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Research article

Further characterizations of the weak group inverse of matrices and the weak group matrix

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Abstract: This paper is devoted to consider some new characteristics and properties of the weak group inverse and the weak group matrix. First, we characterize the weak group inverse of a square matrix based on its range space and null space. Also several different characterizations of the weak group inverse are presented by projection and the Bott-Duffin inverse. Then by using the core-EP decomposition, we investigate the relationships between weak group inverse and other generalized inverses. And some new characterizations of weak group matrix are obtained.

Keywords: weak group inverse; weak group matrix; core-EP decomposition; rang space; null space
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1. Introduction

Throughout this paper, we denote the set of $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. And denote the identity matrix of order n by I_n, the range space, the null space , the conjugate transpose and the rank of the matrix $A \in \mathbb{C}^{m\times n}$ by $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* and $r(A)$, respectively. The index of $A \in \mathbb{C}^{n\times n}$, denoted by $Ind(A)$, is the smallest nonnegative integer k such that $r(A^k) = r(A^{k+1})$. The subset of $\mathbb{C}^{n\times n}$ with index k will be denoted by $\mathbb{C}^{n\times n}_k$. $P_{L,M}$ stands for the projector (idempotent) on the subspace L along the subspace M. For $A \in \mathbb{C}^{n\times n}$, P_A and P_{A^*} represent the orthogonal projections onto $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ respectively, i.e. $P_A = AA^\dagger$ and $P_{A^*} = A^\dagger A$.

For the readers' convenience, we will first recall the definitions of some generalized inverses. For $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ of A is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations [4,18]:

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA.$$

A matrix $X \in \mathbb{C}^{n\times m}$ satisfying (2) is called an outer inverse of A and is denoted by $A^{(2)}$. For $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ satisfying $XAX = X$, $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$, is denoted by $A^{(2)}_{T,S}$ [6], where T is a subspace of $\mathbb{C}^n$ and S is a subspace of $\mathbb{C}^m$. If $A^{(2)}_{T,S}$ exists then it is unique.

The Drazin inverse of $A \in \mathbb{C}^{n\times n}_k$, denoted by A^D, is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations [4, 9]:

$$(5)\ AX^2 = X, \quad (6)\ AX = XA, \quad (7)\ XA^{k+1} = A^k.$$

In particular, if $Ind(A) = 1$, then the Drazin inverse of A is called the group inverse of A and is denoted by $A^{\#}$. The core-inverse of $A \in \mathbb{C}^{n\times n}$, denoted by $A^{\textcircled{\#}}$ is defined in [2] as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$. $A^{\textcircled{\#}}$ exists if and only if $Ind(A) = 1$ [2, 20]. The core-EP inverse of $A \in \mathbb{C}^{n\times n}_k$, denoted by $A^{\textcircled{\dagger}}$ is defined in [19] as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$ [11, 19, 21, 30]. The BT-inverse of $A \in \mathbb{C}^{n\times n}$, denoted by $A^{\diamond}$, which is defined in [1] written by $(AP_A)^{\dagger}$ [1, 13]. The DMP-inverse of $A \in \mathbb{C}^{n\times n}_k$, written by $A^{D,\dagger}$ is defined in [14] as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying $XAX = X, XA = A^DA$ and $A^kX = A^kA^{\dagger}$. Moreover, it was proved that $A^{D,\dagger} = A^DAA^{\dagger}$. Also, the dual DMP-inverse of A was introduced in [14], namely $A^{\dagger,D} = A^{\dagger}AA^D$ [14, 31]. The CMP-inverse of $A \in \mathbb{C}^{n\times n}_k$, written by $A^{C,\dagger}$ is defined in [15] as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying $XAX = X, AX = AA^DAA^{\dagger}$ and $XA = A^{\dagger}AA^DA$. Moreover, it was proved that $A^{C,\dagger} = A^{\dagger}AA^DAA^{\dagger}$ [15, 24]. The (B, C)-inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{(B,C)}$ [8], is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying $XAB = B, CAX = C$, $\mathcal{N}(X) = \mathcal{N}(C)$ and $\mathcal{R}(X) = \mathcal{R}(B)$, where B, $C \in \mathbb{C}^{n\times m}$ [3, 8].

The weak group inverse of $A \in \mathbb{C}^{n\times n}_k$ was defined for the first time by Wang and Chen [22]:

Definition 1.1 [22] Let $A \in \mathbb{C}^{n\times n}_k$. The weak group inverse $A^{\textcircled{w}}$ of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$AX^2 = X, \qquad AX = A^{\textcircled{\dagger}}A.$$

The weak group matrix was defined for the first time by Wang and Liu [23]:

Definition 1.2 [23] Let $A \in \mathbb{C}^{n\times n}_k$. We say a weak group matrix if $A \in \mathbb{C}^{WG}_n$, in which

$$\mathbb{C}^{WG}_n = \{A | AA^{\textcircled{w}} = A^{\textcircled{w}}A,\ A \in \mathbb{C}^{n\times n}_k\}.$$

Recently, the study of the weak group inverse and the generalized weak group inverse has received a lot of attention (see e.g. [16, 17, 26–28]).

In this paper, we discuss some new characterizations and properties of the weak group inverse and the weak group matrix. First, we characterize the weak group inverse of a square matrix based on its range space and null space. Several different characterizations of the weak group inverse are presented by projection and the Bott-Duffin inverse. We also give the limit representations for the weak group inverse. Then we study the relationships between $A^{\textcircled{w}}$ and other generalized inverses such as $A^{\#}, A^{\textcircled{\#}}, A^{\dagger}, A^{\textcircled{\dagger}}, A^D, A^{D,\dagger}$. And some characterizations of the weak group matrix are obtained.

The research is arranged as follows. In Section 2, some indispensable matrix classes and lemmas are given. In Section 3, some characterizations of $A^{\textcircled{w}}$ are presented. In Section 4, we discuss the relationships between $A^{\textcircled{w}}$ and other generalized inverses by core-EP decomposition. Finally, we study the characterizations of the weak group matrix.

2. Preliminaries

For convenience, some matrix classes will be given as follows.

These symbols $\mathbb{C}_n^{CM}, \mathbb{C}_n^{P}, \mathbb{C}_n^{OP}, \mathbb{C}_n^{EP}, \mathbb{C}_n^{U}$ and $\mathbb{C}_n^{H}$ will stand for the subsets of $\mathbb{C}^{n\times n}$ consisting of core matrices, projectors (idempotent matrices), orthogonal projectors (Hermitian idempotent matrices), EP-matrices (Range-Hermitian matrices), unitary matrices and Hermitian matrices, respectively, i.e.,

$$\begin{aligned}
&\mathbb{C}_n^{CM} = \{A|A \in \mathbb{C}^{n\times n}, r(A^2) = r(A)\},\\
&\mathbb{C}_n^{P} = \{A|A \in \mathbb{C}^{n\times n}, A^2 = A\},\\
&\mathbb{C}_n^{OP} = \{A|A \in \mathbb{C}^{n\times n}, A^2 = A = A^*\},\\
&\mathbb{C}_n^{EP} = \{A|A \in \mathbb{C}^{n\times n}, AA^\dagger = A^\dagger A\} = \{A|A \in \mathbb{C}^{n\times n}, \mathcal{R}(A) = \mathcal{R}(A^*)\},\\
&\mathbb{C}_n^{U} = \{A|A \in \mathbb{C}^{n\times n}, AA^* = A^*A = I_n\},\\
&\mathbb{C}_n^{H} = \{A|A \in \mathbb{C}^{n\times n}, A = A^*\}.
\end{aligned}$$

In order to study the characterizations and properties of the weak group inverse and the weak group matrix, we need to recall the core-EP decomposition of A, which plays an important role in this paper.

