Research article

Fixed points of nonlinear contractions with applications

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Abstract: The aim of this paper is to initiate a new concept of nonlinear contraction under the name \( r \)-hybrid \( \psi \)-contraction and establish some fixed point results for such mappings in the setting of complete metric spaces. The presented ideas herein unify and extend a number of well-known results in the corresponding literature. A few of these special cases are pointed out and analysed. From application point of view, we investigate the existence and uniqueness criteria of solutions to certain functional equation arising in dynamic programming and integral equation of Volterra type. Nontrivial illustrative examples are provided to show the generality and validity of our obtained results.

Keywords: fixed point; \( \psi \)-contraction; \( r \)-hybrid \( \psi \)-contraction; dynamic programming; integral equation

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1. Introduction

Fixed point\((f_p)\) theory is the epicenter of modern functional analysis with interesting applications in the study of various significant nonlinear phenomena, including convex optimization and minimization [1, 2], variational inequalities [3], fractional calculus [4–8], homotopy perturbation theory [9, 10], analytical chemistry [11], integral inequalities [12–16], Nash equilibrium problems as well as in network bandwidth allocation [17]. In \( f_p \) theory, the contractive conditions on underlying mappings play an important role in finding solutions of \( f_p \) problems. The Banach contraction
principle \((BC_p)\) [18] is one of the most known applicable results on \(f_p\) of contraction mappings. This highly celebrated theorem \((th \text{r}_m)\), which is an essential tool in several areas of mathematical analysis, surfaced in 1922 in Banach thesis. Due to its usefulness and simplicity, many authors have come up with diverse extensions of the \((BC_p)\) (e.g. [19–21]). In 2012, Wardowski [22] brought up a notion of contraction, called \(\psi\)-contraction and coined a \(f_p\) \(th\text{r}_m\) which refined the \((BC_p)\). Wardowski and Van Dung [23] initiated the idea of \(\psi\)-weak contraction and obtained a refinement of \(\psi\)-contraction. In [24], Secelean opined that condition \((\psi_2)\) in Wardowski’s definition of \(\psi\)-contraction can be replaced with an equivalent and subtle one given by \((\psi_2')\) : \(\inf \psi = -\infty\). Piri and Kumam [25] launched a variant of Wardowski’s \(th\text{r}_m\) by using the condition \((\psi_2)\). Cosentino and Vetro [26] toed the direction of \(\psi\)-contraction and proved \(f_p\) results of Hardy-Rogers-type. On the other hand, one of the active subfields of \(f_p\) theory that is also presently attracting the foci of investigators is the examination of hybrid contractions. The idea has been shaped in two lanes, viz. first, hybrid contraction deals with those contractions involving both single-valued and multi-valued mappings [mpn] and the second combines both linear and nonlinear contractions. For some articles in this direction, we refer [27–29]. Recently, Karapinar et al. [30] launched the notion of \(p\)-hybrid Wardowski contractions. Their results unified and extended several known fixed point theorems due to Wardowski [22], and related results. For other modifications of \(\psi\)-contractions and related fixed point theorems, the reader may consult [31–36].

The focus of this article is to bring up a notion called \(r\)-hybrid \(\psi\)-contraction and establish novel \(f_p, th\text{r}_m\) in the realm of complete metric space. Our results include as special cases, the \(f_p, th\text{r}_m\) due to Wardowski [22], Cosentino and Vetro [26], Karapinar [19], Reich [21], and a few others in the corresponding literature. A nontrivial example is provided to indicate the generality of our ideas herein. Moreover, two applications of certain functional \(eq_n\) arising in dynamic programming and integral \(eq_n\) of Volterra type are provided to show possible usability of our results.

2. Preliminaries

In this section, a handful concepts and results needed in the sequel are recalled. Throughout the article, denote by \(\mathbb{R}, \mathbb{R}_+\) and \(\mathbb{N}\) are the set of real numbers, nonnegative reals and the set of natural numbers, respectively. Moreover, we denote a metric space and a complete metric space by \(M\) and \(CM\), respectively.

\(U_\psi\) represents the family of functions \((fnx) \psi : \mathbb{R}_+ \rightarrow \mathbb{R}\):

\((\psi_1)\) \(\psi\) is strictly increasing, that is, for all \(h, \varrho \in (0, \infty)\), if \(h < \varrho\) then, \(\psi(h) < \psi(\varrho)\);

\((\psi_2)\) for every sequence \((seq)\) \(\{h_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+\), \(\lim_{n \rightarrow \infty} h_n = 0\) if and only if \(\lim_{n \rightarrow \infty} \psi(h_n) = -\infty\);

\((\psi_3)\) there exists \(\xi \in (0, 1)\) such that \(\lim_{n \rightarrow \infty} h^n \psi(h) = 0\).

**Definition 2.1.** [22] Let \((\Upsilon, \mu)\) be a \(M\). A mapping \((mpn)\) \(\Upsilon \rightarrow \Upsilon\) is called a \(\psi\)-contraction if there exist \(\sigma > 0\) and a \(fnx\) \(\psi \in U_\psi\) such that for all \(\varsigma, \zeta \in \Upsilon, \mu(\Upsilon \varsigma, \Upsilon \zeta) > 0\) implies

\[\sigma + \psi(\mu(\Upsilon \varsigma, \Upsilon \zeta)) \leq \psi(\mu(\varsigma, \zeta)).\]  

**Example 2.2.** [22] Let \(\psi : \mathbb{R}_+ \rightarrow \mathbb{R}\) be defined by \(\psi(h) = \ln h, h > 0\). Clearly, \(\psi\) satisfies \((\psi_1) - (\psi_3)\). Each \(mpn\) \(\Upsilon \rightarrow \Upsilon\) satisfying (2.1) is a \(\psi\)-contraction such that for all \(\varsigma, \zeta \in \Upsilon\) with \(\Upsilon \varsigma \neq \Upsilon \zeta\),

\[\mu(\Upsilon \varsigma, \Upsilon \zeta) \leq e^{-\sigma} \mu(\varsigma, \zeta).\]
It is obvious that for all $\zeta, \xi \in \Upsilon$ such that $\mathcal{F}_\zeta = \mathcal{F}_\xi$, the inequality (2.2) also holds; that is, $\mathcal{F}$ is a $BC_p$.

**Example 2.3.** [22] Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be defined by $\psi(h) = \ln h + h$, $h > 0$, then $\psi$ satisfies $(\psi_1) - (\psi_3)$. Therefore, from Condition (2.1), the mpn $\mathcal{F} : \Upsilon \rightarrow \Upsilon$ is of the form

$$\frac{\mu(\mathcal{F}_\zeta, \mathcal{F}_\xi)}{\mu(\zeta, \xi)} e^{\mu(\mathcal{F}_\zeta, \mathcal{F}_\xi) - \mu(\zeta, \xi)} \leq e^{-\sigma},$$

for all $\zeta, \xi \in \Upsilon$, $\mathcal{F}_\zeta \neq \mathcal{F}_\xi$.

**Remark 1.** From $(\psi_1)$ and (2.1), it is easy to see that if $\mathcal{F}$ is a $\psi$-contraction, then $\mu(\mathcal{F}_\zeta, \mathcal{F}_\xi) < \mu(\zeta, \xi)$ for all $\zeta, \xi \in \Upsilon$ such that $\mathcal{F}_\zeta \neq \mathcal{F}_\xi$, that is, $\mathcal{F}$ is a contractive mapping. Hence, every $\psi$-contraction is a continuous mpn.

**Theorem 2.4.** [22] Let $(\Upsilon, \mu)$ be a $CM_p$ and $\mathcal{F} : \Upsilon \rightarrow \Upsilon$ be a $\psi$-contraction. Then, $\mathcal{F}$ has a unique $f_p$, $u \in \Upsilon$, and for each $\zeta \in \Upsilon$, the set $\{\mathcal{F}^n\zeta\}_{n \in \mathbb{N}}$ converges (cvg) to $u$.

We design the set of all $f_p$ of a mpn $\mathcal{F} \psi_{i_\mathcal{F}}(\mathcal{F})$.

