Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 9277–9289.
DOI:10.3934/math.2021539
Received: 08 April 2021
Accepted: 02 June 2021
Published: 22 June 2021

Research article

Infinitely many solutions for a class of fractional Robin problems with variable exponents

Ramzi Alsaedi*

Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: ramzialsaedi@yahoo.co.uk; Tel: +966555519822.

Abstract: In this paper, we are concerned with a class of fractional Robin problems with variable exponents. Their main feature is that the associated Euler equation is driven by the fractional $p(\cdot)-$Laplacian operator with variable coefficient while the boundary condition is of Robin type. This paper is a continuation of the recent work established by A. Bahrouni, V. Radulescu and P. Winkert [5].

Keywords: fracional Sobolev spaces; variable exponents; Robin; variational methods
Mathematics Subject Classification: 35D30, 35J20, 35J60, 35P15, 35P30, 35R11, 46E35

1. Introduction

Fractional Sobolev spaces have major applications to various nonlinear problems, including phase transitions, thin obstacle problem, anomalous diffusion, crystal dislocation, semipermeable membranes and flame propagation, ultra-relativistic limits of quantum mechanics, minimal surfaces, water waves, etc. For more details, we refer the readers to Di Nezza, Palatucci and Valdinoci [21]. More recently, the works of Caffarelli et al. [9–11], led to a large amount of papers involving the fractional diffusion operator $(-\Delta)^s$ $(0 < s < 1)$. The cited results turn out to be very fruitful in order to recover an elliptic PDE approach in a nonlocal framework, and they have recently been used very often, see [1,6,7,16,18,24,26,27]. We mention that there are also a great number of results which do not survive in the fractional framework, such as the ones mentioned in [13,14].

On the other hand, the study of PDE's involving variable exponents has become very attractive in recent decades, see [15,17,19,23,25,30] and the references therein.

It is therefore a natural question to see which results "survive" when the $p(x)-$Laplacian is replaced by the fractional $p(x)-$Laplacian.

As far as we know, the first result about the fractional Sobolev spaces with variable exponent of the form $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ and the fractional $p(x)-$Laplacian is obtained by Kaufmann-Rossi-Vidal in

[22]. In particular it is shown that theses spaces are compactly embedded into variable exponent Lebesgue spaces. They also study the existence existence of solution for nonlocal problems involving the fractional $p(\cdot,\cdot)-$Laplacian. Bahrouni-Radulescu [2] obtained some further qualitative properties of the fractional Sobolev spaces and the fractional $p(\cdot,\cdot)-$Laplacian. Further developments have been done by Bahrouni, Ho, Biswas, Chung, Zhang, see [3–5, 8, 12, 20, 29].

The main goal of this paper is to study the existence of infinitely many solutions for fractional $p(\cdot,\cdot)-$Laplacian equation with nonlocal Robin boundary condition. Precisely, we deal with the following problem

$$\begin{aligned} (-\Delta)^s_{p(\cdot,\cdot)} u + |u|^{\overline{p}(x)-2}u &= a(x)|u|^{q(x)-2}u && \text{in } \Omega, \\ \mathcal{N}_{s,p(\cdot,\cdot)}u + \beta(x)|u|^{\overline{p}(x)-2}u &= 0 && \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N > 1$, is a bounded domain with Lipschitz boundary, $a, q \in L^\infty(\Omega)$, $s \in (0,1)$, $p\colon \mathbb{R}^{2N} \to (1,+\infty)$ is a symmetric, continuous function bounded away from 1, $\overline{p}(\cdot) = p(\cdot,\cdot)$, $\beta \in L^\infty(\mathbb{R}^N \setminus \Omega)$ with $\beta \geq 0$ in $\mathbb{R}^N \setminus \Omega$ and $(-\Delta)^s_{p(\cdot,\cdot)}$ stands for the fractional $p(\cdot,\cdot)-$Laplacian which is given by

$$(-\Delta)^s_{p(\cdot,\cdot)} u(x) = \text{p. v.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\, dy \quad \text{for } x \in \Omega. \tag{1.2}$$

Furthermore, $\mathcal{N}_{s,p(\cdot,\cdot)}$ is defined by

$$\mathcal{N}_{s,p(\cdot,\cdot)}u(x) = \int_{\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\, dy \quad \text{for } x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{1.3}$$

and denotes the nonlocal normal $p(\cdot,\cdot)$-derivative (or $p(\cdot,\cdot)-$Neumann boundary condition) and describes the natural Neumann boundary condition in presence of the fractional $p(\cdot,\cdot)-$Laplacian. We would like to mention that the nonlocal normal derivative was introduced for the first time by A. Bahrouni, V. Radulescu and P. Winkert in [5]. This paper can be considered as a continuation of this study. Precisely, using variational methods, we will prove the existence of infinitely many solutions of Eq (1.1).