According to Theorem 2.2 in [21], every matrix $A \in \mathbb{C}_k^{n\times n}$ can be represented in the form:

$$A = A_1 + A_2 = U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}U^*, \; A_1 := U\begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}U^*, \; A_2 := U\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}U^*, \tag{2.1}$$

where $T \in \mathbb{C}^{p\times p}$ is nonsingular with $p := r(T) = r(A^k)$, N is nilpotent of index k, and $U \in \mathbb{C}^{n\times n}$ is unitary. The representation of A given in (2.1) satisfies $Ind(A_1) \le 1$, $A_2^k = 0$, and $A_1^*A_2 = A_2A_1 = 0$ [21, Theorem 2.1]. Moreover, it is unique [21, Theorem 2.4] and is called the core-EP decomposition of A.

Lemma 2.1 *Let $A \in \mathbb{C}_k^{n\times n}$ be as in (2.1). Then*

$$r(A) = r(A^2) \Longleftrightarrow N = 0.$$

In which case, we have

$$A^\# = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^*, \; A^{\text{#}\!\!\!\bigcirc} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*. \tag{2.2}$$

Proof. The proof is easy.

In [7, 10, 13, 21, 29], some generalized inverses such as $A^\dagger, A^D, A^{\dagger\!\!\!\bigcirc}, A^\diamond, A^{D,\dagger}, A^{\dagger,D}, A^{C,\dagger}$ can be represented by the core-EP decomposition of A. We list these results in the lemmas below.

Lemma 2.2 [7, 10, 13, 21, 29] *Let $A \in \mathbb{C}_k^{n\times n}$ be as in (2.1). Then*

$$(a) \qquad A^\dagger = U\begin{pmatrix} T^*\Delta & -T^*\Delta S N^\dagger \\ (I_{n-p} - N^\dagger N)S^*\Delta & N^\dagger - (I_{n-p} - N^\dagger N)S^*\Delta S N^\dagger \end{pmatrix}U^*; \tag{2.3}$$

$$(b) \qquad A^D = U\begin{pmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T} \\ 0 & 0 \end{pmatrix}U^*; \tag{2.4}$$

$$(c) \qquad A^{\text{Ⓣ}} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*; \tag{2.5}$$

$$(d) \qquad A^{\diamond} = U\begin{pmatrix} T^*\Delta_1 & -T^*\Delta_1 S N^{\diamond} \\ (P_N - P_{N^{\diamond}})S^*\Delta_1 & N^{\diamond} - (P_N - P_{N^{\diamond}})S^*\Delta_1 S N^{\diamond} \end{pmatrix}U^*,$$

$$\textit{where } \Delta_1 = [TT^* + S(P_N - P_{N^{\diamond}})S^*]^{-1}; \tag{2.6}$$

$$(e) \qquad A^{D,\dagger} = U\begin{pmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T}NN^{\dagger} \\ 0 & 0 \end{pmatrix}U^*; \tag{2.7}$$

$$(f) \qquad A^{\dagger,D} = U\begin{pmatrix} T^*\Delta & T^*\Delta T^{-k}\widetilde{T} \\ (I_{n-p} - N^{\dagger}N)S^*\Delta & (I_{n-p} - N^{\dagger}N)S^*\Delta T^{-k}\widetilde{T} \end{pmatrix}U^*; \tag{2.8}$$

$$(g) \qquad A^{C,\dagger} = U\begin{pmatrix} T^*\Delta & T^*\Delta T^{-k}\widetilde{T}NN^{\dagger} \\ (I_{n-p} - N^{\dagger}N)S^*\Delta & (I_{n-p} - N^{\dagger}N)S^*\Delta T^{-k}\widetilde{T}NN^{\dagger} \end{pmatrix}U^*, \tag{2.9}$$

$$\textit{where} \quad \widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j} \quad \textit{and} \quad \Delta = [TT^* + S(I_{n-p} - N^{\dagger}N)S^*]^{-1}. \tag{2.10}$$

The notation $\widetilde{T}$ and Δ are often used as follows.

Lemma 2.3 [10, 21, 22] *Let $A \in \mathbb{C}_k^{n\times n}$ be as in (2.1). Then*

$$(a) \qquad AA^{\dagger} = U\begin{pmatrix} I_p & 0 \\ 0 & NN^{\dagger} \end{pmatrix}U^*; \tag{2.11}$$

$$(b) \qquad AA^{\text{Ⓣ}} = P_{A^k} = U\begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}U^*; \tag{2.12}$$

$$(c) \qquad A^{\text{Ⓦ}} = (A^{\text{Ⓣ}})^2 A = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^*. \tag{2.13}$$

Lemma 2.4 [25] *Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times l}$ and $Y \in \mathbb{C}^{l\times m}$. Then the following conditions are equivalent:*

(a) $\lim\limits_{\lambda\to 0} X(\lambda I_l + YAX)^{-1}Y$ *exists;*

(b) $r(XYAXY) = r(XY)$;

(c) $A^{(2)}_{\mathcal{R}(XY),\mathcal{N}(XY)}$ *exists,*

in which case,

$$\lim_{\lambda\to 0} X(\lambda I_l + YAX)^{-1}Y = A^{(2)}_{\mathcal{R}(XY),\mathcal{N}(XY)}.$$

Lemma 2.5 [16] *Let $A \in \mathbb{C}_k^{n\times n}$. Then*

$$A^{\text{Ⓦ}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)} = A^{(2)}_{\mathcal{R}(A^k(A^k)^*A),\mathcal{N}(A^k(A^k)^*A)}.$$

Lemma 2.6 [17] *Let $A \in \mathbb{C}_k^{n\times n}$. Then the following statements hold:*

(a) $AA^{\text{Ⓦ}} = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$;

(b) $A^{\text{Ⓦ}}A = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)}$.

3. Some characterizations of the weak group inverse

Using the results of Lemma 2.5, we have $\mathcal{R}(A^{\textcircled{W}}) = \mathcal{R}(A^k)$ and $\mathcal{N}(A^{\textcircled{W}}) = \mathcal{N}((A^k)^*A)$. Now, we will give several different characterizations of the weak group inverse for a matrix A.

Theorem 3.1 *Let $A \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*
(a) $X = A^{\textcircled{W}}$;
(b) $\mathcal{R}(X) = \mathcal{R}(A^k), \mathcal{N}(X) = \mathcal{N}((A^k)^*A)$ and $AX^2 = X$;
(c) $\mathcal{R}(X) = \mathcal{R}(A^k), \mathcal{N}(X) = \mathcal{N}((A^k)^*A)$ and $XA^{k+1} = A^k$;
(d) $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $AX = A^{\textcircled{\dagger}}A$;
(e) $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $A^2X = P_{A^k}A$;
(f) $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $(A^k)^*A^2X = (A^k)^*A$.

Proof. $(a) \Rightarrow (b)$. The assertion follows directly from Lemma 2.5 and the definition of $A^{\textcircled{W}}$.
$(b) \Rightarrow (c)$. From (2.1), we can calculate that

$$A^k = U\begin{pmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{pmatrix}U^*,$$

where $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$.

