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Research article

Asymptotic behavior for a class of viscoelastic equations with memory lacking instantaneous damping

Jiangwei Zhang and Yongqin Xie*

School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, 410114, China

* **Correspondence:** Email: xieyqmath126.com.

Abstract: In this paper, we mainly investigate long-time behavior for viscoelastic equation with fading memory

$$u_{tt} - \Delta u_{tt} - \nu\Delta u + \int_0^{+\infty} k'(s)\Delta u(t-s)ds + f(u) = g(x).$$

The main feature of the above equation is that the equation doesn't contain $-\Delta u_t$, which contributes to a strong damping. The existence of global attractors is obtained by proving asymptotic compactness of the semigroup generated by the solutions for the viscoelastic equation. In addition, the upper semicontinuity of global attractors also is obtained.

Keywords: viscoelastic equation; contractive function; global attractor; memory
Mathematics Subject Classification: 35K57, 35B40, 35B41

1. Introduction

In this paper, we mainly study the following initial-boundary value problem for viscoelastic equation with hereditary memory:

$$\begin{cases} u_{tt} - \Delta u_{tt} - \nu\Delta u - \displaystyle\int_0^{+\infty} k'(s)\Delta u(t-s)ds + f(u) = g(x), & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,\ t)|_{\partial\Omega} = 0, & \forall\, t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, $\nu > 0$, and the forcing term $g = g(x) \in L^2(\Omega)$ is given.

Next, we establish the following hypotheses for the kernel function $k(s)$

(H_1) Let $\mu(s) = -k'(s)$, and assume

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \mu(s) \geq 0, \mu'(s) \leq 0, \forall s \in \mathbb{R}^+, \tag{1.2}$$

and there exists $\delta > 0$, such that

$$\mu'(s) + \delta\mu(s) \leqslant 0, \forall s \in \mathbb{R}^+, \tag{1.3}$$

and let

$$m_0 = \int_0^\infty \mu(s)ds.$$

(H_2) The nonlinearity $f \in C^1$ satisfies $f(0) = 0$ and also fulfills the following conditions

$$|f'(s)| \leq c(1 + |s|^4), \ \forall s \in \mathbb{R} \tag{1.4}$$

and

$$\lim_{|s|\to\infty} \inf \frac{f(s)}{s} > -\lambda_1. \tag{1.5}$$

where c, λ_1 are positive constants and λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with Dirichlet boundary condition. From (1.5), it's easy to get that there exist $\lambda (0 < \lambda < \lambda_1)$ and $c_f \geq 0$; such that

$$f(s)s \geq -\lambda s^2 - C. \tag{1.6}$$

Let $F(s) = \int_0^s f(\sigma)d\sigma$, then

$$F(u) \geq -\frac{1}{2}\lambda|u|^2 - c_f \quad and \quad f(u)u \geq F(u) - \frac{1}{2}\lambda|u|^2 - c_f. \tag{1.7}$$

The equation associated with Eq (1.1) is as follows

$$u_{tt} - u_{xx} - u_{xxtt} = 0,$$

which mainly describes a pure dispersion wave process, such as the motion equation of strain-arc wave of linear elastic rod considering transverse inertia and ion-acoustic wave propagation equation in space transformation with weak nonlinear effects (see e.g., [1–4]).

In recent years, the following types of equations have been studied by many scholars (see e.g., [5–18] and the references therein)

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \gamma\Delta u_t - \alpha\Delta u + \int_0^t g(u - t)\Delta u(s)ds + \nu f(u_t) + \mu g(u) = 0.$$

Many researchers considered different kinds of cases, respectively, when the parameters $\rho, \gamma, \alpha, \nu, \mu = 0$ or $\rho, \gamma, \alpha, \nu, \mu \neq 0$ under different situations. However, they only obtained global existence of solutions and the energy decay results [6–8, 15, 16]. In particular, in [10, 11, 17], the scholars only proved blow-up result, decay result and global existence result of solutions under various kinds of conditions and when the dispersion term and dissipative term don't be contained. Next, we analyze several key results in detail. Araújo et al. [18] established well-posedness result when $\gamma \geq 0, \nu, \mu = 0$ and proved the existence of global attractor when $\nu, \mu = 0$ and $-\Delta u_t$ was included.

Qin et al. [19] proved the existence of uniform attractors in non-autonomous case by improving the result of [18] when $-\Delta u_t$ was still included. Recently, Conti et al. [20] obtained the existence of global attractors and optimal regularity of global attractors for the following equation when the nonlinearity f meets critical growth

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u + \int_0^{+\infty} \mu(s)\Delta u(t-s)ds + f(u) = h.$$

with $\rho \in [0, 4]$ and $\rho < 4$, respectively. Therefore, based on the above existing research, we devote to obtain the existence of global attractors in higher regular space for the problem (1.1) which doesn't contain the strong damping $-\Delta u_t$ in this paper.

Firstly, because the Eq (1.1) doesn't contain strong dissipative term $-\Delta u_t$, which makes that the Eq (1.1) is different from usual viscoelastic equations. Next, for the Eq (1.1), its dissipation is only generated by memory term with weaker dissipation rather than the strong dissipative term $-\Delta u_t$, which leads to the need of higher regularity to ensure compactness, so the multiplier $A^\kappa u_t$ will be used to obtain our result. We use new analytical techniques to obtain the upper semicontinuity of global attractors. Thus, our results complement the existing conclusions because we only use the memory dissipation to prove the existence and the semicontinuity of global attractors.

In addition, to the best of our knowledge, the key point for proving the existence of global attractors is to verify the existence of bounded absorbing set and the compactness of the semigroup in some sense. However, the absence of term $-\Delta u_t$ causes that energy dissipation of Eq (1.1) is lower than usual viscoelastic equation, and its dissipation only is presented by the memory term. Hence, this will lead to two main difficulties. On the one hand, the absence of term $-\Delta u_t$ makes the equation lacks strong structural damping. On the other hand, to ensure strong convergence of the solution in $L^2(0, T; H_0^1(\Omega))$, how to obtain higher regularity of solutions. Thereby, for obtaining dissipative and compactness of semigroup, we will use analysis techniques and the ideas in [21, 22] to overcome these difficulties.

The plan of this paper is as follows. In Section 2, we recall some basic concepts and useful results that will be used later. In Section 3, firstly, the bounded absorbing set is obtained. Secondly, we verify asymptotic compact of semigroup by contractive function method [23, 24]. Finally, the existence of global attractors $\mathscr{A}$ is proved in $H_0^1(\Omega) \times H_0^1(\Omega) \times L^2_\mu(\mathbb{R}^+; H_0^1(\Omega))$. In section 4, we obtain the upper semicontinuity of global attractors.

2. Preliminaries

Following the Dafermos' idea of introducing an additional variable η^t, the past history of u, whose evolution is ruled by a first-order hyperbolic equation (see e.g., [25] and references therein). Thus the original problem (1.1) can be translated into a dynamical system on a phase space with two components (see [26]). In particular, in the following, we introduce the past history of u in the, i.e.

$$\eta^t = \eta^t(x, s) := u(x, t) - u(x, t-s), s \in \mathbb{R}^+, \tag{2.1}$$

Provided that let $\eta^t_t = \frac{\partial}{\partial t}\eta^t, \quad \eta^t_s = \frac{\partial}{\partial s}\eta^t$, then we have

$$\eta^t_t = -\eta^t_s + u_t \qquad \forall (x, s) \in \Omega \times \mathbb{R}^+, t \geq 0. \tag{2.2}$$

Historical variable $u_0(\cdot,-s)$ of u satisfies the following condition

$$\int_0^\infty e^{-\sigma s}\|u_0(-s)\|_0^2 ds \le \mathfrak{R}. \tag{2.3}$$

where $\mathfrak{R} > 0$ and $\sigma \le \delta$ (δ is from (1.3)).