Now, we recall some results obtained by U. Kaufmann et al. [22]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $s \in (0,1)$, $q \in C(\overline{\Omega}, \mathbb{R})$, and $p \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$. Throughout this paper, we assume that

$$1 < p(x,y) = p(y,x) < \frac{N}{s}, \ \forall\, (x,y) \in \overline{\Omega} \times \overline{\Omega} \tag{P'}$$

and

$$1 < q(x) < \frac{Np(x,x)}{N - sp(x,x)} =: p_s^*(x), \ \forall\, x \in \overline{\Omega}. \tag{Q'}$$

We define the fractional Sobolev space with variable exponents $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ as

$$W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) = \left\{ u \in L^{q(\cdot)}(\Omega) : \ \exists\, \lambda > 0, \ \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy < \infty \right\}.$$

Let

$$[u]_{s,p(\cdot,\cdot),\Omega} = \inf \left\{ \lambda > 0 : \ \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy \leq 1 \right\}$$

be the corresponding variable exponent Gagliardo seminorm. For brevity, we denote $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ by E for a general $q \in C(\overline{\Omega}, \mathbb{R})$ satisfying (Q') and by $W^{s,p(\cdot,\cdot)}(\Omega)$ when $q(x) = p(x, x)$ on $\overline{\Omega}$. We equip E with the norm

$$\|u\|_E = [u]_{s,p(\cdot,\cdot),\Omega} + \|u\|_{L^{q(\cdot)}(\Omega)}.$$

Then, E becomes a reflexive and separable Banach space.
Now, we are ready to recall a crucial theorem which prove some embedding results was obtained in [22] for the case $q(x) > p(x, x)$ on $\overline{\Omega}$ and then was refined in [20, 29].

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $s \in (0, 1)$. Let $p \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$ and $q \in C(\overline{\Omega}, \mathbb{R})$ satisfy (P') and (Q') with $q(x) \geq p(x, x)$ for all $x \in \overline{\Omega}$. Let $r \in C(\overline{\Omega}, \mathbb{R})$ satisfy*

$$1 < r(x) < p_s^*(x), \ \forall x \in \overline{\Omega}. \tag{R}$$

Then, there exists a constant $C = C(N, s, p, q, r, \Omega)$ such that

$$\|f\|_{L^{r(\cdot)}(\Omega)} \leq C\|f\|_E, \ \forall f \in E.$$

Thus, E is continuously embedded in $L^{r(\cdot)}(\Omega)$. Moreover, this embedding is compact.

From Theorem 1.1 and using assumptions (P') and (Q') with $q(x) \geq p(x, x)$ for all $x \in \overline{\Omega}$, we can deduce that spaces E and $W^{s,p(\cdot,\cdot)}(\Omega)$ actually coincide. Evidently, E is not suitable for studying the fractional $p(., .)$−Laplacian problem with Robin boundary condition and hence, we need to introduce another space as our solution space.

We suppose the following assumptions:

(A) $a \in L^\infty(\Omega)$ and $a > 0$ in Ω.

(S) $s \in \mathbb{R}$ with $s \in (0, 1)$;

(P) $p\colon \mathbb{R}^{2N} \to (1, +\infty)$ is a symmetric, continuous function bounded away from 1, that is,

$$p(x, y) = p(y, x) \quad \text{for all } x, y \in \mathbb{R}^{2N}$$

with

$$1 < p^- := \min_{(x,y)\in\mathbb{R}^{2N}} p(x, y) \leq p(x, y) \leq p^+ := \max_{(x,y)\in\mathbb{R}^{2N}} p(x, y).$$

and $sp^+ < N$;

(β) $\beta \in L^\infty(\mathbb{R}^N \setminus \Omega)$ and $\beta \geq 0$ in $\mathbb{R}^N \setminus \Omega$;

Let $u\colon \mathbb{R}^N \to \mathbb{R}$ be a measurable function and let $\overline{p}(x) = p(x, x)$ for all $x \in \mathbb{R}^{2N}$. We set

$$\|u\|_X := [u]_{s,p(\cdot,\cdot),\mathbb{R}^{2N}\setminus(C\Omega)^2} + \|u\|_{L^{\overline{p}(\cdot)}(\Omega)} + \left\|\beta^{\frac{1}{\overline{p}(\cdot)}} u\right\|_{L^{\overline{p}(\cdot)}(C\Omega)},$$

where $C\Omega = \mathbb{R}^N \setminus \Omega$ and

$$X := \left\{u\colon \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \ \|u\|_X < \infty\right\}.$$

$(X, \|\cdot\|_X)$ is a reflexive and separable Banach space, see [5]. Let us recall the compact embedding result introduced in [22].

