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Research article

Metric dimension and edge metric dimension of windmill graphs

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Abstract: Graph invariants provide an amazing tool to analyze the abstract structures of graphs. Metric dimension and edge metric dimension as graph invariants have numerous applications, among them are robot navigation, pharmaceutical chemistry, etc. In this article, we compute the metric and edge metric dimension of two classes of windmill graphs such as French windmill graph and Dutch windmill graph, and also certain generalizations of these graphs.

Keywords: resolving set; metric dimension; edge metric dimension; edge metric basis; French windmill graph; Dutch windmill graph
Mathematics Subject Classification: 05C12, 05C40, 05C76, 05C85

1. Introduction

In recent years, there is an increased demand for the application of mathematics. Graph theory has proven to be particularly useful to a large number of rather diverse fields. As a useful tool for dealing with relations of events, graph theory has rapidly grown in theoretical results as well as its applications to real-life problems. One concept that pervades all the graph theory is that of distance, and distance is used in isomorphism testing, graph operations, maximal and minimal problems on connectivity and diameter. Several parameters related to distances in graphs are highly attracting the attention of several researchers. One of them, namely, the metric dimension, has specifically centered several investigations.

The concept of metric dimension was introduced by Slater [23] in 1975 in which he used the term locating set in connection with some location problems in graphs. Harary and Melter [7] also studied the same concept and used the term resolving set. After these two pioneering papers, a lot of work on this invariant has been done concerning applications as well as theory. The families of graphs with constant metric dimensions have been characterized by many different authors, one can see [9, 10]. There are a lot of variants of the standard metric dimension, such as local metric dimension, strong metric dimension, fractional metric dimension, fault-tolerant metric dimension and many more have been studied in [1, 8, 15, 16, 18]. In 2000, Chartrand et al. [3] determined all connected graphs of order n having metric dimension 1, $n-1$ or $n-2$. Some other interesting results on the metric dimension and references can be found in [5, 11, 17, 19, 21].

In 2018, the new parameter edge metric dimension has been introduced by Kelenc [12] in which they determined the value for wheel graphs and fan graphs. They studied that the wheel graphs of order $n \geq 6$ and the fan graphs of order $n \geq 5$ have edge metric dimension $n-2$. Nasir et al. [14] also determined the edge metric dimension for two families when G is an n-sunlet graph and a prism graph. Zubrilina et al. [25] characterized the graphs for which the edge metric dimension of graphs is $n-1$. They also proposed an open problem: For which graphs G of order n, the edge metric dimension is $n-2$? Recently, Wei et al. [24] gave the characterization of all connected bipartite graphs with edge metric dimension $n-2$, which partially answers an open problem of Zubrilina et al. [25]. They also investigated the relationship between the edge metric dimension and the clique number of a graph G.

In this paper, we consider the problem of computing the metric dimension and edge metric dimension of windmill graphs and certain generalizations of these graphs. Applications of this optimization problem arise in diverse areas. See [3] for an application of this problem in pharmaceutical chemistry, [22] for coin weighing problem, [13] for robot navigation, [2] for network discovery and verification, [20] for connected joins in graphs, and [4] for strategies for the mastermind game. See [6] for an application of windmill graphs in networks.

A *graph* $G = (V, E)$ is an ordered pair consisting of a nonempty set $V = V(G)$ of elements called *vertices* and a set $E = E(G)$ of unordered pairs of vertices called *edges*. For distinct vertices v_1 and v_2, the distance between v_1 and v_2, denoted by $d(v_1, v_2)$, is the length of the shortest path connecting v_1 and v_2. Let $d(v, e)$ denotes the distance between edge e and vertex v, defined as $d(v, e) = min\{d(v, a), d(v, b)\}$, where $e = ab$. A vertex v distinguishes two edges e_1 and e_2, if $d(e_1, v) \neq d(e_2, v)$. G is called a *complete graph* if every pair of vertices is joined by exactly one edge. A complete graph of order n is denoted by K_n. A graph G with n vertices ($n \geq 3$) and n edges is called a *cyclic graph* if all its edges form a cycle of length n. It is denoted by C_n. G is a *bipartite* graph means that vertex sets can be partitioned into two subsets U and W, called *partite sets*, such that every edge of G joins a vertex of U and a vertex of W. If every vertex of U is adjacent to every vertex of W, G is called a *complete bipartite graph*, where U and W are independent. A *star graph* is a complete bipartite graph in which $(n-1)$ vertices have a degree 1 and a single vertex has a degree $(n-1)$. It is denoted by S_n.

Let $R = \{r_1, r_2, \ldots, r_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|R)$ of v with respect to R is the k-tuple $(d(v, r_1), d(v, r_2), \ldots, d(v, r_k))$. If distinct vertices of G have distinct representation with respect to R, then R is called a *resolving set* for G. A resolving set of minimum cardinality is a *metric basis* for G, and its cardinality is called the *metric dimension* of G, denoted by $dim(G)$.

Let $R_E = \{x_1, x_2, \ldots, x_k\}$ be an ordered set in $V(G)$ and let $e \in E(G)$. The representation $r(e|R_E)$ of e with respect to R_E is the k-tuple $(d(e, x_1), d(e, x_2), ..., d(e, x_k))$. If distinct edges of G have distinct representation with respect to R_E, then R_E is called an *edge metric generator* for G. An edge metric generator of minimum cardinality is an *edge metric basis* for G, and its cardinality is called *edge metric dimension* of G, denoted by $edim(G)$.

Throughout this paper, the French star windmill graph is denoted by SW_n^m, French cyclic windmill graph by CW_n^m and French complete windmill graph by KW_n^m wherein the shared vertex of French windmill graph (W_n^m) is replaced by star graph, cyclic graph and complete graph respectively. We also denote the Dutch star windmill graph by SD_n^m, Dutch cyclic windmill graph by CD_n^m and Dutch complete windmill graph by KD_n^m wherein the shared vertex of Dutch windmill graph (D_n^m) is replaced by star graph, cyclic graph, and complete graph respectively.

2. French windmill graph

In this section we discuss French windmill graph (see, Figure 1) and certain generalizations of this graph.

The *French windmill graph*, W_n^m , $n \geq 3, m \geq 2$ is the graph obtained by taking m copies of the complete graph (K_n) joined at a shared universal vertex. It has $m(n-1)+1$ vertices and $mn(n-1)/2$ edges. For our purpose, we denote the complete subgraphs of W_n^m by $W_n^i, i = 1, 2, \ldots m$, the shared vertex by c, the vertices of W_n^i by $\{a_1^i, a_2^i, \ldots, a_{n-1}^i, c\}$, V_i, n^*, n^{**} by $\{a_1^i, a_2^i, \ldots, a_{n-1}^i\}$, $\underbrace{n, n, \ldots, n}_{n-2 \text{ times}}$, $\underbrace{n, n, \ldots, n}_{n-1 \text{ times}}$ respectively.