**Definition 2.5.** [30] Let $M$ be the family of functions $\psi : (0, \infty) \rightarrow \mathbb{R}$:

$(\psi_a)$ $\psi$ is strictly increasing;

$(\psi_b)$ there exists $\sigma > 0$ such that for every $\varpi_0 > 0$,

$$\sigma + \lim_{\varpi \rightarrow \varpi_0} \inf \psi(\varpi) > \lim_{\varpi \rightarrow \varpi_0} \sup \psi(\varpi).$$

3. **Main results**

In this section, we launch a new form of nonlinear contraction called $r$-hybrid $\psi$-contraction and establish the corresponding $f_p$ results. Let $(\Upsilon, \mu)$ be a metric space and $\mathcal{F} : \Upsilon \rightarrow \Upsilon$ be a single-valued mpn. For $r \geq 0$ and $a_i \geq 0$ ($i = 1, 2, 3, 4, 5$) such that $\sum_{i=1}^{5} a_i = 1$, we define:

$$\Omega_3(\zeta, \xi) = \begin{cases} [G(\zeta, \xi)]^3, & \text{for } r > 0, \xi, \zeta \in \Upsilon, \\ H(\zeta, \xi), & \text{for } r = 0, \zeta, \xi \in \Upsilon \setminus \psi_{i_0}(\mathcal{F}), \end{cases} \quad (3.1)$$

where

$$G(\zeta, \xi) = a_1(\mu(\zeta, \xi))^{\gamma} + a_2(\mu(\zeta, \mathcal{F}_\xi))^{\gamma} + a_3(\mu(\zeta, 3\xi))^{\gamma}$$

$$a_4 \left( \frac{\mu(\xi, \mathcal{F}_\xi)(1 + \mu(\zeta, \mathcal{F}_\xi))}{1 + \mu(\xi, \xi)} \right)^\gamma$$

$$a_5 \left( \frac{\mu(\xi, \mathcal{F}_\xi)(1 + \mu(\zeta, \mathcal{F}_\xi))}{1 + \mu(\xi, \xi)} \right)^\gamma$$

and

$$H(\zeta, \xi)$$

$$= (\mu(\zeta, \xi))^{a_1}(\mu(\zeta, 3\xi))^{a_2}(\mu(\zeta, \mathcal{F}_\xi))^{a_3} \left( \frac{\mu(\xi, \mathcal{F}_\xi)(1 + \mu(\zeta, \mathcal{F}_\xi))}{1 + \mu(\xi, \xi)} \right)^{a_4} \left( \frac{\mu(\xi, \mathcal{F}_\xi) + \mu(\zeta, \mathcal{F}_\xi)}{2} \right)^{a_5}. \quad (3.3)$$
**Definition 3.1.** Let \((\Upsilon, \mu)\) be a \(M_n\). A mpn \(\mathcal{S} : \Upsilon \rightarrow \Upsilon\) is called an \(r\)-hybrid \(\psi\)-contraction if there exist \(\psi \in M\) and \(\sigma > 0\) such that for each \(r > 0\), \(\mu(\mathcal{S}\psi, \mathcal{S}\xi) > 0\) implies

\[
\sigma + \psi(\mu(\mathcal{S}\psi, \mathcal{S}\xi)) \leq \psi(\Omega_3^r(\psi, \xi)).
\]

(3.4)

In particular, if (3.4) holds for \(r = 0\), we say that the mpn \(\mathcal{S}\) is a 0-hybrid \(\psi\)-contraction.

**Remark 2.** Every \(\psi\)-contraction is an \(r\)-hybrid contraction, but the converse is not always true (see Example 3.3). In other words, the class of \(r\)-hybrid \(\psi\)-contractions is richer.

**Theorem 3.2.** Let \((\Upsilon, \mu)\) be a CM, and \(\mathcal{S} : \Upsilon \rightarrow \Upsilon\) be an \(r\)-hybrid \(\psi\)-contraction for \(r > 0\). Then, \(\mathcal{S}\) has a unique \(f_\mu\) in \(\Upsilon\).

**Proof.** Let \(\varsigma_0 \in \Upsilon\) be arbitrary, and rename it as \(\varsigma_0 := \varsigma\). Note that if \(\varsigma_0 = \mathcal{S}\varsigma_0\), the proof is finished. We develop an iterative sequence \(\{\varsigma_n\}_{n \in \mathbb{N}}\) given by \(\varsigma_n = \mathcal{S}\varsigma_{n-1}, n \geq 1\). Without loss of generality, let

\[
0 < \mu(\varsigma_{n+1}, \varsigma_n) = \mu(\mathcal{S}\varsigma_n, \mathcal{S}\varsigma_{n-1}) \text{ if and only if } \varsigma_{n+1} \neq \varsigma_n, n \in \mathbb{N}.
\]

(3.5)

Taking \(\varsigma = \varsigma_{n-1}\) and \(\zeta = \varsigma_n\) in (3.1) with \(r > 0\), we have

\[
\Omega_3^r(\varsigma_{n-1}, \varsigma_n) = [G(\varsigma_{n-1}, \varsigma_n)]^\frac{1}{2} = \left[a_1(\mu(\varsigma_{n-1}, \varsigma_n)) + a_2(\mu(\varsigma_{n-1}, \mathcal{S}\varsigma_{n-1}))\right]^r
\]

\[+ a_3(\mu(\varsigma_n, \mathcal{S}\varsigma_n)) + a_4\left(\frac{\mu(\varsigma_n, \mathcal{S}\varsigma_n)(1 + \mu(\varsigma_{n-1}, \mathcal{S}\varsigma_{n-1}))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^r
\]

\[+ a_5\left(\frac{\mu(\varsigma_n, \varsigma_{n-1})(1 + \mu(\varsigma_{n-1}, \varsigma_{n-1}))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^r\]

\[= \left[a_1(\mu(\varsigma_{n-1}, \varsigma_n))^r + a_2(\mu(\varsigma_{n-1}, \varsigma_n))^r
\]

\[+ a_3(\mu(\varsigma_n, \varsigma_{n+1})) + a_4\left(\frac{\mu(\varsigma_n, \varsigma_{n+1})(1 + \mu(\varsigma_{n-1}, \varsigma_{n+1}))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^r
\]

\[+ a_5\left(\frac{\mu(\varsigma_n, \varsigma_{n+1})(1 + \mu(\varsigma_{n-1}, \varsigma_{n+1}))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^r\]

\[= \left[(a_1 + a_2)(\mu(\varsigma_{n-1}, \varsigma_n))^r + (a_3 + a_4)(\mu(\varsigma_n, \varsigma_{n+1}))^r\right]^\frac{1}{2}.
\]

(3.6)

From (3.4) and (3.6), we have

\[
\sigma + \psi(\mu(\mathcal{S}\varsigma_{n-1}, \mathcal{S}\varsigma_n)) \leq \psi(\Omega_3^r(\varsigma_{n-1}, \varsigma_n)),
\]

that is,

\[
\psi(\mu(\mathcal{S}\varsigma_{n-1}, \mathcal{S}\varsigma_n)) \leq \psi(\Omega_3^r(\varsigma_{n-1}, \varsigma_n)) - \sigma
\]

\[= \psi \left[\left((a_1 + a_2)(\mu(\varsigma_{n-1}, \varsigma_n))^r + (a_3 + a_4)(\mu(\varsigma_n, \varsigma_{n+1}))^r\right)^\frac{1}{2}\right] - \sigma.
\]

(3.7)
Suppose that $\mu(\xi_{n-1}, \xi_n) \leq \mu(\xi_n, \xi_{n+1})$, then, from (3.7),

$$
\psi(\mu(\mathfrak{S}_{n-1}, \mathfrak{S}_n)) \leq \psi\left(\left[(a_1 + a_2 + a_3 + a_4)(\mu(\xi_n, \xi_{n+1}))^2\right]^{\frac{1}{2}} - \sigma\right)
$$

$$
\leq \psi((\mu(\xi_n, \xi_{n+1}))^2 - \sigma)
$$

$$
= \psi(\mu(\mathfrak{S}_{n-1}, \mathfrak{S}_n)) - \sigma
$$

$$
< \psi(\mu(\mathfrak{S}_{n-1}, \mathfrak{S}_n)),
$$

which is invalid. Therefore, $\max\{\mu(\xi_{n-1}, \xi_n), \mu(\xi_n, \xi_{n+1})\} = \mu(\xi_{n-1}, \xi_n)$, and there exists $b \geq 0$ such that

$$
\lim_{n \to \infty} \mu(\xi_{n-1}, \xi_n) = b.
$$

(3.8)