Proposition 1.2. *Assume that (S), (P) and (β) hold. Then, for any $r \in C(\overline{\Omega})$ with $1 < r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, there exists a constant $\alpha > 0$ such that*

$$\|u\|_{L^{r(\cdot)}(\Omega)} \leq \alpha \|u\|_X \quad \textit{for all } u \in X.$$

Moreover, this embedding is compact.

Now we give our main result.

Theorem 1.3. *Assume that $q(x) \in (1, p^-)$, for all $x \in \Omega$ and conditions (A), (S), (P) and (β) are fulfilled. Then problem* (1.1) *has infinitely many solutions.*

This paper is organized as follows. In Section 2 we recall some definitions and fundamental properties of the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. In Section 3 we give the proof of Theorem 1.3.

2. Variable exponent spaces and preliminary results

In this section, we recall some definition and basic properties concerning the basic function spaces with variable exponent. We refer to [5, 15, 17, 23, 25, 30] and the references therein.

We start by giving a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$. Next, we consider the following set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}, \mathbb{R}) : \ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, denote

$$p^+ = \sup_{x \in \Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x \in \Omega} p(x)$$

and recall the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ as

$$L^{p(\cdot)}(\Omega) = \left\{ u : \ u \text{ is measurable real-valued function, } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

which is endowed with the following Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \mu > 0 : \ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

It is well known that $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}(\Omega)})$ is a separable reflexive Banach space.
The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

with the norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)} + \|u\|_{p(\cdot)}.$$

Let $L^{q(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, that is, $1/p(x) + 1/q(x) = 1$ for all $x \in \overline{\Omega}$. If $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, then the Hölder-type inequality

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}$$

is satisfied.

Defining the modular function $\rho\colon L^{p(\cdot)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_{\Omega} |u|^{p(x)}\, dx.$$

Then, we have the following crucial result which will be useful in the sequel.

Proposition 2.1. *Assume that $u \in L^{p(\cdot)}(\Omega)$. Then:*

(i) $\|u\|_{p(\cdot)} < 1\ (= 1, > 1) \iff \rho(u) < 1\ (= 1,\ 1)$*;*

(ii) $\|u\|_{p(\cdot)} > 1 \ \Rightarrow\ \|u\|_{p(\cdot)}^{p^-} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p^+}$*;*

(iii) $\|u\|_{p(\cdot)} < 1 \ \Rightarrow\ \|u\|_{p(\cdot)}^{p^+} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p^-}$.

Proposition 2.2. *Assume that $u, u_n \in L^{p(\cdot)}(\Omega)$ with $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\lim_{n\to+\infty} \|u_n - u\|_{p(\cdot)} = 0$*;*

(ii) $\lim_{n\to+\infty} \rho(u_n - u) = 0$*;*

(iii) $u_n(x) \to u(x)$ *a. e. in* Ω *and* $\lim_{n\to+\infty} \rho(u_n) = \rho(u)$.

Now, we introduce the variational setting for problem (1.1). We define the functional $I\colon X \to \mathbb{R}$ by

$$I(u) = \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u(x)-u(y)|^{p(x,y)}}{2p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy + \int_{\Omega} \frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}\, dx \\ + \int_{C\Omega} \frac{\beta(x)|u|^{\overline{p}(x)}}{\overline{p}(x)}\, dx - \int_{\Omega} \frac{a(x)}{q(x)}|u|^{q(x)}\, dx,$$

which is well defined and of class C^1 on X. Clearly, the weak solutions of our main problem (1.1) are exactly the critical points of the Euler-Lagrange functional I.