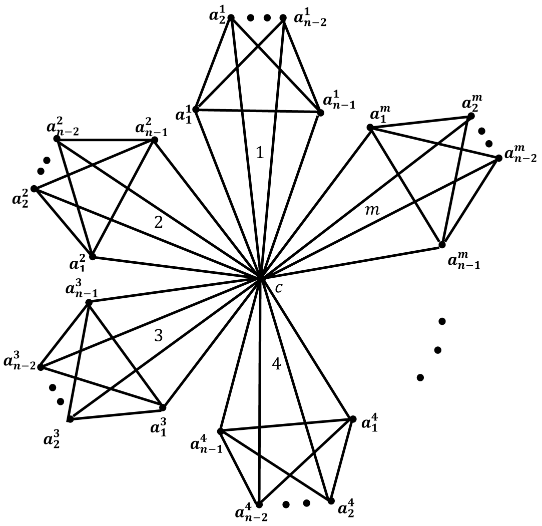

Figure 1. French windmill graph(W_n^m).

In the following theorems, we compute the metric dimension and edge metric dimension of French windmill graph.

Theorem 2.1. *The metric dimension of French windmill graph is $m(n-2)$.*

Proof. Let R be the resolving set of French windmill graph (W_n^m). We can assume that there exist some $x, y \in V_i$ for some i, such that $x, y \notin R$. Then $r(x|R) = r(y|R)$, which is a contradiction. Now, let $R_0 = \{a_1^i, a_2^i, \ldots, a_{n-2}^i\},\ i = 1, 2, \ldots, m$. Representation of vertices of W_n^m with respect to R_0 are

$$\begin{aligned} r(a_{n-1}^i|R_0) &= (2^*, \ldots, \underbrace{1^*}_{i^{th} \text{ tuple}}, 2^*, \ldots, 2^*), \\ r(c|R_0) &= (1^*, 1^*, \ldots, 1^*). \end{aligned}$$

Since there is no vertex having same representation, so R_0 is a resolving set of W_n^m . Hence $dim(W_n^m) = m(n-2)$. □

Theorem 2.2. *The edge metric dimension of French windmill graph is $m(n-1)-1$.*

Proof. Let R_E be the edge metric basis of W_n^m . We claim that it contains all vertices from $\bigcup V_i$ except one. Suppose on the contrary that there exist $x, y \in \bigcup V_i$ such that $x, y \notin R_E$. We have two cases:
1) when $x \in V_i$ and $y \in V_j$, then $r(cx|R_E) = r(cy|R_E)$
2) when $x, y \in V_i$, then $r(cx|R_E) = r(cy|R_E)$
which is a contradiction. Now, let $R_E^{'} = \bigcup_i V_i \setminus \{a_1^1\}$. Representation of edges of W_n^m with respect to $R_E^{'}$ are

$$\begin{aligned} r(a_j^1 a_1^1|R_E^{'}) &= (\underbrace{1,\ldots,0,1,\ldots,1}_{1^{st}\text{ tuple}}, 2^{**},\ldots,2^{**}),\\ r(a_j^i c|R_E^{'}) &= (1^{*},\ldots,\underbrace{1,\ldots,0,1,\ldots,1}_{i^{th}\text{ tuple}}, 1^{**},\ldots,1^{**}),\\ r(a_1^1 c|R_E^{'}) &= (1^{*},1^{**},\ldots,1^{**}). \end{aligned}$$

Since there is no edge having same representation, so $R_E^{'}$ is an edge metric generator of W_n^m. Hence $edim(W_n^m) = m(n-1)-1$. □

The following lemmas show that the metric dimension and edge metric dimension of generalizations of French windmill graph (discussed in this section) is at least $m(n-2)$.

Lemma 2.3. *The metric dimension of SW_n^m, CW_n^m, and KW_n^m is at least $m(n-2)$.*

Proof. We can assume that there exist some $x, y \in V_i$ for some i, such that $x, y \notin R$. Then $r(x|R) = r(y|R)$, which is a contradiction. □

Lemma 2.4. *The edge metric dimension of SW_n^m, CW_n^m, and KW_n^m is at least $m(n-2)$.*

Proof. We can assume that there exist $x, y \in V_i$ for some i, such that $x, y \notin R_E$. Then $r(a_n^i x|R_E) = r(a_n^i y|R_E)$, which is a contradiction. □

2.1. French star windmill graph

Let SW_n^m be a graph obtained by replacing the shared vertex of French windmill graph with a star graph S_m (see, Figure 2). It has $mn+1$ vertices and $\frac{mn(n-1)}{2}+m$ edges. For our sake, we denote the complete subgraphs of SW_n^m by W_n^i and its vertices as $\{a_1^i, a_2^i, \ldots, a_{n-1}^i, a_n^i\}$. In the following results, we compute the metric dimension and edge metric dimension of this graph.

Theorem 2.5. *The metric dimension of SW_n^m is $m(n-2)$.*

Proof. By Lemma 2.3, we have $dim(SW_n^m) \geq m(n-2)$. Now, let $R_0 = \{a_1^i, a_2^i, \ldots, a_{n-2}^i\},\ i = 1, 2, \ldots, m$. Representation of vertices of SW_n^m with respect to R_0 are

$$r(a_{n-1}^i|R_0) = (4^{*},\ldots,\underbrace{1^{*}}_{i^{th}\text{ tuple}},4^{*},\ldots,4^{*}),$$

$$r(a_n^i|R_0) = (3^*, \ldots, \underbrace{1^*}_{i^{th}\text{ tuple}}, 3^*, \ldots, 3^*),$$
$$r(c|R_0) = (2^*, 2^*, \ldots, 2^*).$$

Therefore R_0 is a resolving set of SW_n^m. Hence $dim(SW_n^m) = m(n-2)$. □

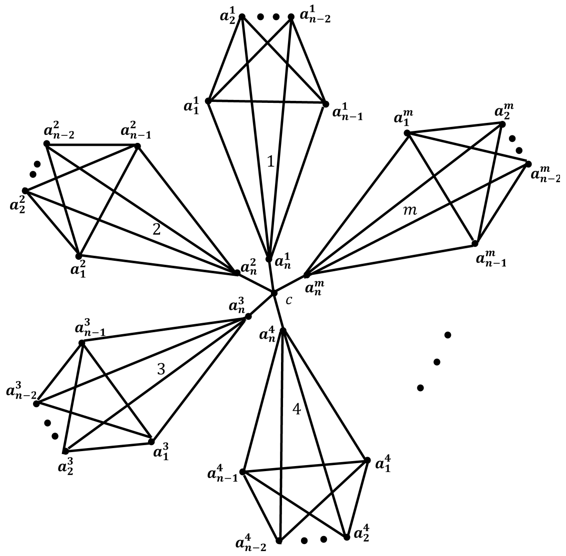

Figure 2. French star windmill graph (SW_n^m).