Assuming that $b > 0$, we have $\lim_{n \to \infty} \Omega^*(\xi_{n-1}, \xi_n) = b$, and by $\psi(b)$, we get

$$
\sigma + \psi(b) \leq \psi(b),
$$

(3.9)

from which we have $\psi(b) \leq \psi(b) - \sigma < \psi(b)$, a contradiction. Consequently,

$$
\lim_{n \to \infty} \mu(\xi_{n-1}, \xi_n) = 0.
$$

(3.10)

Next, we argue that $\{\xi_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{T}$. For this, assume that there exists $\epsilon > 0$ and such positive integers satisfying $n_\epsilon(l) > m_\epsilon(l)$ with

$$
\mu(\xi_{n_\epsilon(l)}, \xi_{m_\epsilon(l)}) \geq \epsilon
$$

(3.11)

$$
\mu(\xi_{m_\epsilon(l)}, \xi_{n_\epsilon(l)-1}) < \epsilon.
$$

for all $l \in \mathbb{N}$. Hence, we obtain

$$
\epsilon \leq \mu(\xi_{n_\epsilon(l)}, \xi_{m_\epsilon(l)}) \leq \mu(\xi_{n_\epsilon(l)}, \xi_{n_\epsilon(l)-1}) + \mu(\xi_{n_\epsilon(l)-1}, \xi_{m_\epsilon(l)})
$$

(3.12)

$$
< \mu(\xi_{n_\epsilon(l)}, \xi_{n_\epsilon(l)-1}) + \epsilon.
$$

Letting $n \to \infty$ in (3.12), and using (3.10), yields

$$
\lim_{n \to \infty} \mu(\xi_{n_\epsilon(l)}, \xi_{m_\epsilon(l)}) = \epsilon.
$$

(3.13)

By triangle inequality on $\mathcal{T}$, we get

$$
0 \leq \left|\mu(\xi_{n_\epsilon(l)+1}, \xi_{m_\epsilon(l)+1}) - \mu(\xi_{n_\epsilon(l)}, \xi_{m_\epsilon(l)})\right|
$$

$$
\leq \mu(\xi_{n_\epsilon(l)+1}, \xi_{n_\epsilon(l)}) - \mu(\xi_{m_\epsilon(l)}, \xi_{m_\epsilon(l)+1}).
$$

Hence,

$$
\lim_{l \to \infty} \left|\mu(\xi_{n_\epsilon(l)+1}, \xi_{m_\epsilon(l)+1}) - \mu(\xi_{n_\epsilon(l)}, \xi_{m_\epsilon(l)})\right|
$$

(3.14)

$$
\leq \lim_{l \to \infty} \left[\mu(\xi_{n_\epsilon(l)+1}, \xi_{n_\epsilon(l)}) - \mu(\xi_{m_\epsilon(l)}, \xi_{m_\epsilon(l)+1})\right] = 0.
It comes up that
\[
\lim_{l \to \infty} \mu(s_{n,(l)+1}, s_{m,(l)+1}) = \lim_{l \to \infty} \mu(s_{n,(l)}, s_{m,(l)}) = \epsilon > 0.
\] (3.15)

In addition, since
\[
\epsilon = \mu(s_{n,(l)}, s_{m,(l)}) \leq \mu(s_{n,(l)}, s_{n,(l)+1}) + \mu(s_{n,(l)+1}, s_{m,(l)})
\]
\[
\leq \mu(s_{n,(l)}, s_{n,(l)+1}) + \mu(s_{m,(l)}, s_{n,(l)+1}),
\]
then,
\[
\lim_{n \to \infty} \mu(s_{n,(l)}, s_{m,(l)+1}) = \lim_{n \to \infty} \mu(s_{m,(l)}, s_{n,(l)+1}) = \epsilon.
\]

Thus, for all \( l \geq n_0 \), we get
\[
\mu(\mathcal{I}s_{n,(l)}, \mathcal{I}s_{m,(l)}) = \mu(s_{m,(l)+1}, s_{m,(l)+1}).
\]

Therefore, by (3.4), there exists \( \sigma > 0 \) such that
\[
\sigma + \psi(\mu(s_{n,(l)+1}, s_{m,(l)+1})) \leq \psi\left(\Omega^L_G(s_{n,(l)}, s_{m,(l)})\right),
\] (3.16)
where
\[
\Omega^L_G(s_{n,(l)}, s_{m,(l)}) = \left[ a_1(\mu(s_{n,(l)}, s_{m,(l)}) + a_2(\mu(s_{n,(l)}, s_{n,(l)+1}) + a_3(\mu(s_{m,(l)}, s_{m,(l)+1}) + a_4\left(\mu(s_{m,(l)}, s_{n,(l)+1})(1 + \mu(s_{n,(l)+1} s_{n,(l)+1})) \right)^{1/2}
\right.
\]
\[
\left. + a_5\left(\mu(s_{m,(l)}, s_{m,(l)+1})(1 + \mu(s_{n,(l)+1} s_{m,(l)+1})) \right)^{1/2}
\right].
\] (3.17)

Moreover, since the \( f \in X \psi \) is nondecreasing, we have
\[
\sigma + \liminf_{l \to \infty} \psi(\mu(s_{n,(l)+1}, s_{m,(l)+1})) \\
\leq \sigma + \liminf_{l \to \infty} \psi(\mu(\mathcal{I}s_{n,(l)}, \mathcal{I}s_{m,(l)})) \\
\leq \liminf_{l \to \infty} \psi\left(\Omega^L_G(s_{n,(l)}, s_{m,(l)})\right) \\
\leq \limsup_{l \to \infty} \psi\left(\Omega^L_G(s_{n,(l)}, s_{m,(l)})\right). 
\] (3.18)

From (3.18), we have \( \sigma + \psi(\epsilon) \leq \psi(\epsilon) \), that is, \( \psi(\epsilon) \leq \psi(\epsilon) - \sigma < \psi(\epsilon) \), a contradiction. This proves that \( \{s_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{I} \). Since \( \mathcal{I} \) is a \( CM_s \), there exists \( u \in \mathcal{I} \) such that...
Hence, from (3.20) and (3.22), we get
\[ \mu(u, \mathcal{I}u) = \lim_{n \to \infty} \mu(\mathcal{I}^n u, \mathcal{I}u) = \lim_{n \to \infty} \mu(\mathcal{I}\mathcal{I}^n u, \mathcal{I}u) = 0, \]
which implies that
\[ u = \mathcal{I}u. \] (3.19)

Assume that (3.19) is not true. Then, there is a number \( n_0 \in \mathbb{N} \) such that \( \mu(\mathcal{I}\mathcal{I}^n u, \mathcal{I}u) > 0 \) for all \( n \geq n_0 \).

Now, using (3.4) with \( \varsigma = \mathcal{I} \) and \( \varsigma = u \) with \( r > 0 \), we have
\[ \sigma + \psi(\mu(\mathcal{I}\mathcal{I}^{n_0-1} u, \mathcal{I}u)) \leq \psi(\Omega_3(\varsigma^{n_0-1}, u)), \] (3.20)
where
\[ \Omega_3'(\varsigma^{n_0-1}, u) = \left[ a_1(\mu(\varsigma^{n_0-1}, u))^r + a_2(\mu(\varsigma^{n_0-1}, \mathcal{I}\mathcal{I}^{n_0-1} u))^r + a_3(\mu(u, \mathcal{I}u))^r \right. \]
\[ + a_4\left( \frac{\mu(u, \mathcal{I}u)(1 + \mu(\varsigma^{n_0-1}, \mathcal{I}\mathcal{I}^{n_0-1} u))}{1 + \mu(\varsigma^{n_0-1}, u)} \right)^r \]
\[ + a_5\left( \frac{\mu(u, \mathcal{I}\mathcal{I}^{n_0-1} u)(1 + \mu(\varsigma^{n_0-1}, \mathcal{I}u))}{1 + \mu(\varsigma^{n_0-1}, u)} \right)^r \]
\[ = \left[ a_1(\mu(\varsigma^{n_0-1}, u))^r + a_2(\mu(\varsigma^{n_0-1}, \mathcal{I}u))^r + a_3(\mu(u, \mathcal{I}u))^r \right. \]
\[ + a_4\left( \frac{\mu(u, \mathcal{I}u)(1 + \mu(\varsigma^{n_0-1}, \mathcal{I}u))}{1 + \mu(\varsigma^{n_0-1}, u)} \right)^r \]
\[ + a_5\left( \frac{\mu(u, \mathcal{I}u)(1 + \mu(\varsigma^{n_0-1}, \mathcal{I}u))}{1 + \mu(\varsigma^{n_0-1}, u)} \right)^r \] (3.21)

