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Research article

Geometric characterizations of canal surfaces with Frenet center curves

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Abstract: In this work, we study the canal surfaces foliated by pseudo hyperbolic spheres $\mathbb{H}_0^2$ along a Frenet curve in terms of their Gauss maps in Minkowski 3-space. Such kind of surfaces with pointwise 1-type Gauss maps are classified completely. For example, an oriented canal surface that has proper pointwise 1-type Gauss map of the first kind satisfies $\Delta\mathbb{G} = -2K\mathbb{G}$, where K and $\mathbb{G}$ is the Gaussian curvature and the Gauss map of the canal surface, respectively.

Keywords: canal surface; Gauss map; Laplace operator; pseudo hyperbolic sphere; Minkowski 3-space
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1. Introduction

The idea of finite type immersion of Riemannian manifolds into Euclidean space (resp. pseudo Euclidean space) was introduced by B.Y. Chen in the late 1970's, which was extended to the differential maps on the submanifolds such as the Gauss maps. A submanifold $\mathbb{M}$ in Euclidean space (resp. pseudo Euclidean space) whose Gauss map $\mathbb{G}$ satisfies $\Delta\mathbb{G} = f(\mathbb{G}+C)$ is said to have a proper pointwise 1-type Gauss map for a non-zero smooth function f and a constant vector C, where Δ is the Laplacian defined on $\mathbb{M}$ and in local coordinates given by

$$\Delta = -\frac{1}{\sqrt{|det(g_{ij})|}}\sum_{i,j}\frac{\partial}{\partial x^i}\left(\sqrt{|det(g_{ij})|}g^{ij}\frac{\partial}{\partial x^j}\right), \tag{1.1}$$

where g^{ij} are the components of the inverse matrix of the first fundamental form of $\mathbb{M}$. Specially, it is said to be of the first kind or the second kind when the vector C is zero or non-zero, respectively. Furthermore, $\mathbb{G}$ is said to be of proper pointwise 1-type if the function f is not constant, otherwise a non-proper pointwise 1-type Gauss map is just of ordinary 1-type. When the smooth function f vanishes, $\mathbb{G}$ is said to be harmonic [1, 2, 5].

In the theory of surfaces, a canal surface is formed by moving a family of spheres whose centers lie on a space curve in Euclidean 3-space. The geometric characteristics of such surfaces have been studied by many experts and geometers [4, 8, 13]. For example, the authors of [4] investigated the geometric properties of such surfaces, including the Gaussian curvature, the mean curvature and their relationships. In recent years, the construction idea of canal surfaces in Euclidean 3-space is extended to Lorentz-Minkowski space. In Minkowski 3-space, a canal surface can be formed as the envelope of a family of pseudo-Riemannian space forms, i.e., pseudo spheres $\mathbb{S}^2_1$, pseudo hyperbolic spheres $\mathbb{H}^2_0$ and lightlike cones $\mathbb{Q}^2$ [3, 9, 12]. Let p be a fixed point, $r > 0$ be a constant in $\mathbb{E}^3_1$. The pseudo-Riemannian space forms, i.e., the de-Sitter space $\mathbb{S}^2_1(p, r)$, the hyperbolic space $\mathbb{H}^2_0(p, r)$ and the lightlike cone $\mathbb{Q}^2_1(p)$ are defined by

$$\mathbb{M}^2(\epsilon) = \{x \in \mathbb{E}^3_1 : \langle x - p, x - p\rangle = \epsilon r^2\} = \begin{cases} \mathbb{S}^2_1(p, r) \mid \epsilon = 1; \\ \mathbb{H}^2_0(p, r) \mid \epsilon = -1; \\ \mathbb{Q}^2_1(p) \quad \mid \epsilon = 0. \end{cases}$$

When $r = 1$ and the center p is the origin, we write them by $\mathbb{S}^2_1$, $\mathbb{H}^2_0$ and $\mathbb{Q}^2$, simply. According to the classification of curves in Minkowski space, there are nine types of canal surfaces in Minkowski 3-space whose fundamental geometric properties have been achieved by discussing the linear Weingarten canal surfaces in [3, 9].

Based on the conclusions obtained in [4], a canal surface with pointwise 1-type Gauss map is discussed in [8]. In order to do further geometric investigation for canal surfaces in Minkowski 3-space, in this work we study surfaces foliated by pseudo hyperbolic spheres $\mathbb{H}^2_0$ along Frenet curves. In section 2, the Frenet formulas of Frenet curves, the parameterized equations and the relationships between the Gaussian curvatures and the mean curvatures of three types of canal surfaces are recalled. In section 3, three types of canal surfaces with pointwise 1-type Gauss maps are classified completely.

The surfaces which are discussed here are smooth, regular and topologically connected unless otherwise stated.

2. Preliminaries

Let $\mathbb{E}^3_1$ be a Minkowski 3-space with natural Lorentzian metric

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

in terms of the natural coordinate system (x_1, x_2, x_3). It is well known that a vector $\upsilon \in \mathbb{E}^3_1$ is called to be spacelike if $\langle \upsilon, \upsilon\rangle > 0$ or $\upsilon = 0$; timelike if $\langle \upsilon, \upsilon\rangle < 0$; null (lightlike) if $\langle \upsilon, \upsilon\rangle = 0$, respectively. The norm of a vector υ is given by $\|\upsilon\| = \sqrt{|\langle \upsilon, \upsilon\rangle|}$. The timelike or lightlike vector is said to be causal [6]. Due to the causal character of the tangent vectors, the curves are classified into spacelike curves, timelike curves or lightlike (null) curves. What's more, the spacelike curves are classified into the first and the second kind of spacelike curves or the null type spacelike curves (pseudo null curves) according to their normal vectors are spacelike, timelike or lightlike, respectively.