By (H_1) and (2.1), (2.2), we get

$$-\int_0^\infty k'(s)\Delta u(t-s)ds = \int_0^\infty \mu(s)\Delta u ds - \int_0^\infty \mu(s)\Delta\eta^t(s)ds \tag{2.4}$$

$$= m_0\Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds. \tag{2.5}$$

Thus, if we assume $\nu - m_0 = 1$, then the system (1.1) can be rewrite as

$$\begin{cases} u_{tt} - \Delta u_{tt} - \Delta u - \displaystyle\int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = g(x), \\ \eta_t^t = -\eta_s^t + u_t. \end{cases} \tag{2.6}$$

with initial-boundary condition

$$\begin{cases} u(x,t)|_{\partial\Omega} = 0, \eta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+} = 0, \quad t \ge 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \eta^0(x,s) = \displaystyle\int_0^s u_0(x,-r)dr, \quad (x,s) \in \Omega\times\mathbb{R}^+. \end{cases} \tag{2.7}$$

In the whole paper, unless otherwise stated, $z(t) = (u(t), u_t(t), \eta^t)$ is the solution of systems (2.6), (2.7) with initial value $z_0 = (u_0, u_1, \eta^0)$.

For conveniences, hereafter let $|\cdot|_p$ be the norm of $L^p(\Omega)(p \ge 1)$. Let $\langle\cdot,\cdot\rangle$ be the inner product of $L^2(\Omega)$, $\|\cdot\|_0^2$ be the equivalent norm $H_0^1(\Omega)$. Denote $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Denoting the weight space $\mathcal{V}_1 = L_\mu^2(\mathbb{R}^+; H_0^1(\Omega))$, $\mathcal{V}_2 = L_\mu^2(\mathbb{R}^+; D(A))$ and its inner product and norm are

$$\langle\psi,\eta\rangle_{\mu,0} = \int_0^\infty \mu(s)\langle\nabla\psi,\nabla\eta\rangle ds; \ \left\|\eta^t\right\|_{\mu,0}^2 = \int_0^\infty \mu(s)\left\|\eta^t\right\|_0^2 ds.$$

Then phase spaces of the Eq (2.6) are

$$\mathcal{M}_1 = H_0^1 \times H_0^1 \times \mathcal{V}_1,$$

and their corresponding norms are

$$\|\cdot\|_{\mathcal{M}_1}^2 = \|\cdot\|_0^2 + \|\cdot\|_0^2 + \|\cdot\|_{\mathcal{V}_1}^2,$$

In addition, denote $\mathcal{V}_\kappa = D(A^{\frac{\kappa+1}{2}})(\kappa \in (0,\frac{1}{2}))$ and let $\|\cdot\|_\kappa$ be the norm of $\mathcal{V}_\kappa$. Then we can also define phase space of the Eq (1.1) is

$$\mathcal{M}_\kappa = \mathcal{V}_\kappa \times \mathcal{V}_\kappa \times L_\mu^2(\mathbb{R}^+; \mathcal{V}_\kappa).$$

and the corresponding norm is $\|\cdot\|_{\mathcal{M}_\kappa}^2 = \|\cdot\|_\kappa^2 + \|\cdot\|_\kappa^2 + \|\cdot\|_{\mu,\kappa}^2$.

And there exists the following compact embedding

$$D(A^{\frac{s}{2}}) \hookrightarrow D(A^{\frac{r}{2}}), \quad \forall s > r. \tag{2.8}$$

Definition 2.1. *Let X be a Banach spaces and $\mathcal{X}$ be a family of operators defined on it. We say that $\{S(t)\}_{t\geq 0}$ is a continuous semigroup on X if $\{S(t)\}_{t\geq 0}$ fulfills*

$$S(t) : \mathcal{X} \to \mathcal{X}, \quad \forall t \geq 0.$$

and satisfies

(i)$S(0) = Id$(Identity operator);

(ii)$S(t + s) = S(t)S(s), \forall t, s \geq 0$.

The main results of this paper (the existence of global attractors) can be obtained by the following definitions and theorem. Next, let's talk about it (it's similar to [14, 23, 26]).

Definition 2.2. *Let X, Y be two Banach spaces and B be a bounded subset of $X \times Y$. We call a function $\phi(\cdot, \cdot; \cdot, \cdot)$, defined on $(X \times X) \times (Y \times Y)$, to be a contractive function if for any sequence $\{(x_n, y_n)\}_{n=1}^{\infty} \subset B$, there is a subsequence $\{(x_{n_k}, y_{n_k})\}_{k=1}^{\infty} \subset \{(x_n, y_n)\}_{n=1}^{\infty}$ satisfies*

$$\lim_{k\to\infty} \lim_{l\to\infty} \phi(x_{n_k}, x_{n_l}; y_{n_k}, y_{n_l}) = 0. \tag{2.9}$$

We denote the set of all contractive functions on $B \times B$ by $\mathfrak{C}(B)$.

Lemma 2.3. *Let X, Y be two Banach spaces and B be a bounded subset of $X \times Y$, $\{S(t)\}_{t\geq 0}$ is semigroup with a bounded absorbing set B_0 on $X \times Y$. Moreover, assume that for any $\varepsilon > 0$ there exist $T = T(B; \varepsilon)$ and $\phi_T(\cdot, \cdot; \cdot, \cdot) \in \mathfrak{C}(B)$ such that*

$$\|S(T)z_1 - S(T)z_2\|_X \leq \varepsilon + \phi_T(x_1, x_2; y_1, y_2), \qquad \forall (x_i, y_i) \in B (i = 1, 2).$$

where ϕ_T depends on T. Then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically compact in $X \times Y$.

In the following theorem, we give a method to verify the asymptotically compactness of a semigroup generated by the Eq (1.1), which will be used in our later discussion.

Theorem 2.4. *Let X, Y be two Banach spaces and $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on $X \times Y$. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in $X \times Y$. Provided that the following conditions hold:*
(i) $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set B_0 on $X \times Y$;
(ii) $\{S(t)\}_{t\geq 0}$ is a contractive semigroup on $X \times Y$.

Lemma 2.5. *Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0, T; X)$ and du_n/dt is uniformly bounded in $L^p(0, T; Y)$, for some $p > 1$. Then there is a subsequence of u_n that converges strongly in $L^2(0, T; H)$.*

3. Global attractors in $\mathcal{M}_1$

Throughout the paper, we assume that $\Omega \subset \mathbb{R}^n (n = 3)$ be bounded smooth domain, the kernel function μ and the nonlinearity satisfy (H_1) and (H_2) respectively, and $g \in L^2(\Omega)$.

Firstly, the well-posedness result for the Eq (1.1) can be obtained by the *Faedo-Galerkin* method (see e.g., [18]). Thereout, we only give the final result.

Lemma 3.1. *For any $T > 0$ and $z_0 = (u_0, u_1, \eta^0) \in \mathcal{M}_1$, the problem* (1.1) *has unique weak solution $z = (u(x,t), u_t(x,t), \eta^t)$ satisfying*

$$z \in C([0,T]; \mathcal{M}_1), .$$

and

$$u \in L^\infty([0,T]; H_0^1(\Omega)), \quad u_t \in L^\infty([0,T]; H_0^1(\Omega)),$$
$$u_{tt} \in L^2([0,T]; H_0^1(\Omega)), \quad \eta^t \in L^\infty([0,T], L^2_\mu(\mathbb{R}^+; H_0^1(\Omega))).$$

By Lemma 3.1, the semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{M}_1$ will be defined as the following:

$$S(t) : \mathcal{M}_1 \to \mathcal{M}_1,$$
$$z_0 \to z(t) = S(t)z_0,$$

and it is a strongly continuous semigroup on $\mathcal{M}_1$.

