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Research article

Coefficient bounds for certain two subclasses of bi-univalent functions

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Abstract: In this paper, coefficient bounds of bi-univalent functions in certain two subclasses, which are defined by subordination are estimated. Some special outcomes of the main results are also presented. Moreover, it is remarked that the given bounds improve and generalize some of the pervious results.

Keywords: univalent functions, bi-univalent functions, coefficient bounds, Faber polynomial expansion, subordination
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1. Introduction and Preliminaries

Suppose that $\mathcal{A}$ is the category of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and assume that $\mathcal{S}$ is the subset of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$.

It is known that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by $f^{-1}(f(z)) = z$ $(z \in \mathbb{U})$ and $f\left(f^{-1}(w)\right) = w$ $(|w| < r_0(f);\ r_0(f) \geq 1/4)$ where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots =: w + \sum_{n=2}^{\infty} c_n w^n. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the family of bi-univalent functions in $\mathbb{U}$. Lewin [20] studied the bi-univalent function family Σ and obtained the bound for the second Taylor-Maclaurin coefficient $|a_2|$. A brief summery of functions in the family Σ can be found in the study of Srivastava *et al.* [30], which is a basic research on the bi-univalent function family Σ and in the references cited therein. In a number of sequels to [30], bounds for the first two coefficients $|a_2|$ and $|a_3|$ of different subclasses of bi-univalent functions were given, for example, see [6,15,24,28,35]. However, determination of extremal functions for bi-univalent functions remains a challenge. In fact, the study of analytic and bi-univalent functions was successfully revived by the pioneering research of Srivastava *et al.* [30] in recent years regarding the numerous papers on the subject. There are also several papers dealing with bi univalent functions defined by subordination, for example, see [3, 10, 36].

In addition, in a survey-cum-expository article [23] by Srivastava, it was indicated that the recent and future applications and importance of the classical q-calculus and the fractional q-calculus in geometric function theory of complex analysis motivate researchers to study many of these and other related subjects in this filed. It is notable, the fact that the so-called (p, q)-results are no more general than the corresponding q-results because the additional parameter p is obviously redundant (see [23]). For example, in [19] researchers defined a generalized subclass of analytic and bi-univalent functions associated with a certain q-integral operator in the open unit disk $\mathbb{U}$ and estimated bounds on the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for the functions belonging to this subclass.

Obtaining the upper bound for coefficients have been one of the main research areas in geometric function theory as it renders several meaningful features of functions. Individually, bound for the second coefficient renders growth and distortion theorems for functions in the family $\mathcal{S}$. According to [30], many authors put effort to review and study various subclasses of the class Σ of bi-univalent functions in recent years, for example, see [15, 24, 27, 28, 32, 35]. In the literature, several researchers applied the Faber polynomial expansions to determine the general coefficient bounds of $|a_n|$ for the analytic bi-univalent functions [4, 7, 9, 16, 17, 26, 29, 31–34, 38, 39]. It is remarkable that Faber polynomials play an important role in geometric function theory, introduced by Faber [14].

In this paper, let ϑ is an analytic function, which is characterized by positive real part in $\mathbb{U}$ and $\vartheta(\mathbb{U})$ is symmetric with respect to the real axis, satisfying $\vartheta(0) = 1,\ \vartheta'(0) > 0$ such that it has series expansion of the form

$$\vartheta(z) = 1 + G_1 z + G_2 z^2 + G_3 z^3 + \ldots \quad (G_1 > 0). \tag{1.3}$$

A function $\omega : \mathbb{U} \to \mathbb{C}$ is said *Schwarz function* if ω is a analytic function in $\mathbb{U}$ with conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. The family of all Schwarz functions on $\mathbb{U}$ is denoted by Ω. Clearly, a Schwarz function ω has the form

$$\omega(z) = w_1 z + w_2 z^2 + \ldots.$$

Recently, Srivastava and Bansal [25] (see, also [11]) introduced a subclass of analytic bi-univalent functions and obtained non-sharp estimates of the first two coefficients of functions in this class as follows:

Definition 1. Let $0 \le \rho \le 1$ and $\varsigma \in \mathbb{C} \backslash \{0\}$. A function $f \in \Sigma$ is said to be in the subclass $\Sigma(\varsigma, \rho, \vartheta)$ if each of the next conditions holds true:

$$1 + \frac{1}{\varsigma}[f'(z) + \rho z f''(z) - 1] \prec \vartheta(z) \qquad (z \in \mathbb{U})$$

and

$$1 + \frac{1}{\varsigma}[g'(w) + \rho w g''(w) - 1] \prec \vartheta(w) \qquad (w \in \mathbb{U}),$$

where $g = f^{-1}$ is given by (1.2).