Remark 2.1. *[7] Timelike curves and spacelike curves with spacelike or timelike normal vectors are called Frenet curves in Minkowski space.*

Proposition 2.2. *[4] Let $c(s) : \mathbf{I} \to \mathbb{E}_1^3$ be a Frenet curve parameterized by arclength s with Frenet frame $\{T(s), N(s), B(s)\}$. Then the following Frenet equations are satisfied*

$$\begin{cases} T'(s) = \kappa(s)N(s), \\ N'(s) = -\epsilon_1\kappa(s)T(s) + \epsilon_2\tau(s)B(s), \\ B'(s) = \tau(s)N(s), \end{cases}$$

where T is the tangent vector, N and B is the normal vector and the binormal vector of $c(s)$, respectively. When $c(s)$ is a timelike curve, $\epsilon_1 = \epsilon_2 = -1$; when $c(s)$ is a spacelike curve of the first kind, $\epsilon_1 = \epsilon_2 = 1$; when $c(s)$ is a spacelike curve of the second kind, $\epsilon_1 = -\epsilon_2 = -1$. The function $\kappa(s)$, $\tau(s)$ is called the curvature, the torsion of $c(s)$, respectively.

Definition 1. [3] A surface $\mathbb{M}$ in $\mathbb{E}_1^3$ is called a canal surface which is formed as the envelope of a family of pseudo hyperbolic spheres $\mathbb{H}_0^2$ (resp. pseudo spheres $\mathbb{S}_1^2$ or lightlike cones $\mathbb{Q}^2$) whose centers lie on a space curve $c(s)$ framed by $\{T, N, B\}$. Then $\mathbb{M}$ can be parameterized by

$$x(s,\theta) = c(s) + \lambda(s,\theta)T(s) + \mu(s,\theta)N(s) + \omega(s,\theta)B(s),$$

where λ, μ and ω are differential functions of s and θ, $\|x(s,\theta) - c(s)\|^2 = \epsilon r^2(s)$, ($\epsilon = \pm 1$ or 0). The curve $c(s)$ is called the center curve and $r(s)$ is called the radial function of $\mathbb{M}$.

Precisely, if $\mathbb{M}$ is foliated by pseudo hyperbolic spheres $\mathbb{H}_0^2$ (resp. pseudo spheres $\mathbb{S}_1^2$ or lightlike cones $\mathbb{Q}^2$), then $\epsilon = -1$(resp. 1 or 0) and $\mathbb{M}$ is said to be of type $\mathbb{M}_-$ (resp. $\mathbb{M}_+$ or $\mathbb{M}_0$). As well, the canal surfaces of type $\mathbb{M}_-$ can be classified into $\mathbb{M}_-^1$ (resp. $\mathbb{M}_-^2$ or $\mathbb{M}_-^3$) when $c(s)$ is spacelike (resp. timelike or null). Moreover, when $c(s)$ is the first kind spacelike curve, the second kind spacelike curve and the pseudo null curve, $\mathbb{M}_-^1$ is denoted by $\mathbb{M}_-^{11}$, $\mathbb{M}_-^{12}$ and $\mathbb{M}_-^{13}$, respectively. Similarly, the canal surfaces $\mathbb{M}_+$ (resp. $\mathbb{M}_0$) can be classified into $\mathbb{M}_+^1$, $\mathbb{M}_+^2$ and $\mathbb{M}_+^3$ (resp. $\mathbb{M}_0^1$, $\mathbb{M}_0^2$ or $\mathbb{M}_0^3$). Naturally, $\mathbb{M}_+^1$ (resp. $\mathbb{M}_0^1$) can be divided into $\mathbb{M}_+^{11}$, $\mathbb{M}_+^{12}$ and $\mathbb{M}_+^{13}$ (resp. $\mathbb{M}_0^{11}$, $\mathbb{M}_0^{12}$ or $\mathbb{M}_0^{13}$) [9].

Remark 2.3. *In the present work, we consider the canal surfaces foliated by pseudo hyperbolic spheres $\mathbb{H}_0^2$ which have Frenet curves as center curves, i.e., the canal surfaces of type $\mathbb{M}_-^{11}$, $\mathbb{M}_-^{12}$ and $\mathbb{M}_-^2$.*

The canal surfaces $\mathbb{M}_-^{11}$, $\mathbb{M}_-^{12}$ and $\mathbb{M}_-^2$ are expressed as [9]

$$\mathbb{M}_-^{11} : x(s,\theta) = c(s) + r(s)(r'(s)T + \sqrt{1 + r'^2(s)}\sinh\theta N + \sqrt{1 + r'^2(s)}\cosh\theta B);$$

$$\mathbb{M}_-^{12} : x(s,\theta) = c(s) + r(s)(r'(s)T + \sqrt{1 + r'^2(s)}\cosh\theta N + \sqrt{1 + r'^2(s)}\sinh\theta B);$$

$$\mathbb{M}_-^{2} : x(s,\theta) = c(s) + r(s)(-r'(s)T + \sqrt{r'^2(s) - 1}\cos\theta N + \sqrt{r'^2(s) - 1}\sin\theta B).$$

Without loss of generality, the authors assumed $r'(s) = \sinh\varphi$ for $\mathbb{M}_-^{11}$ and $\mathbb{M}_-^{12}$, $-r'(s) = \cosh\varphi$ for $\mathbb{M}_-^2$, where $\varphi = \varphi(s)$ is a smooth function, then the canal surfaces $\mathbb{M}_-^{11}$, $\mathbb{M}_-^{12}$ and $\mathbb{M}_-^2$ can be rewritten by

$$\mathbb{M}_-^{11} : x(s,\theta) = c(s) + r(s)(\sinh\varphi(s)T + \cosh\varphi(s)\sinh\theta N + \cosh\varphi(s)\cosh\theta B); \tag{2.1}$$

$$\mathbb{M}_-^{12} : x(s,\theta) = c(s) + r(s)(\sinh\varphi(s)T + \cosh\varphi(s)\cosh\theta N + \cosh\varphi(s)\sinh\theta B); \tag{2.2}$$

$$\mathbb{M}_-^{2} : x(s,\theta) = c(s) + r(s)(\cosh\varphi(s)T + \sinh\varphi(s)\cos\theta N + \sinh\varphi(s)\sin\theta B). \tag{2.3}$$

Proposition 2.4. *[9] For the canal surface $\mathbb{M}_-^{11}$ (resp. $\mathbb{M}_-^{12}, \mathbb{M}_-^{2}$), the Gaussian curvature K and the mean curvature H satisfy*

$$H = -\frac{1}{2}(Kr - \frac{1}{r}).$$

Remark 2.5. *By Proposition 2.4, the principal curvatures κ_1, κ_2 of the canal surface $\mathbb{M}_-^{11}$ (resp. $\mathbb{M}_-^{12}, \mathbb{M}_-^{2}$) are given by*

$$\kappa_1 = Kr,\ \kappa_2 = -\frac{1}{r}.$$

From now on, we concern on the classifications of three kinds of canal surfaces in terms of their Gauss maps. We only prove the results for $\mathbb{M}_-^{11}$ and omit the proofs for $\mathbb{M}_-^{12}$ and $\mathbb{M}_-^{2}$ since they can be similarly done to those of $\mathbb{M}_-^{11}$.