Theorem 2.6. *The edge metric dimension of SW_n^m is $m(n-2)$.*

Proof. By Lemma 2.4, we have $edim(SW_n^m) \geq m(n-2)$. Now, let $R_E^{'} = \{a_1^i, a_2^i, \ldots, a_{n-2}^i\}$, $i = 1, 2, \ldots, m$. Representation of edges of SW_n^m with respect to $R_E^{'}$ are

$$r(a_j^i a_{n-1}^i|R_E^{'}) = (4^*, \ldots, \underbrace{1, \ldots, 0, 1, \ldots, 1}_{i^{th}\text{ tuple}}, 4^*, \ldots, 4^*),$$
$$r(a_j^i a_n^i|R_E^{'}) = (3^*, \ldots, \underbrace{1, \ldots, 0, 1, \ldots, 1}_{i^{th}\text{ tuple}}, 3^*, \ldots, 3^*),$$
$$r(a_{n-1}^i a_n^i|R_E^{'}) = \{3^*, \ldots, \underbrace{1^*}_{i^{th}\text{ tuple}}, 3^*, \ldots, 3^*\},$$
$$r(ca_n^i|R_E^{'}) = \{2^*, \ldots, \underbrace{1^*}_{i^{th}\text{ tuple}}, 2^*, \ldots, 2^*\}.$$

which implies that $edim(SW_n^m) \leq m(n-2)$. Therefore, $edim(SW_n^m) = m(n-2)$. □

2.2. *French cycle windmill graph*

Let CW_n^m be a graph obtained by replacing the shared vertex of French windmill graph with a cycle graph (see, Figure 3). It has mn vertices and $\frac{mn(n-1)}{2} + m$ edges. For our sake, we denote the complete subgraphs of CW_n^m by W_n^i and its vertices as $a_1^i, a_2^i, \ldots, a_{n-1}^i, a_n^i$. Now we determine the metric dimension and edge metric dimension of this graph.

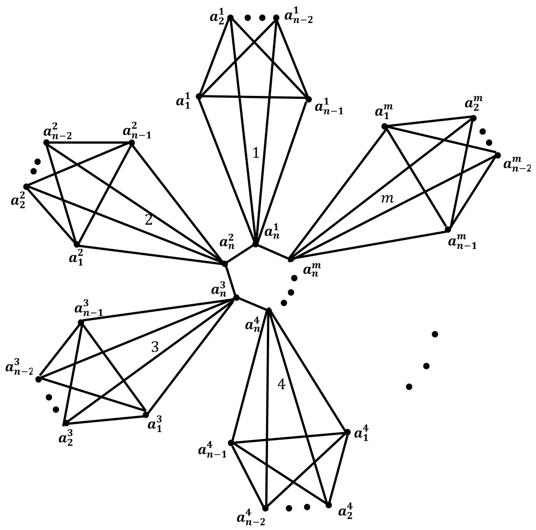

Figure 3. French cycle windmill graph (CW_n^m).

Theorem 2.7. *The metric dimension of CW_n^m is $m(n-2)$.*

Proof. Let $R_0 = \{a_1^i, a_2^i, \ldots, a_{n-2}^i\},\ i = 1, 2, \ldots, m$ and $x_k = d(a_n^i|a_n^k)$. Then representation of vertices of CW_n^m are

$$\begin{aligned} r(a_{n-1}^i|R_0) &= ((x_1+2)^*, (x_2+2)^*, \ldots, \underbrace{1^*}_{i^{th}\text{ tuple}}, \ldots, (x_m+2)^*), \\ r(a_n^i|R_0) &= ((x_1+1)^*, (x_2+1)^*, \ldots, \underbrace{1^*}_{i^{th}\text{ tuple}}, \ldots, (x_m+1)^*). \end{aligned}$$

It implies that $dim(CW_n^m) \leq m(n-2)$. Also, from Lemma 2.3, we have $dim(CW_n^m) \geq m(n-2)$. Therefore, $dim(CW_n^m) = m(n-2)$. □

Theorem 2.8. *The edge metric dimension of CW_n^m is $m(n-2)$.*

Proof. Let $R_E' = \{a_1^i, a_2^i, \ldots, a_{n-2}^i\},\ i = 1, 2, \ldots, m$. Representation of edges of CW_n^m with respect to R_E' are as follows

$$\begin{aligned} r(a_{n-1}^i a_n^i|R_E') &= ((x_1+1)^*, (x_2+1)^*, \ldots, \underbrace{1^*}_{i^{th}\text{ tuple}}, \ldots, (x_m+1)^*), \\ r(a_n^i a_n^{i+1}|R_E') &= ((x_1'+1)^*, (x_2'+1)^*, \ldots, \underbrace{1^*, 1^*}_{i^{th}\text{ and }(i+1)^{th}\text{ tuple}}, \ldots, (x_m'+1)^*), \end{aligned}$$

where $x_k' = min\{(d(a_n^i|a_n^k)), (d(a_n^{i+1}|a_n^k))$. It implies that $edim(CW_n^m) \leq m(n-2)$. Also, from Lemma 2.4, we have $edim(CW_n^m) \geq m(n-2)$. Therefore, $edim(CW_n^m) = m(n-2)$. □

2.3. French complete windmill graph

Let KW_n^m be a graph obtained by replacing the shared vertex of windmill graph with a complete graph K_m (see, Figure 4). It has mn vertices and $\frac{mn(n-1)}{2} + \frac{m(m-1)}{2}$ edges. For our sake, we denote the complete subgraphs of KW_n^m by W_n^i and its vertices as $a_1^i, a_2^i, \ldots, a_{n-1}^i, a_n^i$. Now we determine the metric dimension and edge metric dimension of this graph.

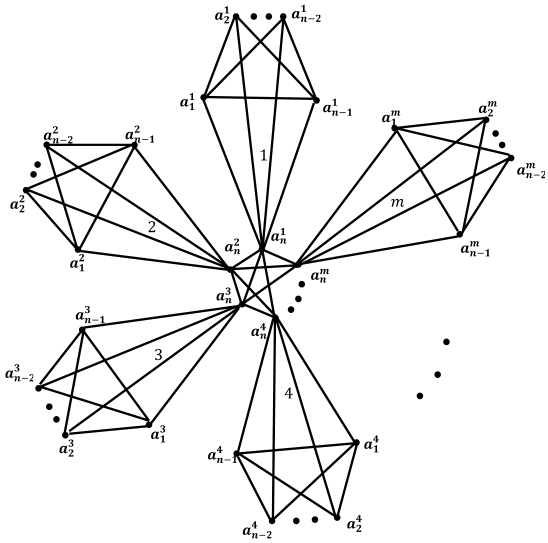

Figure 4. French complete windmill graph (KW_n^m).

