denotes the solution of

$$\frac{\partial G}{\partial m}(m, n) = r.$$

It is easy to deduce that

$$\frac{\partial s}{\partial r}(r, n) = \frac{\partial G}{\partial m}\frac{\partial m}{\partial r} - m(r, n) - r\frac{\partial m}{\partial r} = -m(r, n).$$

So we can obtain

$$\begin{aligned} H(n) : = H_{r,t,q}(n) &= s(r+t, n) - s(r, n) \\ &= \int_r^{r+t} \frac{\partial s}{\partial u}(u, n)du = -\int_r^{r+t} m(u, n)du, \end{aligned}$$

which implies that $|H^{(j)}| \sim tMN^{-j}$, $(j = 0, 1, 2, 3, 4, 5, 6)$. Denote by $I(r, t)$ the interval $N < n \le 2N$, $r_1(n) < n \le r_2(n) - t$. Then we have

$$\Sigma_2 \ll \sum_{t=1}^{T} \sum_{r\sim R} \left| \sum_{n\in I(r,t)} \varphi(n, r+t)\varphi(n, r)e(s(r+t, n) - s(r, n)) \right|.$$

Thus using partial summation, we get

$$\begin{aligned} \Sigma_2 &\ll \sum_{t=1}^{T} \sum_{r\sim R} \left(\frac{tM}{N}\right)^{\kappa} N^{\lambda} \\ &\ll RM^{\kappa} N^{\lambda-\kappa} T^{1+\kappa} \\ &\ll NR \end{aligned} \tag{3.18}$$

with the exponent pair (κ, λ), if we note that $M \gg X^{26-\frac{53}{3}\frac{(1+2\kappa)c+\lambda}{1+\kappa}} F^{\frac{53}{9}}$. From (3.15)–(3.18) we get that for any $1 \le q \le Q$,

$$E_q \ll \frac{MN \log^{2B+1} X}{q} + N \log^{2B+1} X + MN^{\frac{3}{2}} (Fq)^{-\frac{1}{2}} \log^{2B} X. \tag{3.19}$$

Now this lemma follows from inserting (3.19) into (3.14). □

Lemma 3.3. *For $\tau \le \alpha \le 1-\tau$, we have*

$$S(\alpha) \ll P^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}+4\varepsilon},$$

where (κ,λ) is any exponent pair.

Proof. Throughout the proof of this lemma, we write $H = X^{1-\frac{(1+2\kappa)c+\lambda}{2+2\kappa}}$ for convenience. By a dissection argument we only need to prove that

$$\sum_{X<n\le 2X} \Lambda(n)e(\alpha[n^c]) \ll X^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}+3\varepsilon} \tag{3.20}$$

holds for $P^{\frac{5}{6}} \le X \le P$ and $\tau \le \alpha \le 1-\tau$. According to Lemma 2.3, we have

$$\sum_{X<n\le 2X} \Lambda(n)e(\alpha[n^c]) = \sum_{|h|\le H} c_h(\alpha) \sum_{X<n\le 2X} \Lambda(n)e((h+\alpha)n^c) + O\left((\log X)\sum_{X<n\le 2X}\min\left(1,\frac{1}{H\|n^c\|}\right)\right). \tag{3.21}$$

By the expansion

$$\min\left(1,\frac{1}{H\|n^c\|}\right) = \sum_{h=-\infty}^{\infty} a_h e(hn^c),$$

where

$$|a_h| \le \min\left(\frac{\log 2H}{H},\frac{1}{|h|},\frac{H}{h^2}\right),$$

we get

$$\begin{aligned}
\sum_{X<n\le 2X}\min\left(1,\frac{1}{H\|n^c\|}\right) &\le \sum_{h=-\infty}^{\infty} a_h \left|\sum_{X<n\le 2X} e(hn^c)\right| \\
&\ll \frac{X\log 2H}{H} + \sum_{1\le h\le H}\frac{1}{h}\left((hX^c)^{\frac12}+\frac{X}{hX^c}\right) \\
&\quad + \sum_{h\ge H}\frac{H}{h^2}\left((hX^c)^{\frac12}+\frac{X}{hX^c}\right) \\
&\ll X^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}}\log X,
\end{aligned} \tag{3.22}$$

where we estimated the sum over n by Lemma 2.2 with the exponent pair $\left(\frac12,\frac12\right)$.