3. Proof of main result

In this section, we investigate the existence of infinitely many solutions for problem (1.1). It is known that, by [19], there exist $(e_n) \subset X$ and $e_n^* \subset X^*$ such that

$$e_n^*(e_m) = 1 \quad \text{if} \ \ n = m \quad \text{and} \quad e_n^*(e_m) = 0 \ \ \text{if} \ \ n \neq m.$$

It follows that

$$X = \overline{\operatorname{span}}\{e_n,\ n \geq 1\} \quad \text{and} \quad X^* = \overline{\operatorname{span}}\{e_n^*,\ n \geq 1\}.$$

For any integer $k \geq 1$, denote

$$E_k = \operatorname{span}\{e_k\},\quad Y_k = \oplus_{j=1}^{k} E_j \ \ \text{and} \ \ Z_k = \overline{\oplus_{j=k}^{\infty} E_j}.$$

Consider now the functional

$$I_\lambda(u) = J(u) - \lambda K(u),$$

where

$$J(u) = \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u(x)-u(y)|^{p(x,y)}}{2p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega \frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}\,dx$$
$$+ \int_{C\Omega} \frac{\beta(x)|u|^{\overline{p}(x)}v}{\overline{p}(x)}\,dx$$

and

$$K(u) = \int_\Omega a(x)\frac{|u(x)|^{q(x)}}{q(x)}dx.$$

An important ingredient in the proof of Theorem 1.3 is the following version of the fountain theorem, see Zou [31].

Theorem 3.1. *Suppose that the functional I_λ defined above satisfies the following conditions:*
(T_1) *I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$;*
(T_2) *$K(u) \geq 0$, $K(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of X;*
(T_3) *there exist $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq 0 > b_k(\lambda) = \max_{u\in Y_k, \|u\|=r_k} I_\lambda(u) \quad \text{for } \lambda \in [1,2]\,,$$

$$d_k(\lambda) = \inf_{u\in Z_k, \|u\|\leq\rho_k} I_\lambda(u) \to 0 \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

Then there exist a sequence of real numbers (λ_n) converging to 1 and $u(\lambda_n) \in Y_n$ such that $I'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0$ and $(I_{\lambda_n})(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$ as $n \to \infty$. In particular, fixed $k \in \mathbb{N}$, if $(u(\lambda_n))$ has a convergent subsequence to u_k, then I_1 has infinitely many nontrivial critical points $(u_k) \subset X\setminus\{0\}$ satisfying $I_1(u_k) \to 0^-$ as $k \to \infty$.

We start with the following auxiliary property.

Lemma 3.2. *Suppose that condition (A) is satisfied. Then*

$$\beta_k = \sup_{u\in Z_k, \|u\|=1} \int_\Omega a(x)\frac{|u(x)|^{q(x)}}{q(x)}dx \to 0 \ \text{ as } \ k \to +\infty.$$

Proof. It is easy to see that $0 < \beta_{k+1} \leq \beta_k$, so that $\beta_k \to \beta \geq 0$ as $k \to +\infty$. For every $k \geq 0$, by definition of β_k, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\int_\Omega a(x)\frac{|u_k|^{q(x)}}{q(x)}dx > \frac{\beta_k}{2}$. Since $u_k \in Z_k$, it follows that $u_k \rightharpoonup 0$ in X. From Proposition 1.2, we deduce that $\int_\Omega a(x)\frac{|u_k|^{q(x)}}{q(x)}dx \to 0$ as $k \to +\infty$. Thus, $\beta = 0$ and the proof is complete. □

Next, we prove the coercivity of K on finite dimensional subspaces of X.

Lemma 3.3. *Suppose that conditions of Theorem 1.3 are fulfilled. Then $K(u) \to +\infty$ as $\|u\| \to +\infty$ on any finite dimensional subspace of X.*

Proof. Let F be a finite dimensional subspace of X. Put

$$\widetilde{a}(x) = \frac{a(x)}{q(x)}, \quad \forall x \in \Omega.$$

First we show that there exists $\epsilon_1 > 0$ such that

$$m\left\{x \in \Omega;\ \widetilde{a}(x)\,|u|^{q(x)} \geq \epsilon_1\,\|u\|^{q(x)}\right\} \geq \epsilon_1, \quad \forall u \in F\backslash\{0\}. \tag{3.1}$$

Arguing by contradiction, for any positive integer n, there exists $u_n \in F\backslash\{0\}$ such that

$$m\left\{x \in \Omega;\ \widetilde{a}(x)\,|u_n|^{q(x)} \geq \frac{1}{n}\,\|u_n\|^{q(x)}\right\} < \frac{1}{n}. \tag{3.2}$$

Set $v_n(x) = \frac{u_n(x)}{\|u_n\|} \in F\backslash\{0\}$. Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and

$$m\left\{x \in \Omega;\ \widetilde{a}(x)\,|v_n|^{q(x)} \geq \frac{1}{n}\right\} < \frac{1}{n}.$$