3. The canal surfaces of type $\mathbb{M}_-^{11}$

From Eq (2.1), the canal surface $\mathbb{M}_-^{11}$ is parameterized by

$$x(s,\theta) = c(s) + r(s)(\sinh\varphi(s)T + \cosh\varphi(s)\sinh\theta N + \cosh\varphi(s)\cosh\theta B),$$

where $\sinh\varphi(s) = r'(s)$.

Through direct calculations, we have

$$x_s = x_s^1 T + x_s^2 N + x_s^3 B,\ x_\theta = x_\theta^1 N + x_\theta^2 B,$$

where

$$\begin{aligned} x_s^1 &= rr'' - r\kappa\cosh\varphi\sinh\theta + \cosh^2\varphi, \\ x_s^2 &= r'\cosh\varphi\sinh\theta + rr'\kappa + rr'\varphi'\sinh\theta + r\tau\cosh\varphi\cosh\theta, \\ x_s^3 &= r'\cosh\varphi\cosh\theta + r\tau\cosh\varphi\sinh\theta + rr'\varphi'\cosh\theta, \\ x_\theta^1 &= r\cosh\varphi\cosh\theta, \\ x_\theta^2 &= r\cosh\varphi\sinh\theta. \end{aligned} \tag{3.1}$$

Thus, the Gauss map $\mathbb{G}$ of $\mathbb{M}_-^{11}$ is

$$\mathbb{G} = \sinh\varphi T + \cosh\varphi\sinh\theta N + \cosh\varphi\cosh\theta B, \tag{3.2}$$

which point outwards $\mathbb{M}_-^{11}$ and $\langle\mathbb{G},\mathbb{G}\rangle = -1$.

Meanwhile, the component functions of the first fundamental form are obtained as

$$g_{11} = \frac{P_1^2 + r^2R_1^2}{\cosh^2\varphi},\ g_{12} = r^2R_1,\ g_{22} = r^2\cosh^2\varphi, \tag{3.3}$$

the component functions of the second fundamental form are written by

$$h_{11} = -\frac{rR_1^2 + P_1Q_1}{\cosh^2\varphi},\ h_{12} = -rR_1,\ h_{22} = -r\cosh^2\varphi, \tag{3.4}$$

where

$$\begin{aligned} P_1 &= rr'' + \cosh^2\varphi - r\kappa\cosh\varphi\sinh\theta = rQ_1 + \cosh^2\varphi, \\ Q_1 &= r'' - \kappa\cosh\varphi\sinh\theta, \\ R_1 &= r'\kappa\cosh\varphi\cosh\theta + \tau\cosh^2\varphi. \end{aligned} \tag{3.5}$$

From Eqs (3.3) and (3.4), the Gaussian curvature K and the mean curvature H of $\mathbb{M}_{-}^{11}$ are

$$K = -\frac{Q_1}{rP_1},\ H = \frac{2P_1 - \cosh^2\varphi}{2rP_1}. \tag{3.6}$$

Remark 3.1. *From $g_{11}g_{22} - g_{12}^2 = r^2P_1^2$, due to regularity, we see that $P_1 \neq 0$ everywhere.*

Serving the following discussion, the Laplacian of the Gauss map $\mathbb{G}$ of $\mathbb{M}_{-}^{11}$ need to be calculated. First, from the first fundamental form of $\mathbb{M}_{-}^{11}$, we have

$$g^{11} = \frac{\cosh^2\varphi}{P_1^2},\ g^{12} = -\frac{R_1}{P_1^2},\ g^{22} = \frac{P_1^2 + r^2R_1^2}{r^2P_1^2\cosh^2\varphi}. \tag{3.7}$$

Substituting (3.2), (3.3) and (3.7) into (1.1), and by putting

$$U_1 = g_{22}H_s - g_{12}H_\theta,\ V_1 = -g_{12}H_s + g_{11}H_\theta, \tag{3.8}$$

where

$$\begin{aligned} H_s =& \frac{5r^2r'r''\kappa\cosh\varphi\sinh\theta - 2r^2r'\kappa^2\cosh^2\varphi\sinh^2\theta + r^2r'''\cosh^2\varphi - 2rr'r''\cosh^2\varphi - 4r^2r'r''^2}{2r^2P_1^2} + \\ & \frac{(2rr'\kappa - r^2\kappa')\cosh^3\varphi\sinh\theta - r'\cosh^4\varphi}{2r^2P_1^2}, \\ H_\theta =& -\frac{\kappa\cosh^3\varphi\cosh\theta}{2P_1^2}, \end{aligned} \tag{3.9}$$

after tedious tidying up, we get

$$\begin{aligned} \Delta\mathbb{G} = -&\frac{1}{r^2P_1^2}\{[(r^2Q_1^2 + P_1^2)\sinh\varphi - 2x_s^1U_1]T + [(r^2Q_1^2 + P_1^2)\cosh\varphi\sinh\theta - 2(x_s^2U_1 + x_\theta^1V_1)]N + \\ & [(r^2Q_1^2 + P_1^2)\cosh\varphi\cosh\theta - 2(x_s^3U_1 + x_\theta^2V_1)]B\}. \end{aligned} \tag{3.10}$$

Assume a canal surface $\mathbb{M}_{-}^{11}$ satisfies $\Delta\mathbb{G} = f(\mathbb{G} + C)$. Without loss of generality, we may suppose

$$C = C_1T + C_2N + C_3B, \tag{3.11}$$

where $C_1 = \langle C, T\rangle, C_2 = \langle C, N\rangle, C_3 = -\langle C, B\rangle$.