Let $R = \max\left\{X^{3-\frac{(1+2\kappa)c+\lambda}{1+\kappa}}F^{-1}, X^{4-\frac{2(1+2\kappa)c+2\lambda}{1+\kappa}}, X^{26-\frac{53}{3}\frac{(1+2\kappa)c+\lambda}{1+\kappa}}F^{\frac{53}{9}}\right\}$. Recall the definition of Y in Lemma 3.1. Let $U=R$, $V = X^{\frac{(1+2\kappa)c+\lambda}{1+\kappa}-1}$, $Z = [XY^{-1}]+\frac12$. By Lemma 2.4 with $F(n) = e((h+\alpha)n^c)$, then we reduce the estimate of

$$\sum_{|h|\le H} c_h(\alpha) \sum_{X<n\le 2X} \Lambda(n)e((h+\alpha)n^c)$$

to the estimates of sums of type I

$$S_I' = \sum_{|h|\le H} c_h(\alpha) \sum_{M<m\le 2M} a(m) \sum_{N<n\le 2N} e((h+\alpha)(mn)^c),\ N > Z,$$

and sums of type II

$$S'_{II} = \sum_{|h|\le H} c_h(\alpha) \sum_{M<m\le 2M} a(m) \sum_{N<n\le 2N} b(n)e((h+\alpha)(mn)^c),\ U < M < V,$$

where $a(m) \ll m^{\varepsilon},\ b(n) \ll n^{\varepsilon},\ X \ll MN \ll X$. By Lemma 3.1, we get

$$S'_I \ll X^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}+2\varepsilon}. \tag{3.23}$$

By Lemma 3.2, we get

$$S'_{II} \ll X^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}+3\varepsilon}. \tag{3.24}$$

From (3.23) and (3.24) we can obtain

$$\sum_{|h|\le H} c_h(\alpha) \sum_{X<n\le 2X} \Lambda(n)e((h+\alpha)n^c) \ll X^{\frac{(1+2\kappa)c+\lambda}{2+2\kappa}+3\varepsilon}. \tag{3.25}$$

Now (3.20) follows from (3.21), (3.22) and (3.25). Thus we complete the proof of this Lemma. □

4. Proof of the theorem

It is easy to see that

$$\begin{aligned} R(N) &= \int_{-\tau}^{1-\tau} S^3(\alpha)e(-\alpha N)d\alpha \\ &= \int_{-\tau}^{\tau} S^3(\alpha)e(-\alpha N)d\alpha + \int_{\tau}^{1-\tau} S^3(\alpha)e(-\alpha N)d\alpha \\ &= R_1(N) + R_2(N). \end{aligned} \tag{4.1}$$

4.1. Evaluation of $R_1(N)$

Following the argument of Laporta and Tolev [18, pages 928–929], we can get that

$$R_1(N) = \frac{\Gamma^3\left(1+\frac{1}{c}\right)}{\Gamma\left(\frac{3}{c}\right)} N^{\frac{3}{c}-1} + O\left(N^{\frac{3}{c}-1}\exp(-\log^{\frac{1}{3}-\delta} N)\right) \tag{4.2}$$

for $1 < c < \frac{3}{2}$ and any $0 < \delta < \frac{1}{3}$, where the implied constant in the $O-$symbol depends only on c.

4.2. Evaluation of $R_2(N)$

Let

$$S(\alpha, X) = \sum_{X<p\le 2X} e(\alpha[p^c])\log p,\ T(\alpha, X) = \sum_{X<n\le 2X} e(\alpha[n^c]).$$

We can get

$$R_2(N) = \int_{\tau}^{1-\tau} S^3(\alpha)e(-\alpha N)d\alpha$$

$$\ll (\log X) \max_{P^{\frac{5}{6}} \leq X \leq 0.5P} \left| \int_{\tau}^{1-\tau} S^2(\alpha) S(\alpha, X) e(-\alpha N) d\alpha \right| + P^{\frac{11}{6}} \log^2 P, \tag{4.3}$$

where we used

$$\int_{\tau}^{1-\tau} |S^2(\alpha)| d\alpha \ll \int_0^1 |S^2(\alpha)| d\alpha \ll P \log^2 P. \tag{4.4}$$