Lemma 3.2. *For some $\mathcal{R} > 0$ and*

$$\|z_0\|_{\mathcal{M}_1} \leq \mathcal{R},$$

then there exists a constant $\mathcal{R}_1 = \mathcal{R}_1(\mathcal{R})$, such that for any $t \geq 0$, the following estimate holds:

$$|u_t(t)|_2^2 + \|u_t(t)\|_0^2 + \|u(t)\|_0^2 + \|\eta^t\|^2_{\mu,0} \leq \mathcal{R}_1.$$

Proof. Multiplying the first equation of (2.6) by u_t, and integrating over Ω, we obtain that

$$\frac{1}{2}\frac{d}{dt}[|u_t|_2^2 + \|u_t\|_0^2 + \|u\|_0^2 + \|\eta^t\|^2_{\mu,0} - 2\langle F(u), 1\rangle - 2\langle g, u\rangle] + \delta\|\eta^t\|^2_{\mu,0} = 0. \tag{3.1}$$

Next, let $E(t) = |u_t|_2^2 + \|u_t\|_0^2 + \|u\|_0^2 + \|\eta^t\|^2_{\mu,0} - 2\langle F(u), 1\rangle - 2\langle g, u\rangle$, then by (H_2), *Hölder* inequality and Young inequality, we can get that

$$\begin{aligned} E(t) &\geq |u_t|_2^2 + \|u_t\|_0^2 + \frac{1 - \frac{\lambda}{\lambda_1}}{2}\|u\|_0^2 + \|\eta^t\|^2_{\mu,0} - \frac{2}{\lambda_1 - \lambda}|g|_2^2 - 2c_f|\Omega| \\ &\geq \frac{1 - \frac{\lambda}{\lambda_1}}{2}\left[|u_t|_2^2 + \|u_t\|_0^2 + \frac{1}{2}\|u\|_0^2 + \|\eta^t\|^2_{\mu,0}\right] - C(1 + |g|_2^2), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} E(t) &\leq |u_t|_2^2 + \|u_t\|_0^2 + \|u\|_0^2 + \|\eta^t\|^2_{\mu,0} + \frac{\lambda_1}{2}|u|_2^2 + \frac{2}{\lambda_1}|g|_2^2 + C(1 + |u|_6^6) \\ &\leq |u_t|_2^2 + \|u_t\|_0^2 + \frac{3}{2}\|u\|_0^2 + \|\eta^t\|^2_{\mu,0} + \frac{2}{\lambda_1}|g|_2^2 + C(1 + \|u\|_0^6) \end{aligned} \tag{3.3}$$

hold for any $t \geq 0$.

In addition, it's easy to obtain that

$$\frac{d}{dt}E(t) + \delta\|\eta^t\|^2_{\mu,0} \leq 0. \tag{3.4}$$

Integrating (3.4) about t from 0 to t, and combining with (3.2), (3.3), we have

$$|u_t(t)|_2^2 + \|u_t(t)\|_0^2 + \|u(t)\|_0^2 + \|\eta^t\|^2_{\mu,0} + \int_0^t \|\eta^\tau\|^2_{\mu,0} d\tau \leq \mathcal{R}_1. \tag{3.5}$$

where $\mathcal{R}_1 = \mathcal{R}_1(\|z(0)\|_{\mathcal{M}_1})$ depends on $\|z(0)\|_{\mathcal{M}_1}$.

Lemma 3.3. *For any $T > 0$, $z_0 \in \mathcal{M}_1$ and $\|z_0\|_{\mathcal{M}_1} \le \mathcal{R}$, then there exists a constant $\mathcal{K}_1 = \mathcal{K}_1(\mathcal{R}, T)$, it follows that*

$$|u_{tt}|_2^2 + \|u_{tt}(t)\|_0^2 + \int_0^t \|u_{tt}(s)\|_0^2 ds \le \mathcal{K}_1. \tag{3.6}$$

holds for any $t \in [0, T]$.

Proof. Multiplying the first equation of (2.6) by u_{tt}, and integrating over Ω, we obtain that

$$\begin{aligned} |u_{tt}|_2^2 + \|u_{tt}(t)\|_0^2 \le & -\int_\Omega \nabla u \nabla u_{tt} - \int_\Omega f(u) u_{tt} + \int_\Omega g u_{tt} \\ & - \int_0^\infty \mu(s) \int_\Omega \nabla \eta^t(s) \nabla u_{tt} ds. \end{aligned} \tag{3.7}$$

Using Lemma3.2, *Hölder* inequality and Young inequality, then

$$\begin{aligned} & |-\int_\Omega \nabla u \nabla u_{tt} dx| \le 2\|u\|_0^2 + \frac{1}{8}\|u_{tt}\|_0^2. \\ & |\int_\Omega g u_{tt} dx| \le \frac{2}{\lambda_1}|g|_2^2 + \frac{1}{8}\|u_{tt}\|_0^2. \\ & |-\int_0^\infty \mu(s) \int_\Omega \nabla \eta^t(s) \nabla u_{tt} dx ds| \le 2m_0 \|\eta^t\|_{\mu,0}^2 + \frac{1}{8}\|u_{tt}\|_0^2. \\ & |-\int_\Omega f(u) u_{tt} dx| \le c \int_\Omega (1 + |u|^5) u_{tt} dx \le C(1 + \mathcal{R}_1^6) + \frac{1}{2}|u_{tt}|_2^2 + \frac{1}{8}\|u_{tt}\|_0^2 \end{aligned} \tag{3.8}$$

By Lemma 3.2, we have

$$|u_{tt}|_2^2 + \|u_{tt}(t)\|_0^2 \le C[1 + \mathcal{R}_1^6 + |g|_2^2]. \tag{3.9}$$

Combining with (3.8) and $t \in [0, T]$, we get

$$\int_0^t \|u_{tt}(s)\|_0^2 ds \le C[1 + \mathcal{R}_1^6 + |g|_2^2]T. \tag{3.10}$$

Just let $\mathcal{K}_1 = C[1 + \mathcal{R}_1^6 + |g|_2^2](1 + T)$, then (3.6) holds.

Lemma 3.4. *Provided that $(u(t), u_t(t), \eta^t)$ is a sufficiently regular solution of (2.6), (2.7). Then, for the functional*

$$\Lambda_0(t) = -\langle \eta^t, u_t \rangle_{\mathcal{M}_1} - \int_0^\infty \mu(s) \langle \eta^t, u_t \rangle ds,$$

it satisfies the following estimate

$$\begin{aligned} \frac{d}{dt}\Lambda_0(t) + \frac{m_0}{2}[|u_t(t)|_2^2 + \|u_t(t)\|_0^2] \le & (l + C)\|u(t)\|_0^2 + (\frac{m_0}{l} + m_0)\|\eta^t\|_{\mu,0}^2 \\ & + \frac{\mu(0)}{2m_0}(1 + \frac{1}{\lambda_1^2}) \int_0^\infty -\mu'(s)\|\eta^t(s)\|_0^2 ds + \frac{l}{2\lambda_1^2}|g|_2^2. \end{aligned} \tag{3.11}$$

And we can also obtain

$$|\Lambda_0(t)| \le k_0 H(t). \tag{3.12}$$

where $H(t) = \frac{1}{2}|u_t|_2^2 + \frac{1}{2}\|u\|_0^2 + \frac{1}{2}\|u_t\|_0^2 + \frac{1}{2}\|\eta^t\|_{\mu,0}^2$, and $k_0 = k_0(m_0)$ is a positive constant.