We may assume, up to a subsequence, that $v_n \to v_0$ in X for some $v_0 \in F$. Then $\|v_0\| = 1$ and, by Proposition 1.2,

$$\int_\Omega \widetilde{a}(x)\,|v_n - v_0|^{q(x)}\,dx \to 0 \ \text{ as } \ n \to +\infty. \tag{3.3}$$

Claim: There exists $\gamma_0 > 0$ such that

$$m\left\{x \in \Omega;\ \widetilde{a}(x)\,|v_0|^{q(x)} \geq \gamma_0\right\} \geq \gamma_0. \tag{3.4}$$

Otherwise, we have

$$m\left\{x \in \Omega;\ \widetilde{a}(x)\,|v_0|^{q(x)} \geq \frac{1}{n}\right\} = 0, \quad \forall n \in \mathbb{N}.$$

It follows that

$$0 \leq \int_\Omega \widetilde{a}(x)\,|v_0|^{q(x)+1}\,dx < \frac{\|v_0\|_1}{n} \to 0, \ \text{ as } \ n \to +\infty.$$

Hence $v_0 = 0$, which contradicts $\|v_0\| = 1$.

Set

$$\Omega_0 = \left\{x \in \Omega;\ \widetilde{a}(x)\,|v_0|^{q(x)} \geq \gamma_0\right\}, \quad \Omega_n = \left\{x \in \Omega;\ \widetilde{a}(x)\,|v_n|^{q(x)} < \frac{1}{n}\right\}$$

and

$$\Omega_n^c = \left\{x \in \Omega;\ \widetilde{a}(x)\,|v_n|^{q(x)} \geq \frac{1}{n}\right\}.$$

By (3.2) and (3.4), we obtain

$$\begin{aligned} m\left(\Omega_n \cap \Omega_0\right) &= m\left(\Omega_0 \backslash \left(\Omega_n^c \cap \Omega_0\right)\right) \\ &\geq m\left(\Omega_0\right) - m\left(\Omega_n^c \cap \Omega_0\right) \\ &\geq \gamma_0 - \frac{1}{n} > \frac{\gamma_0}{2} \end{aligned}$$

for large enough n. Consequently, for all large n, we have

$$\begin{aligned}\int_\Omega \widetilde{a}(x)\,|v_n - v_0|^{q(x)}\,dx &\ge \int_{\Omega_n\cap\Omega_0} \widetilde{a}(x)\,|v_n - v_0|^{q(x)}\,dx \\ &\ge \frac{1}{2^{q^+-1}}\int_{\Omega_n\cap\Omega_0} \widetilde{a}(x)\,|v_0|^{q(x)}\,dx - \int_{\Omega_n\cap\Omega_0} \widetilde{a}(x)\,|v_n|^{q(x)}\,dx \\ &\ge \left(\frac{\gamma_0}{2^{q^+-1}} - \frac{1}{n}\right) m\,(\Omega_n \cap \Omega_0) \\ &\ge \frac{\gamma_0^2}{2^{q^++1}} > 0,\end{aligned}$$

which is a contradiction to (3.3). Therefore (3.1) holds. For the ϵ_1 given in (3.1), let

$$\Omega_u = \left\{x \in \Omega;\ \widetilde{a}(x)\,|u|^{q(x)} \ge \epsilon_1\,\|u\|^{q(x)}\right\},\quad \forall u \in F\setminus\{0\}.$$

Then

$$m\,(\Omega_u) \ge \epsilon_1 \ \ \forall\ \ u \in F\setminus\{0\}. \tag{3.5}$$

Using (B) and (3.5), for any $u \in F\setminus\{0\}$ with $\|u\| \ge 1$, we infer that

$$\begin{aligned}K(u) = \int_\Omega \widetilde{a}(x)\,|u|^{q(x)}\,dx &\ge \int_{\Omega_u} \widetilde{a}(x)\,|u|^{q(x)}\,dx \\ &\ge \epsilon_1\,\|u\|^{q^-}\,m\,(\Omega_u) \ge \epsilon_1^2\,\|u\|^{q^-}.\end{aligned}$$

This shows that $K(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of X and this gives the proof of our desired result. □

Lemma 3.4. *Suppose that the conditions of Theorem 1.3 are satisfied. Then there exists a sequence $\rho_k \to 0^+$ as $k \to +\infty$ such that*

$$a_k(\lambda) = \inf_{u\in Z_k,\|u\|=\rho_k} I_\lambda(u) \ge 0,\quad \forall k \ge k_1$$

and

$$d_k(\lambda) = \inf_{u\in Z_k,\|u\|\le\rho_k} I_\lambda(u) \to 0 \ \text{ as } \ k \to +\infty \ \text{ uniformly for } \ \lambda \in [1,2].$$