From (3.21), we have
\[ \lim_{n \to \infty} \mu(\varsigma^{n_0-1}, \mathcal{I}u) = \mu(u, \mathcal{I}u) = \lim_{n \to \infty} \Omega_3'(\varsigma^{n_0-1}, u) \]
\[ = \left[ a_2(\mu(u, \mathcal{I}u))^r + a_3(\mu(u, \mathcal{I}u))^r + a_4(\mu(u, \mathcal{I}u))^r \right]^\frac{1}{r} \]
\[ = [(a_2 + a_3 + a_4)(\mu(u, \mathcal{I}u))^r]^\frac{1}{r} \]
\[ = (a_2 + a_3 + a_4)^\frac{1}{r} \mu(u, \mathcal{I}u). \] (3.22)

Hence, from (3.20) and (3.22), we get
\[ \sigma + \lim_{\mathcal{I}u \to \mu(u, \mathcal{I}u)} \inf \psi(\{(a_2 + a_3 + a_4)^\frac{1}{r} \sigma\}) \]
\[ \leq \sigma + \lim_{\mathcal{I}u \to \mu(u, \mathcal{I}u)} \inf \psi(\sigma) \]
\[ < \lim_{\mathcal{I}u \to \mu(u, \mathcal{I}u)} \sup \psi(\{(a_2 + a_3 + a_4)^\frac{1}{r} \sigma\}), \]
which is a contradiction, according to \((\psi_b)\). Thus, \(\mathcal{S}u = u\).

To show that the \(f_p\) of \(\mathcal{S}\) is unique, assume there exists \(u^* \in \Upsilon\) with \(u \neq u^*\) such that \(\mathcal{S}u^* = u^*\) so that \(\mu(u, u^*) = \mu(\mathcal{S}u, \mathcal{S}u^*) > 0\). Then, from (3.4), we have

\[
\sigma + \psi(\mu(u, u^*)) = \sigma + \psi(\mu(\mathcal{S}u, \mathcal{S}u^*)) \leq \psi(\Omega_3(u, u^*))
\]

\[
= \psi\left( \frac{a_1(\mu(u, u^*))^r + a_2(\mu(u, \mathcal{S}u))^r + a_3(\mu(u^*, \mathcal{S}u^*))^r}{1 + \mu(u, u^*)} \right)
\]

\[
+ a_4\left( \frac{\mu(u^*, \mathcal{S}u^*)(1 + \mu(u, \mathcal{S}u))}{1 + \mu(u, u^*)} \right)^{\frac{r}{2}}
\]

\[
+ a_5\left( \frac{\mu(u, \mathcal{S}u)(1 + \mu(u, \mathcal{S}u))}{1 + \mu(u, u^*)} \right)^{\frac{r}{2}}
\]

\[
= \psi\left( \frac{(a_1 + a_3)(\mu(u, u^*))^r}{1 + \mu(u, u^*)} \right)
\]

\[
= \psi\left( \frac{(a_1 + a_3)^2 \mu(u, u^*)}{1 + \mu(u, u^*)} \right)
\]

\[
\leq \psi(\mu(u, u^*))
\]

that is,

\[
\psi(\mu(u, u^*)) \leq \psi(\mu(u, u^*)) - \sigma < \psi(\mu(u, u^*))
\]

a contradiction. Therefore, \(u = u^*\). \(\Box\)

**Example 3.3.** Let \(\Upsilon = [0, 1]\) and \(\mu(\varsigma, \zeta) = |\varsigma - \zeta|\) for all \(\varsigma, \zeta \in \Upsilon\). Then, \((\Upsilon, \mu)\) is a \(CM_s\). Define \(\mathcal{S} : \Upsilon \rightarrow \Upsilon\) by

\[
\mathcal{S}\varsigma = \begin{cases} 
\varsigma, & \text{if } \varsigma \in [0, 1) \\
\frac{1}{2}, & \text{if } \varsigma = \frac{1}{2}.
\end{cases}
\]

Take \(r = 2, \sigma = \ln\left(\frac{3}{2}\right), a_1 = \frac{1}{4}, a_2 = \frac{36}{35}, a_3 = \frac{3}{196}, a_4 = a_5 = 0\) and \(\psi(\sigma) = \ln(\sigma)\) for all \(\sigma > 0\). Then, consider the following cases:

Case 1. For \(\varsigma, \zeta \in [0, 1)\) with \(\varsigma \neq \zeta\), we have \(0 < \mu(\mathcal{S}\varsigma, \mathcal{S}\zeta) = \frac{|\varsigma - \zeta|}{6}\) and

\[
\sigma + \psi(\mu(\mathcal{S}\varsigma, \mathcal{S}\zeta)) = \ln\left(\frac{3}{2}\right) + \psi\left(\frac{|\varsigma - \zeta|}{6}\right)
\]

\[
= \ln\left(\frac{3|\varsigma - \zeta|}{12}\right) < \ln\left(\frac{|\varsigma - \zeta|}{2}\right)
\]

\[
= \ln\left(\frac{|\varsigma - \zeta|^2}{4}\right) = \ln\left(a_1(\mu(\varsigma, \zeta))^2\right)^{\frac{1}{2}}
\]

\[
\leq \ln(\Omega_3^2(\varsigma, \zeta)).
\]
Case 2. For $\varsigma \in [0, 1)$ and $\zeta = 1$, we have $0 < \mu(\mathcal{I}_n, \mathcal{I}_n) = \frac{|\varsigma - 3|}{6}$, and

$$\sigma + \psi(\mu(\mathcal{I}_n, \mathcal{I}_n)) = \ln \left(\frac{3}{2}\right) + \ln \left(\frac{|\varsigma - 3|}{6}\right)$$

$$= \ln \left(\frac{|\varsigma - 3|}{4}\right) \leq \ln \left(\frac{6}{7} \times \frac{1}{2}\right)$$

$$= \ln \left(\frac{36}{49 \mu\left(1, \frac{1}{2}\right)}\right)$$

$$\leq \ln \left(\Omega^2_{\sigma}(\varsigma, \zeta)\right).$$

Hence, all the assertions of $Thr_m$ 3.2 are satisfied. Consequently, $\mathcal{I}$ has a unique $f_{p}$ in $\Upsilon$.

Whereas, with $\varsigma = \frac{5}{6}$, $\zeta = 1$,

$$\mu\left(\mathcal{I}\left(\frac{5}{6}\right), \mathcal{I}(1)\right) = \frac{13}{36} > \frac{1}{6} = \mu\left(\frac{5}{6}, 1\right).$$

And, for each $\psi \in \mathcal{M}$, there exists $\sigma > 0$ such that

$$\sigma + \psi\left(\mu\left(\mathcal{I}\left(\frac{5}{6}\right), \mathcal{I}(1)\right)\right) = \sigma + \ln \left(\frac{13}{36}\right)$$

$$> \ln \left(\frac{1}{6}\right) = \psi\left(\mu\left(\frac{5}{6}, 1\right)\right).$$

Therefore, $\mathcal{I}$ is not a $\psi$-contraction. So, $Thr_m$ 2.4 due to Wardowski [22] is not applicable here.

**Remark 3.** By taking $a_1 = 1, a_2 = a_3 = a_4 = a_5 = 0$ in the contractive condition (3.4), we obtain the contractive inequality (2.1) due to Wardowski [22].

**Theorem 3.4.** Let $(\Upsilon, \mu)$ be a CM, and $\mathcal{I} : \Upsilon \rightarrow \Upsilon$ be a $0$-hybrid $\psi$-contraction. Then, $\mathcal{I}$ has a $f_{p}$ in $\Upsilon$, provided that for each sequence $\{h_n\}_{n \in \mathbb{N}}$ in $(0, \infty)$, $\lim_{n \to \infty} h_n = 0$ if and only if $\lim_{n \to \infty} \psi(h_n) = -\infty$.