Proof. First of all, by Hölder inequality and Young inequality, it's easy to get that

$$
\begin{aligned}
|\Lambda_0(t)| &\le \|u_t\|_0 \int_0^\infty \mu(s)\|\eta^t(s)\|_0 ds + |u_t|_2 \int_0^\infty \mu(s)|\eta^t|_2 ds \\
&\le \sqrt{m_0}\|u_t\|_0\|\eta^t(s)\|_{\mu,0} + \frac{\sqrt{m_0}}{\lambda_1}|u_t|_2\|\eta^t(s)\|_{\mu,0} \\
&\le k_0 H(t).
\end{aligned}
$$

Next, taking the derivative about t for $\Lambda_0(t)$, we have

$$
\begin{aligned}
\frac{d}{dt}\Lambda_0(t) &= \int_0^\infty \mu(s)\langle \eta^t, \Delta u_{tt} - u_{tt}\rangle ds + \int_0^\infty \mu(s)\langle \eta^t_t, \Delta u_t - u_t\rangle ds \\
&= \int_0^\infty \mu(s)\langle \eta^t, -\Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) - g\rangle ds + \int_0^\infty \mu(s)\langle \eta^t_t, \Delta u_t - u_t\rangle ds.
\end{aligned}
\tag{3.13}
$$

Now, we sequentially deal with the two terms on the right of (3.13).

The estimate for the first term is as follows

$$
\begin{aligned}
&|\int_0^\infty \mu(s)\langle \eta^t, -\Delta u\rangle ds| \le l\|u\|_0^2 + \frac{m_0}{4l}\|\eta^t\|_{\mu,0}^2 \\
&|\int_0^\infty \mu(s)\langle \eta^t, -\int_0^\infty \mu(s)\Delta\eta^t ds\rangle ds| \le m_0\|\eta^t\|_{\mu,0}^2 \\
&|\int_0^\infty \mu(s)\langle \eta^t, g\rangle ds| \le \frac{l}{2\lambda_1^2}|g|_2^2 + \frac{m_0}{2l}\|\eta^t\|_{\mu,0}^2; \\
&|\int_0^\infty \mu(s)\langle \eta^t, f(u)\rangle ds| \le m_0^{\frac{1}{2}}|f(u)|_{\frac{6}{5}}\|\eta^t\|_{\mu,0} \le C\|u\|_0^2 + \frac{m_0}{4l}\|\eta^t\|_{\mu,0}^2.
\end{aligned}
\tag{3.14}
$$

The estimate for the second term, by concerning the second equation of (2.6), we obtain

$$
\begin{aligned}
\int_0^\infty \mu(s)\langle \eta^t_t, \Delta u_t\rangle ds &= -\langle \eta^t_t, u_t\rangle_{\mathcal{M}_1} \\
&= \langle \eta^t_s, u_t\rangle_{\mathcal{M}_1} - m_0\|u_t\|_0^2 \\
&\le \sqrt{\mu(0)}\|u_t\|_0(\int_0^\infty -\mu'(s)\|\eta^t(s)\|_0^2 ds)^{\frac{1}{2}} - m_0\|u_t\|_0^2 \\
&\le -\frac{m_0}{2}\|u_t\|_0^2 + \frac{\mu(0)}{2m_0}\int_0^\infty -\mu'(s)\|\eta^t(s)\|_0^2 ds.
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
|\int_0^\infty \mu(s)\langle \eta^t_t, -u_t\rangle ds| &\le \frac{\sqrt{\mu(0)}}{\lambda_1}|u_t|_2(\int_0^\infty -\mu'(s)\|\eta^t(s)\|_0^2 ds)^{\frac{1}{2}} - m_0|u_t|_2^2 \\
&\le -\frac{m_0}{2}|u_t|_2^2 + \frac{\mu(0)}{2\lambda_1^2 m_0}\int_0^\infty -\mu'(s)\|\eta^t(s)\|_0^2 ds.
\end{aligned}
\tag{3.16}
$$

Thus, combining with (3.13)–(3.16), then (3.11) holds.

Lemma 3.5. *Assuming that* $(u(t), u_t(t), \eta^t)$ *is a sufficiently regular solution of* (2.6), (2.7). *Then the functional*

$$\mathcal{N}(t) = \int_\Omega (u_t - \Delta u_t)\, u,$$

fulfills the following control

$$|\mathcal{N}(t)| \le kH(t). \tag{3.17}$$

And we can obtain differential inequality

$$\begin{aligned}\mathcal{N}'(t) \le & \frac{\varepsilon - 1}{2}\|u\|_0^2 + (\varepsilon - 1)H(t) + (k_1 - \frac{\varepsilon - 1}{2})\|u_t\|_0^2 + (\frac{m_0}{4\varepsilon} - \frac{\varepsilon - 1}{2})\|\eta^t\|_{\mu,0}^2 \\ & + \frac{1 - \varepsilon}{2}|u_t|_2^2 + |g|_2|u|_2.\end{aligned} \tag{3.18}$$

where k is a positive constant and $\varepsilon \in (0, 1)$.

Proof. Using *Hölder* inequality, Young inequality and *Poincaré* inequality, it's easy to get

$$\begin{aligned}|\mathcal{N}(t)| &\le \frac{1}{2}(|u_t|_2^2 + |u|_2^2) + \frac{1}{2}(\|u_t\|_0^2 + \|u\|_0^2) \\ &\le k(\|u_t\|_0^2 + \|u\|_0^2) \le kH(t).\end{aligned} \tag{3.19}$$

Furthermore, taking the time-derivative for $\mathcal{N}(t)$ and combining with the first equation (2.6), it follows that

$$\mathcal{N}'(t) + \int_\Omega f(u)udx \le -\int_0^\infty \mu(s)\langle \nabla u, \nabla \eta^t\rangle ds + \langle g, u\rangle - \|u\|_0^2 + k_1\|u_t\|_0^2, \tag{3.20}$$

where $k_1 \ge \frac{1}{\lambda_1} + 1$. Next, we dispose each term on the right side of (3.20), it follows that

$$\begin{aligned}|-\int_0^\infty \mu(s)\langle \nabla u, \nabla \eta^t\rangle ds| &\le \sqrt{m_0}\|u\|_0 \int_0^\infty \mu(s)\|\eta^t(s)\|_0^2 ds \\ &\le \varepsilon\|u\|_0^2 + \frac{m_0}{4\varepsilon}\|\eta^t(s)\|_{\mu,0}^2,\end{aligned} \tag{3.21}$$

and by (1.6)

$$|\int_\Omega gudx| \le |g|_2|u|_2, \tag{3.22}$$

By (3.20)–(3.22) and the definition of $H(t)$, it yields

$$\begin{aligned}&\mathcal{N}'(t) + \int_\Omega f(u)udx \\ &\le \frac{\varepsilon - 1}{2}\|u\|_0^2 + \frac{\varepsilon - 1}{2}[2H(t) - |u_t|_2^2 - \|u_t\|_0^2 - \|\eta^t\|_{\mu,0}^2] + \frac{m_0}{4\varepsilon}\|\eta^t\|_{\mu,0}^2 + |g|_2|u|_2 + k_1\|u_t\|_0^2 \\ &\le \frac{1 - \varepsilon}{2}|u_t|_2^2 + \frac{\varepsilon - 1}{2}\|u\|_0^2 + (\varepsilon - 1)H(t) + (k_1 - \frac{\varepsilon - 1}{2})\|u_t\|_0^2 + (\frac{m_0}{4\varepsilon} - \frac{\varepsilon - 1}{2})\|\eta^t\|_{\mu,0}^2 + |g|_2|u|_2.\end{aligned}$$

Theorem 3.6. *There exists a constant $\mathcal{R}_0$, such that, for any $T_0 = T_0(\|z_0\|_{\mathcal{M}_1}) > 0$, whenever*

$$z_0 \in \mathcal{M}_1,$$

then for all $t \geq T_0$, we have

$$\|S(t)z_0\|_{\mathcal{M}_1} \leq \mathcal{R}_0.$$

Proof. According to the definition of $H(t)$, we can obtain

$$H'(t) + \frac{d}{dt}\langle F(u), 1\rangle + \frac{\delta}{4}\|\eta^t\|^2_{\mu,0} \leq \frac{1}{4}\int_0^\infty \mu'(s)\|\eta^t(s)\|_0^2 ds + \langle g, u_t\rangle. \tag{3.23}$$

In addition, the following functional can be defined

$$\mathcal{L}(t) = CH(t) + \epsilon\mathcal{N}(t) + \epsilon\langle F(u), 1\rangle + \Lambda_0(t).$$