Theorem 2.9. *The metric dimension of KW_n^m is $m(n-2)$.*

Proof. By Lemma 2.3, we have $dim(KW_n^m) \geq m(n-2)$. Also, let $R_0 = \{a_1^i, a_2^i, \ldots, a_{n-2}^i\}$, $i = 1, 2, \ldots, m$. Representation of vertices of KW_n^m with respect to R_0 are

$$\begin{aligned} r(a_{n-1}^i|R_0) &= (3^*, \ldots, \underbrace{1^*}_{i^{th} \text{ tuple}}, 3^*, \ldots, 3^*), \\ r(a_n^i|R_0) &= (2^*, \ldots, \underbrace{1^*}_{i^{th} \text{ tuple}}, 2^*, \ldots, 2^*) \end{aligned}$$

which implies that R_0 is a resolving set and $dim(KW_n^m) \leq m(n-2)$. Therefore, $dim(KW_n^m) = m(n-2)$. □

Theorem 2.10. *The edge metric dimension of KW_n^m is $m(n-2)$.*

Proof. By Lemma 2.4, we have $edim(KW_n^m) \geq m(n-2)$. Also, let $R_E' = \{a_1^i, a_2^i, \ldots, a_{n-2}^i\}$, $i = 1, 2, \ldots, m$, we claim that R_E' is an edge metric generator of KW_n^m. Representation of edges of KW_n^m with respect to R_E' are

$$\begin{aligned} r(a_j^i a_{n-1}^i|R_E') &= (3^*, \ldots, \underbrace{1, \ldots, 0, 1, \ldots, 1}_{i^{th} \text{ tuple}}, 3^*, \ldots, 3^*), \\ r(a_j^i a_n^i|R_E') &= (2^*, \ldots, \underbrace{1, \ldots, 0, 1, \ldots, 1}_{i^{th} \text{ tuple}}, 2^*, \ldots, 2^*), \\ r(a_{n-1}^i a_n^i|R_E') &= (3^*, \ldots, \underbrace{1^*}_{i^{th} \text{ tuple}}, 3^*, \ldots, 3^*), \\ r(a_n^i a_n^k|R_E') &= (2^*, \ldots, \underbrace{1^*}_{i^{th} \text{ tuple}}, 2^*, \ldots, \underbrace{1^*}_{k^{th} \text{ tuple}}, 2^*, \ldots, 2^*). \end{aligned}$$

which implies that $R_E^{'}$ is an edge metric generator and $edim(KW_n^m) \leq m(n-2)$. Therefore, $edim(KW_n^m) = m(n-2)$. □

3. Dutch windmill graph

In this section, Dutch windmill graph (see, Figure 5) and certain generalizations of this graph are discussed.

The *Dutch windmill graph*, D_n^m, $n \geq 3, m \geq 2$ is the graph obtained by taking m copies of the cycle graph C_n joined at a shared universal vertex. It has $m(n-1)+1$ vertices and mn edges. For our purpose, we denote the cycle subgraphs of D_n^m by $D_n^i, i = 1, 2, \ldots m$, the shared vertex by c, and the vertices of D_n^i by $a_1^i, a_2^i, \ldots, a_{n-1}^i, c$. In the following theorems, we compute the metric dimension and edge metric dimension of Dutch windmill graph.

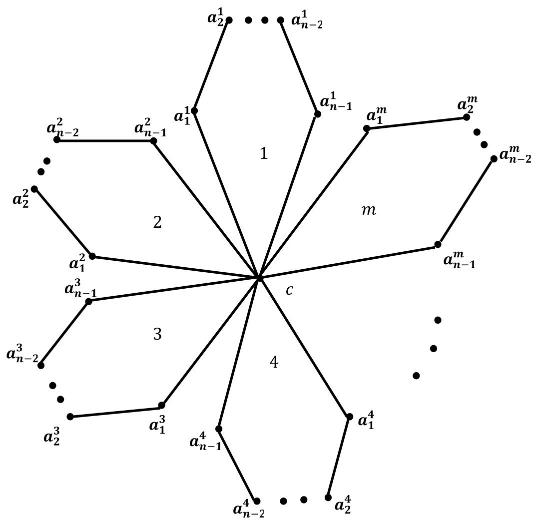

Figure 5. Dutch windmill graph (D_n^m).

Theorem 3.1. *The metric dimension of Dutch windmill graph is:*

$$dim(D_n^m) = \begin{cases} m, & \text{if } n \text{ is odd} \\ 2m-1, & \text{otherwise.} \end{cases}$$

Proof. 1). **When n is odd:** Let R be any resolving set of D_n^m. Assume that there exist some V_i with no vertex in R, then

$$r(a_1^i|R) = r(a_{n-1}^i|R)$$

which is a contradiction. Therefore, $dim(D_n^m) \geq m$. Let $R_0 = \{a^i_{\lceil \frac{n}{2} \rceil}\},\ i = 1, 2, \ldots, m$. We have,

$$d(a_j^i|a^k_{\lceil \frac{n}{2} \rceil}\}) = \begin{cases} \lfloor \frac{n}{2} \rfloor + j, & j < \lceil \frac{n}{2} \rceil \\ n - j + \lfloor \frac{n}{2} \rfloor & j > \lceil \frac{n}{2} \rceil \end{cases}$$

$$d(a_j^i|a^i_{\lceil \frac{n}{2} \rceil}\}) = \begin{cases} \lceil \frac{n}{2} \rceil - j, & \text{if } j < \lceil \frac{n}{2} \rceil \\ j - \lceil \frac{n}{2} \rceil, & \text{if } j > \lceil \frac{n}{2} \rceil \end{cases}$$

Representation of vertices of D_n^m with respect to R_0 are

$$r(a_j^i|R_0) = \begin{cases} (\lfloor \frac{n}{2} \rfloor + j, \ldots, \underbrace{\lceil \frac{n}{2} \rceil - j}_{i^{th}\text{ tuple}}, \lfloor \frac{n}{2} \rfloor + j, \ldots, \lfloor \frac{n}{2} \rfloor + j), & \text{if } j < \lceil \frac{n}{2} \rceil \\ (n - j + \lfloor \frac{n}{2} \rfloor, \ldots, \underbrace{j - \lceil \frac{n}{2} \rceil}_{i^{th}\text{ tuple}}, n - j + \lfloor \frac{n}{2} \rfloor, \ldots, n - j + \lfloor \frac{n}{2} \rfloor), & \text{if } j > \lceil \frac{n}{2} \rceil \end{cases}$$

$$r(c|R_0) = (\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \ldots, \lfloor \frac{n}{2} \rfloor).$$