Let $X = U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}U^*$, from $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $AX^2 = X$, we have

$$X = U\begin{pmatrix} T^{-1} & X_2 \\ 0 & 0 \end{pmatrix}U^*,$$

where $X_2 \in \mathbb{C}^{p\times(n-p)}$, $p = r(A^k)$.

Hence, $XA^{k+1} = U\begin{pmatrix} T^{-1} & X_2 \\ 0 & 0 \end{pmatrix}U^* \cdot U\begin{pmatrix} T^{k+1} & T\widetilde{T} \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{pmatrix}U^* = A^k.$

$(c) \Rightarrow (a)$. By $\mathcal{R}(X) = \mathcal{R}(A^k)$, we have $X = A^kL$ for some $L \in \mathbb{C}^{n\times n}$. Then we get that $XAX = X$ since $XA^{k+1} = A^k$. Hence, $X = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)} = A^{\textcircled{W}}$ by Lemma 2.5.

$(a) \Rightarrow (d)$. It can be obtained directly from Lemma 2.5 and the definition of $A^{\textcircled{W}}$.

$(d) \Rightarrow (e)$. Notice that $AA^{\textcircled{\dagger}} = P_{A^k}$, which is obvious.

$(e) \Rightarrow (f)$. From the condition, it follows that $(A^k)^*A^2X = (A^k)^*P_{A^k}A = (A^k)^*A$.

$(f) \Rightarrow (a)$. From (2.1) and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we can let $X = U\begin{pmatrix} X_1 & X_2 \\ 0 & 0 \end{pmatrix}U^*$, where $X_1 \in \mathbb{C}^{p\times p}$ and $X_2 \in \mathbb{C}^{p\times(n-p)}$, $p = r(A^k)$. And by $(A^k)^*A^2X = (A^k)^*A$, we have $X_1 = T^{-1}, X_2 = T^{-2}S$.

Hence, $X = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = A^{\textcircled{W}}$. □

In the following theorem, we show the other characterizations of weak group inverse by $A^{\textcircled{W}}AA^{\textcircled{W}} = A^{\textcircled{W}}$.

Theorem 3.2 *Let $A \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*
(a) $X = A^{\textcircled{w}}$;
(b) $XAX = X, \mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(X) = \mathcal{N}((A^k)^*A)$;
(c) $XAX = X, \mathcal{R}(XA) = \mathcal{R}(A^k)$ *and* $AX = A^{\textcircled{\dagger}}A$;
(d) $XAX = X, \mathcal{R}(X) = \mathcal{R}(A^k)$ *and* $A^*A^2X \in \mathbb{C}_n^H$.

Proof. $(a) \Rightarrow (b)$. We can obtain the assertion from Lemma 2.5.

$(b) \Rightarrow (c)$. By $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(A^k)$, we can obtain that $\mathcal{R}(AX) = A\mathcal{R}(X) = A\mathcal{R}(A^k) = \mathcal{R}(A^{k+1}) = \mathcal{R}(A^k) = \mathcal{R}(A^{\textcircled{\dagger}}A)$ and $\mathcal{N}(AX) = \mathcal{N}(X) = \mathcal{N}((A^k)^*A) = \mathcal{N}(A^{\textcircled{\dagger}}A)$. Hence $AX = A^{\textcircled{\dagger}}A$, since $AX, A^{\textcircled{\dagger}}A \in \mathbb{C}_n^P$.

From $\mathcal{R}(X) = \mathcal{R}(A^k)$ we get $A^k = P_X A^k$. Hence

$$A^k = P_X A^k = XX^\dagger A^k = XAXX^\dagger A^k = XA^{k+1}.$$

Thus, we have $\mathcal{R}(A^k) = \mathcal{R}(XA^{k+1}) \subseteq \mathcal{R}(XA) \subseteq \mathcal{R}(X) = \mathcal{R}(A^k)$, we get $\mathcal{R}(XA) = \mathcal{R}(A^k)$.

$(c) \Rightarrow (d)$. Notice that $AA^{\textcircled{\dagger}} = P_{A^k} \in \mathbb{C}_n^H$, then $A^*A^2X = A^*(AA^{\textcircled{\dagger}})A \in \mathbb{C}_n^H$. From $XAX = X$ and $\mathcal{R}(XA) = \mathcal{R}(A^k)$, we obtain $\mathcal{R}(X) = \mathcal{R}(A^k)$.

$(d) \Rightarrow (a)$. Using the core-EP decomposition of $A \in \mathbb{C}_k^{n\times n}$ (2.1), we partition X as follows:

$$X = U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}U^*,$$

where $X_1 \in \mathbb{C}^{p\times p}$, $X_2 \in \mathbb{C}^{p\times(n-p)}$, $X_3 \in \mathbb{C}^{(n-p)\times p}$, $X_4 \in \mathbb{C}^{(n-p)\times(n-p)}$ and $p = r(A^k)$.

By $\mathcal{R}(X) = \mathcal{R}(A^k)$, we get that $X = U\begin{pmatrix} X_1 & X_2 \\ 0 & 0 \end{pmatrix}U^*$ and $r(X_1, X_2) = p$, the matrix (X_1, X_2) is full row rank. Thus, we have $X_1T = I_p$ by $XAX = X$. Hence $X_1 = T^{-1}$. Again note that $A^*A^2X \in \mathbb{C}_n^H$, moreover

$$A^*A^2X = U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}^*\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}^2\begin{pmatrix} T^{-1} & X_2 \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^*T & T^*T^2X_2 \\ S^*T & S^*T^2X_2 \end{pmatrix}U^*,$$

we have $S^*T = (T^*T^2X_2)^*$ and $X_2 = T^{-2}S$. Hence $X = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = A^{\textcircled{w}}$. □

From Theorem 3.2, by $\mathcal{R}(A^{\textcircled{w}}) = \mathcal{R}(A^k)$ we know that $A^{\textcircled{w}}$ also satisfies the condition (7) in the definition of Drazin inverse. In the following theorem, we show some characterizations of $A^{\textcircled{w}}$ by $A^{\textcircled{w}}A^{k+1} = A^k$.

Theorem 3.3 *Let $A \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*
(a) $X = A^{\textcircled{w}}$;
(b) $XA^{k+1} = A^k, AX^2 = X$ *and* $A^2X = P_{A^k}A$;
(c) $XA^{k+1} = A^k, AX = A^{\textcircled{\dagger}}A$ *and* $r(X) = r(A^k)$.

Proof. $(a) \Rightarrow (b)$. By (2.13) and simple calculation, it is obvious.

$(b) \Rightarrow (c)$. Since $AX^2 = X$, by induction it then follows that $X = AX^2 = AXX = A(AX^2)X = A^2X^3 = \cdots = A^kX^{k+1}$. Thus, by $A^k = U\begin{pmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{pmatrix}U^*$, where $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$, we can let $X = U\begin{pmatrix} X_1 & X_2 \\ 0 & 0 \end{pmatrix}U^*$, where $X_1 \in \mathbb{C}^{p\times p}$, $X_2 \in \mathbb{C}^{p\times(n-p)}$ and $p = r(A^k)$.

By conditions $XA^{k+1} = A^k$ and $A^2X = P_{A^k}A$, we get

$$X = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^*$$

Thus $AX = A^{\textcircled{\dagger}}$ and $r(X) = r(A^k)$.