By Lemma 3.4 and Lemma 3.5, we get

$$(C - \epsilon k - k_0)H(t) \leq \mathcal{L}(t) - \epsilon\langle F(u), 1\rangle \leq (C + \epsilon k + k_0)H(t). \tag{3.24}$$

Let its perturbation ϵ be small enough and C be sufficiently large, and combining with (H_2), then it yields

$$\frac{C}{2}H(t) + \epsilon\langle F(u), 1\rangle \leq \mathcal{L}(t) \leq \frac{3C}{2}H(t) + \epsilon\langle F(u), 1\rangle. \tag{3.25}$$

However, combining with (H_2), (3.11), (3.18) and (3.23), we have

$$\begin{aligned}
&\mathcal{L}'(t) + \epsilon\langle f(u), u\rangle + C\frac{\delta}{4}\|\eta^t\|^2_{\mu,0} + \frac{m_0}{2}[|u_t|_2^2 + \|u_t\|_0^2] \\
&\leq (\frac{m_0}{4} + \epsilon\frac{1-\varepsilon}{2})|u_t|_2^2 + (\frac{C^2}{m_0} + \frac{1+4\epsilon}{2\lambda_1^2})|g|_2^2 + [l - \frac{\epsilon(3-4\varepsilon)}{8}]\|u\|_0^2 - \epsilon(1-\varepsilon)H(t) \\
&+ \epsilon(k_1 - \frac{\varepsilon - 1}{2})\|u_t\|_0^2 + \epsilon(\frac{m_0}{4\varepsilon} - \frac{\varepsilon-1}{2} + \frac{3m_0}{4l} + m_0)\|\eta^t\|^2_{\mu,0} \\
&+ [\frac{C}{4} - \frac{\mu(0)}{2m_0}(1 + \frac{1}{\lambda_1^2})]\int_0^\infty \mu'(s)\|\eta^t(s)\|_0^2 ds.
\end{aligned} \tag{3.26}$$

i.e.

$$\begin{aligned}
&\mathcal{L}'(t) + \epsilon\langle F(u), 1\rangle + \epsilon(1-\varepsilon)H(t) \\
&\leq (-\frac{m_0}{4} + \epsilon\frac{1-\varepsilon}{2})|u_t|_2^2 + (\frac{C^2}{m_0} + \frac{1+4\epsilon}{2\lambda_1^2})|g|_2^2 + [l + \frac{\epsilon\lambda}{\lambda_1} - \frac{\epsilon(3-4\varepsilon)}{8}]\|u\|_0^2 \\
&+ [\epsilon(k_1 - \frac{\varepsilon-1}{2}) - \frac{m_0}{2}]\|u_t\|_0^2 + [\epsilon(\frac{m_0}{4\varepsilon} - \frac{\varepsilon-1}{2} + \frac{3m_0}{4l} + m_0) - C\frac{\delta}{4}]\|\eta^t\|^2_{\mu,0} \\
&+ [\frac{C}{4} - \frac{\mu(0)}{2m_0}(1 + \frac{1}{\lambda_1^2})]\int_0^\infty \mu'(s)\|\eta^t(s)\|_0^2 ds + \epsilon c_f|\Omega|.
\end{aligned} \tag{3.27}$$

Thus, when δ is fixed, then we can choose appropriate l, ϵ, C, such that

$$l + \frac{\epsilon\lambda}{\lambda_1} - \frac{\epsilon(3-4\varepsilon)}{8} < 0, \quad \epsilon(k_1 - \frac{\varepsilon - 1}{2}) - \frac{m_0}{2} < 0, \quad -\frac{m_0}{4} + \epsilon\frac{1-\varepsilon}{2} < 0,$$

and

$$\epsilon(\frac{m_0}{4\varepsilon} - \frac{\varepsilon - 1}{2}) - C\frac{\delta}{4} < 0, \quad \frac{C}{4} - \frac{\mu(0)}{2m_0}(1 + \frac{1}{\lambda_1^2}) > 0.$$

Furthermore, let $\gamma = \epsilon(1-\varepsilon), \gamma_0 = \max\{\frac{C^2}{m_0} + \frac{1+2\epsilon}{2\lambda_1^2}, \epsilon c_f|\Omega|\}$, then by (3.27), we obtain

$$\mathcal{L}'(t) \le -\gamma(H(t) + \langle F(u), 1\rangle) + \gamma_0(|g|_2^2 + 1). \tag{3.28}$$

Using (3.25), we have

$$\mathcal{L}'(t) \le -\frac{2\gamma}{3C}\mathcal{L}(t) + \gamma_0(|g|_2^2 + 1). \tag{3.29}$$

From Gronwall Lemma, it's easy to obtain

$$\mathcal{L}(t) \le \mathcal{L}(0)e^{-\frac{2\gamma}{3C}t} + \frac{3\gamma_0 C}{2\gamma}(|g|_2^2 + 1). \tag{3.30}$$

Using (3.25) again, we have

$$H(t) \le \frac{2\mathcal{L}(0)}{C}e^{-\frac{2\gamma}{3C}t} + \frac{3\gamma_0}{\gamma}(|g|_2^2 + 1) + \frac{2\epsilon c_f}{C}|\Omega|. \tag{3.31}$$

Hence, for any $t \ge T_0 = \frac{3C}{2\gamma}\ln\frac{2\gamma\mathcal{L}(0)}{3\gamma_0(|g|_2^2+1)+\frac{2\epsilon c_f}{C}|\Omega|}$, we obtain

$$\|S(t)z_0\|_{\mathcal{M}_1} \le \mathcal{R}_0.$$

where $\mathcal{R}_0 = \frac{12\gamma_0}{\gamma}(|g|_2^2 + 1) + \frac{8\epsilon c_f}{C}|\Omega|$.

Therefore, we can know that the set

$$B_0 = \{(u, u_t, \eta^t) \in \mathcal{M}_1 : \|z(t)\|_{\mathcal{M}_1} \le \mathcal{R}_0\}$$

is a bounded absorbing set for semigroup $\{S(t)\}_{t\ge0}$ on $\mathcal{M}_1$.

Corollary 3.7. *There exists a constant $C_{\mathcal{R}_0}$, such that, for all $t \ge T_0$, we have*

$$\int_t^{t+1}(|u_t(s)|_2^2 + \|u(s)\|_0^2 + \|u_t(s)\|_0^2)ds \le C_{\mathcal{R}_0}. \tag{3.32}$$

Proof. Integrating (3.29) about t from t to $t+1$, and combining with (3.25) and Lemma 3.7, the above estimate is easily obtained.

Lemma 3.8. *For any $T > 0$, there exists a constant $\mathcal{R}_3 > 0$, such that, whenever*

$$\|z(0)\|_{\mathcal{M}_1} \le \mathcal{R}_1,$$

it follows that

$$|A^{\frac{\kappa}{2}}u_t(t)|_2^2 + \|u_t(t)\|_\kappa^2 + \|u(t)\|_\kappa^2 + \|\eta^t\|_{\mu,\kappa}^2 \le \mathcal{R}_3 \quad \forall t \in [0, T].$$

Proof. Multiplying the first equation of (2.6) by $A^{\kappa}u_t$, and integrating over Ω, we obtain that

$$\frac{d}{dt}E_1(t) + \langle f(u), A^{\kappa}u_t\rangle + \frac{\delta}{2}\|\eta^t\|^2_{\mu,\kappa} \leq \langle g, A^{\kappa}u_t\rangle. \tag{3.33}$$

where $E_1(t) = \frac{1}{2}[|A^{\frac{\kappa}{2}}u_t|_2^2 + \|u_t\|^2_{\kappa} + \|u\|^2_{\kappa} + \|\eta^t\|^2_{\mu,\kappa}]$.
Due to

$$\langle g, A^{\kappa}u_t\rangle \leq h|A^{-\frac{1}{2}}g|_2|A^{\frac{1+\kappa}{2}}u_t|_2 \leq h|g|_2^2 + h|A^{\frac{1+\kappa}{2}}u_t|_2^2, \tag{3.34}$$

and by (H_2) and Lemma 3.2, we obtain

$$\langle f(u), A^{\kappa}u_t\rangle \leq C(1 + \|u_t\|^2_{\kappa}), \tag{3.35}$$

Then by (3.33)–(3.35), we have

$$\frac{d}{dt}E_1(t) \leq h_1E_1 + h|g|_2^2. \tag{3.36}$$

where h, h_1 are positive constant.