Let $a_{j_1}, a_{j_2} \in D_n^m$ be any two vertices, then we have the following cases:
Case 1. When the vertices belong to different sets, say V_i and V_k, then $r(a_{j_1}^i|R_0) \neq r(a_{j_2}^k|R_0)$
Case 2. When vertices belong to same sets, then we have two subcases
a) when $j_1, j_2 < \lceil \frac{n}{2} \rceil (or > \lceil \frac{n}{2} \rceil)$ then $r(a_{j_1}^i|R_0) \neq r(a_{j_2}^i|R_0)$
b) when $j_1 < \lceil \frac{n}{2} \rceil$, $j_2 > \lceil \frac{n}{2} \rceil$. Suppose $r(a_{j_1}^i|R_0) = r(a_{j_2}^i|R_0)$, then $\lfloor \frac{n}{2} \rfloor + j_1 = n - j_2 + \lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil - j_1 = j_2 - \lceil \frac{n}{2} \rceil$, which implies $n = j_1 + j_2$ and $n = j_1 + j_2 - 1$, which is a contradiction. Therefore $r(a_{j_1}^i|R_0) \neq r(a_{j_2}^i|R_0)$ for all j_1, j_2. Since every vertex of D_n^m has unique representation with respect to R_0 which implies that $dim(D_n^m) \leq m$. Therefore, $dim(D_n^m) = m$.
2). **When n is even:** First, we claim that any resolving set R of D_n^m contains at least two vertices from each set V_i except one. Suppose on the contrary that there exist two sets say V_i and V_j with only one vertex in R. Without loss of generality, suppose $a_1^i, a_1^j \in R$, then $r(a_{n-1}^i|R) = r(a_{n-1}^j|R)$, which is a contradiction. Therefore, $dim(D_n^m) \geq 2m - 1$. Let $R_0 = \{a_1^1, a_1^2, \ldots, a_1^m, a_{n-1}^1, a_{n-1}^2, \ldots, a_{n-1}^{m-1}\}$. We show that R_0 is resolving set of D_n^m. Representation of vertices of D_n^m with respect to R_0 are

$$r(a_j^i|R_0) = \begin{cases} (j + 1, \ldots, \underbrace{j - 1}_{i^{th}\text{ tuple}}, j + 1, \ldots, j + 1), & \text{if } j < \frac{n}{2} \\ (\frac{n}{2} + 1, \ldots, \underbrace{\frac{n}{2} - 1}_{i^{th}\text{ tuple}}, \frac{n}{2} + 1, \ldots, \underbrace{\frac{n}{2} - 1}_{(m+i)^{th}\text{ tuple}}, \frac{n}{2} + 1, \ldots, \frac{n}{2} + 1), & \text{if } j = \frac{n}{2} \\ (n - j + 1, \ldots, \underbrace{n - j - 1}_{(m+i)^{th}\text{ tuple}}, n - j + 1, \ldots, n - j + 1), & \text{if } j > \frac{n}{2} \end{cases}$$

$$r(a_{n-1}^m|R_0) = (2, 2, \ldots, 2),$$
$$r(c|R_0) = (1, 1, \ldots, 1).$$

Let $a_{j_1}, a_{j_2} \in D_n^m$ be any two vertices, then we have the following cases:
Case 1. When the vertices belong to different sets, say V_i and V_k, then $r(a_{j_1}^i|R_0) \neq r(a_{j_2}^k|R_0)$.
Case 2. When vertices belong to same sets, then we have two subcases:
a) when $j_1, j_2 < \frac{n}{2} (or > \frac{n}{2})$ then $r(a_{j_1}^i|R_0) \neq r(a_{j_2}^i|R_0)$
b) when $j_1 < \frac{n}{2}$, $j_2 > \frac{n}{2}$. Suppose $r(a_{j_1}^i|R_0) = r(a_{j_2}^i|R_0)$, then $j_1 - 1 = n - j_2 + 1$ and $j_1 + 1 = n - j_2 + 1$ which implies $n = j_1 + j_2$ and $n = j_1 + j_2 - 2$, which is a contradiction. Therefore, $r(a_{j_1}^i|R_0) \neq r(a_{j_2}^i|R_0)$ for all j_1, j_2. Since every vertex of D_n^m has unique representation with respect to R_0 which implies that $dim(D_n^m) \leq 2m - 1$. Therefore, $dim(D_n^m) = 2m - 1$. □

Theorem 3.2. *The edge metric dimension of D_n^m is $2m - 1$.*

Proof. Assume that there exist two sets say V_i and V_j with only one vertex in R_E. Without loss of generality, suppose $a_1^i, a_1^j \in R_E$. Then $r(a_{n-1}^i a_n^i | R_E) = r(a_{n-1}^j a_n^j | R_E)$, which is a contradiction. Therefore $edim(D_n^m) \geq 2m-1$. Let $R_E^{'} = \{a_1^1, a_1^2, \ldots, a_1^m, a_{n-1}^1, a_{n-1}^2, \ldots, a_{n-1}^{m-1}\}$. We show that $R_E^{'}$ is an edge metric generator of D_n^m. Representation of edges of D_n^m with respect to $R_E^{'}$ are

When n is odd:

$$r(a_j^i a_{j+1}^i | R_E^{'}) = \begin{cases} (j+1, \ldots, \underbrace{j-1}_{i^{th}\text{ tuple}}, j+1, \ldots, j+1), & \text{if } j < \lfloor \frac{n}{2} \rfloor \\ (\lfloor \frac{n}{2} \rfloor + 1, \ldots, \underbrace{\lfloor \frac{n}{2} \rfloor - 1}_{i^{th}\text{ tuple}}, \lfloor \frac{n}{2} \rfloor + 1, \ldots, \underbrace{\lfloor \frac{n}{2} \rfloor - 1}_{(m+i)^{th}\text{ tuple}}, \lfloor \frac{n}{2} \rfloor + 1, \ldots, \lfloor \frac{n}{2} \rfloor + 1), & \text{if } j = \lfloor \frac{n}{2} \rfloor \\ (n-j, \ldots, \underbrace{n-j-2}_{(m+i)^{th}\text{ tuple}}, n-j, \ldots, n-j), & \text{if } j > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

$$r(a_{n-1}^m c | R_E^{'}) = (1, 1, \ldots, 1).$$

When n is even:

$$r(a_j^i a_{j+1}^i | R_E^{'}) = \begin{cases} (j+1, \ldots, \underbrace{j-1}_{i^{th}\text{ tuple}}, j+1, \ldots, j+1), & \text{if } j < \frac{n}{2} - 1 \\ (\frac{n}{2}, \ldots, \underbrace{\frac{n}{2} - 2}_{i^{th}\text{ tuple}}, \frac{n}{2}, \ldots, \underbrace{\frac{n}{2} - 1}_{(m+i)^{th}\text{ tuple}}, \frac{n}{2}, \ldots, \frac{n}{2}), & \text{if } j = \frac{n}{2} - 1 \\ (\frac{n}{2}, \ldots, \underbrace{\frac{n}{2} - 1}_{i^{th}\text{ tuple}}, \frac{n}{2}, \ldots, \underbrace{\frac{n}{2} - 2}_{(m+i)^{th}\text{ tuple}}, \frac{n}{2}, \ldots, \frac{n}{2}), & \text{if } j = \frac{n}{2} \\ (n-j, \ldots, \underbrace{n-j-2}_{(m+i)^{th}\text{ tuple}}, n-j, \ldots, n-j), & \text{if } j > \frac{n}{2}. \end{cases}$$

$$r(a_{n-1}^m c | R_E^{'}) = (1, 1, \ldots, 1).$$

Therefore, $edim(D_n^m) = 2m - 1$. □

The following two lemmas show that the metric dimension and edge metric dimension of generalizations of Dutch windmill graph (as discussed in this section) is at least m.

Lemma 3.3. *The metric dimension of SD_n^m, CD_n^m, and KD_n^m is at least m.*

Proof. We can assume that there exist some V_i with no vertex in R, then $r(a_1^i | R) = r(a_{n-1}^i | R)$, which is a contradiction. □

Lemma 3.4. *The metric dimension of SD_n^m, CD_n^m, and KD_n^m is at least m.*

Proof. We can assume that there exist a set V_i with no vertex in R_E, then $r(a_{n-1}^i a_n^i | R_E) = r(a_1^i a_n^i | R_E)$, which is a contradiction. Therefore, $edim(SD_n^m) \geq m$. □

3.1. Dutch star windmill graph

Let SD_n^m be a graph obtained by replacing the shared vertex of Dutch windmill graph with a star graph S_m (see, Figure 6). It has $mn + 1$ vertices and $m(n + 1)$ edges. For our purpose, we denote the cycle subgraphs of SD_n^m by D_n^i , its vertices by $a_1^i, a_2^i, \ldots, a_{n-1}^i, a_n^i$ and vertices of S_m by $a_n^1, a_n^2, \ldots, a_n^m, c$. In the following results, we discuss the metric dimension and edge metric dimension of this graph.

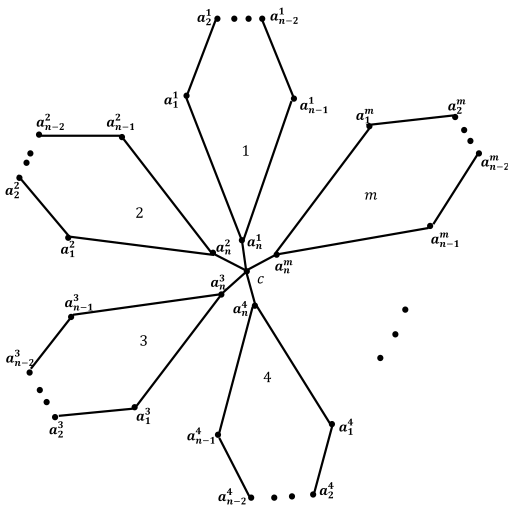

Figure 6. Dutch star windmill graph (SD_n^m).

Theorem 3.5. *The metric dimension of SD_n^m is m.*

Proof. By Lemma 3.3, we have $dim(SD_n^m) \geq m$. Also, let $R_0 = \{a_1^i\},\ i = 1, 2, \ldots, m$. We claim that R_0 is a resolving set of SD_n^m. Now,

$$d(a_j^i|a_1^k\}) = \begin{cases} j + 3, & j < \lceil \frac{n}{2} \rceil \\ n - j + 3 & j \geq \lceil \frac{n}{2} \rceil. \end{cases}$$

$$d(a_j^i|a_1^i\}) = \begin{cases} j - 1, & \text{if } j \leq \lceil \frac{n}{2} \rceil \\ n - j + 1, & \text{if } j > \lceil \frac{n}{2} \rceil. \end{cases}$$

Representation of vertices of SD_n^m with respect to R_0 are:

$$r(a_j^i|R_0) = \begin{cases} (j + 3, \ldots, \underbrace{j - 1}_{i^{th} \text{ tuple}}, j + 3, \ldots, j + 3), & \text{if } j < \lceil \frac{n}{2} \rceil \\ (n - j + 3, \ldots, \underbrace{j - 1}_{i^{th} \text{ tuple}}, n - j + 3, \ldots, n - j + 3), & \text{if } j = \lceil \frac{n}{2} \rceil \\ (n - j + 3, \ldots, \underbrace{n - j + 1}_{i^{th} \text{ tuple}}, n - j + 3, \ldots, n - j + 3), & \text{if } j > \lceil \frac{n}{2} \rceil. \end{cases}$$

$$r(c|R_0) = (2, \ldots, 2).$$

Clearly, we have $dim(SD_n^m) \leq m$. Therefore, $dim(SD_n^m) = m$. □

Theorem 3.6. *The edge metric dimension of SD_n^m is m.*

Proof. By Lemma 3.4, we have $edim(SD_n^m) \geq m$. Also, let $R_E' = \{a_1^i\},\ i = 1, 2, \ldots, m$. We claim that R_E' is an edge metric generator of SD_n^m. Representation of edges of SD_n^m with respect to R_E' are:

$$r(a_j^i a_{j+1}^i | R_E') = \begin{cases} (j+3, \ldots, \underbrace{j-1}_{i^{th} \text{ tuple}}, j+3, \ldots, j+3), & \text{if } j < \lceil \frac{n}{2} \rceil \\ (n-j+2, \ldots, \underbrace{j-1}_{i^{th} \text{ tuple}}, n-j+2, \ldots, n-j+2), & \text{if } j = \lceil \frac{n}{2} \rceil \\ (n-j+2, \ldots, \underbrace{n-j}_{i^{th} \text{ tuple}}, n-j+2, \ldots, n-j+2), & \text{if } j > \lceil \frac{n}{2} \rceil. \end{cases}$$

$$r(ca_n^i | R_E') = (2, \ldots, \underbrace{1}_{i^{th} \text{ tuple}}, 2, \ldots, 2).$$

Clearly, we have $edim(SD_n^m) \leq m$. Hence $edim(SD_n^m) = m$. □

3.2. Dutch cycle windmill graph

Let CD_n^m be a graph obtained by replacing the shared vertex of Dutch windmill graph with a cycle graph (C_m) (see, Figure 7). It has mn vertices and $m(n+1)$ edges. For our sake, we denote the cycle subgraphs of CD_n^m by D_n^i and its vertices as $a_1^i, a_2^i, \ldots, a_{n-1}^i, a_n^i$. Now, we determine the metric dimension and edge metric dimension of this graph.