Now, we start to estimate the absolute value on the right hand in (4.3). By Cauchy's inequality we have

$$\begin{aligned}
& \left| \int_{\tau}^{1-\tau} S^2(\alpha) S(\alpha, X) e(-\alpha N) d\alpha \right| \\
= & \left| \sum_{X<p\leq 2X} (\log p) \int_{\tau}^{1-\tau} S^2(\alpha) e(\alpha[p^c] - \alpha N) d\alpha \right| \\
\leq & \sum_{X<p\leq 2X} (\log p) \left| \int_{\tau}^{1-\tau} S^2(\alpha) e(\alpha[p^c] - \alpha N) d\alpha \right| \\
\ll & (\log X) \sum_{X<n\leq 2X} \left| \int_{\tau}^{1-\tau} S^2(\alpha) e(\alpha[n^c] - \alpha N) d\alpha \right| \\
\ll & X^{\frac{1}{2}} (\log X) \left(\sum_{X<n\leq 2X} \left| \int_{\tau}^{1-\tau} S^2(\alpha) e(\alpha[n^c] - \alpha N) d\alpha \right|^2 \right)^{\frac{1}{2}} \\
= & X^{\frac{1}{2}} (\log X) \left(\int_{\tau}^{1-\tau} \overline{S^2(\beta) e(-\beta N)} d\beta \int_{\tau}^{1-\tau} S^2(\alpha) T(\alpha - \beta, X) e(-\alpha N) d\alpha \right)^{\frac{1}{2}} \\
\ll & X^{\frac{1}{2}} (\log X) \left(\int_{\tau}^{1-\tau} |S(\beta)|^2 d\beta \int_{\tau}^{1-\tau} |S(\alpha)|^2 |T(\alpha - \beta, X)| d\alpha \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

Then we have

$$\begin{aligned}
\int_{\tau}^{1-\tau} |S(\alpha)|^2 |T(\alpha - \beta, X)| d\alpha \ll & \int_{\substack{\tau<\alpha<1-\tau \\ |\alpha-\beta|\leq X^{-c}}} |S(\alpha)|^2 |T(\alpha - \beta, X)| d\alpha \\
& + \int_{\substack{\tau<\alpha<1-\tau \\ |\alpha-\beta|> X^{-c}}} |S(\alpha)|^2 |T(\alpha - \beta, X)| d\alpha.
\end{aligned} \tag{4.6}$$

By Lemma 3.3, we have

$$\begin{aligned}
\int_{\substack{\tau<\alpha<1-\tau \\ |\alpha-\beta|\leq X^{-c}}} |S(\alpha)|^2 |T(\alpha - \beta, X)| d\alpha \ll & X \max_{\alpha\in(\tau,1-\tau)} |S(\alpha)|^2 \int_{|\alpha-\beta|\leq X^{-c}} 1 d\alpha \\
\ll & X^{1-c} P^{\frac{(1+2\kappa)c+\lambda}{1+\kappa}+8\varepsilon},
\end{aligned} \tag{4.7}$$

where we used the trivial bound $T(\alpha, X) \ll X$. By Lemma 2.9, Lemma 3.3 and (4.4), we get

$$\int_{\substack{\tau<\alpha<1-\tau \\ |\alpha-\beta|> X^{-c}}} |S(\alpha)|^2 |T(\alpha - \beta, X)| d\alpha$$

$$\ll \int_{\substack{\tau<\alpha<1-\tau \\ |\alpha-\beta|>X^{-c}}} |S(\alpha)|^2 \left(X^{\frac{\kappa c+\lambda}{1+\kappa}} \log X + \frac{X}{|\alpha-\beta|X^c} \right) d\alpha$$

$$\ll X^{\frac{\kappa c+\lambda}{1+\kappa}} (\log X) \int_{\tau}^{1-\tau} |S(\alpha)|^2 d\alpha + \max_{\alpha\in(\tau,1-\tau)} |S(\alpha)|^2 \int_{|\alpha-\beta|>X^{-c}} \frac{X}{|\alpha-\beta|X^c} d\alpha$$

$$\ll X^{\frac{\kappa c+\lambda}{1+\kappa}} P \log^3 P + X^{1-c} P^{\frac{(1+2\kappa)c+\lambda}{1+\kappa}+9\varepsilon}. \tag{4.8}$$