Substituting (3.2), (3.10) and (3.11) into $\Delta\mathbb{G} = f(\mathbb{G} + C)$, we obtain the following equation system

$$\begin{cases} (r^2Q_1^2 + P_1^2)\sinh\varphi - 2x_s^1U_1 = -r^2P_1^2(\sinh\varphi + C_1)f, \\ (r^2Q_1^2 + P_1^2)\cosh\varphi\sinh\theta - 2(x_s^2U_1 + x_\theta^1V_1) = -r^2P_1^2(\cosh\varphi\sinh\theta + C_2)f, \\ (r^2Q_1^2 + P_1^2)\cosh\varphi\cosh\theta - 2(x_s^3U_1 + x_\theta^2V_1) = -r^2P_1^2(\cosh\varphi\cosh\theta + C_3)f. \end{cases} \tag{3.12}$$

From the last two equations of (3.12), we have

$$\begin{aligned}&2U_1(r\tau\cosh^2\varphi + rr'\kappa\cosh\varphi\cosh\theta - x_s^3C_2 + x_s^2C_3) + 2V_1(r\cosh^2\varphi - x_\theta^2C_2 + x_\theta^1C_3)\\&=\cosh\varphi(2P_1^2 - 2P_1\cosh^2\varphi + \cosh^4\varphi)(C_3\sinh\theta - C_2\cosh\theta).\end{aligned}\tag{3.13}$$

Rearranging (3.13) with the help of (3.1), (3.5) and (3.8), we get

$$\begin{aligned}&2(g_{22}H_s - g_{12}H_\theta)(r\tau\cosh^2\varphi + rr'\kappa\cosh\varphi\cosh\theta - x_s^3C_2 + x_s^2C_3) + 2(-g_{12}H_s + g_{11}H_\theta)\\&(r\cosh^2\varphi - r\cosh\varphi\sinh\theta C_2 + r\cosh\varphi\cosh\theta C_3)\\&=\cosh\varphi[(rr'' - r\kappa\cosh\varphi\sinh\theta)^2 + (rr'' + \cosh^2\varphi - r\kappa\cosh\varphi\sinh\theta)^2](C_3\sinh\theta - C_2\cosh\theta).\end{aligned}\tag{3.14}$$

Since $\{\cosh(m\theta), \sinh(m\theta)|m \in \mathbb{N}\}$ constructs a set of linearly independent functions, in view of the coefficients of $\sinh 5\theta$ and $\cosh 5\theta$ in (3.14) by the aid of (3.1), (3.3) and (3.9), we have

$$\begin{cases} r^6\kappa^4\cosh^5\varphi C_3 = 0,\\ r^6\kappa^4\cosh^5\varphi C_2 = 0.\end{cases}\tag{3.15}$$

From (3.15), we consider a non-empty open subset $O = \{p \in \mathbb{M}_-^{11} \mid \kappa(p) \neq 0\}$ of $\mathbb{M}_-^{11}$. Since $r \neq 0$, $\cosh\varphi \neq 0$, then we have $C_2 = C_3 = 0$ on O. However, if $C_2 = C_3 = 0$, (3.14) gives

$$-r^2R_1^2 + g_{11}\cosh^2\varphi = P_1^2 = 0$$

which contradicts to the regularity of $\mathbb{M}_-^{11}$. Therefore, $\kappa \equiv 0$, $\mathbb{M}_-^{11}$ is a surface of revolution.

Let $c(s) = (s, 0, 0)$ and $T = (1, 0, 0), N = (0, 1, 0), B = (0, 0, 1)$, then $\mathbb{M}_-^{11}$ can be represented as

$$x(s,\theta) = (s + r(s)\sinh\varphi, r(s)\cosh\varphi\sinh\theta, r(s)\cosh\varphi\cosh\theta).$$

Furthermore, when $\kappa = 0$, the first equation of (3.12) gives

$$f = \frac{2r^2\cosh^2\varphi P_1H_s - (2P_1^2 - 2P_1\cosh^2\varphi + \cosh^4\varphi)\sinh\varphi}{r^2P_1^2(\sinh\varphi + C_1)}.\tag{3.16}$$

Because P_1, H_s are all functions of s when $\kappa = 0$, Equation (3.16) yields $f = f(s)$. Then, by the last two equations of (3.12), we have

$$\begin{cases}(r^2Q_1^2 + P_1^2)\cosh\varphi\sinh\theta - 2r^2r'\cosh^2\varphi H_s(\cosh\varphi + r\varphi')\sinh\theta = -r^2P_1^2(\cosh\varphi\sinh\theta + C_2)f,\\(r^2Q_1^2 + P_1^2)\cosh\varphi\cosh\theta - 2r^2r'\cosh^2\varphi H_s(\cosh\varphi + r\varphi')\cosh\theta = -r^2P_1^2(\cosh\varphi\cosh\theta + C_3)f.\end{cases}\tag{3.17}$$

Because $r \neq 0$, $P_1 \neq 0$ and $f = f(s)$, Equation system (3.17) implies $C_2 = C_3 = 0$. And

$$f(s) = \frac{2r^2H_sP_1\sinh\varphi - (2P_1^2 - 2P_1\cosh^2\varphi + \cosh^4\varphi)}{r^2P_1^2}.\tag{3.18}$$

Combining (3.16) and (3.18), we get

$$C_1(2P_1^2 - 2P_1\cosh^2\varphi + \cosh^4\varphi) + 2r^2H_sP_1(1 - C_1r') = 0,\tag{3.19}$$

substituting (3.19) into (3.18), we have

$$f(s) = \frac{2H_s}{C_1 P_1} = \frac{4H^2 + 2K}{C_1 r' - 1}. \tag{3.20}$$

Considering the principal curvatures are given by

$$\kappa_1 = -\frac{r''}{rr'' + r'^2 + 1}, \ \kappa_2 = -\frac{1}{r} \tag{3.21}$$

when $\kappa = 0$, thus the Gaussian curvature K and the mean curvature H are

$$K = -\frac{r''}{r(rr'' + r'^2 + 1)}, \ H = \frac{2rr'' + r'^2 + 1}{2r(rr'' + r'^2 + 1)}. \tag{3.22}$$

Due to $f \neq 0$, the mean curvature cannot be constant. With the help of (3.6), Equation (3.19) can be rewritten as

$$2r^2 H_s (1 - rH)(C_1 r' - 1) = C_1 (2r^2 H^2 - 2rH + 1)(r'^2 + 1). \tag{3.23}$$

Simplifying Eq (3.23) with the help of (3.22), the radial function $r(s)$ satisfies

$$\kappa_1 \left(\kappa_1 - \frac{1}{r}\right)' = (\ln |C_1 r' - 1|)' \left(\kappa_1^2 + \frac{1}{r^2}\right), \tag{3.24}$$

where κ_1 is stated as (3.21).