Hence, using Gronwall lemma, we can obtain that

$$E_1(t) \leq \mathcal{R}_4(\mathcal{R}_1)(1 + |g|_2^2)e^{\ell t}. \tag{3.37}$$

holds for any $t \in [0, T]$. This proof is finished.

Lemma 3.9. *For any $t \in [0, T]$, there exists a constant $\mathcal{R}_5 > 0$, such that, whenever*

$$\|z_0\|_{\mathcal{M}_1} \leq \mathcal{R}_1,$$

it follows that

$$\int_t^{t+1} \|u(s)\|^2_{\kappa} ds \leq \mathcal{R}_5.$$

Proof. Firstly, the first equation of the system (2.6), it can be rewritten

$$u_{tt} + (1-\delta_1)Av + Av_t + (1 - \delta_1 + \delta_1^2)Au - Au_t + \int_0^{\infty} \mu(s)A\eta^t(s)ds + f(u) = g. \tag{3.38}$$

Next, let $v = u_t + \delta_1 u$, and multiplying (3.38) by $A^{\kappa}v$, and integrating over Ω, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|v(t)\|^2_{\kappa} + (1 - 2\delta_1 + \delta_1^2)\|u(t)\|^2_{\kappa} + \|\eta^t\|^2_{\mu,\kappa}] + (1-\delta_1)\|v(t)\|^2_{\kappa} + \frac{\delta}{2}\|\eta^t\|^2_{\mu,\kappa} \\
&+ \delta_1(1 - \delta_1 + \delta_1^2)\|u(t)\|^2_{\kappa} + \langle f(u), A^{\kappa}v\rangle \\
&\leq \|u_t\|^2_{\kappa} + \langle g, A^{\kappa}v\rangle - \langle u_{tt}, A^{\kappa}u_t\rangle - \delta_1\int_0^{\infty} \mu(s)\langle A\eta^t, A^{\kappa}u\rangle ds.
\end{aligned} \tag{3.39}$$

In addition, we deal with each term on the right of (3.39), it yields by using *Minkowski* inequality

$$|\langle g, A^\kappa v\rangle| \leq h_2|g|_2^2 + \frac{\delta_1}{2}\|u_t\|_\kappa^2 + \frac{\delta_1}{4}\|u\|_\kappa^2 \tag{3.40}$$
$$\leq h_2|g|_2^2 + \frac{\delta_1}{2}\|v\|_\kappa^2 + \frac{2\delta_1^2+\delta_1}{4}\|u\|_\kappa^2,$$

and

$$|-\delta_1\int_0^\infty \mu(s)\langle A\eta^t, A^\kappa u\rangle ds| \leq \delta_1\sqrt{m_0}\|u\|_\kappa(\int_0^\infty \mu(s)\|\eta\|_\kappa^2 ds)^{\frac{1}{2}} \tag{3.41}$$
$$\leq \frac{\delta_1}{2}\|u\|_\kappa^2 + \frac{m_0\delta_1}{2}\|\eta\|_{\mu,\kappa}^2,$$

$$|-\langle u_{tt}, A^\kappa u_t\rangle| \leq h_0(\|u_{tt}\|_0^2 + \|u_t\|_0^2), \tag{3.42}$$

next, we deal with the nonlinear term by using Hölder inequality and Sobolev embedding theorem, it yields

$$|\langle f(u), A^\kappa v\rangle| \leq c\int_\Omega (1+|u|^5)|A^\kappa v|dx$$
$$\leq c\int_\Omega |A^\kappa v|dx + c\int_\Omega |u|^5|A^\kappa v|dx \tag{3.43}$$
$$\leq C + \frac{\delta_1}{8}(\|u_t\|_\kappa^2 + \|u\|_\kappa^2),$$

which, together with (3.39)–(3.42), obtains

$$\frac{1}{2}\frac{d}{dt}[\|v(t)\|_\kappa^2 + (1-2\delta_1+\delta_1^2)\|u(t)\|_\kappa^2 + \|\eta^t\|_{\mu,\kappa}^2] + (1-\frac{\delta_1}{3})\|v(t)\|_\kappa^2$$
$$+\delta_1(\frac{1}{8}-\frac{3}{2}\delta_1+\delta_1^2)\|u(t)\|_\kappa^2 + \frac{\delta - m_0\delta_1}{2}\|\eta^t\|_{\mu,\kappa}^2 \tag{3.44}$$
$$\leq (1+\frac{\delta_1}{8})\|u_t\|_\kappa^2 + h_2|g|_2^2 + h_0(\|u_{tt}\|_0^2 + \|u_t\|_0^2).$$

Where h_0, h_2 are positive constant.

Let δ_1 be small enough, such that

$$\beta_1 = \min\{1, 1-2\delta_1+\delta_1^2\} > 0 \quad and \quad \beta_0 = \min\{1-\frac{\delta_1}{3}, \delta_1(\frac{1}{8}-\frac{3}{2}\delta_1+\delta_1^2), \frac{\delta - m_0\delta_1}{2}\} > 0.$$

Then combining Lemma 3.1, Lemma 3.2, Lemma 3.8 and Gronwall lemma, we get

$$\|v(t)\|_\kappa^2 + \|u(t)\|_\kappa^2 + \|\eta^t\|_{\mu,\kappa}^2 \leq \frac{Q(\|z_0\|_\kappa)}{\beta_1}e^{-2\beta_0 t} + \frac{2}{\beta_0\beta_1}[(1+\frac{\delta_1}{8})\mathcal{R}_3 + h_0(\mathcal{R}_1+\mathcal{K}_1) + h_2|g|_2^2].$$

Moreover, integrating (3.41) about t from t to t+1, then we have

$$\int_t^{t+1} \|v(s)\|_\kappa^2 + \|u(s)\|_\kappa^2 + \|\eta^s\|_{\mu,\kappa}^2 ds \leq \mathcal{R}_5.$$

where $\mathcal{R}_5 = \mathcal{R}_5(Q(\|z_0\|_\kappa), \beta_0, \beta_1, \delta_1, \mathcal{R}_1, \mathcal{R}_3, \mathcal{K}_1, |g|_2)$.

In order to prove the existence of global attractor for $\{S(t)\}_{t\geq 0}$ on $\mathcal{M}_1$, we have to verify some compactness for the semigroup $\{S(t)\}_{t\geq 0}$. For further purpose, we will give asymptotically compact theorem of the semigroup on $\mathcal{M}_1$.

Theorem 3.10. *The semigroup* $\{S(t)\}_{t\geq 0}$ *associated with problem* (2.6), (2.7) *is asymptotically compact on* $\mathcal{M}_1$.