**Proof.** On the same steps as in $Thr_m$ 3.2, we presume that for each $n \in \mathbb{N}$,

$$0 < \mu(\varsigma_{n+1}, \varsigma_n) = \mu(\mathcal{I}_n, \mathcal{I}_{n-1})$$

if and only if $\varsigma_n \neq \varsigma_{n+1}$. Setting $\varsigma = \varsigma_{n-1}$ and $\zeta = \varsigma_n$ in (3.3), we have

$$\Omega^0_{\sigma}(\varsigma_n, \varsigma_n) = H(\varsigma_{n-1}, \varsigma_n) = (\mu(\varsigma_{n-1}, \varsigma_n))^{a_0} (\mu(\varsigma_{n-1}, \mathcal{I}_n(\varsigma_{n-1})))^{a_2}$$

$$\cdot (\mu(\varsigma_n, \mathcal{I}_n))^{a_3} \left(\frac{\mu(\varsigma_n, \mathcal{I}_n)(1 + \mu(\varsigma_{n-1}, \mathcal{I}_n)))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^{a_4}$$

$$\cdot \left(\frac{\mu(\varsigma_{n-1}, \mathcal{I}_n) + \mu(\varsigma_n, \mathcal{I}_n))}{2}\right)^{a_5}$$

$$= (\mu(\varsigma_{n-1}, \varsigma_n))^{a_0} (\mu(\varsigma_{n-1}, \varsigma_n))^{a_2} (\mu(\varsigma_n, \varsigma_{n+1}))^{a_3}$$

$$\cdot \left(\frac{\mu(\varsigma_{n-1}, \varsigma_{n+1})(1 + \mu(\varsigma_{n-1}, \varsigma_n))}{1 + \mu(\varsigma_{n-1}, \varsigma_n)}\right)^{a_4} \left(\frac{\mu(\varsigma_{n-1}, \varsigma_{n+1}) + \mu(\varsigma_n, \varsigma_n)}{2}\right)^{a_5}$$

(3.23)

$$= (\mu(\varsigma_{n-1}, \varsigma_n))^{a_0 + a_2} (\mu(\varsigma_n, \varsigma_{n+1}))^{a_3 + a_4} \left(\frac{\mu(\varsigma_{n-1}, \varsigma_n) + \mu(\varsigma_n, \varsigma_{n+1})}{2}\right)^{a_5}.$$
Combining (3.4) and (3.23), we get
\[
\sigma + \psi(\mu(\mathcal{I}_0, \mathcal{I}_n)) \leq \psi\left(\Omega_3^0(\mathcal{I}_0, \mathcal{I}_n)\right)
\]
\[
\leq \psi\left(\mu(\mathcal{I}_0, \mathcal{I}_n)^{a_1+a_2} \left(\mu(\mathcal{I}_0, \mathcal{I}_n) + \mu(\mathcal{I}_n, \mathcal{I}_{n+1})\right)^{a_3}\right)
\]
\[
\cdot \left(\frac{\mu(\mathcal{I}_0, \mathcal{I}_n) + \mu(\mathcal{I}_n, \mathcal{I}_{n+1})}{2}\right)^{a_5}
\]
(3.24)

Assume that \(\mu(\mathcal{I}_0, \mathcal{I}_n) \leq \psi(\mathcal{I}_0, \mathcal{I}_{n+1})\), then, (3.24) gives
\[
\psi(\mu(\mathcal{I}_0, \mathcal{I}_{n+1})) \leq \psi\left(\left(\mu(\mathcal{I}_0, \mathcal{I}_{n+1})\right)^{\frac{\sum a_i}{n}}\right) - \sigma
\]
\[
= \psi(\mu(\mathcal{I}_0, \mathcal{I}_{n+1})) - \sigma
\]
\[
< \psi(\mu(\mathcal{I}_0, \mathcal{I}_{n+1})),
\]
a contradiction. Hence, \(\mu(\mathcal{I}_0, \mathcal{I}_{n+1}) < \mu(\mathcal{I}_0, \mathcal{I}_n)\), for each \(n \in \mathbb{N}\), and there exists \(b \geq 0\) such that \(\lim_{n \to \infty} \mu(\mathcal{I}_0, \mathcal{I}_n) = b\). We claim that \(b = 0\). Otherwise, if \(b > 0\), then, letting \(n \to \infty\) in (3.25), yields \(\psi(b) < \psi(b)\), which is not possible. It comes up that
\[
\lim_{n \to \infty} \mu(\mathcal{I}_0, \mathcal{I}_n) = 0.
\]
(3.26)

Now, for each \(n \in \mathbb{N}\) and \(i \geq 1\), we have
\[
\Omega_3^0(\mathcal{I}_0, \mathcal{I}_{n+1}) = (\mu(\mathcal{I}_0, \mathcal{I}_{n+1}))^{a_1} (\mu(\mathcal{I}_0, \mathcal{I}_n))^{a_2} (\mu(\mathcal{I}_0, \mathcal{I}_{n+1}))^{a_3}
\]
\[
= \frac{\mu(\mathcal{I}_0, \mathcal{I}_{n+1})(1 + \mu(\mathcal{I}_0, \mathcal{I}_n))}{1 + \mu(\mathcal{I}_0, \mathcal{I}_{n+1})}
\]
\[
\cdot \left(\frac{\mu(\mathcal{I}_0, \mathcal{I}_{n+1}) + \mu(\mathcal{I}_0, \mathcal{I}_n)}{2}\right)^{a_5}
\]
\[
\cdot \frac{\mu(\mathcal{I}_0, \mathcal{I}_{n+1}) + \mu(\mathcal{I}_0, \mathcal{I}_n)}{2}.
\]
Using (3.26), we obtain
\[
\lim_{n \to \infty} \Omega_3^0(\mathcal{I}_0, \mathcal{I}_{n+1}) = 0.
\]
(3.27)

Consequently, by hypotheses, \(\lim_{n \to \infty} \psi(\Omega_3^0(\mathcal{I}_0, \mathcal{I}_{n+1})) = -\infty\), and, since
\[
\sigma + \lim_{n \to \infty} \psi(\mu(\mathcal{I}_0, \mathcal{I}_{n+1})) \leq \lim_{n \to \infty} \psi(\Omega_3^0(\mathcal{I}_0, \mathcal{I}_{n+1})),
\]
we get \(\lim_{n \to \infty} \psi(\mu(\mathcal{I}_0, \mathcal{I}_{n+1})) = -\infty\), from which it follows that \(\lim_{n \to \infty} \mu(\mathcal{I}_0, \mathcal{I}_{n+1}) = 0\). This shows that \(\{\mathcal{I}_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \(\mathcal{F}\). Since \(\mathcal{F}\) is a CM space, there exists \(u \in \mathcal{F}\) such that \(\mathcal{I}_n \to u\) as \(n \to \infty\). Moreover, it is a routine to check that for \(\zeta = \mathcal{I}_n\) and \(\zeta = u\) in (3.3), we have \(\Omega_3^0(\mathcal{I}_0, u) \to 0\) as \(n \to \infty\). If we assume that there exists a subsequence \(\{\mathcal{I}_{n_i}\}\) of \(\{\mathcal{I}_n\}_{n \in \mathbb{N}}\) such that \(\mathcal{I}_{n_i} = \mathcal{F} u\), then,
\[
0 = \lim_{n \to \infty} \mu(\mathcal{F} \mathcal{I}_n, \mathcal{F} u) = \lim_{n \to \infty} \mu(\mathcal{I}_{n+1}, \mathcal{F} u)
\]
\[
= \mu(u, \mathcal{F} u),
\]
(3.28)
that is, \( u = \mathcal{I}u \). Hence, let \( \mu(\mathcal{I}_\varsigma, \mathcal{I}u) > 0 \) for each \( n \in \mathbb{N} \). Then, from (3.4), we obtain
\[
\sigma + \psi(\mu(\mathcal{I}_\varsigma, \mathcal{I}u)) \leq \psi \left( \Omega_3^\sigma(\varsigma, u) \right).
\] (3.29)
Letting \( n \to \infty \) in (3.29), we have \( \lim_{n \to \infty} \psi(\mu(\mathcal{I}_\varsigma, \mathcal{I}u)) = -\infty \). Hence, \( \mathcal{I}u = u \), since
\[
\mu(u, \mathcal{I}u) = \lim_{n \to \infty} \mu(\mathcal{I}_\varsigma, \mathcal{I}u) = 0.
\]
\[\square\]

In what follows, we derive a few immediate consequences of Theorem 3.2 and 3.4.