Conversely, if $\mathbb{M}_-^{11}$ is a surface of revolution satisfying (3.24), $\Delta\mathbb{G} = f(\mathbb{G} + C)$ can be satisfied for a non-zero function f as stated by (3.20) and a constant vector $C = (C_1, 0, 0)$ in which C_1 is a non-zero constant.

Theorem 3.2. *A canal surface $\mathbb{M}_-^{11}$ has proper pointwise 1-type Gauss map of the second kind if and only if it is a surface of revolution with the following form*

$$x(s, \theta) = (s + r(s)\sinh\varphi, r(s)\cosh\varphi\sinh\theta, r(s)\cosh\varphi\cosh\theta),$$

where $r(s)$ satisfies (3.24).

Corollary 3.3. *A canal surface $\mathbb{M}_-^{11}$ with proper pointwise 1-type Gauss map of the second kind satisfies $\Delta\mathbb{G} = f(\mathbb{G} + C)$ for a constant vector $C = (C_1, 0, 0)$ and non-zero smooth function*

$$f(s) = \frac{4H^2 + 2K}{C_1 r' - 1},$$

where H and K are given by (3.22), C_1 is a non-zero constant.

Corollary 3.4. *A canal surface $\mathbb{M}_-^{11}$ has 1-type Gauss map of the second kind if and only if it is a surface of revolution represented as*

$$x(s, \theta) = (s + r(s)\sinh\varphi, r(s)\cosh\varphi\sinh\theta, r(s)\cosh\varphi\cosh\theta),$$

where $r(s)$ satisfies (3.26).

Proof of Corollary 3.4. When a canal surface $\mathbb{M}_{-}^{11}$ satisfies $\Delta\mathbb{G} = \lambda(\mathbb{G} + C)$, $(\lambda \in \mathbb{R} - \{0\}, C \neq 0)$, by Theorem 3.2, $\mathbb{M}_{-}^{11}$ is a surface of revolution satisfying (3.24). By Corollary 3.3, we get

$$\kappa_1 \left(\kappa_1 - \frac{1}{r}\right)' = \lambda C_1 r''. \tag{3.25}$$

From (3.24) and (3.25), we have

$$\kappa_1^2 + \frac{1}{r^2} = \lambda(C_1 r' - 1), \tag{3.26}$$

where λ and C_1 are non-zero constants, κ_1 is stated as (3.21). The converse is straightforward. □

Theorem 3.5. *A canal surface $\mathbb{M}_{-}^{11}$ has proper pointwise 1-type Gauss map of the first kind if and only if it is minimal. Precisely, it is a part of a surface of revolution as*

$$x(s,\theta) = (s + r(s)\sinh\varphi, r(s)\cosh\varphi\sinh\theta, r(s)\cosh\varphi\cosh\theta),$$

where $r(s)$ satisfies (3.29).

Proof of Theorem 3.5. A canal surface $\mathbb{M}_{-}^{11}$ has proper pointwise 1-type Gauss map of the first kind, i.e., $\Delta\mathbb{G} = f\mathbb{G}$ for a smooth function f. From Equation (3.10), we have

$$\begin{cases} (r^2Q_1^2 + P_1^2)\sinh\varphi - 2x_s^1U_1 = -r^2P_1^2\sinh\varphi f, \\ (r^2Q_1^2 + P_1^2)\cosh\varphi\sinh\theta - 2(x_s^2U_1 + x_\theta^1V_1) = -r^2P_1^2\cosh\varphi\sinh\theta f, \\ (r^2Q_1^2 + P_1^2)\cosh\varphi\cosh\theta - 2(x_s^3U_1 + x_\theta^2V_1) = -r^2P_1^2\cosh\varphi\cosh\theta f. \end{cases} \tag{3.27}$$

From the last two equations of (3.27), we get

$$(g_{22}H_s - g_{12}H_\theta)(\tau\cosh^2\varphi + r'\kappa\cosh\varphi\cosh\theta) + (-g_{12}H_s + g_{11}H_\theta)\cosh^2\varphi = 0. \tag{3.28}$$

With the help of Eq (3.3), we obtain $P_1^2H_\theta = 0$. Therefore, $H_\theta = 0$ due to $P_1 \neq 0$. Furthermore, from the first two equations of (3.27), we get $2r^2P_1H_s = 0$. It is obvious that $H_s = 0$. Then, the mean curvature of $\mathbb{M}_{-}^{11}$ is constant. By the Corollary 2 of [9], i.e., the canal surface $\mathbb{M}_{-}^{11}$ with non-zero constant mean curvature does not exist, thus the canal surface $\mathbb{M}_{-}^{11}$ is minimal. From the Theorem 4 of [9], it is a part of a surface of revolution with the following form

$$x(s,\theta) = (s + r(s)\sinh\varphi, r(s)\cosh\varphi\sinh\theta, r(s)\cosh\varphi\cosh\theta),$$

where $r(s)$ satisfies

$$s = c_2 \pm \int \sqrt{\frac{r}{c_1 - r}}dr, \ (c_1 > r, c_2 \in \mathbb{R}). \tag{3.29}$$

Looking back the Eq (3.27) with the conclusions obtained above, we have

$$f(s) = -2K = -\frac{2}{r^2}. \tag{3.30}$$

Conversely, suppose that $\mathbb{M}_{-}^{11}$ is a surface of revolution satisfying (3.29), $\mathbb{M}_{-}^{11}$ is minimal from the Theorem 4 of [9] and $\Delta\mathbb{G} = f\mathbb{G}$ is satisfied for a non-zero function f given by (3.30). □

Corollary 3.6. *A canal surface $\mathbb{M}_{-}^{11}$ with proper pointwise 1-type Gauss map of the first kind satisfies*

$$\Delta\mathbb{G} = -2K\mathbb{G} = -\frac{2}{r^2}\mathbb{G}.$$

Assume that a canal surface $\mathbb{M}_{-}^{11}$ satisfies $\Delta\mathbb{G} = \lambda\mathbb{G}$, $(\lambda \in \mathbb{R} - \{0\})$. By Corollary 3.6, we have $\lambda = -\frac{2}{r^2}$ is a constant, i.e., r is a constant. Thus, we have the following result.