Proof. Firstly, Let $z^1(t) = (u^1(t), u_t^1(t), \eta_1^t), z^2(t) = (u^2(t), u_t^2(t), \eta_2^t)$ are two solutions of (2.6) corresponding with the initial data $z_0^1 = (u_0^1, u_1^1, \eta_1^0), z_0^2 = (u_0^2, u_1^2, \eta_2^0)$ respectively. Setting $z(t) = (\omega(t), \omega_t(t), \theta^t) = (u^1(t) - u^2(t), u_t^1(t) - u_t^2(t), \eta_1^t - \eta_2^t)$, then $z(t)$ satisfies the following equation

$$\begin{cases} \omega_{tt} - \Delta\omega_{tt} - \Delta\omega - \int_0^{+\infty} \mu(s)\Delta\theta^t(s)ds + f(u^1) - f(u^2) = 0, \\ \theta_t^t + \theta_s^t = \omega_t. \end{cases} \tag{3.45}$$

with initial-boundary conditions

$$\begin{cases} \omega(x,t)|_{\partial\Omega} = 0, \theta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+} = 0, \\ \omega(x,0) = u_0^1 - u_0^2,\ \omega_t(x,0) = u_1^1 - u_1^2,\ \theta^0(x,s) = \eta_1^0 - \eta_2^0. \end{cases} \tag{3.46}$$

Similar to the definition of $H(t)$, we let $\mathcal{H}_\omega(t) = \frac{1}{2}|\omega_t(t)|_2^2 + \frac{1}{2}\|\omega(t)\|_0^2 + \frac{1}{2}\|\omega_t(t)\|_0^2 + \frac{1}{2}\|\theta^t\|_{\mu,0}^2$, then according to Lemma 3.4 and Lemma 3.5, we obtain
(i) Let $\Lambda_\omega(t) = -\langle\theta^t, \omega_t\rangle_{\mathcal{M}_1} - \int_0^\infty \mu(s)\langle\theta^t, \omega_t\rangle ds$, we have

$$\begin{aligned} \frac{d}{dt}\Lambda_\omega(t) + \frac{m_0}{2}[|\omega_t(t)|_2^2 + \|\omega_t(t)\|_0^2] \leq C\|\omega(t)\|_0^2 + 2m_0\|\theta^t\|_{\mu,0}^2 \\ + \frac{\mu(0)}{2m_0}(1 + \frac{1}{\lambda_1^2})\int_0^\infty -\mu'(s)\|\theta^t(s)\|_0^2 ds. \end{aligned} \tag{3.47}$$

and

$$|\Lambda_\omega(t)| \leq k_0\mathcal{H}_\omega(t). \tag{3.48}$$

(ii) Assuming that $\mathcal{N}_\omega(t) = \int_\Omega \omega_t\omega dx + \int_\Omega \nabla\omega_t\nabla\omega dx$, we can also obtain that

$$\begin{aligned} \mathcal{N}_\omega'(t) \leq & \frac{\varepsilon - 1}{2}\|\omega\|_0^2 + \frac{1-\varepsilon}{2}|\omega_t| + (\varepsilon - 1)H_\omega(t) + \frac{1-\varepsilon}{2}|\omega_t|_2^2 \\ & + (k_1 - \frac{\varepsilon - 1}{2})\|\omega_t\|_0^2 + (\frac{m_0}{4\varepsilon} - \frac{\varepsilon - 1}{2})\|\theta^t\|_{\mu,0}^2. \end{aligned} \tag{3.49}$$

and

$$|\mathcal{N}_\omega(t)| \leq k\mathcal{H}_\omega(t). \tag{3.50}$$

(iii) Obviously, we can get it easily

$$\mathcal{H}_\omega'(t) + \langle f(u^1) - f(u^2), \omega_t\rangle + \frac{\delta}{4}\|\theta^t(s)\|_{\mu,0}^2 \leq \frac{1}{4}\int_0^\infty \mu'(s)\|\theta^t(s)\|_0^2 ds. \tag{3.51}$$

By (H_2), it's easy to obtain

$$\begin{aligned} |\langle f(u^1) - f(u^2), \omega_t\rangle| &\leq C\int_\Omega (1 + |u_1|^4 + |u|^4)|\omega||\omega_t|dx \\ &\leq \epsilon Q(\frac{1}{\lambda})\|\omega_t\|_0^2 + Q(\frac{1}{\lambda\epsilon})\|\omega\|_0^2, \end{aligned} \tag{3.52}$$

where $Q(\cdot)$ denotes monotonically increasing function.

Combining with (3.51) and (3.52), we have

$$\mathcal{H}'_\omega(t) + \frac{\delta}{4}\|\theta^t(s)\|^2_{\mu,0} \leq \frac{1}{4}\int_0^\infty \mu'(s)\|\theta^t(s)\|_0^2 ds + \epsilon Q(\frac{1}{\lambda})\|\omega_t\|_0^2 + Q(\frac{1}{\lambda\epsilon})\|\omega\|_0^2. \tag{3.53}$$

Secondly, we can define the following functional

$$\mathcal{L}_\omega(t) = C_1\mathcal{H}_\omega(t) + \epsilon\mathcal{N}_\omega(t) + \Lambda_\omega(t).$$

By (3.48) and (3.50), we get

$$(C_1 - \epsilon k - k_0)\mathcal{H}_\omega(t) \leq \mathcal{L}_\omega(t) \leq (C_1 + \epsilon k + k_0)\mathcal{H}_\omega(t). \tag{3.54}$$

Next, let its perturbation ϵ be small enough and C_1 be sufficiently large, then yields

$$\frac{C_1}{2}\mathcal{H}_\omega(t) \leq \mathcal{L}_\omega(t) \leq \frac{3C_1}{2}\mathcal{H}_\omega(t). \tag{3.55}$$

Therefore, we can also deduce easily that

$$\begin{aligned}\mathcal{L}'_\omega(t) \leq\ & \epsilon(\varepsilon-1)\mathcal{H}_\omega(t) - [\frac{\delta C_1}{4} - \frac{5m_0}{4} - \epsilon(\frac{m_0}{4\varepsilon} - \frac{\varepsilon-1}{2})]\|\theta^t(s)\|^2_{\mu,0} \\ & + \frac{\epsilon(\varepsilon-1)}{2}\|\omega\|_0^2 + Q(\frac{1}{\epsilon\lambda})\|\omega\|_0^2 + [\epsilon(k_1 - \frac{\varepsilon-1}{2} + Q(\frac{1}{\lambda})) - \frac{m_0}{2}]\|\omega_t\|_0^2 \\ & - (\frac{m_0}{2} - \epsilon\frac{1-\varepsilon}{2})|\omega_t|_2^2 + [\frac{C_1}{4} - \frac{\mu(0)}{2m_0}(1 + \frac{1}{\lambda_1^2})]\int_0^\infty \mu'(s)\|\theta^t(s)\|_0^2 ds.\end{aligned} \tag{3.56}$$

In the same way, let $\varepsilon > 0$ be small enough and C_1 is sufficiently large, such that

$$\epsilon(k_1 - \frac{\varepsilon-1}{2} + Q(\frac{1}{\lambda})) - \frac{m_0}{2} < 0, \qquad \frac{C_1}{4} - \frac{\mu(0)}{2m_0}(1 + \frac{1}{\lambda_1^2}) > 0.$$

and

$$\frac{\delta C_1}{4} - \frac{5m_0}{4} - \epsilon(\frac{m_0}{4\varepsilon} - \frac{\varepsilon-1}{2}) > 0, \qquad \frac{m_0}{2} - \epsilon\frac{1-\varepsilon}{2} > 0.$$

and let $\alpha_0 = \epsilon(1-\varepsilon)$, then (3.56) becomes

$$\begin{aligned}\mathcal{L}'_\omega(t) &\leq -\alpha_0\mathcal{H}_\omega(t) + Q(\frac{1}{\epsilon\lambda})\|\omega\|_0^2 \\ &\leq -\frac{2\beta}{3C}\mathcal{L}_\omega(t) + Q(\frac{1}{\epsilon\lambda})\|\omega\|_0^2.\end{aligned} \tag{3.57}$$

Using Gronwall lemma, we can deduce that

$$\mathcal{L}_\omega(T) \leq \mathcal{L}_\omega(0)e^{-\frac{2\beta}{3C}T} + \frac{2\beta\alpha_1}{3C}Q(\frac{1}{\epsilon\lambda})\int_0^T \|\omega(s)\|_0^2 ds, \tag{3.58}$$

holds for any $T > \frac{3C}{2\beta}\ln\frac{2\mathcal{L}_\omega(0)}{C\varepsilon}$.