Thus, combining (4.6)–(4.8) we obtain

$$\int_{\tau}^{1-\tau} |S(\alpha)|^2 |T(\alpha-\beta, X)| d\alpha \ll X^{\frac{\kappa c+\lambda}{1+\kappa}} P \log^3 P + X^{1-c} P^{\frac{(1+2\kappa)c+\lambda}{1+\kappa}+9\varepsilon}. \tag{4.9}$$

By (4.3), (4.5) and (4.9), we can obtain

$$R_2(N) \ll P^{3-c-\varepsilon}. \tag{4.10}$$

Now putting (4.1), (4.2) and (4.10) into together, we have

$$R(N) = \frac{\Gamma^3\left(1+\frac{1}{c}\right)}{\Gamma\left(\frac{3}{c}\right)} N^{\frac{3}{c}-1} + O\left(N^{\frac{3}{c}-1} \exp\left(-\log^{\frac{1}{3}-\delta} N\right)\right)$$

follows for any $0 < \delta < \frac{1}{3}$, where the implied constant in the O−symbol depends only on c. Thus we complete the proof of Theorem 1.1.

Acknowledgment

The authors would like to thank the referees for their many useful comments. This work is supported by National Natural Science Foundation of China (Grant Nos. 11801328 and 11771256).

Conflict of interest

The authors declare no conflict of interest.

References

1. G. I. Arkhipov, A. N. Zhitkov, On the Waring problem with non-integer degrees, *Izv. Akad. Nauk SSSR*, **48** (1984), 1138–1150.
2. K. Buriev, *Additive problems with prime numbers*, Moscow University Thesis, 1989.
3. E. P. Balanzario, M. Z. Garaev, R. Zuazua, Exceptional set of a representation with fractional powers, *Acta Math. Hung.*, **114** (2007), 103–115.
4. Y. C. Cai, On a Diophantine equation involving primes, *Ramanujan J.*, **50** (2019), 151–162.
5. J. M. Deshouillers, Problème de waring avec exposants non entiers, *B. Soc. Math. Fr.*, **101** (1973), 285–295.

6. S. W. Graham, G. Kolesnik, *Van der Corput's method of exponential sums*, London: Cambridge University Press, 1991.
7. X. Han, H. F. Liu, D. Y. Zhang, A system of two Diophantine inequalities with primes, *J. Math.*, **2021**, 6613947.
8. X. Han, X. F. Yan, D. Y. Zhang, On Fourier coefficients of the symmetric Square *L*-function at Piatetski-Shapiro prime twins, *Mathematics*, **9** (2021), 1254.
9. D. R. Heath-Brown, The Pjateckiĭ-Šapiro prime number theorem, *J. Number Theory*, **16** (1983), 242–266.
10. H. F. Liu, J. Huang, Diophantine approximation with mixed powers of primes, *Taiwanese J. Math.*, **23** (2019), 1073–1090.
11. J. Huang, W. G. Zhai, D. Y. Zhang, Ω-result for the index of composition of an integral ideal, *AIMS Mathematics*, **6** (2021), 4979–4988.
12. J. Huang, W. G. Zhai, D. Y. Zhang, Diophantine inequalities over Piatetski-Shapiro primes, *Front. Math. China*, **16** (2021), DOI: 10.1007/s11464-021-0916-7.
13. C. H. Jia, The distribution of square-free numbers, *Sci. China Ser. A*, **36** (1993), 154–169.
14. A. Kumchev, A Diophantine inequality involving prime powers, *Acta Arith.*, **89** (1999), 311–330.
15. A. V. Kumchev, A binary additive equation involving fractional powers, *Int. J. Number Theory.*, **5** (2009), 281–292.
16. A. Kumchev, T. Nedeva, On an equation with prime numbers, *Acta Arith.*, **83** (1998), 117–126.
17. M. Laporta, On a binary problem with prime numbers, *Math. Balkanica (N. S.)*, **13** (1999), 119–123.
18. M. B. S. Laporta, D. I. Tolev, On an equation with prime numbers, *Mat. Zametki*, **57** (1995), 926–929 (In Russian), translated in *Math. Notes*, **57** (1995), 654–657.
19. T. Y. Li, H. F. Liu, Diophantine approximation over Piatetski-Shapiro primes, *J. Number Theory*, **211** (2020), 184–198.
20. H. F. Liu, A divisor problem attached to regular quadratic forms, *Lith. Math. J.*, **59** (2019), 169–184.
21. H. F. Liu, Diophantine approximation with one prime, two squares of primes and powers of two, *Ramanujan J.*, **51** (2020), 85–97.
22. H. F. Liu, S. Li, D. Y. Zhang, Power moments of automorphic L-function attached to Maass formsx, *Int. J. Number Theory*, **12** (2016), 427–443.
23. H. Q. Liu, On the number of abelian groups of a given order, *Acta Arith.*, **59** (1991), 261–277.
24. L. Ma, X. F. Yan, Resonance between the representation function and exponential functions over arithemetic progression, *J. Math.*, **2021**, 6616348.
25. I. I. Piatetski-Shapiro, On a variant of the Waring-Goldbach problem, *Mat. Sb. (N. S.)*, **30** (1952), 105–120.
26. P. Sargos, Points entiers au voisinage d'une courbe, sommes trigonometriques courtes et paires d'exposants, *P. Lond. Math. Soc.*, **70** (1995), 285–312.