$(c) \Rightarrow (a)$. From $XA^{k+1} = A^k$ and $r(X) = r(A^k)$, we have $\mathcal{R}(X) = \mathcal{R}(A^k)$. $XAX = X$ and $\mathcal{R}(XA) = \mathcal{R}(A^k)$. Thus, by (c) of Theorem 3.2, we obtain that $X = A^{\textcircled{W}}$. □

Remark 3.4 Notice that the condition $XA^{k+1} = A^k$ in items (b) and (c) of Theorem 3.3 can be replaced by $XA^{l+1} = A^l (l \geq k)$. Also the condition $AX^2 = X$ in items (b) of Theorem 3.3 can be replaced by $X = A^tX^{t+1} (t \geq 1)$.

By Lemma 2.6, it is clear that $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)}$ when $X = A^{\textcircled{W}}$. However, the conditions $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)}$ can not deduce that $X = A^{\textcircled{W}}$. We will present the following example to illustrate that.

Example 3.5 Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{then} \quad A^{\textcircled{W}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $k = Ind(A) = 2$, $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}$ and $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)}$. However, $X \neq A^{\textcircled{W}}$.

But we have the following result.

Theorem 3.6 *Let $A \in \mathbb{C}_k^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$. Then $X = A^{\textcircled{W}}$ if and only if X satisfies the following:*

$$AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}, \ XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)} \ \text{ and } \ r(X) = r(A^k).$$

Proof. If $X = A^{\textcircled{W}}$, we can obtain the assertion directly by Lemma 2.6 and Theorem 3.3.

Conversely, by conditions $\mathcal{R}(XA) = \mathcal{R}(A^k)$ and $r(X) = r(A^k)$, we get $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $XAX = X$. Thus, $\mathcal{N}(X) = \mathcal{N}(AX) = \mathcal{N}((A^k)^*A)$. Therefore, $X = A^{\textcircled{W}}$ by Lemma 2.5. □

The definition of $A^{\textcircled{W}}$ has been introduced from an algebraic approach. In [17], Dijana Mosic and Daochang Zhang characterized $A^{\textcircled{W}}$ by $AX = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A)}, \mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. In the following, we characterize $A^{\textcircled{w}}$ by the condition $XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)}$.

Theorem 3.7 *Let $A \in \mathbb{C}_k^{n\times n}$. Then $A^{\textcircled{w}}$ is the unique matrix X that satisfies:*

$$XA = P_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*A^2)}, \quad \mathcal{N}(X) \supseteq \mathcal{N}((A^k)^*A). \tag{3.1}$$

Proof. $A^{\textcircled{w}}$ satisfies the two equations in (3.1) by Lemma 2.5 and Lemma 2.6. It remains to prove the uniqueness.

Suppose that X_1, X_2 satisfy (3.1). Then $X_1A = X_2A$, $\mathcal{N}(X_1) \supseteq \mathcal{N}((A^k)^*A)$ and $\mathcal{N}(X_2) \supseteq \mathcal{N}((A^k)^*A)$, we first show that $\mathcal{N}((A^k)^*) \cap \mathcal{R}(A^*A^k) = \{0\}$. For any $\eta \in \mathcal{N}((A^k)^*) \cap \mathcal{R}(A^*A^k)$, we get $(A^k)^*\eta = 0$, $\eta = A^*A^k\xi$ for some $\xi \in \mathbb{C}^n$. Since $Ind(A) = k$, then $A^k\xi = A^{k+1}\xi_0$ for some $\xi_0 \in \mathbb{C}^n$. Since $0 = (A^k)^*\eta = (A^{k+1})^*A^{k+1}\xi_0$, we get $A^{k+1}\xi_0 = 0$, that is $\eta = A^*A^k\xi = A^*A^{k+1}\xi_0 = 0$. Hence, $\mathcal{N}((A^k)^*) \cap \mathcal{R}(A^*A^k) = \{0\}$.

From $(X_1 - X_2)A = 0$, we get $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{N}(A^*) \subseteq \mathcal{N}((A^k)^*)$. From $\mathcal{N}(X_1) \supseteq \mathcal{N}((A^k)^*A)$ and $\mathcal{N}(X_2) \supseteq \mathcal{N}((A^k)^*A)$, we get $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{R}(A^*A^k)$, that is $\mathcal{R}(X_1^* - X_2^*) \subseteq \mathcal{N}((A^k)^*) \cap \mathcal{R}(A^*A^k) = \{0\}$. Hence, $X_1^* = X_2^*$ and $X_1 = X_2$. $\square$

Bott-Duffin defined the B-D inverse of $A \in \mathbb{C}^{n\times n}$ by $A_{(L)}^{(-1)} = P_L(AP_L + P_{L^\perp})^{-1} = P_L(AP_L + I - P_L)^{-1}$ when $AP_L + P_{L^\perp}$ is nonsingular (see [5]). In [12], $A^{\textcircled{\dagger}}$ is expressed by the B-D inverse. In the following, we use a special B-D inverse of A^2 to express the weak group inverse of A.

Theorem 3.8 *Let $A \in \mathbb{C}_k^{n\times n}$. Then*

$$A^{\textcircled{w}} = (A^2)_{(\mathcal{R}(A^k))}^{(-1)}A = (P_{A^k}A^2P_{A^k})^+A.$$

Proof. By (2.1), we have $A^k = U\begin{pmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{pmatrix}U^*$, where $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$. Then $P_{A^k} = A^k(A^k)^+ = U\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}U^*$. Thus

$$\begin{aligned}
(A^2)_{(\mathcal{R}(A^k))}^{(-1)}A &= P_{A^k}(A^2P_{A^k} + I - P_{A^k})^{-1}A \\
&= U\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T^2 & 0 \\ 0 & I \end{pmatrix}^{-1} \cdot \begin{pmatrix} T & S \\ 0 & N \end{pmatrix}U^* \\
&= U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = A^{\textcircled{w}}.
\end{aligned}$$

By direct calculation, we can get $A^{\textcircled{w}} = (P_{A^k}A^2P_{A^k})^+A$. $\square$

Example 3.9 (see [16], Example 3.1) Let $A = \begin{pmatrix} 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{pmatrix}$.

Since $Ind(A) = 2$, by Theorem 2.1 of [16] the error-free weak group inverse of A is

$$A^{\textcircled{w}} = A^2(A^4)^+A = \begin{bmatrix} \frac{205}{338} & \frac{205}{338} & -\frac{705}{1352} & -\frac{895}{676} & -\frac{645}{1352} & \frac{125}{338} \\ -\frac{23}{338} & -\frac{23}{338} & \frac{10501}{104104} & \frac{11351}{52052} & \frac{10873}{104104} & -\frac{239}{26026} \\ -\frac{4}{13} & -\frac{4}{13} & \frac{313}{1001} & \frac{1501}{2002} & \frac{332}{1001} & -\frac{173}{2002} \\ -\frac{5}{26} & -\frac{5}{26} & \frac{21}{104} & \frac{25}{52} & \frac{17}{104} & -\frac{2}{13} \\ -\frac{1}{13} & -\frac{1}{13} & \frac{365}{4004} & \frac{212}{1001} & -\frac{19}{4004} & -\frac{443}{2002} \\ \frac{1}{26} & \frac{1}{26} & -\frac{157}{8008} & -\frac{229}{4004} & -\frac{1385}{8008} & -\frac{289}{1001} \end{bmatrix}.$$