Deniz *et al.* [12], by a class of functions defined in [11], introduced the next comprehensive family of analytic functions

$$\mathscr{S}(\nu, \rho; \vartheta) = \left\{ f \in \mathcal{A} : \ 1 + \frac{1}{\rho}\left(\frac{zf'(z) + \nu z^2 f''(z)}{(1-\nu)f(z) + \nu z f'(z)} - 1\right) \prec \vartheta(z); \ z \in \mathbb{U}, \ 0 \le \nu \le 1, \ \rho \in \mathbb{C} \setminus \{0\} \right\}.$$

As particular cases of the family $\mathscr{S}(\nu, \rho; \vartheta)$ we have $\mathscr{S}(0, 1; \vartheta) = \mathcal{S}^*(\vartheta)$ and $\mathscr{S}(1, 1; \vartheta) = \mathcal{C}(\vartheta)$ that these categories are called Ma-Minda starlike and convex, respectively [21]. A function $f \in \mathcal{A}$ is said to be generalized bi-subordinate of complex order ρ and type ν if both f and $g = f^{-1}$ are in $\mathscr{S}(\nu, \rho; \vartheta)$.

Theorem 1. [5] *For* $0 \le \rho \le 1$ *and* $\varsigma \in \mathbb{C} \setminus \{0\}$, *let the function* $f \in \Sigma(\varsigma, \rho, \vartheta)$ *be given by* (1.1). *If* $a_k = 0$ *for* $2 \le k \le n-1$, *then*

$$|a_n| \le \frac{|\varsigma| \, G_1}{n\,[1 + \rho(n-1)]} \qquad (n \ge 3)\,. \tag{1.4}$$

Theorem 2. [12] *Let* $0 \le \nu \le 1$ *and* $\rho \in \mathbb{C} \setminus \{0\}$. *If both functions* f *and* $g = f^{-1}$ *are given by* (1.1) *and* (1.2), *respectively, be in* $\mathscr{S}(\nu, \rho; \vartheta)$ *and* $a_k = 0$ *for* $2 \le k \le n-1$, *then*

$$|a_n| \le \frac{|\rho| \, G_1}{(n-1)\,[1 + \nu(n-1)]} \qquad (n \ge 3)\,. \tag{1.5}$$

The present paper is motivated essentially by the recent works [5, 8, 12] and the aim of this paper is to study the coefficient estimates of two subclasses $\Sigma(\varsigma, \rho, \vartheta)$ and $\mathscr{S}(\nu, \rho; \vartheta)$ of bi-univalent functions. We apply the Faber polynomial expansions to get bounds for the coefficients $|a_n|$ for the functions of the general classes that our results improve some of the previously ones.

2. Coefficient bounds

In this section, we obtain a smaller upper bound with precise estimation of coefficients $|a_n|$ of analytic bi-univalent functions in the subclasses $\Sigma(\varsigma, \rho, \vartheta)$ and $\mathscr{S}(\nu, \rho; \vartheta)$. To establish the outcomes, we need the following lemmas:

Lemma 1. [1, 2] *Let* $f \in \mathcal{S}$ *be given by* (1.1). *Then the coefficients of its inverse map* $g = f^{-1}$ *are given in terms of the Faber polynomials of* f *with*

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^n, \tag{2.1}$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3$$

$$+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5+(-n+2)a_3^2\right]$$
$$+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6+(-2n+5)a_3a_4\right]+\sum_{j\geq 7}a_2^{n-j}V_j$$

such that V_j $(7 \leq j \leq n)$ *is a homogeneous polynomial in the variables* $a_2, a_3, \ldots, a_n$*, and the expressions such as (for example)* $(-m)!$ *are to be interpreted* symbolically *by*

$$(-m)! \equiv \Gamma(1-m) := (-m)(-m-1)(-m-2)\ldots, \quad \textit{with} \quad m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\},\ \mathbb{N} := \{1,2,3,\ldots\}.$$

We note that the first three terms of K_{n-1}^{-n} are given by

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3\left(2a_2^2 - a_3\right) \quad \text{and} \quad K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

Generally, for every integer number p the expansion of K_n^p is given below (see for details, [1]; see also [2, p. 349])

$$K_n^p = pa_{n+1} + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!3!}D_n^3 + \ldots + \frac{p!}{(p-n)!n!}D_n^n, \tag{2.2}$$

where $D_n^p = D_n^p(a_2, a_3, \ldots, a_{n+1})$ (see for details [37]). We also have

$$D_n^m(a_2, a_3, \ldots, a_{n+1}) = \sum_{n=1}^{\infty} \frac{m!(a_2)^{\mu_1}\cdot\ldots\cdot(a_{n+1})^{\mu_n}}{\mu_1!\cdot\ldots\cdot\mu_n!}, \tag{2.3}$$

where the sum is taken over all nonnegative integers $\mu_1, \ldots, \mu_n$ holding the conditions

$$\begin{cases} \mu_1 + \mu_2 + \ldots + \mu_n = m \\ \mu_1 + 2\mu_2 + \ldots + n\mu_n = n. \end{cases}$$

Obviously, $D_n^n(a_2, a_3, \ldots, a_{n+1}) = a_2^n$.