**Corollary 1.** Let \( \langle \Theta, \mu \rangle \) be a CM, and \( \mathcal{I} : \Theta \to \Theta \) be a single-valued mpn. If there exist \( \psi \in \mathfrak{M} \) and \( \sigma > 0 \) such that for all \( \varsigma, \zeta \in \Theta \) with \( \varsigma \neq \mathcal{I}\varsigma, \mu(\mathcal{I}_\varsigma, \mathcal{I}\zeta) > 0 \) implies
\[
\sigma + \psi(\mu(\mathcal{I}_\varsigma, \mathcal{I}\zeta)) \leq \psi \left( \frac{\mu(\varsigma, \mathcal{I}\zeta) + \mu(\varsigma, \mathcal{I}_\varsigma)}{2} \right),
\]
then, there exists \( u \in \Theta \) such that \( \mathcal{I}u = u \).

**Proof.** Take \( r = 1, a_1 = a_2 = \frac{1}{2} \) and \( a_3 = a_4 = a_5 = 0 \) in Theorem 3.2. \[\square\]

**Corollary 2.** Let \( \langle \Theta, \mu \rangle \) be a CM, and \( \mathcal{I} : \Theta \to \Theta \) be a single-valued mapping. If there exist \( \psi \in \mathfrak{M} \) and \( \sigma > 0 \) such that for all \( \varsigma, \zeta \in \Theta \setminus \psi_m(\mathcal{I}) \), \( \mu(\mathcal{I}_\varsigma, \mathcal{I}\zeta) > 0 \) implies
\[
\sigma + \psi(\mu(\mathcal{I}_\varsigma, \mathcal{I}\zeta)) \leq \psi \left( \frac{\mu(\varsigma, \mathcal{I}\zeta) + \mu(\varsigma, \mathcal{I}_\varsigma)}{2} \right),
\]
then, there exists \( u \in \Theta \) such that \( \mathcal{I}u = u \).

**Proof.** Put \( a_1 = a_4 = a_5 = 0, a_2 = a_3 = \frac{1}{2} \) and \( r = 1 \) in Theorem 3.2. \[\square\]

**Corollary 3.** Let \( \langle \Theta, \mu \rangle \) be a CM, and \( \mathcal{I} : \Theta \to \Theta \) be a single-valued mpn. If there exist \( \psi \in \mathfrak{M} \) and \( \sigma > 0 \) such that for all \( \varsigma, \zeta \in \Theta \), \( \mu(\mathcal{I}_\varsigma, \mathcal{I}\zeta) > 0 \) implies
\[
\sigma + \psi(\mu(\mathcal{I}_\varsigma, \mathcal{I}\zeta)) \leq \psi(\mu(\varsigma, \mathcal{I}\zeta) + a_2 \mu(\varsigma, \mathcal{I}_\varsigma) + a_3 \mu(\varsigma, \mathcal{I}_\varsigma))
\]
(3.30)
where \( \sum_{i=1}^{3} a_i = 1 \), then, \( \mathcal{I} \) has a unique fp, in \( \Theta \).

**Proof.** Take \( r = 1 \) and \( a_4 = a_5 = 0 \) in Theorem 3.2. \[\square\]

**Corollary 4.** Let \( \langle \Theta, \mu \rangle \) be a CM, and \( \mathcal{I} : \Theta \to \Theta \) be a single-valued mpn. If there exist \( \psi \in \mathfrak{M} \), \( \gamma \in (0, 1) \) and \( \sigma > 0 \) such that for all \( \varsigma, \zeta \in \Theta \), with \( \varsigma \neq \mathcal{I}\varsigma, \mu(\mathcal{I}_\varsigma, \mathcal{I}\zeta) > 0 \) implies
\[
\sigma + \psi(\mu(\mathcal{I}_\varsigma, \mathcal{I}\zeta)) \leq \psi \left( \mu(\varsigma, \mathcal{I}\zeta)^\gamma \mu(\varsigma, \mathcal{I}_\varsigma)^{1-\gamma} \right)
\]
then, there exists \( u \in \Theta \) such that \( \mathcal{I}u = u \).

**Proof.** Set \( a_1 = \gamma, a_2 = 1 - \gamma \) and \( a_3 = a_4 = a_5 = 0 \) in Theorem 3.4. \[\square\]

**Remark 4.** Following Corollaries 1–4, it is obvious that more particular cases of Theorem 3.2 and 3.4 can be pointed out.
4. Further consequences

In this section, we show that some well-known $f_p$ thr $m$ with metric structure in the existing literature can be deduced as special cases of our results.

**Corollary 5.** [26] Let $(Y, \mu)$ be a CM, and $\mathcal{F} : Y \to Y$ be a single-valued mpn, If there exist $\sigma > 0$ and a mpn $\psi : \mathbb{R}_+ \to \mathbb{R}$ such that for each $\xi, \zeta \in Y$, $\mu(\mathcal{F}_\xi, \mathcal{F}_\zeta) > 0$ implies

$$\sigma + \psi(\mu(\mathcal{F}_\xi, \mathcal{F}_\zeta)) \leq \psi(\Lambda_1 \mu(\xi, \zeta) + \Lambda_2 \mu(\xi, \zeta))$$

for all nonnegative real numbers $\Lambda_1, \Lambda_2 \in [0, 1]$ with $\sum_{i=1}^{2} \Lambda_i = 1$, then, $\mathcal{F}$ has a $f_p$ in $Y$.

**Proof.** Put $a_1 = 0$ and $a_2 = \Lambda_1, a_3 = \Lambda_2$ in Corollary 3. \hfill $\square$

**Definition 4.1.** [21] Let $(Y, \mu)$ be a $M$. A single-valued mpn $\mathcal{F} : Y \to Y$ is called Reich contraction if there exist $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathbb{R}_+$ with $\Lambda_1 + \Lambda_2 + \Lambda_3 < 1$ such that for all $\xi, \zeta \in Y$,

$$\mu(\mathcal{F}_\xi, \mathcal{F}_\zeta) \leq \Lambda_1 \mu(\xi, \zeta) + \Lambda_2 \mu(\xi, \zeta) + \Lambda_3 \mu(\xi, \zeta)$$

(4.1)

**Corollary 6.** [21] Let $(Y, \mu)$ be a CM, and $\mathcal{F} : Y \to Y$ be a Reich contraction. Then, $\mathcal{F}$ has a unique $f_p$ in $Y$.

**Proof.** Take $\psi(\sigma) = \ln(\sigma)$ for all $\sigma > 0$ and $\Lambda_i = a_i e^{-\varphi}$ in Corollary 3. \hfill $\square$

**Definition 4.2.** [19] Let $(Y, \mu)$ be a $M$. A mpn $\mathcal{F} : Y \to Y$ is called an interpolative Kannan contraction if there exist $\gamma \in (0, 1)$ and $\lambda \in (0, 1)$ such that for all $\xi, \zeta \in Y \setminus \psi_\mu(\mathcal{F})$,

$$\mu(\mathcal{F}_\xi, \mathcal{F}_\zeta) \leq \lambda \left[\mu(\xi, \mathcal{F}_\xi)\gamma \mu(\zeta, \mathcal{F}_\zeta)^{1-\gamma}\right]$$

(4.2)

**Corollary 7.** [19] Let $(Y, \mu)$ be a CM, and $\mathcal{F}$ be an interpolative Kannan contraction. Then, $\mathcal{F}$ has a $f_p$ in $Y$.

**Proof.** From (4.2), for all $\xi, \zeta \in Y \setminus \psi_\mu(\mathcal{F})$ with $\mu(\mathcal{F}_\xi, \mathcal{F}_\zeta) > 0$, we have

$$\sigma + \ln(\mu(\mathcal{F}_\xi, \mathcal{F}_\zeta)) \leq \ln\left(\left[\mu(\xi, \mathcal{F}_\xi)\gamma \mu(\zeta, \mathcal{F}_\zeta)^{1-\gamma}\right]\right).$$

(4.3)

By taking $\psi(\sigma) = \ln(\sigma)$ for all $\sigma > 0$, (4.3) becomes

$$\sigma + \psi(\mu(\mathcal{F}_\xi, \mathcal{F}_\zeta)) \leq \psi\left(\left[\mu(\xi, \mathcal{F}_\xi)\gamma \mu(\zeta, \mathcal{F}_\zeta)^{1-\gamma}\right]\right) \leq \psi\left(\Omega^0_{\gamma}(\xi, \zeta)\right),$$

where $\sigma = \ln\left(\frac{1}{\lambda}\right)$. Therefore, putting $a_1 = a_4 = a_5 = 0, a_2 = \gamma$ and $a_3 = 1 - \gamma$, Theorem 3.4 can be applied to find $u \in Y$ such that $\mathcal{F}u = u$. \hfill $\square$
5. An application in dynamic programming

Mathematical optimization is one of the areas in which the techniques of \( f_n \) theory are generously used. It is a known fact that dynamic programming provides important tools for mathematical optimization and computer programming. In this direction, the problem of dynamic programming with regards to multistage process reduces to solving the functional \( eq_n \):

\[
    h(\varsigma) = \sup_{\varsigma \in G} \{ g(\varsigma, \zeta) + S(\varsigma, \zeta, h(b(\varsigma, \zeta))) \}, \ \varsigma \in L, \tag{5.1}
\]

where \( b : L \times G \rightarrow L, \ g : L \times G \rightarrow \mathbb{R} \) and \( S : L \times G \times \mathbb{R} \rightarrow \mathbb{R} \).