Corollary 3.7. *A canal surface $\mathbb{M}_{-}^{11}$ has 1-type Gauss map of the first kind if and only if it is a circular cylinder.*

From Corollary 3.6, the following conclusion is straightforward since $-\frac{2}{r^2} \neq 0$.

Corollary 3.8. *The canal surface $\mathbb{M}_{-}^{11}$ with harmonic Gauss map does not exist.*

4. The canal surfaces of type $\mathbb{M}_{-}^{12}$

From Eq (2.2), the canal surface $\mathbb{M}_{-}^{12}$ is parameterized by

$$x(s,\theta) = c(s) + r(s)(\sinh\varphi(s)T + \cosh\varphi(s)\cosh\theta N + \cosh\varphi(s)\sinh\theta B),$$

where $\sinh\varphi(s) = r'(s)$.

Through direct calculations, we have

$$x_s = x_s^1 T + x_s^2 N + x_s^3 B,\ x_\theta = x_\theta^1 N + x_\theta^2 B,$$

where

$$\begin{aligned} x_s^1 &= rr'' + \cosh^2\varphi + r\kappa\cosh\varphi\cosh\theta; \\ x_s^2 &= r'\cosh\varphi\cosh\theta + rr'\kappa + rr'\varphi'\cosh\theta + r\tau\cosh\varphi\sinh\theta; \\ x_s^3 &= r'\cosh\varphi\sinh\theta + r\tau\cosh\varphi\cosh\theta + rr'\varphi'\sinh\theta; \\ x_\theta^1 &= r\cosh\varphi\sinh\theta; \\ x_\theta^2 &= r\cosh\varphi\cosh\theta. \end{aligned}$$

Then, the Gauss map $\mathbb{G}$ of $\mathbb{M}_{-}^{12}$ is

$$\mathbb{G} = -\sinh\varphi T - \cosh\varphi\cosh\theta N - \cosh\varphi\sinh\theta B, \tag{4.1}$$

which point outwards $\mathbb{M}_{-}^{12}$ and $\langle\mathbb{G},\mathbb{G}\rangle = -1$.

Meanwhile, the component functions of the first fundamental form are given by

$$g_{11} = \frac{P_2^2 + r^2R_2^2}{\cosh^2\varphi},\ g_{12} = r^2R_2,\ g_{22} = r^2\cosh^2\varphi, \tag{4.2}$$

the component functions of the second fundamental form are written by

$$h_{11} = -\frac{rR_2^2 + P_2Q_2}{\cosh^2\varphi},\ h_{12} = -rR_2,\ h_{22} = -r\cosh^2\varphi, \tag{4.3}$$

where

$$P_2 = rr'' + \cosh^2\varphi + r\kappa\cosh\varphi\cosh\theta = rQ_2 + \cosh^2\varphi,$$
$$Q_2 = r'' + \kappa\cosh\varphi\cosh\theta,$$
$$R_2 = \tau\cosh^2\varphi - r'\kappa\cosh\varphi\sinh\theta.$$

From Eqs (4.2) and (4.3), the Gaussian curvature K and the mean curvature H of $\mathbb{M}_-^{12}$ are

$$K = -\frac{Q_2}{rP_2}, \ \ H = \frac{2P_2 - \cosh^2\varphi}{2rP_2}.$$

Remark 4.1. *From $g_{11}g_{22} - g_{12}^2 = r^2P_2^2$, due to regularity, we see that $P_2 \neq 0$ everywhere.*

Next, we compute the Laplacian of the Gauss map $\mathbb{G}$ of $\mathbb{M}_-^{12}$. First, from the first fundamental form of $\mathbb{M}_-^{12}$, we have

$$g^{11} = \frac{\cosh^2\varphi}{P_2^2}, \ g^{12} = -\frac{R_2}{P_2^2}, \ g^{22} = \frac{r^2R_2^2 + P_2^2}{r^2P_2^2\cosh^2\varphi}. \tag{4.4}$$

Substituting (4.1), (4.2) and (4.4) into (1.1), and by putting

$$U_2 = g_{22}H_s - g_{12}H_\theta, \ V_2 = -g_{12}H_s + g_{11}H_\theta,$$

where

$$H_s = \frac{2rr'r''\cosh^2\varphi + r^2r'''\cosh^2\varphi - 4r^2r'r''^2 - 2r^2r'\kappa^2\cosh^2\varphi\cosh^2\theta - 5r^2r'r''\kappa\cosh\varphi\cosh\theta}{2r^2P_2^2} - \frac{(2rr'\kappa - r^2\kappa')\cosh^3\varphi\cosh\theta + r'\cosh^4\varphi}{2r^2P_2^2},$$

$$H_\theta = \frac{\kappa\cosh^3\varphi\sinh\theta}{2P_2^2},$$

after tedious tidying up, we get

$$\Delta\mathbb{G} = \frac{1}{r^2P_2^2}\{[(r^2Q_2^2 + P_2^2)\sinh\varphi + 2x_s^1U_2]T + [(r^2Q_2^2 + P_2^2)\cosh\varphi\cosh\theta + 2(x_s^2U_2 + x_\theta^1V_2)]N + [(r^2Q_2^2 + P_2^2)\cosh\varphi\sinh\theta + 2(x_s^3U_2 + x_\theta^2V_2)]B\}.$$

Do discussions similar to those of $\mathbb{M}_-^{11}$, we have the following conclusions directly.