Combining with (3.55) and (3.58), then we have

$$\mathcal{H}_\omega(T) \leq \varepsilon + \phi^T(z^1, z^2). \tag{3.59}$$

where

$$\phi^T(z^1, z^2) = \frac{4\beta\alpha_1}{3C^2} Q(\frac{1}{\epsilon\lambda}) \int_{T_1}^T \|\omega(s)\|_0^2 ds.$$

Next, for all $T \geq T_1 > 0$, we prove $\phi^T(z^1, z^2) \in \mathfrak{E}(B)$.

Indeed, let $z_n(t) = (u_n(t), u_{nt}(t), \eta_n^t) \in B_0$ be the solution with initial value $z_n(0) = z_0^n = (u_0^n, u_1^n, \eta_n^0) \in B_0$. According to Corollary 3.11 and Lemma 3.8, we know that the sequence $\{(u_n(t), u_{nt}(t), \eta_n^t)\}$ is uniformly bounded in $\mathcal{M}_1$. That is

$$\begin{aligned} &u_n \text{ is uniformly bounded in } L^2(0, T; D(A^{\frac{1+\kappa}{2}})),\\ &u_{nt} \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega))). \end{aligned}$$

Because we know that the embedding $D(A^{\frac{1+\kappa}{2}}) \hookrightarrow D(A^{\frac{1}{2}})$ is compact by (2.8) and $D(A^{\frac{1}{2}}) \hookrightarrow L^2(\Omega)$ is continuous, then combining with Lemma 2.5, it's easy to get that there exists subsequence (still note $\{(u_n(t), u_{nt}(t), \eta_n^t)\}$) of $\{(u_n(t), u_{nt}(t), \eta_n^t)\}$ such that

$$\lim_{n\to\infty} \lim_{m\to\infty} \int_{T_1}^T \|u_n(s) - u_m(s)\|_0^2 ds = 0.$$

Thanks to Theorem 3.7 and Theorem 3.10, we can deduce the main result of this paper as the following theorem:

Theorem 3.11. *The semigroup $\{S(t)\}_{t\geq 0}$ for the problem* (2.6), (2.7) *possesses a global attractor $\mathcal{A}$ in $\mathcal{M}_1$; and $\mathcal{A}$ is non-empty, compact, invariant in $\mathcal{M}_1$ and attracts any bounded set of $\mathcal{M}_1$ with respect to $\mathcal{M}_1$-norm.*

4. Upper semicontinuity of attractors in $\mathcal{M}_1$

Let's consider the following equations

$$\begin{cases} v_{tt} - \Delta v_{tt} - \Delta v + \omega v_t - \int_0^\infty \mu(s) \Delta \xi^t(s) ds + f(v) = g(x), \\ \xi_t^t = -\xi_s^t + v_t. \end{cases} \tag{4.1}$$

with initial-boundary condition

$$\begin{cases} v(x,t)|_{\partial\Omega} = 0, \xi^t(x,s)|_{\partial\Omega\times\mathbb{R}^+} = 0, \quad t \geq 0, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ \xi^0(x,s) = \int_0^s v_0(x,-r)dr, \quad (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{4.2}$$

where $\omega \in [0, 1]$ is a disturbance parameter.

Let $\omega = 0$, then the above equation is transformed into Eq (2.6) with (2.7). Using the proof method of above section word by word, we have the following lemma:

Lemma 4.1. *The semigroup* $\{S_\omega(t)\}_{t\geq 0}$ *associated with* Eq (4.1) *with* (4.2) *possesses a compact global attractor* $\mathcal{A}_\omega$ *for any* $\omega \geq 0$.

Then $\mathcal{A}_0 = \mathcal{A}$($\mathcal{A}$ from Theorem 3.11).

Lemma 4.2. *For any* $T \geq 0$ *and* $z_0 \in \mathcal{A}_\omega$*, then we have*

$$\|S_\omega(t)z_0 - S_0(t)z_0\|_{\mathcal{M}_1} \leq C\omega,$$

for $0 \leq t \leq T$ *holds, where C is independently of* ω.

Proof. Let $z = (u, u_t, \eta^t)$ and $z^\omega = (v, v_t, \xi^t)$ are unique solutions of Eqs (2.6) and (4.1) with initial value $z_0 \in \mathcal{A}_\omega$ respectively. Setting $w = u - v, \zeta^t = \eta^t - \xi^t$, then (w, w_t, ζ^t) is a unique solution of the following equations

$$\begin{cases} w_{tt} - \Delta w_{tt} - \Delta w - \int_0^\infty \mu(s)\Delta\zeta^t(s)ds + f(u) - f(v) = \omega v_t, \\ \zeta_t^t = -\zeta_s^t + w_t. \end{cases} \tag{4.3}$$

with initial-boundary condition

$$\begin{cases} w(x,t)|_{\partial\Omega} = 0, \zeta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+} = 0, \quad t \geq 0, \\ w(x,0) = 0,\ w_t(x,0) = 0,\ \zeta^0(x,s) = 0, \quad (x,s) \in \Omega \times \mathbb{R}^+. \end{cases} \tag{4.4}$$

Multiplying the first equation of (4.3) by w_t in $L^2(\Omega)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|w_t|_2^2 + \|w_t\|_0^2 + \|\zeta^t\|_{\mu,0}^2\right) + \delta\|\zeta^t\|_{\mu,0}^2 \leq \int_\Omega |f(u) - f(v)||w_t| + \omega\int_\Omega |v_t||w_t|$$

According to Hölder inequality, $z^\omega \in \mathcal{A}_\omega$ and Lemma 3.2, there exist a constant $\alpha > 0$ such that

$$\frac{d}{dt}\left(|w_t|_2^2 + \|w_t\|_0^2 + \|\zeta^t\|_{\mu,0}^2\right) - \alpha\left(|w_t|_2^2 + \|w_t\|_0^2 + \|\zeta^t\|_{\mu,0}^2\right) \leq Q_{\mathcal{R}_0}\omega, \tag{4.5}$$

where $Q_{\mathcal{R}_0} = Q_{\mathcal{R}_0}(\mathcal{R}_0)$ is independently ω.

Let $C = \frac{Q_{\mathcal{R}_0}}{\alpha}$ and applying Gronwall Lemma, we have

$$|w_t|_2^2 + \|w_t\|_0^2 + \|\zeta^t\|_{\mu,0}^2 \leq C\omega. \tag{4.6}$$

Theorem 4.3. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary, and we assume that f satisfies* (1.4)–(1.7) *and* (1.2), (1.3) *holds, given* $g \in L^2(\Omega)$*, then*

$$\lim_{\omega\to 0} dist_{\mathcal{M}_1}(\mathcal{A}_\omega, \mathcal{A}) = 0.$$

Proof. For any $\varepsilon > 0$, since $\mathcal{A}_\omega$ is an universal bounded subset of $\mathcal{M}_1$ for any $\omega \in [0,1]$, and $\mathcal{A}$ is a compact attracting set for $\{S(t)\}_{t\geq 0}$ on $\mathcal{M}_1$. So there exists $T > 0$ such that $S(t)(T)\mathcal{A}_\omega \subset N(\mathcal{A}, \frac{\varepsilon}{2})$. On the other hand, associating with the invariance of $\mathcal{A}_\omega$ and Lemma (4.2), for any $t \geq T$, we have

$$\mathcal{A}_\omega = S_\omega(t)\mathcal{A}_\omega \subset N(S(t)\mathcal{A}_\omega, \frac{\varepsilon}{2}),$$

as ω small enough. Setting $\varepsilon = \frac{\omega}{C}$, so we have

$$\mathcal{A}_\omega \subset N(\mathcal{A}, \frac{\omega}{\nu})$$

here $\nu \geq C$ is a constant, which completes the proof of the desired results.

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Conflict of interest

The authors declare that they have no competing interests.

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