Assume that \( K \) and \( W \) are Banach spaces, \( L \subseteq K \) is a state space and \( G \subseteq W \) is a decision space. Precisely, the studied process consists of:

(i) a state space, which is the set of initial state, action and transition model of the process;
(ii) a decision space, which is the set of possible actions that are allowed for the process.

For further details of functional \( eq_n \) arising in dynamic programming, the interested reader may consult Bellman and Lee [37]. In this section, we investigate the existence of bounded solution to the functional \( eq_n \) (5.1). Let \( \Upsilon = B(L) \) be the set of all bounded real-valued functions on \( L \) and, for an arbitrary element \( p \in \Upsilon \), take \( \|p\| = \sup_{\varsigma \in L} |p(\varsigma)| \). Obviously, \( (\Upsilon, \|\cdot\|) \) equipped with the metric \( \mu \) induced by the norm \( \|\cdot\| \), via:

\[
    \mu(p, q) = \|p - q\| = \sup_{\varsigma \in L} |p(\varsigma) - q(\varsigma)| \tag{5.2}
\]

for all \( p, q \in \Upsilon \), is a Banach space. In fact, the convergence (cvgnce) in \( \Upsilon \) with respect to \( \|\cdot\| \) is uniform. Hence, if we consider a Cauchy seq \( \{ p_n \}_{n \in \mathbb{N}} \) in \( \Upsilon \), then, \( \{ p_n \}_{n \in \mathbb{N}} \) cvg uniformly to a fnx say \( p^* \), that is also bounded and so \( p^* \in \Upsilon \).

Consider a mpn \( \mathfrak{J} : \Upsilon \rightarrow \Upsilon \) defined by

\[
    \mathfrak{J}(p)(\varsigma) = \sup_{\varsigma \in G} \{ g(\varsigma, \zeta) + S(\varsigma, \zeta, p(b(\varsigma, \zeta))) \} \tag{5.3}
\]

for all \( p \in \Upsilon \) and \( \varsigma \in L \). Clearly, if the fnx \( g \) and \( S \) are bounded, then \( \mathfrak{J} \) is well-defined.

For all \( p, q \in \Upsilon \), let

\[
    G^*(p, q) = \left[ \Lambda_1(\mu(p, q))^r + \Lambda_2(\mu(p, \mathfrak{J} p))^r + \Lambda_3(\mu(q, \mathfrak{J} q))^r \right]^{1/r} + \Lambda_4 \left( \frac{\mu(q, \mathfrak{J} q)(1 + \mu(p, \mathfrak{J} p))}{1 + \mu(p, q)} \right)^r + \Lambda_5 \left( \frac{\mu(q, \mathfrak{J} q)(1 + \mu(p, \mathfrak{J} q))}{1 + \mu(p, q)} \right)^r, \tag{5.4}
\]

where \( \Lambda_i \ (i = 1, 2, 3, 4, 5) \) are nonnegative real numbers satisfying \( \sum_{i=1}^5 \Lambda_i = 1 \).

**Theorem 5.1.** Let \( \mathfrak{J} : \Upsilon \rightarrow \Upsilon \) be a mpn represented in (5.3) and suppose that:

\( (D_1) \) \( S : L \times G \times \mathbb{R} \rightarrow \mathbb{R} \) and \( g : L \times G \rightarrow \mathbb{R} \) are continuous;
\( (D_2) \) there exists \( \sigma > 0 \) such that

\[
    |S(\varsigma, \zeta, p(\varsigma)) - S(\varsigma, \zeta, q(\varsigma))| \leq e^{-\sigma t} G^*(p, q),
\]

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for all \( p, q \in \Upsilon \), where \( \varsigma \in L \) and \( \zeta \in G \).

Then, the functional eqn \((5.1)\) has a bounded solution in \( \Upsilon \).

**Proof.** First, note that \((\Upsilon, \mu)\) is a \(CM\), where the metric \( \mu \) is given by \((5.2)\). Let \( \varrho > 0 \) be an arbitrary real number, \( \varsigma \in L \) and \( p_1, p_2 \in \Upsilon \). then, there exist \( \zeta_1, \zeta_2 \in G \) such that

\[
\mathfrak{I}(p_1)(\varsigma) < g(\varsigma, \zeta_1) + S(\varsigma, \zeta_1, p_1(b(\varsigma, \zeta_1))) + \varrho, \tag{5.5}
\]

\[
\mathfrak{I}(p_2)(\varsigma) < g(\varsigma, \zeta_2) + S(\varsigma, \zeta_2, p_2(b(\varsigma, \zeta_2))) + \varrho, \tag{5.6}
\]

\[
\mathfrak{I}(p_1)(\varsigma) \geq g(\varsigma, \zeta_2) + S(\varsigma, \zeta_2, p_1(b(\varsigma, \zeta_2))), \tag{5.7}
\]

\[
\mathfrak{I}(p_2)(\varsigma) \geq g(\varsigma, \zeta_1) + S(\varsigma, \zeta_1, p_2(b(\varsigma, \zeta_1))). \tag{5.8}
\]

Hence, it follows from \((5.5)\) and \((5.8)\) that

\[
\mathfrak{I}(p_1)(\varsigma) - \mathfrak{I}(p_2)(\varsigma) < S(\varsigma, \zeta_1, p_1(b(\varsigma, \zeta_1))) - S(\varsigma, \zeta_1, p_2(b(\varsigma, \zeta_1))) + \varrho \leq |S(\varsigma, \zeta_1, p_1(b(\varsigma, \zeta_1))) - S(\varsigma, \zeta_1, p_2(b(\varsigma, \zeta_1)))| + \varrho \leq e^{-\sigma}G^*(p_1, p_2) + \varrho,
\]

that is,

\[
\mathfrak{I}(p_1)(\varsigma) - \mathfrak{I}(p_2)(\varsigma) \leq e^{-\sigma}G^*(p_1, p_2) + \varrho. \tag{5.9}
\]

On similar steps, using \((5.6)\) and \((5.7)\), we get

\[
\mathfrak{I}(p_2)(\varsigma) - \mathfrak{I}(p_1)(\varsigma) \leq e^{-\sigma}G^*(p_1, p_2) + \varrho. \tag{5.10}
\]

Therefore, from \((5.9)\) and \((5.10)\), we have

\[
|\mathfrak{I}(p_1)(\varsigma) - \mathfrak{I}(p_2)(\varsigma)| \leq e^{-\sigma}G^*(p_1, p_2) + \varrho. \tag{5.11}
\]

Taking supremum over all \( \varsigma \in L \) in \((5.11)\), yields

\[
\mu(\mathfrak{I}(p_1), \mathfrak{I}(p_2)) \leq e^{-\sigma}G^*(p_1, p_2) + \varrho. \tag{5.12}
\]

Given that \( \varrho > 0 \) is arbitrary, then, we deduce from \((5.12)\) that

\[
\mu(\mathfrak{I}(p_1), \mathfrak{I}(p_2)) \leq e^{-\sigma}G^*(p_1, p_2). \tag{5.13}
\]

So, passing to logarithms in \((5.13)\), gives

\[
\sigma + \ln(\mu(\mathfrak{I}(p_1), \mathfrak{I}(p_2))) \leq \ln(G^*(p_1, p_2)). \tag{5.14}
\]

By defining the \( fnx \psi : (0, \infty) \to \mathbb{R} \) as \( \psi(\varpi) = \ln(\varpi) \) for all \( \varpi > 0 \), \((5.14)\) becomes

\[
\sigma + \psi(\mathfrak{I}(p_1), \mathfrak{I}(p_2)) \leq \psi(G^*(p_1, p_2)).
\]

Thus, \( \mathfrak{I} \) is an \( r \)-hybrid \( \psi \)-contraction. Consequently, as an application of \( Thr_m \) 3.2, we conclude that \( \mathfrak{I} \) has a \( f_p \) in \( \Upsilon \), which corresponds to a solution of the functional eqn \((5.1)\). \( \square \)
6. Applications to nonlinear Volterra integral equations

$F_{pt}$ for contractive operators on metric spaces are commonly investigated and have gained enormous applications in the theory of differential and integral $eq_n$ (see [34, 36] and references therein). In this subsection, we apply $Thr_m$ 3.2 to discuss the existence and uniqueness of a solution to the following integral $eq_n$ of Volterra type:

$$\varsigma(\varpi) = f(\varpi) + \int_0^{\varpi} L(\varpi, s, \varsigma(s)) \mu s, \; \varpi \in [0, \delta] = J,$$

(6.1)

where $\delta > 0$, $L : J \times J \times \mathbb{R} \rightarrow \mathbb{R}$ and $f : J \rightarrow \mathbb{R}$.