Theorem 4.2. *A canal surface $\mathbb{M}_-^{12}$ has proper pointwise 1-type Gauss map of the second kind if and only if it is a surface of revolution with the following form*

$$x(s,\theta) = (r(s)\sinh\varphi + s, r(s)\cosh\varphi\sinh\theta, r(s)\cosh\varphi\cosh\theta),$$

in which $r(s)$ satisfies

$$\kappa_1\left(\kappa_1 - \frac{1}{r}\right)' = -(\ln|1 + C_1r'|)'\left(\kappa_1^2 + \frac{1}{r^2}\right),$$

where C_1 is a non-zero constant and κ_1 is the principal curvature as

$$\kappa_1 = -\frac{r''}{rr'' + r'^2 + 1}.$$

Corollary 4.3. *A canal surface $\mathbb{M}_-^{12}$ with proper pointwise 1-type Gauss map of the second kind satisfies $\Delta\mathbb{G} = f(\mathbb{G} + C)$ for a constant vector $C = (C_1, 0, 0)$ and non-zero smooth function*

$$f(s) = -\frac{4H^2 + 2K}{1 + C_1 r'},$$

where C_1 is a non-zero constant, H and K are given by

$$K = -\frac{r''}{r(rr'' + r'^2 + 1)},\ H = \frac{2rr'' + r'^2 + 1}{2r(rr'' + r'^2 + 1)}.$$

Corollary 4.4. *A canal surface $\mathbb{M}_-^{12}$ has 1-type Gauss map of the second kind if and only if it is a surface of revolution represented as*

$$x(s, \theta) =(s + r(s)\sinh\varphi, r(s)\cosh\varphi\sinh\theta, r(s)\cosh\varphi\cosh\theta),$$

in which $r(s)$ satisfies

$$\kappa_1^2 + \frac{1}{r^2} = -\lambda(1 + C_1 r'),$$

where λ and C_1 are non-zero constants and

$$\kappa_1 = -\frac{r''}{rr'' + r'^2 + 1}.$$

Theorem 4.5. *A canal surface $\mathbb{M}_-^{12}$ has proper pointwise 1-type Gauss map of the first kind if and only if it is minimal. Precisely, it is a part of a surface of revolution as*

$$x(s, \theta) =(s + r(s)\sinh\varphi(s), r(s)\cosh\varphi(s)\sinh\theta, r(s)\cosh\varphi(s)\cosh\theta),$$

in which $r(s)$ satisfies

$$s = c_2 \pm \int \sqrt{\frac{r}{c_1 - r}}dr,\ (c_1 > r, c_2 \in \mathbb{R}).$$

Corollary 4.6. *A canal surface $\mathbb{M}_-^{12}$ with proper pointwise 1-type Gauss map of the first kind satisfies*

$$\Delta\mathbb{G} = -2K\mathbb{G} = -\frac{2}{r^2}\mathbb{G}.$$

Corollary 4.7. *A canal surface $\mathbb{M}_-^{12}$ has 1-type Gauss map of the first kind if and only if it is a circular cylinder.*

Corollary 4.8. *The canal surface $\mathbb{M}_-^{12}$ with harmonic Gauss map does not exist.*

5. The canal surfaces of type $\mathbb{M}_-^2$

From Eq (2.3), the canal surface $\mathbb{M}_-^2$ is parameterized by

$$x(s, \theta) =c(s) + r(s)(\cosh\varphi(s)T + \sinh\varphi(s)\cos\theta N + \sinh\varphi(s)\sin\theta B),$$

where $-\cosh\varphi(s) = r'(s)$.

Through direct calculations, we have

$$x_s = x_s^1 T + x_s^2 N + x_s^3 B,\ x_\theta = x_\theta^1 N + x_\theta^2 B,$$

where

$$\begin{aligned}
x_s^1 &= -rr'' + r\kappa \sinh\varphi\cos\theta - \sinh^2\varphi;\\
x_s^2 &= r'\sinh\varphi\cos\theta - rr'\kappa - rr'\varphi'\cos\theta + r\tau\sinh\varphi\sin\theta;\\
x_s^3 &= r'\sinh\varphi\sin\theta - r\tau\sinh\varphi\cos\theta - rr'\varphi'\sin\theta;\\
x_\theta^1 &= -r\sinh\varphi\sin\theta;\\
x_\theta^2 &= r\sinh\varphi\cos\theta.
\end{aligned}$$

Then, the Gauss map $\mathbb{G}$ of $\mathbb{M}_-^2$ is

$$\mathbb{G} = -\cosh\varphi T - \sinh\varphi\cos\theta N - \sinh\varphi\sin\theta B, \tag{5.1}$$

which point outwards $\mathbb{M}_-^2$ and $\langle\mathbb{G},\mathbb{G}\rangle = -1$.

Meanwhile, the component functions of the first fundamental form are obtained as

$$g_{11} = \frac{P_3^2 + r^2R_3^2}{\sinh^2\varphi},\ g_{12} = r^2R_3,\ g_{22} = r^2\sinh^2\varphi, \tag{5.2}$$

the component functions of the second fundamental form are written by

$$h_{11} = -\frac{rR_3^2 + P_3Q_3}{\sinh^2\varphi},\ h_{12} = -rR_3,\ h_{22} = -r\sinh^2\varphi, \tag{5.3}$$

where

$$\begin{aligned}
P_3 &= rr'' - r\kappa\sinh\varphi\cos\theta + \sinh^2\varphi = rQ_3 + \sinh^2\varphi,\\
Q_3 &= r'' - \kappa\sinh\varphi\cos\theta,\\
R_3 &= \tau\sinh^2\varphi + r'\kappa\sinh\varphi\sin\theta.
\end{aligned}$$

From Eqs (5.2) and (5.3), the Gaussian curvature K and the mean curvature H of $\mathbb{M}_-^2$ are

$$K = -\frac{Q_3}{rP_3},\ \ H = \frac{2P_3 - \sinh^2\varphi}{2rP_3}. \tag{5.4}$$

Remark 5.1. *From $g_{11}g_{22} - g_{12}^2 = r^2P_3^2$, due to regularity, we see that $P_3 \neq 0$ everywhere.*

In the following, the Laplacian of the Gauss map $\mathbb{G}$ of $\mathbb{M}_-^2$ is to be calculated. First, from the first fundamental form of $\mathbb{M}_-^2$, we have

$$g^{11} = \frac{\sinh^2\varphi}{P_3^2},\ g^{12} = -\frac{R_3}{P_3^2},\ g^{22} = \frac{P_3^2 + r^2R_3^2}{r^2P_3^2\sinh^2\varphi}. \tag{5.5}$$