On the other hand,

$$A^2 = \begin{pmatrix} 65 & 65 & 60 & 50 & 30 & 10 \\ 56 & 56 & 52 & 44 & 26 & 8 \\ 44 & 44 & 41 & 35 & 20 & 5 \\ 29 & 29 & 27 & 23 & 14 & 5 \\ 14 & 14 & 13 & 11 & 8 & 5 \\ -1 & -1 & -1 & -1 & 2 & 5 \end{pmatrix},$$

$$(A^2)^+ = \begin{pmatrix} 6003/21868 & -421/5467 & -323/1735 & -329/2482 & -376/4763 & -802/31663 \\ 6003/21868 & -421/5467 & -323/1735 & -329/2482 & -376/4763 & -802/31663 \\ -117/21868 & 41/10934 & 169/19880 & 69/31240 & -17/4163 & -167/16095 \\ -508/899 & 275/1664 & 1030/2589 & 697/2565 & 772/5301 & 179/9165 \\ -235/994 & 67/994 & 305/1988 & 35/284 & 185/1988 & 125/1988 \\ 196/2125 & -109/3579 & -476/5231 & -143/5662 & 164/4051 & 227/2137 \end{pmatrix},$$

$$P_{A^2} = A^2(A^2)^+ = \begin{pmatrix} 145/154 & 15/77 & -3/28 & -3/44 & -9/308 \\ 15/77 & 27/77 & 5/14 & 5/22 & 15/154 \\ -3/28 & 5/14 & 141/280 & 11/40 & 13/280 \\ -3/44 & 5/22 & 11/40 & 97/440 & 73/440 \\ -9/308 & 15/154 & 13/280 & 73/440 & 502/1759 \\ 3/308 & -5/154 & -51/280 & 49/440 & 1247/3080 \end{pmatrix},$$

$$(A^2P_{A^2} + I - P_{A^2})^{-1}$$
$$= \begin{pmatrix} 1483/1250 & -167/321 & -1068/1565 & -1304/2775 & -1271/4938 & -179/3989 \\ -383/1053 & 197/281 & -393/1691 & -554/3763 & -185/2982 & 107/4623 \\ -601/1171 & -174/1001 & 687/728 & 19/1144 & 160/1787 & 380/2339 \\ -373/1144 & -16/143 & 1/1040 & 1601/1649 & -157/2653 & -298/3339 \\ -139/1001 & -50/1001 & 53/910 & -107/1430 & 823/1039 & -775/2273 \\ 387/8008 & 12/1001 & 291/2519 & -774/6421 & -332/931 & 389/955 \end{pmatrix},$$

$$(P_{A^2}A^2P_{A^2})^{+}$$
$$=\begin{pmatrix} 1525/1352 & -55/169 & -424/537 & -671/1247 & -456/1591 & -95/2704 \\ -888/5257 & -134/2591 & 59/473 & 193/2411 & 317/8964 & -190/20381 \\ -621/1001 & 367/2002 & 407/910 & 417/1430 & 665/4891 & -197/10010 \\ -41/104 & 3/26 & 287/1040 & 199/1040 & 111/1040 & 23/1040 \\ -673/4004 & 95/2002 & 381/3640 & 521/5720 & 467/6026 & 184/2879 \\ 158/2721 & -41/2002 & -97/1456 & -21/2288 & 773/16016 & 163/1542 \end{pmatrix}.$$

After simplification, it follows that $A^{(2)}_{(\mathcal{R}(A^2))}A = A^{ⓦ}$ and $(P_{A^2}A^2P_{A^2})^{+}A = A^{ⓦ}$.

In Theorem 2.2 of [16] the authors proved that $A^{ⓦ} = (A+P)^{-1}(I-Q) = (A-P)^{-1}(I-Q)$ by using $P = I - A^{ⓦ}A$ and $Q = I - AA^{ⓦ}$. Now we generalize this equation by other ways.

Theorem 3.10 *Let $A \in \mathbb{C}_k^{n\times n}$, $a \neq 0$, $P = I - A^{ⓦ}A$ and $Q = I - AA^{ⓦ}$. Then the matrices $A + aP$ and $A + aQ$ are invertible. In addition, the following identities hold:*
(a) $A^{ⓦ} = (A + aP)^{-1}(I - Q)$;
(b) $A^{ⓦ} = (I - P)(A + aQ)^{-1}$.

Proof. (a) Firstly, we show that $A + aP$ is invertible by (2.1) and (2.13).
Let $\alpha = U\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \in \mathbb{C}^n$, where $\alpha_1 \in \mathbb{C}^p$, $\alpha \in \mathcal{N}(A + aP)$, then

$$\begin{pmatrix} T & S \\ 0 & N\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = -a\begin{pmatrix} 0 & -T^{-1}S - T^{-2}SN \\ 0 & I\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$$

Thus, $\alpha_2 = 0$ and $\alpha_1 = 0$ as $a \neq 0$, N is nilpotent and T is nonsingular.
By $(A + aP)A^{ⓦ} = [A + a(I - A^{ⓦ}A)]A^{ⓦ} = AA^{ⓦ} = I - Q$, we get $A^{ⓦ} = (A + aP)^{-1}(I - Q)$.
(b) It is similar to the proof of (a).

Remark 3.11 In the part (a) of Theorem 3.10, let $a = \pm 1$, we have Theorem 2.2 of [16].

Example 3.12 Let $A, A^{ⓦ}$ be the same as in example 3.9, $P = I - A^{ⓦ}A$ and $Q = I - AA^{ⓦ}$. Then

$$(A + \frac{1}{2}P)^{-1}$$
$$=\begin{pmatrix} 361/338 & -315/338 & 938/2279 & -1544/1713 & -2419/8788 & 3083/8788 \\ -179/338 & 497/338 & -1031/701 & 742/1147 & 117/1013 & -868/2087 \\ -4/13 & -4/13 & 511/338 & -287/338 & -23/338 & 241/338 \\ -5/26 & -5/26 & -203/676 & 1003/676 & -227/676 & -105/676 \\ -1/13 & -1/13 & -19/169 & -31/169 & 236/169 & -173/169 \\ 1/26 & 1/26 & 51/676 & 101/676 & -589/676 & 73/676 \end{pmatrix},$$

$$(A - \frac{1}{3}Q)^{-1} = \begin{pmatrix} -29/338 & 985/338 & 2763/1352 & -4897/676 & -788/375 & 514/169 \\ 211/338 & -803/338 & -4896/4535 & 485/113 & 1015/801 & -1255/714 \\ -4/13 & -4/13 & -1766/1001 & 7045/2002 & 1025/1001 & -787/535 \\ -5/26 & -5/26 & 81/104 & -41/52 & 101/104 & -7/26 \\ -1/13 & -1/13 & 1289/4004 & 905/1001 & -1147/551 & 650/697 \\ 1/26 & 1/26 & -201/1489 & -699/1733 & 399/461 & -1733/2002 \end{pmatrix},$$

By direct calculation, we have $(A + \frac{1}{2}P)^{-1}(I - Q) = A^{ⓦ}$ and $(I - P)(A - \frac{1}{3}Q)^{-1} = A^{ⓦ}$.

In the following theorem we present a connection between (B, C)-inverse and weak group inverse, it shows that a weak group inverse of a matrix $A \in \mathbb{C}_k^{n\times n}$ is its $(A^k, (A^k)^*A)$-inverse.