Proof. By Propositions 1.2 and 2.1, we deduce that for any $u \in Z_k$ with $\|u\| < 1$, we have

$$\begin{aligned}I_\lambda(u) \ &\ge \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u(x)-u(y)|^{p(x,y)}}{2p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega \frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}\,dx \\ &\quad + \int_{C\Omega} \frac{\beta(x)|u|^{\overline{p}(x)}v}{\overline{p}(x)}\,dx - \lambda\int_\Omega \frac{a(x)}{q(x)}|u(x)|^{q(x)}dx \\ &\ge \frac{1}{3^{p^+-1}p^+}\|u\|^{p^+} - \lambda\|u\|^{q^+}\int_\Omega \frac{a(x)}{q(x)}\Big(\frac{|u(x)|}{\|u\|}\Big)^{q(x)}dx \\ &\ge \frac{1}{3^{p^+-1}p^+}\|u\|^{p^+} - \frac{2\beta_k}{q^-}\|u\|^{q^+}.\end{aligned} \tag{3.6}$$

We denote $\rho_k = \big(\frac{3^{p^+-1}(4p^+)\beta_k}{q^-}\big)^{\frac{1}{p^+-q^+}}$. By Lemma 3.2 we deduce that $\rho_k \to 0$ as $k \to +\infty$. Then there exists $k_1 \in \mathbb{N}$ such that $\rho_k \le \frac{1}{3^{p^+-1}p^+}$ for all $k \ge k_1$. Relation (3.6) implies that

$$a_k(\lambda) = \inf_{u\in Z_k,\|u\|=\rho_k} I_\lambda(u) \ge \frac{1}{2.3^{p^+-1}p^+}\rho_k^{p^++1},\ \text{ for all }\ k \ge k_1.$$

Furthermore, by (3.6), we have

$$0 \geq \inf_{u\in Z_k,\|u\|\leq\rho_k} I_\lambda(u) \geq -\frac{2\beta_k}{q^-}\|u\|^{q^-}, \quad \forall k \geq k_1.$$

Since $\beta_k \to 0$ as $k \to +\infty$, we deduce that

$$d_k(\lambda) = \inf_{u\in Z_k,\|u\|=\rho_k} I_\lambda(u) \to 0 \text{ as } k \to +\infty \text{ uniformly for } \lambda \in [1.2].$$

This completes the proof. □

Lemma 3.5. *Assume that hypotheses of Theorem 1.3 are fulfilled. Then, for the sequence obtained in Lemma 3.4, there exists* $0 < r_k < \rho_k$ *for all* $k \in \mathbb{N}$ *such that*

$$b_k(\lambda) = \max_{u\in Y_k,\|u\|=r_k} I_\lambda(u) < 0 \ \textit{for all} \ \lambda \in [1,2].$$

Proof. Let $u \in Y_k$ with $\|u\| < 1$ and $\lambda \in [1,2]$. By (A), (P) and (3.1), there exists $\epsilon_k > 0$ such that

$$\begin{aligned}
I_\lambda(u) = &\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|u(x)-u(y)|^{p(x,y)}}{2p(x,y)|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_\Omega \frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}\,dx \\
&+ \int_{C\Omega} \frac{\beta(x)|u|^{\overline{p}(x)}v}{\overline{p}(x)}\,dx - \lambda\int_\Omega a(x)\frac{|u(x)|^{q(x)}}{q(x)}dx \\
\leq &\frac{3}{p^-}\|u\|^{p^-} - \epsilon_k\|u\|^{q^-}m(\Omega_u) \\
\leq &\frac{3}{p^-}\|u\|^{p^-} - \epsilon_k^2\|u\|^{q^-}.
\end{aligned}$$

Since $0 < q^- < q^+ < p^- < p^+$, we deduce that for small $\|u\| = r_k$ we have

$$b_k(\lambda) < 0, \quad \forall k \in \mathbb{N}.$$

This completes the proof of our lemma. □

Proof of Theorem 1.3 completed. It is cleat that condition (T_1) in Theorem 3.1 holds. Combining Lemmas 3.3, 3.4 and 3.5, we concludethat conditions (T_2) and (T_3) in Theorem 3.1 are satisfied. Then, by Theorem 3.1 there exist $\lambda_n \to 1$ and $u(\lambda_n) \in Y_n$ such that

$$I'_{\lambda_n}|Y_n(u(\lambda_n)) = 0, \quad I_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$$

as $n \to +\infty$.