27. B. I. Segal, On a theorem similar to the Waring theorem, *Dokl. Akad. Nauk SSSR*, **1** (1933), 47–49.
28. B. I. Segal, The Waring theorem with fractional and irrational degrees, *Trudy Mat. Inst. Stekov*, **5** (1933), 73–86.
29. P. Song, W. G. Zhai, D. Y. Zhang, Power moments of Hecke eigenvalues for congruence group, *J. Number Theory*, **198** (2019), 139–158.
30. Y. K. Sui, W. G. Zhai, D. Y. Zhang, Ω-Result for the index of composition of an integer, *Int. J. Number Theory*, **14** (2018), 339–348.
31. Q. F. Sun, D. Y. Zhang, Sums of the triple divisor function over values of a ternary quadratic form, *J. Number Theory*, **168** (2016), 215–246.
32. R. Zhang, X. Han, D. Y. Zhang, Power moments of the Riesz mean error term of symmetric square L-function in short intervals, *Symmetry*, **12** (2020), 2036.
33. D. Y. Zhang; M. M. Lü, W. G. Zhai, On the mean value of the index of composition of an integer *II*, *Int. J. Number Theory*, **9** (2013), 431–445.
34. D. Y. Zhang, Y. N. Wang, Higher-power moments of Fourier coefficients of holomorphic cusp forms for the congruence subgroup $\Gamma_0(N)$, *Ramanujan J.*, **47** (2018), 685–700.
35. D. Y. Zhang, Y. N. Wang, Ternary quadratic form with prime variables attached to Fourier coefficients of primitive holomorphic cusp form, *J. Number Theory*, **176** (2017), 211–225.
36. D. Y. Zhang, W. G. Zhai, On the Waring-Goldbach problem in thin sets of primes (II), *Acta Math. Sin. (Chin. Ser.)*, **48** (2005), 809–816.
37. D. Y. Zhang, W. G. Zhai, On the mean value of the index of composition of an integral ideal (*II*), *J. Number Theory*, **133** (2013), 1086–1110.
38. D. Y. Zhang, W. G. Zhai, On the distribution of Hecke eigenvalues over Piatetski-Shapiro prime twins, *Acta Math. Sin. (Engl. Ser.)*, 2021, In presss.
39. W. G. Zhai, X. D. Cao, A Diophantine equation with prime numbers, *Acta Math. Sin. (Chin. Ser.)*, **45** (2002), 443–454.
40. L. Zhu, An additive equation involving fractional powers, *Acta Math. Hungar.*, **159** (2019), 174–186.
41. W. G. Zhai, On a system of two diophantine inequalities with prime numbers, *Acta Arith.*, **92** (2000), 31–46.
42. W. G. Zhai, On the simultaneous distribution of the fractional parts of different powers of prime numbers, *J. Number Theory*, **86** (2001), 133–155.

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