Theorem 3.13 *Let $A \in \mathbb{C}_k^{n\times n}$. Then $A^{ⓦ} = A^{(A^k,(A^k)^*A)}$.*

Proof. From the properties of weak group inverse (Lemma 2.5, Lemma 2.6 and Theorem 3.3), it follows that

$$A^{ⓦ}A \cdot A^k = A^{ⓦ}A^{k+1} = A^k, \;\; (A^k)^*AAA^{ⓦ} = (A^k)^*A, \;\; \mathcal{R}(A^{ⓦ}) = \mathcal{R}(A^k), \;\; \mathcal{N}(A^{ⓦ}) = \mathcal{N}((A^k)^*A).$$

Hence $A^{ⓦ} = A^{(A^k,(A^k)^*A)}$. □

In Theorem 3.1 of [16], the limit representation for the weak group inverse is derived using the limit representation of MP-inverse. In the following, the weak group inverse can also be characterized by Lemma 2.4.

Theorem 3.14 *Let $A \in \mathbb{C}_k^{n\times n}$. Then*
(a) $A^{ⓦ} = \lim_{\lambda\to 0} A^k[\lambda I_n + (A^k)^*A^{k+2}]^{-1}(A^k)^*A$;
(b) $A^{ⓦ} = \lim_{\lambda\to 0} A^k(A^k)^*A[\lambda I_n + A^{k+1}(A^k)^*A]^{-1}$;
(c) $A^{ⓦ} = \lim_{\lambda\to 0}[\lambda I_n + A^k(A^k)^*A^2]^{-1}A^k(A^k)^*A$;
(d) $A^{ⓦ} = \lim_{\lambda\to 0} A^k(A^k)^*[\lambda I_n + A^{k+2}(A^k)^*]^{-1}A$.

Proof. In the proof of the theorem, the results of Lemma 2.4 and Lemma 2.5 are used frequently.
(a). Let $X = A^k$, $Y = (A^k)^*A$. We have

$$A^{ⓦ} = A^{(2)}_{\mathcal{R}(A^k(A^k)^*A),\mathcal{N}(A^k(A^k)^*A)} = \lim_{\lambda\to 0} A^k[\lambda I_n + (A^k)^*A^{k+2}]^{-1}(A^k)^*A.$$

(b). Let $X = A^k(A^k)^*A$, $Y = I_n$. Then

$$A^{ⓦ} = A^{(2)}_{\mathcal{R}(A^k(A^k)^*A),\mathcal{N}(A^k(A^k)^*A)} = \lim_{\lambda\to 0} A^k(A^k)^*A[\lambda I_n + A^{k+1}(A^k)^*A]^{-1}.$$

(c). Let $X = I_n$, $Y = A^k(A^k)^*A$. Then

$$A^{ⓦ} = A^{(2)}_{\mathcal{R}(A^k(A^k)^*A),\mathcal{N}(A^k(A^k)^*A)} = \lim_{\lambda\to 0}[\lambda I_n + A^k(A^k)^*A^2]^{-1}A^k(A^k)^*A.$$

(d). Let $X = A^k(A^k)^*$, $Y = A$. Then

$$A^{\textcircled{w}} = A^{(2)}_{\mathcal{R}(A^k(A^k)^*A),\mathcal{N}(A^k(A^k)^*A)} = \lim_{\lambda\to 0} A^k(A^k)^*[\lambda I_n + A^{k+2}(A^k)^*]^{-1}A.$$

□

Example 3.15 Let $A, A^{\textcircled{w}}$ be the same as in example 3.9, $M = A^2[\lambda I_6+(A^2)^*A^4]^{-1}(A^2)^*A = (m_{ij})_{6\times 6}$, where

$$m_{11} = (10\cdot(18079\lambda^2 + 5552072\lambda + 71727040))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{12} = (10\cdot(18079\lambda^2 + 5552072\lambda + 71727040))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{13} = (40\cdot(4131\lambda^2 + 1170248\lambda - 15416940))\ /\ (\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{14} = (10\cdot(13623\lambda^2 + 3506196\lambda - 156574880))\ /\ (\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{15} = (80\cdot(1079\lambda^2 + 406501\lambda - 7052430))\ /\ (\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{16} = (10\cdot(3641\lambda^2 + 2997820\lambda + 43736000))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440);$$

$$m_{21} = (4\cdot(39125\lambda^2 + 10701612\lambda - 20118560))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{22} = (4\cdot(39125\lambda^2 + 10701612\lambda - 20118560))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{23} = (16\cdot(8940\lambda^2 + 2590793\lambda + 7455710))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{24} = (8\cdot(14741\lambda^2 + 4352547\lambda + 32236840))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{25} = (8\cdot(9341\lambda^2 + 8242848\lambda + 15439660))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{26} = (8\cdot(3941\lambda^2 + 12133149\lambda - 1357520))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440);$$

$$m_{31} = (2\cdot(61511\lambda^2 + 13065592\lambda - 181941760))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{32} = (2\cdot(61511\lambda^2 + 13065592\lambda - 181941760))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{33} = (2\cdot(56221\lambda^2 + 15339222\lambda + 184895360))\ /\ (\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{34} = (2\cdot(46351\lambda^2 + 13767572\lambda + 443335360))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{35} = (2\cdot(29377\lambda^2 + 66532814\lambda + 196119040))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{36} = (2\cdot(12403\lambda^2 + 119298056\lambda - 51097280))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440);$$

$$m_{41} = (32\cdot(2547\lambda^2 + 817790\lambda - 7107100))\ /\ (\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{42} = (32\cdot(2547\lambda^2 + 817790\lambda - 7107100))\ /\ (\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{43} = (2\cdot(37247\lambda^2 + 12070490\lambda + 119399280))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{44} = (8\cdot(7677\lambda^2 + 2633554\lambda + 71071000))\ /\ (\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{45} = (2\cdot(19455\lambda^2 - 10346054\lambda + 96656560))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$$
$$m_{46} = -(4\cdot(-4101\lambda^2 + 15613162\lambda + 45485440))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440);$$

$m_{51} = (2\cdot(19993\lambda^2 + 13103688\lambda - 45485440))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{52} = (2\cdot(19993\lambda^2 + 13103688\lambda - 45485440))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{53} = (2\cdot(18273\lambda^2 + 8801758\lambda + 53903200))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{54} = (10\cdot(3013\lambda^2 + 1460172\lambda + 25046528))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{55} = -(2\cdot(-9533\lambda^2 + 87224922\lambda + 2805920))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{56} = -(2\cdot(-4001\lambda^2 + 181750704\lambda + 130844480))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440);$

$m_{61} = (4\cdot(-383\lambda^2 + 6561368\lambda + 11371360))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{62} = (4\cdot(-383\lambda^2 + 6561368\lambda + 11371360))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{63} = -(2\cdot(701\lambda^2 - 5533026\lambda + 11592880))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{64} = -(4\cdot(289\lambda^2 - 2033752\lambda + 16909360))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{65} = -(2\cdot(389\lambda^2 + 164103790\lambda + 102268400))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440),$
$m_{66} = -(8\cdot(50\lambda^2 + 83068771\lambda + 42679520))/(\lambda^3 + 7321998\lambda^2 + 2183534272\lambda + 1182621440);$

By direct calculation, we get

$$\lim_{\lambda\to 0} M = \lim_{\lambda\to 0} A^2[\lambda I_6 + (A^2)^* A^4]^{-1}(A^2)^* A = A^{Ⓦ}.$$

4. Some properties of weak group inverse

In this section, we first give some properties of weak group inverse by core-EP decomposition.