For the sake of notational simplicity, we always set in what follows $u_n = u(\lambda_n)$ for all $n \in \mathbb{N}$.

Claim: the sequence (u_n) is bounded in X.

Otherwise, we can assume that (u_n) is unbounded in X. Without loss of generality, we can assume that $\|u_n\| > 1$ for all $n \geq 1$.

First, we can observe that there exists $c > 0$ such that for large enough n,

$$\langle I'_{\lambda_n}(u_n), u_n\rangle \leq \|u_n\| \text{ and } |I_{\lambda_n}(u_n)| \leq c. \tag{3.7}$$

Using relation (3.7), we have

$$\begin{aligned} c \quad &\geq I_{\lambda_n}(u_n) \geq \frac{1}{2p^+}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \frac{1}{p^+}\int_\Omega |u_n|^{\overline{p}(x)}\,dx \\ &+\tfrac{1}{p^+}\int_{C\Omega}\beta(x)|u_n|^{\overline{p}(x)}\,dx - \tfrac{1}{q^-}\int_\Omega a(x)|u_n(x)|^{q(x)}dx. \end{aligned} \tag{3.8}$$

From Proposition 2.1, relation (3.8) and since $q^+ < p^-$, we get that (u_n) is bounded in X. This proves that our claim is true. So, by Proposition 1.2 and up to a subsequence, we suppose that

$$u_n \rightharpoonup u_0 \ \text{ in } \ X$$

and

$$u_n \to u_0 \ \text{ in } \ L^{q(x)}(\Omega).$$

In what follows we show that

$$u_n \to u_0 \ \text{ in } \ X.$$

Recalling that (u_n) is a bounded sequence, we get

$$\lim_{n\to+\infty}\langle I'_{\lambda_n}(u_n) - I'_{\lambda_n}(u_0), u_n - u_0\rangle = 0. \tag{3.9}$$

Hence, (3.9) and Proposition 1.2 give as $n \to +\infty$

$$\begin{aligned} o(1) &= \langle I'_{\lambda_n}(u_n) - I'_{\lambda_n}(u_0), u_n - u_0\rangle \\ &= \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}A(x,y)\,dx\,dy \\ &+ \int_\Omega [|u_n|^{\overline{p}(x)-2}u_n - |u|^{\overline{p}(x)-2}u](u_n-u)\,dx \quad + \int_{C\Omega}\beta(x)[|u_n|^{\overline{p}(x)-2}u_n - |u|^{\overline{p}(x)-2}u](u_n-u)\,dx \end{aligned}$$

where $A(x,y) = (u_n(x) - u(x) - u_n(y) + u(y))$.
We have for all $n \in \mathbb{N}$

$$\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}A(x,y)\,dx\,dy \geq 0,$$

$$\int_\Omega [|u_n|^{\overline{p}(x)-2}u_n - |u|^{\overline{p}(x)-2}u](u_n-u)\,dx \quad \geq 0,$$

and

$$\int_{C\Omega}\beta(x)[|u_n|^{\overline{p}(x)-2}u_n - |u|^{\overline{p}(x)-2}u](u_n-u)\,dx \geq 0$$

Therefore

$$\lim_{n\to+\infty}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}A(x,y)\,dx\,dy = 0, \tag{3.10}$$

$$\lim_{n\to+\infty}\int_\Omega [|u_n|^{\overline{p}(x)-2}u_n - |u|^{\overline{p}(x)-2}u](u_n-u)\,dx = 0, \tag{3.11}$$

and

$$\lim_{n\to+\infty}\int_{C\Omega}\beta(x)[|u_n|^{\overline{p}(x)-2}u_n - |u|^{\overline{p}(x)-2}u](u_n - u)\,dx = 0. \tag{3.12}$$

Let us now recall the Simon inequalities [28, formula 2.2]

$$\begin{cases} |x-y|^p \le c_p\left(|x|^{p-2}\,x - |y|^{p-2}\,y\right).(x-y) & \text{for } \ p \ge 2 \\ |x-y|^p \le C_p\left[\left(|x|^{p-2}\,x - |y|^{p-2}\,y\right).(x-y)\right]^{\frac{p}{2}}(|x|^p+|y|^p)^{\frac{2-p}{2}} & \text{for } \ 1 < p < 2, \end{cases} \tag{3.13}$$

for all $x, y \in \mathbb{R}^N$, where c_p and C_p are positive constants depending only on p. Combining (3.10), (3.11), (3.12) and (3.13), we conclude that

$$\lim_{n\to+\infty}\|u_n - u_0\| = 0.$$

Now, by Theorem 3.1, we conclude the proof of Theorem 1.3.