Let $Y = C(J, \mathbb{R})$ be the space of all continuous real-valued $fnx$ defined on $J$. And, for arbitrary $\varsigma \in Y$, define $\|\varsigma\|_\sigma = \sup_{\varpi \in J} |\varsigma(\varpi)| e^{-\sigma \varpi}$, where $\sigma > 0$. It is well-known that $\|\|_\sigma$ is a norm equivalent to the supremum norm, and $Y$ equipped with the metric $\mu_\sigma$ defined by

$$\mu_\sigma(\varsigma, \zeta) = \sup \{ |\varsigma(\varpi) - \zeta(\varpi)| e^{-\sigma \varpi} \},$$

(6.2)

for all $\varsigma, \zeta \in Y$, is a Banach space.

**Theorem 6.1.** Suppose that:

(C1) $L : J \times J \times \mathbb{R} \rightarrow \mathbb{R}$ and $f : J \rightarrow \mathbb{R}$ are continuous;

(C2) there exists $\sigma > 0$ such that for all $s, \varpi \in J$ and $\varsigma, \zeta \in \mathbb{R}$,

$$|L(\varpi, s, \varsigma) - L(\varpi, s, \zeta)| \leq \sigma e^{-\sigma} |\varsigma - \zeta|.$$  

Then, the integral $eq_n$ (6.1) has a unique solution in $Y$.

**Proof.** Note that $(Y, \mu_\sigma)$ is a $CM_s$, where the metric $\mu_\sigma$ is given by (6.2). Consider a mpn $\mathcal{H} : Y \rightarrow Y$ defined by

$$\mathcal{H}(\varsigma)(\varpi) = f(\varpi) + \int_0^{\varpi} L(\varpi, s, \varsigma(s)) \mu s, \; \varsigma \in Y, \; \varpi \in J.$$  

(6.3)

Let $\varsigma, \zeta \in Y$ such that $\mathcal{H}\varsigma \neq \mathcal{H}\zeta$. Then,

$$\left| \mathcal{H}(\varsigma)(\varpi) - \mathcal{H}(\zeta)(\varpi) \right| \leq \int_0^{\varpi} |L(\varpi, s, \varsigma(s)) - L(\varpi, s, \zeta(s))| \mu s$$

$$\leq \int_0^{\varpi} \sigma e^{-\sigma} |\varsigma(s) - \zeta(s)| \mu s$$

$$= \int_0^{\varpi} \sigma e^{-\sigma} |\varsigma(s) - \zeta(s)| e^{-\sigma s} e^{\sigma s} \mu s$$

$$\leq \int_0^{\varpi} \sigma e^{\sigma s} e^{-\sigma s} |\varsigma(s) - \zeta(s)| e^{-\sigma s} \mu s$$

$$\leq \sigma e^{-\sigma} \|\varsigma - \zeta\|_\sigma \int_0^{\varpi} e^{\sigma s} \mu s$$

$$\leq \sigma e^{-\sigma} \|\varsigma - \zeta\|_\sigma \frac{e^{\sigma \varpi}}{\sigma}.$$  

(6.4)
It follows from (6.4) that
\[ |\mathcal{I}(\zeta)(\sigma) - \mathcal{I}(\zeta)(\sigma)| e^{-\sigma m} \leq e^{-\sigma} \|\zeta - \zeta\|_{C}. \] (6.5)

Taking supremum over all \( \sigma \in J \) in (6.5), produces
\[ \mu_{\sigma}(\mathcal{I}(\zeta), \mathcal{I}(\zeta)) \leq e^{-\sigma} \mu_{\sigma}(\zeta, \zeta). \] (6.6)

Passing to logarithms in (6.6), yields
\[ \sigma + \ln(\mu_{\sigma}(\mathcal{I}(\zeta), \mathcal{I}(\zeta))) \leq \ln(\mu_{\sigma}(\zeta, \zeta)). \] (6.7)

By defining the function \( \psi : (0, \infty) \rightarrow \mathbb{R} \) as \( \psi(\sigma) = \ln(\sigma) \) for all \( \sigma > 0 \), (6.7) can be rewritten as:
\[ \sigma + \psi(\mu_{\sigma}(\mathcal{I}(\zeta), \mathcal{I}(\zeta))) \leq \psi(\mu_{\sigma}(\zeta, \zeta)). \]

Hence, all the hypotheses of \( Thr_{m} \) 3.2 are satisfied with \( a_{1} = 1 \) and \( a_{2} = a_{3} = a_{4} = a_{5} = 0 \). Consequently, \( \mathcal{I} \) has a \( f_{p} \) in \( \Upsilon \), which is the unique solution of the integral \( eq_{n} \) (6.1).

**Example 6.2.** Consider the Volterra integral \( eq_{n} \) of the form
\[ \varsigma(\sigma) = \frac{\sigma}{1 + \sigma^{2}} + \int_{0}^{\sigma} \frac{\varsigma(s)}{25 + (\varsigma(s))^{2}} \mu_{s}, \ \sigma \in [0, \delta], \delta > 0. \] (6.8)

From (6.1) and (6.8), we note that \( f(\sigma) = \frac{\sigma}{1 + \sigma^{2}} \) and \( L(\sigma, s, \varsigma(s)) = \frac{\varsigma(s)}{25 + (\varsigma(s))^{2}} \) are continuous; that is, condition (C1) of \( Thr_{m} \) 6.1 holds. Moreover,
\[ |L(\sigma, s, \varsigma(s)) - L(\sigma, s, \zeta(s))| \leq \frac{1}{25} |\varsigma(s) - \zeta(s)| \]
\[ \leq (1)e^{-|\varsigma(s) - \zeta(s)|} \leq (\sigma)e^{-\sigma} |\varsigma(s) - \zeta(s)|. \]

Hence, condition (C2) is verified. By \( Thr_{m} \) 6.1, it comes up that (6.8) has a unique solution in \( \Upsilon = C([0, \delta], \mathbb{R}) \).

**7. Conclusions**

In this work, a novel concept called \( r \)-hybrid \( \psi \)-contraction has been introduced and some \( f_{p} \) results for such mpn in the framework of \( CM_{t} \) have been presented. The established \( f_{p} \), \( thr_{m} \) merge and extend a number of well-known concepts in the corresponding literature. A few of these particular cases have been highlighted and discussed. An example is designed to show the generality and authenticity of our points. From application perspective, we investigated the existence and uniqueness conditions of solutions to certain functional equation arising in dynamic programming and integral equation of Volterra type.

It is noteworthy that the idea of this paper, being established in the setting of a \( M_{t} \), is fundamental. Hence, it can be improved upon when examined in the structure of \( b-M_{s}, F-M_{s}, G-M_{s} \), modular \( M_{s} \), and some other pseudo or quasi \( M_{s} \). It is a popular fact that construction of \( f_{p} \) has lots of usefulness; in particular, in transition operators for Cauchy problems of differential equations of either integer and non-integer order. In this direction, the contractive inequalities and functional equations presented here can be studied within the domains of variational inequality and fractional calculus. Furthermore, it is natural to extend the single-valued mappings herein to set-valued mpn within the outlines of either fuzzy or classical mathematics.
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Conflict of interest

The authors declare that they have no competing interests.

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