Theorem 4.1 *Let $A \in \mathbb{C}_k^{n\times n}$ be as in (2.1). Then the following statements hold:*

(a) $A^{Ⓦ} = 0 \Longleftrightarrow A$ *is nilpotent;*
(b) $A^{Ⓦ} = A \Longleftrightarrow A^3 = A;$
(c) $A^{Ⓦ} = A^* \Longleftrightarrow T \in \mathbb{C}_p^U$ *and* $A \in \mathbb{C}_n^{EP};$
(d) $A^{Ⓦ} = P_A \Longleftrightarrow A \in \mathbb{C}_n^{OP}.$
(e) $A^{Ⓦ} = P_{A^*} \Longleftrightarrow A \in \mathbb{C}_n^{OP}.$

Proof. Using the core-EP decomposition of A and (2.13).

$$(a).\quad A^{Ⓦ} = 0 \Longleftrightarrow r(A^k) = p = 0$$
$$\Longleftrightarrow A \text{ is nilpotent.}$$
$$(b).\quad A^{Ⓦ} = A \Longleftrightarrow \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T & S \\ 0 & N \end{pmatrix}$$
$$\Longleftrightarrow T^2 = I_p \text{ and } N = 0$$
$$\Longleftrightarrow A^3 = A.$$
$$(c).\quad A^{Ⓦ} = A^* \Longleftrightarrow \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* & 0 \\ S^* & N^* \end{pmatrix}$$

$$
\begin{aligned}
&\iff T^{-1} = T^*, S = 0 \text{ and } N = 0\\
&\iff T \in \mathbb{C}_p^U \text{ and } A \in \mathbb{C}_n^{EP}.
\end{aligned}
$$

$$
\begin{aligned}
(d).\quad A^{\textcircled{W}} = P_A &\iff A^{\textcircled{W}} = AA^\dagger\\
&\iff \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & NN^\dagger \end{pmatrix}\\
&\iff T = I_p, S = 0 \text{ and } N = 0\\
&\iff A \in \mathbb{C}_n^{OP}.\\
(e).\quad A^{\textcircled{W}} = P_{A^*} &\iff A^{\textcircled{W}} = A^\dagger A\\
&\iff \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^*\Delta T & T^*\Delta S - T^*\Delta S N^\dagger N \\ (I_{n-p}-N^\dagger N)S^*\Delta T & N^\dagger N + (I_{n-p}-N^\dagger N)S^*\Delta S(I_{n-p}-N^\dagger N) \end{pmatrix}\\
&\iff S = SN^\dagger N, N^\dagger N = 0, T^{-1} = T^*\Delta T \text{ and } T^{-2}S = T^*\Delta S\\
&\iff N = 0, S = 0 \text{ and } T = I_p\\
&\iff A \in \mathbb{C}_n^{OP}.
\end{aligned}
$$

where $\Delta = [TT^* + S(I_{n-p} - N^\dagger N)S^*]^{-1}$. □

It is well-known that $A^\dagger = A^\#$ if and only if A is EP matrix. By the core-EP decomposition we will give the conditions which ensure that $A^{\textcircled{W}} = X$, where X is one of $A^\#, A^{\textcircled{\#}}, A^\dagger, A^{\textcircled{\dagger}}, A^D, A^\diamond, A^{D,\dagger}, A^{C,\dagger}$ and $A^{\dagger,D}$.

Theorem 4.2 *Let $A \in \mathbb{C}_k^{n\times n}$. Then the following statements hold:*
(a) $A^{\textcircled{W}} = A^\# \iff A \in \mathbb{C}_n^{CM}$;
(b) $A^{\textcircled{W}} = A^{\textcircled{\#}} \iff A \in \mathbb{C}_n^{EP}$;
(c) $A^{\textcircled{W}} = A^\dagger \iff A \in \mathbb{C}_n^{EP}$.

Proof. The proof is based on the core-EP decomposition of A.
(a). From (2.2) and (2.13), we have

$$
\begin{aligned}
A^{\textcircled{W}} = A^\# &\iff \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix} \text{ and } N = 0\\
&\iff A \in \mathbb{C}_n^{CM}.
\end{aligned}
$$

(b). From (2.2) and (2.13), we get that

$$
\begin{aligned}
A^{\textcircled{W}} = A^{\textcircled{\#}} &\iff \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } N = 0\\
&\iff N = 0 \text{ and } S = 0\\
&\iff A \in \mathbb{C}_n^{EP}.
\end{aligned}
$$

(c). From (2.3) and (2.13), we have

$$
\begin{aligned}
A^{\textcircled{w}} = A^{\dagger} &\iff U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^*\Delta & -T^*\Delta S N^{\dagger} \\ (I_{n-p} - N^{\dagger}N)S^*\Delta & N^{\dagger} - (I_{n-p} - N^{\dagger}N)S^*\Delta S N^{\dagger} \end{pmatrix}U^* \\
&\iff T^*\Delta = T^{-1}, -T^*\Delta S N^{\dagger} = T^{-2}S, S = SN^{\dagger}N \text{ and } N^{\dagger} = 0 \\
&\iff N = 0 \text{ and } S = 0 \\
&\iff A \in \mathbb{C}_n^{EP},
\end{aligned}
$$

where $\Delta = [TT^* + S(I_{n-p} - N^{\dagger}N)S^*]^{-1}$. □

Theorem 4.3 *Let $A \in \mathbb{C}_k^{n\times n}$ be as in (2.1). Then the following statements hold:*
(a) $A^{\textcircled{w}} = A^{\textcircled{\dagger}} \iff S = 0$;
(b) $A^{\textcircled{w}} = A^{D} \iff SN = 0$;
(c) $A^{\textcircled{w}} = A^{\diamond} \iff S = 0$ *and* $N^2 = 0$;
(d) $A^{\textcircled{w}} = A^{D,\dagger} \iff SN^2 = 0$ *and* $S = SNN^{\dagger}$;
(e) $A^{\textcircled{w}} = A^{\dagger,D} \iff S = SN^{\dagger}N$ *and* $SN = 0$;
(f) $A^{\textcircled{w}} = A^{C,\dagger} \iff S = SN^{\dagger}N, SN^2 = 0$ *and* $S = SNN^{\dagger}$.

Proof. The core-EP decomposition of A is still our main tool.
(a). From (2.5) and (2.13), the result is obvious.
(b). See [10, Lemma 4.2, Theorem 4.3].
(c). By (2.3), (2.11) and (2.13), we have

$$
\begin{aligned}
A^{\textcircled{w}} = A^{\diamond} &\iff (A^{\textcircled{\dagger}})^2A = (AP_A)^{\dagger} \\
&\iff ((A^{\textcircled{\dagger}})^2A)^{\dagger} = AP_A \\
&\iff \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}^{\dagger} = \begin{pmatrix} T & S \\ 0 & N \end{pmatrix}\begin{pmatrix} I_p & 0 \\ 0 & NN^{\dagger} \end{pmatrix} \\
&\iff \begin{pmatrix} (T^{-1})^*\Delta_2 & 0 \\ (T^{-2}S)^*\Delta_2 & 0 \end{pmatrix} = \begin{pmatrix} T & SNN^{\dagger} \\ 0 & N^2N^{\dagger} \end{pmatrix} \\
&\iff S = 0 \text{ and } N^2N^{\dagger} = 0 \\
&\iff S = 0 \text{ and } N^2 = 0,
\end{aligned}
$$

where $\Delta_2 = [T^{-1}(T^{-1})^* + T^{-2}S(T^{-2}S)^*]^{-1}$.
(d). By (2.7) and (2.13), we get that

$$
\begin{aligned}
A^{\textcircled{w}} = A^{D,\dagger} &\iff U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T}NN^{\dagger} \\ 0 & 0 \end{pmatrix}U^* \\
&\iff T^{-2}S = (T^{k+1})^{-1}\widetilde{T}NN^{\dagger} \\
&\iff SN^2 = 0 \text{ and } S = SNN^{\dagger},
\end{aligned}
$$

where $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$.