Substituting (5.1), (5.2) and (5.5) into (1.1), and by putting

$$U_3 = g_{22}H_s - g_{12}H_\theta,\ V_3 = -g_{12}H_s + g_{11}H_\theta,$$

where

$$H_s = \frac{-2r^2 r'\kappa^2 \sinh^2\varphi\cos^2\theta + r^2 r' r''\kappa\sinh\varphi\cos\theta - 2rr'r''\sinh^2\varphi + r^2 r'''\sinh^2\varphi - 4r^2 r' r''^2}{2r^2 P_3^2} + \frac{(2rr'\kappa - r^2\kappa')\sinh^3\varphi\cos\theta - r'\sinh^4\varphi}{2r^2 P_3^2},$$

$$H_\theta = \frac{\kappa\sinh^3\varphi\sin\theta}{2P_3^2},$$

after complicated arrangements, we get

$$\Delta\mathbb{G} = \frac{1}{r^2 P_3^2}\{[(r^2 Q_3^2 + P_3^2)\cosh\varphi + 2x_s^1 U_3]T + [(r^2 Q_3^2 + P_3^2)\sinh\varphi\cos\theta + 2(x_s^2 U_3 + x_\theta^1 V_3)]N + [(r^2 Q_3^2 + P_3^2)\sinh\varphi\sin\theta + 2(x_s^3 U_3 + x_\theta^2 V_3)]B\}.$$

Do discussions similar to those of $\mathbb{M}_-^{11}$ and $\mathbb{M}_-^{12}$, the following results for $\mathbb{M}_-^2$ can be given directly.

Theorem 5.2. *A canal surface $\mathbb{M}_-^2$ has proper pointwise 1-type Gauss map of the second kind if and only if it is a surface of revolution with the following form*

$$x(s,\theta) = (r(s)\sinh\varphi\sin\theta, r(s)\sinh\varphi\cos\theta, s + r(s)\cosh\varphi),$$

in which $r(s)$ satisfies

$$\kappa_1\left(\kappa_1 - \frac{1}{r}\right)' = -(\ln|1 + C_1 r'|)'\left(\kappa_1^2 + \frac{1}{r^2}\right),$$

where C_1 is a non-zero constant and κ_1 is the principal curvature as

$$\kappa_1 = -\frac{r''}{rr'' + r'^2 - 1}.$$

Corollary 5.3. *A canal surface $\mathbb{M}_-^2$ with proper pointwise 1-type Gauss map of the second kind satisfies $\Delta\mathbb{G} = f(\mathbb{G} + C)$ for a constant vector $C = (C_1, 0, 0)$ and non-zero smooth function*

$$f(s) = -\frac{4H^2 + 2K}{1 + C_1 r'},$$

where C_1 is a non-zero constant, H and K are given by

$$K = -\frac{r''}{r(rr'' + r'^2 - 1)},\ H = \frac{2rr'' + r'^2 - 1}{2r(rr'' + r'^2 - 1)}.$$

Corollary 5.4. *A canal surface $\mathbb{M}_-^2$ has 1-type Gauss map of the second kind if and only if it is a surface of revolution represented as*

$$x(s,\theta) = (r(s)\sinh\varphi\sin\theta, r(s)\sinh\varphi\cos\theta, s + r(s)\cosh\varphi),$$

in which $r(s)$ satisfies

$$\kappa_1^2 + \frac{1}{r^2} = -\lambda(1 + C_1 r'),$$

where λ and C_1 are non-zero constants and

$$\kappa_1 = -\frac{r''}{rr'' + r'^2 - 1}.$$

Theorem 5.5. *A canal surface $\mathbb{M}^2_-$ has proper pointwise 1-type Gauss map of the first kind if and only if it is minimal. Precisely, it is a part of a surface of revolution as*

$$x(s,\theta) = (r(s)\sinh\varphi\sin\theta, r(s)\sinh\varphi\cos\theta, s + r(s)\cosh\varphi),$$

in which $r(s)$ satisfies

$$s = c_2 \pm \int \sqrt{\frac{r}{c_1 + r}}dr, \ (c_1 > -r, c_2 \in \mathbb{R}).$$

Corollary 5.6. *A canal surface $\mathbb{M}^2_-$ with proper pointwise 1-type Gauss map of the first kind satisfies*

$$\Delta\mathbb{G} = -2K\mathbb{G} = -\frac{2}{r^2}\mathbb{G}.$$

Corollary 5.7. *The canal surface $\mathbb{M}^2_-$ with 1-type Gauss map of the first kind does not exist.*

Proof. Assume that a canal surface $\mathbb{M}^2_-$ satisfies $\Delta\mathbb{G} = \lambda\mathbb{G}$, $(\lambda \in \mathbb{R} - \{0\})$. By Corollary 5.6, we have $\lambda = -\frac{2}{r^2}$ is a constant, i.e., r is a constant. Notice that the radial function $r(s)$ can't be constant for $\mathbb{M}^2_-$, therefore, we get contradiction. □

Corollary 5.8. *The canal surface $\mathbb{M}^2_-$ with harmonic Gauss map does not exist.*

6. Conclusions

Until now, the canal surfaces $\mathbb{M}^{11}_-$, $\mathbb{M}^{12}_-$ and $\mathbb{M}^2_-$ foliated by pseudo hyperbolic spheres $\mathbb{H}^2_0$ along the first kind spacelike curve, the second kind spacelike curve and a timelike curve, respectively have been classified in terms of their Gauss maps. The similar works for the canal surfaces $\mathbb{M}^{11}_+$, $\mathbb{M}^{12}_+$ and $\mathbb{M}^2_+$ have been done in another recent work. The canal surfaces $\mathbb{M}^{13}_-$, $\mathbb{M}^3_-$ ($\mathbb{M}^{13}_+$, $\mathbb{M}^3_+$) foliated by pseudo hyperbolic spheres $\mathbb{H}^2_0$ (resp. pseudo spheres $\mathbb{S}^2_1$) along a null type spacelike curve or a null curve are to be investigated in the continued works.

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Conflict of interest

The authors declare that there is no conflict of interest.

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