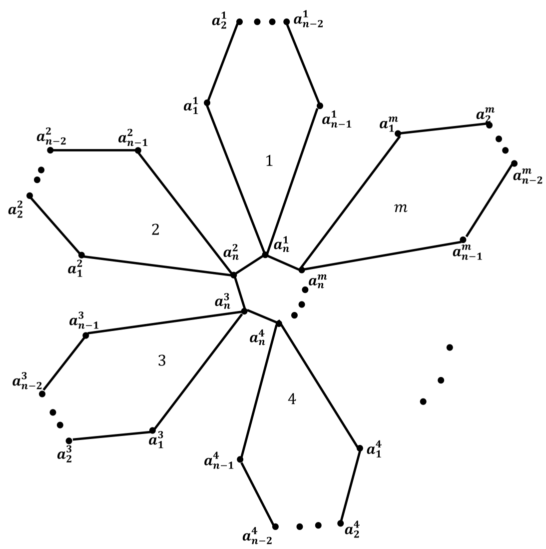

Figure 7. Dutch cycle windmill graph (CD_n^m).

Theorem 3.7. *The metric dimension of CD_n^m is m.*

Proof. By Lemma 3.3, we have $dim(CD_n^m) \geq m$. Also, let $R_0 = \{a_1^i\},\ i = 1, 2, \ldots, m$. We show that R_0

is a resolving set of CD_n^m. Now,

$$d(a_j^i|a_1^i) = \begin{cases} j-1, & \text{if } j < \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{if } j = \lfloor \frac{n}{2} \rfloor \\ n-j+1, & \text{if} j > \lfloor \frac{n}{2} \rfloor \end{cases}$$

$$d(a_j^i|a_1^k) = \begin{cases} j+x_k+1, & \text{if } j < \lfloor \frac{n}{2} \rfloor \\ \lceil \frac{n}{2} \rceil + x_k + 1, & \text{if } j = \lfloor \frac{n}{2} \rfloor \\ n-j+x_k+1, & \text{if } j > \lfloor \frac{n}{2} \rfloor \end{cases}$$

where $x_k = d(a_n^i|a_n^k)$. Representation of vertices with respect to R_0 are

$$= \begin{cases} (j+x_1+1, j+x_2+1, \ldots, \underbrace{j-1}_{i^{th} \text{ tuple}}, \ldots, j+x_m+1), & \text{if } j < \lfloor \frac{n}{2} \rfloor \\ (n-j+x_1+1, n-j+x_2+1, \ldots, \underbrace{n-j+1}_{i^{th} \text{ tuple}}, \ldots, n-j+x_m+1), & \text{if } j > \lfloor \frac{n}{2} \rfloor \\ (\lceil \frac{n}{2} \rceil + x_1+1, \lceil \frac{n}{2} \rceil + x_2+1, \ldots, \underbrace{\lfloor \frac{n}{2} \rfloor - 1}_{i^{th} \text{ tuple}}, \ldots, \lceil \frac{n}{2} \rceil + x_m+1), & \text{if } j = \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Since every vertex of CD_n^m has unique representation with respect to R_0. Therefore, $dim(CD_n^m) \leq m$. Hence, $dim(CD_n^m) = m$. □

Theorem 3.8. *The edge metric dimension of CD_n^m is m.*

Proof. By Lemma 3.4, we have $edim(CD_n^m) \geq m$. Also, let $R_E^{'} = \{a_1^i\},\ i = 1, 2, \ldots, m$. Representation of edges of CD_n^m with respect to $R_E^{'}$ are

$$r(a_j^i a_{j+1}^i|R_E^{'}) = \begin{cases} (j+x_1+1, j+x_2+1, \ldots, \underbrace{j-1}_{i^{th} \text{ tuple}}, \ldots, j+x_m+1), & \text{if } j < \lfloor \frac{n}{2} \rfloor \\ (n-j+x_1, n-j+x_2, \ldots, \underbrace{n-j}_{i^{th} \text{ tuple}}, \ldots, n-j+x_m), & \text{if } j > \lfloor \frac{n}{2} \rfloor \\ (\lceil \frac{n}{2} \rceil + x_1, \lceil \frac{n}{2} \rceil + x_2, \ldots, \underbrace{\lfloor \frac{n}{2} \rfloor - 1}_{i^{th} \text{ tuple}}, \ldots, \lceil \frac{n}{2} \rceil + x_m), & \text{if } j = \lfloor \frac{n}{2} \rfloor. \end{cases}$$

$$r(a_n^i a_n^{i+1}|R_E^{'}) = (x_1+1, x_2+1, \ldots, \underbrace{1, 1}_{i^{th} \text{ and } (i+1)^{th} \text{ tuple}}, \ldots, x_m+1).$$

where $x_k = min\{d(a_n^i|a_n^k), d(a_n^{i+1}|a_n^k)\}, k = 1, 2, \ldots m$. Clearly, representation of every edge of CD_n^m with respect to $R_E^{'}$ is different. Therefore, $edim(CD_n^m) \leq m$. Hence $edim(CD_n^m) = m$ □

3.3. Dutch complete windmill graph

Let KD_n^m be a graph obtained by replacing the shared vertex of Dutch windmill graph with a complete graph(K_m) (see, Figure 8). It has mn vertices and $mn + \frac{m(m-1)}{2}$ edges. For our sake, we denote the cycle subgraphs of KD_n^m by D_n^i and its vertices as $a_1^i, a_2^i, \ldots, a_{n-1}^i, a_n^i$. Now, we obtain the metric dimension and edge metric dimension of this graph.

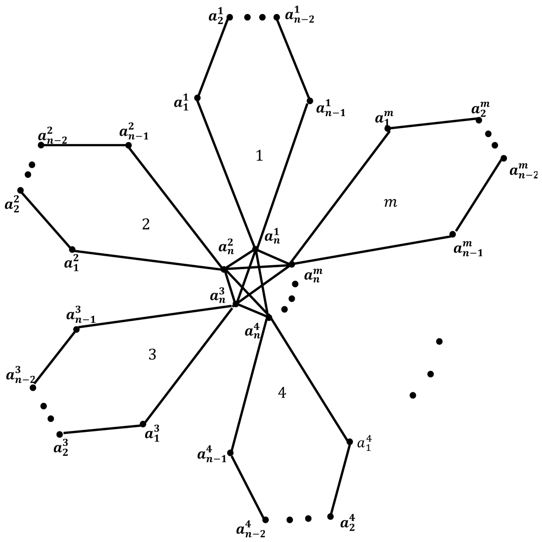

Figure 8. Dutch complete windmill graph (KD_n^m).