(e). By (2.8) and (2.13), we have

$$\begin{aligned} A^{\text{ⓦ}} = A^{\dagger,D} &\iff U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^*\Delta & T^*\Delta T^{-k}\widetilde{T} \\ (I_{n-p}-N^{\dagger}N)S^*\Delta & (I_{n-p}-N^{\dagger}N)S^*\Delta T^{-k}\widetilde{T} \end{pmatrix}U^* \\ &\iff T^{-1} = T^*\Delta, S = SN^{\dagger}N \text{ and } T^{-2}S = T^*\Delta T^{-k}\widetilde{T} \\ &\iff S = SN^{\dagger}N \text{ and } T^{k-1}S = \widetilde{T} \\ &\iff S = SN^{\dagger}N \text{ and } SN = 0, \end{aligned}$$

where $\Delta = [TT^* + S(I_{n-p} - N^{\dagger}N)S^*]^{-1}$ and $\widetilde{T} = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$.

(f). By (2.9) and (2.13), we get that

$$\begin{aligned} A^{\text{ⓦ}} = A^{C,\dagger} &\iff U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} T^*\Delta & T^*\Delta(T^k)^{-1}\widetilde{T}NN^{\dagger} \\ (I_{n-p}-N^{\dagger}N)S^*\Delta & (I_{n-p}-N^{\dagger}N)S^*\Delta(T^k)^{-1}\widetilde{T}NN^{\dagger} \end{pmatrix}U^* \\ &\iff S = SN^{\dagger}N \text{ and } T^{k-1}S = \widetilde{T}NN^{\dagger} \\ &\iff S = SN^{\dagger}N, SN^2 = 0 \text{ and } S = SNN^{\dagger}, \end{aligned}$$

where Δ and $\widetilde{T}$ are the same as in (e). □

5. Different characterizations of weak group matrix

In [23], Wang and Liu introduced the weak group matrix defined by the commutability: $AA^{\text{ⓦ}} = A^{\text{ⓦ}}A$. In this section we shall give different characterizations of weak group matrix by using the core-EP decomposition.

For convenience, we introduce a necessary lemma.

Lemma 5.1 [23] *Let $A \in \mathbb{C}_k^{n\times n}$ be as in (2.1). Then the following statements are equivalent:*
(a) $A \in \mathbb{C}_n^{WG}$;
(b) $SN = 0$;
(c) $(A^2)^{\text{ⓦ}} = (A^{\text{ⓦ}})^2$;
(d) $A^{\text{ⓣ}}A$ *commutes with* $A^{\text{ⓣ}}A^2$;
(e) $r(A^*A^k, (A^*)^2A^k) = r(A^k)$.

Remark 5.2 For (e) of Lemma 5.1, we give a short proof by using the properties of the weak group inverse.

Since $r(A^*A^k) = r((A^*)^2A^k) = r(A^k)$, we get that $r(A^*A^k, (A^*)^2A^k) = r(A^k)$ can be equivalently expressed as $\mathcal{R}(A^*A^k) = \mathcal{R}((A^*)^2A^k)$, that is: $\mathcal{N}((A^k)^*A) = \mathcal{N}((A^k)^*A^2)$.

Hence, $AA^{\text{ⓦ}} = A^{\text{ⓦ}}A$ if and only if $r(A^*A^k, (A^*)^2A^k) = r(A^k)$ by Lemma 2.6.

Theorem 5.3 *Let $A \in \mathbb{C}_k^{n\times n}$. Then the following conditions are equivalent:*
(a) $A \in \mathbb{C}_n^{WG}$;
(b) $(A^{\text{ⓣ}})^tA = (A^{\text{ⓣ}})^{t+1}A^2$ $(t \geq 1)$;

(c) $(A^{\textcircled{\dagger}})^t A$ *commutes with* $(A^{\textcircled{\dagger}})^t A^2$ $(t \geq 1)$.

Proof. By the core-EP decomposition of A, we have

$$A^2 = U \begin{pmatrix} T^2 & TS + SN \\ 0 & N^2 \end{pmatrix} U^* \text{ and } A^{\textcircled{\dagger}} = U \begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

Then, it is easy to prove these conclusions. □

Remark 5.4 Let $t = 1$ in (c) of Theorem 5.3, we get (d) of Lemma 5.1.

Theorem 5.5 *Let* $A \in \mathbb{C}_k^{n\times n}$. *Then the following conditions are equivalent:*
(a) $A \in \mathbb{C}_n^{WG}$;
(b) $A^k A^{\dagger} = A^k A^{\textcircled{\dagger}}$;
(c) $A^k A^{\textcircled{\dagger}} = A^k A^{D,\dagger}$;
(d) $A^{\textcircled{\dagger}} A^k = A^k A^{\textcircled{w}}$;
(e) $AA^D = A^{\textcircled{\dagger}} A$;
(f) $AA^D = AA^{\textcircled{w}}$;
(g) $A^{\textcircled{\dagger}} A = A^{D,\dagger} A$.

Proof. $(a) \Longleftrightarrow (c)$. By (2.5) and (2.7), we get that

$$\begin{aligned}
A^k A^{\textcircled{\dagger}} = A^k A^{D,\dagger} &\Longleftrightarrow \begin{pmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^k & \widetilde{T} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T}NN^+ \\ 0 & 0 \end{pmatrix} \\
&\Longleftrightarrow \begin{pmatrix} T^{k-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^{k-1} & T^{-1}\widetilde{T}NN^+ \\ 0 & 0 \end{pmatrix} \\
&\Longleftrightarrow \widetilde{T}NN^+ = 0 \\
&\Longleftrightarrow SN = 0 \\
&\Longleftrightarrow A \in \mathbb{C}_n^{WG},
\end{aligned}$$

where $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$.

$(a) \Longleftrightarrow (f)$. By (2.4) and (2.13), we have

$$\begin{aligned}
AA^D = AA^{\textcircled{w}} &\Longleftrightarrow \begin{pmatrix} T & S \\ 0 & N \end{pmatrix} \begin{pmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T & S \\ 0 & N \end{pmatrix} \begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix} \\
&\Longleftrightarrow \begin{pmatrix} I & T^{-k}\widetilde{T} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & T^{-1}S \\ 0 & 0 \end{pmatrix} \\
&\Longleftrightarrow T^{-k}\widetilde{T} = T^{-1}S \\
&\Longleftrightarrow SN = 0 \\
&\Longleftrightarrow A \in \mathbb{C}_n^{WG},
\end{aligned}$$

where $\widetilde{T} = \sum_{j=0}^{k-1} T^j S N^{k-1-j}$.

The rest of the proof is similar. □

6. Conclusions

Our goal is to provide some new characterizations and properties of the weak group inverse and the weak group matrix by range space, null space, projection and the Bott-Duffin inverse. We also study the relationships between the weak group inverse and other generalized inverses such as $A^{\#}, A^{\textcircled{\#}}, A^{\dagger}, A^{D}, A^{D,\dagger}$.

We believe that research about the weak group inverse will be very popular in the next years. Some further investigations are proposed as follows:

1. Considering the weak group inverse of finite potent endomorphisms.
2. The applications of the weak group inverse in linear equations and matrix equations.

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Conflict of interest

All authors read and approved the final manuscript. The authors declare no conflict of interest.

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