Conflict of interest

The author declares no conflict of interest.

References

1. A. Bahrouni, Trudinger-Moser type inequality and existence of solution for perturbed non-local elliptic operators with exponential nonlinearity, *Commun. Pure Appl. Anal.*, **16** (2017), 243–252.
2. A. Bahrouni, V. Rădulescu, On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent, *Discrete Contin. Dyn. S*, **11** (2018), 379–389.
3. A. Bahrouni, Comparaison and sub-supersolution principles for the fractional $p(x)$-Laplacian, *J. Math. Anal. Appl.*, **458** (2018), 1363–1372.
4. A. Bahrouni, K. Ho, Remarks on eigenvalue problems for fractional $p(\cdot)$-Laplacian, *Asymptotic Anal.*, **123** (2021), 139–156.
5. A. Bahrouni, V. Radulescu, P. Winkert, Robin fractional problems with symmetric variable growth, *J. Math. Phys.*, **61** (2020), 101503.
6. G. Molica Bisci, V. Rădulescu, Ground state solutions of scalar field fractional Schrödinger equations, *Calc. Var.*, **54** (2015), 2985–3008.
7. G. Molica Bisci, V. Rădulescu, R. Servadei, *Variational methods for nonlocal fractional problems*, Cambridge: Cambridge University Press, 2016.
8. R. Biswas, S. Tiwari, Variable order nonlocal Choquard problem with variable exponents, *Complex Var. Elliptic*, **66** (2021), 853–875.
9. L. Caffarelli, J. M. Roquejoffre, Y. Sire, Variational problems for free boundaries for the fractional Laplacian, *J. Eur. Math. Soc.*, **12** (2010), 1151–1179.
10. L. Caffarelli, S. Salsa, L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, *Invent. Math.*, **171**, (2008), 425–461.

11. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Commun. Part. Diff. Eq.*, **32** (2007), 1245–1260.

12. N. T. Chung, H. Q. Toan, On a class of fractional Laplacian problems with variable exponents and indefinite weights, *Collect. Math.*, **71** (2020), 223–237.

13. S. Dipierro, O. Savin, E. Valdinoci, All functions are locally sharmonic up to a small error, *J. Eur. Math. Soc.*, **19** (2017), 957–966.

14. S. Dipierro, O. Savin, E. Valdinoci, Boundary behavior of nonlocal minimal surfaces, *J. Funct. Anal.*, **272** (2017), 1791–1851.

15. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Heidelberg: Springer-Verlag, 2011.

16. F. Kamache, R. Guefaifia, S. Boulaaras, Existence of three solutions for perturbed nonlinear fractional p-Laplacian boundary value systems with two control parameters, *J. Pseudo-Differ. Oper. Appl.*, **11** (2020), 1781–1803.

17. X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446.

18. R. Guefaifia, S. Boulaaras, C. Bahri, R. Taha, Infinite existence solutions of fractional systems with Lipschitz nonlinearity, *J. Funct. Space.*, **2020** (2020), 6679101.

19. P. Hajek, V. M. Santalucia, J. Vanderwerff, V. Zizler, *Biorthogonal systems in Banach spaces*, Springer, 2008.

20. K. Ho, Y. H. Kim, A-priori bounds and multiplicity of solutions for nonlinear elliptic problems involving the fractional $p(\cdot)$-Laplacian, *Nonlinear Anal.*, **188** (2019), 179–201.

21. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573.

22. U. Kaufmann, J. D. Rossi, R. Vidal, Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians, *Electron. J. Qual. Theo.*, **76** (2017), 1–10.

23. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czech. Math. J., **41** (1991), 592–618.

24. P. Pucci, M. Xiang, B. Zhang, Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations, *Adv. Nonlinear Anal.*, **5** (2016), 27–55.

25. V. D. Rădulescu, D. D. Repovš, Partial differential equations with variable exponents, Boca Raton, FL: CRC Press, 2015.

26. R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, **389** (2012), 887–898.

27. R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, *DCDS*, **33** (2013), 2105–2137.

28. J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, In: *Lecture Notes in Math. Volume 665*, Berlin: Springer, 1978, 205–227.

29. C. Zhang, X. Zhang, Renormalized solutions for the fractional $p(x)$-Laplacian equation with L^1, *Nonlinear Anal.*, **190** (2020), 111610.

30. V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.*, **29** (1987), 33–66.

31. W. Zou, Variant fountain theorems and their applications, *Manuscripta Math.*, **104** (2001), 343–358.