Theorem 3.9. *The metric dimension of KD_n^m is m.*

Proof. Let $R_0 = \{a_1^i\},\ i = 1, 2, \ldots, m$. Representation of vertices of KD_n^m with respect to R_0 are

$$r(a_j^i|R_0) = \begin{cases} (j+2, \ldots, \underbrace{j-1}_{i^{th} \text{ tuple}}, j+2, \ldots, j+2), & \text{if } j < \lceil \frac{n}{2} \rceil \\ (n-j+2, \ldots, \underbrace{j-1}_{i^{th} \text{ tuple}}, n-j+2, \ldots, n-j+2), & \text{if } j = \lceil \frac{n}{2} \rceil \\ (n-j+2, \ldots, \underbrace{n-j+1}_{i^{th} \text{ tuple}}, n-j+2, \ldots, n-j+2), & \text{if } j > \lceil \frac{n}{2} \rceil. \end{cases}$$

which implies that R_0 is a resolving set and $dim(KD_n^m) \leq m$. Also, from Lemma 3.3, $dim(KD_n^m) \geq m$. Therefore, $dim(KD_n^m) = m$. □

Theorem 3.10. *The edge metric dimension of KD_n^m is m.*

Proof. Let $R_E' = \{a_1^i\},\ i = 1, 2, \ldots, m$. We claim that R_E' is an edge metric generator of KD_n^m. Representation of edges of KD_n^m with respect to R_E' are

$$r(a_j^i a_{j+1}^i|R_E') = \begin{cases} (j+2, \ldots, \underbrace{j-1}_{i^{th} \text{ tuple}}, j+2, \ldots, j+2), & \text{if } j < \lceil \frac{n}{2} \rceil \\ (n-j+1, \ldots, \underbrace{j-1}_{i^{th} \text{ tuple}}, n-j+1, \ldots, n-j+1), & \text{if } j = \lceil \frac{n}{2} \rceil \\ (n-j+1, \ldots, \underbrace{n-j}_{i^{th} \text{ tuple}}, n-j+1, \ldots, n-j+1), & \text{if } j > \lceil \frac{n}{2} \rceil. \end{cases}$$

$$r(a_n^i a_n^k|R_E') = (2, \ldots, \underbrace{1}_{i^{th} \text{ tuple}}, 2, \ldots, \underbrace{1}_{k^{th} \text{ tuple}}, 2, \ldots, 2).$$

which implies that $edim(KD_n^m) \leq m$. Also from Lemma 3.4, $edim(KD_n^m) \geq m$. Hence $edim(KD_n^m) = m$. □

4. Conclusions and future work

In this paper we have computed the metric dimension and edge metric dimension of French windmill and Dutch windmill graphs wherein the shared vertex is replaced by star graph, cyclic graph and complete graph. We have found that the metric dimension and edge metric dimension of generalizations of French windmill graph and Dutch windmill graph are same. In future, we would extend our work to Fault-tolerant metric dimension and Fractional metric dimension of French windmill and Dutch windmill graphs.

Conflict of interest

The authors declare no conflict of interest.

References

1. S. Arumugam, V. Mathew, The fractional metric dimension of graphs, *Discret. Math.*, **312** (2012), 1584–1590.
2. Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Hoffman, M. Mihalak, et al., Network discovery and verification, *IEEE J. Sel. Area Comm.*, **24** (2006), 2168–2181.
3. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discret. Appl. Math.*, **105** (2000), 99–113.
4. V. Chvátal, Mastermind, *Combinatorica*, **3** (1983), 325–329.
5. A. Estrada-Moreno, On the (k, t)-Metric dimension of a Graph, Ph.D. thesis, Universitat Rovira i Virgili, 2016.
6. E. Estrada, When local and global clustering of networks diverge, *Linear Algebra Appl.*, **488** (2016), 249–263.
7. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.*, **2** (1976), 191–195.
8. S. Hayat, A. Khan, M. Y. H. Malik, M. Imran, M. K. Siddiqui, Fault-Tolerant Metric Dimension of Interconnection Networks, *IEEE Access*, **8** (2020), 145435–145445.
9. M. Imran, On the metric dimension of barycentric subdivision of Cayley graphs, *Acta Math. Appl. Sinica*, **32** (2016), 1067–1072.
10. I. Javaid, M. T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, *Util. Math.*, **75** (2008), 21–33.
11. A. Kelenc, D. Kuziak, A. Taranenko, I. G. Yero, On the mixed metric dimension of graphs, *Appl. Math. Comput.*, **314** (2017), 429–438.
12. A. Kelenc, N. Tratnik, I. G. Yero, Uniquely identifying the edges of a graph: the edge metric dimension, *Discret. Appl. Math.*, **251** (2018), 204–220.
13. S. Khuller, B. Raghavachari, A. Rosenfield, Landmarks in graphs, *Discret. Appl. Math.*, **70** (1996), 217–229.
14. R. Nasir, S. Zafar, Z. Zahid, Edge metric dimension of graphs, *Ars Comb.*, **147** (2018), 143–156.

15. O. R. Oellermann, J. P. Fransen, The strong metric dimension of graphs and digraphs, *Discret. Appl. Math.*, **155** (2007), 356–364.

16. F. Okamoto, B. Phinezy, P. Zhang, The local metric dimension of a graph, *Math. Bohem.*, **135** (2010), 239–255.

17. H. Raza, S. Hayat, X. F. Pan, On the fault-tolerant metric dimension of convex polytopes, *Appl. Math. Comput.*, **339** (2018), 172–185.

18. H. Raza, S. Hayat, M. Imran, X. F. Pan, Fault-tolerant resolvability and extremal structures of graphs, *Mathematics*, **7** (2019), 78.

19. H. Raza, S. Hayat, X. F. Pan, On the fault-tolerant metric dimension of certain interconnection networks, *J. Appl. Math. Comput.*, **60** (2019), 517–535.

20. A. Sebo, E. Tannier, On metric generators of graphs, *Math. Oper. Res.*, **29** (2004), 383–393.

21. H. M. A. Siddiqui, S. Hayat, A. Khan, M. Imran, A. Razzaq, J. B. Liu, Resolvability and fault-tolerant resolvability structures of convex polytopes, *Theor. Comput. Sci.*, **796** (2019), 114—128.

22. H. Shapiro, S. Soderberg, A combinatory detection problem, *Am. Math. Mon.*, **70** (1963), 1066–1070.

23. P. J. Slater, Leaves of trees, *Congr. Numer.*, **14** (1975), 549–559.

24. M. Wei, J. Yue, X. zhu, On the edge metric dimension of graphs, *AIMS Mathematics*, **5** (2020), 4459–4465.

25. N. Zubrilina, On the edge dimension of a graph, *Discret. Math.*, **341** (2018), 2083–2088.