Lemma 2. [39] *Let* $f \in \Sigma(\varsigma, \rho, \vartheta)$*. Then we have the following expansion:*

$$1 + \frac{1}{\varsigma}[f'(z) + \rho z f''(z) - 1] = 1 + \sum_{n=2}^{\infty}\frac{1}{\varsigma}[1 + \rho(n-1)]na_nz^{n-1}$$

and

$$1 + \frac{1}{\varsigma}[g'(w) + \rho w g''(w) - 1] = 1 + \sum_{n=2}^{\infty}\frac{1}{\varsigma}[1 + \rho(n-1)]nc_nw^{n-1},$$

where $c_n = \frac{1}{n}K_{n-1}^{-n}(a_2, a_3, \ldots, a_n)$ *and* K_{n-1}^{-n} *is given by Lemma 1.*

Lemma 3. [39] *Let* $f \in \Sigma(\varsigma, \rho, \vartheta)$*. Then*

$$\frac{1}{\varsigma}[1 + \rho(n-1)]na_n = \sum_{k=1}^{n-1} G_kD_{n-1}^k(p_1, p_2, \ldots, p_{n-1}) \qquad (n \geq 2)$$

and

$$\frac{1}{\varsigma}[1+\rho(n-1)]K_{n-1}^{-n}(a_2,a_3,\ldots,a_n)=\sum_{k=1}^{n-1}G_kD_{n-1}^{k}(q_1,q_2,\ldots,q_{n-1})\qquad(n\geq 2),$$

where K_{n-1}^{-n} *and* D_{n-1}^{k} *are given by Lemma 1 and* $u(z)=\sum_{n=1}^{\infty}p_nz^n,\ v(z)=\sum_{n=1}^{\infty}q_nz^n\in\Omega$.

Lemma 4. [8] *Let* $f(z)=z+\sum_{k=n}^{\infty}a_kz^k;\ (n\geq 2)$ *be a univalent function in* $\mathbb{U}$ *and*

$$f^{-1}(w)=w+\sum_{k=n}^{\infty}c_kw^k\quad(|w|<r_0(f);\ r_0(f)\geq 1/4).$$

Then

$$c_{2n-1}=na_n^2-a_{2n-1}\qquad\text{and}\qquad c_k=-a_k\quad\text{for}\quad(n\leq k\leq 2n-2).$$

Lemma 5. [8] *Let* $f(z)=z+\sum_{k=n}^{\infty}a_kz^k;\ (n\geq 2)$ *be a univalent function in* $\mathbb{U}$ *and*

$$f^{-1}(w)=w+\sum_{k=n}^{\infty}c_kw^k\quad(|w|<r_0(f);\ r_0(f)\geq 1/4).$$

Then

$$|a_n|\leq\sqrt{\frac{|a_{2n-1}|+|c_{2n-1}|}{n}}.$$

Theorem 3. *Let* $f(z)=z+\sum_{k=n}^{\infty}a_kz^k\in\Sigma(\varsigma,\rho,\vartheta);(n\geq 2)$ *with* $|G_2|\leq G_1$*, then*

(i)

$$|a_n|\leq\min\left\{\frac{|\varsigma|\,G_1}{n\,[1+\rho(n-1)]},\sqrt{\frac{2\,|\varsigma|\,G_1}{n(2n-1)[1+\rho(2n-2)]}}\right\},\tag{2.4}$$

(ii)

$$|na_n^2-a_{2n-1}|\leq\frac{|\varsigma|\,G_1}{(2n-1)\,[1+\rho(2n-2)]}.$$

Proof. Let $f(z)=z+\sum_{k=n}^{\infty}a_kz^k\in\Sigma(\varsigma,\rho,\vartheta);\ (n\geq 2)$. Then by the definition of subordination there are two functions $u,v\in\Omega$ with $u(z)=\sum_{k=1}^{\infty}p_kz^k$ and $v(z)=\sum_{k=1}^{\infty}q_kz^k$, respectively, such that

$$1+\frac{1}{\varsigma}[f'(z)+\rho zf''(z)-1]=\vartheta(u(z))$$

and

$$1+\frac{1}{\varsigma}[g'(w)+\rho wg''(w)-1]=\vartheta(v(w)).$$

Applying the relations (1.3) and (2.3) we have

$$\vartheta(u(z)) = 1 + G_1 p_1 z + (G_1 p_2 + G_2 p_1^2) z^2 + \ldots = 1 + \sum_{k=1}^{\infty} \sum_{i=1}^{k} G_i D_k^i (p_1, p_2, \ldots, p_k) z^k$$

and

$$\vartheta(v(w)) = 1 + \sum_{k=1}^{\infty} \sum_{i=1}^{k} G_i D_k^i (q_1, q_2, \ldots, q_k) w^k.$$

Since $a_k = 0$ for $2 \le k \le n-1$ and $G_1 > 0$ from Lemma 3, we obtain

$$p_1 = \ldots = p_{n-2} = 0, \quad q_1 = \ldots = q_{n-2} = 0.$$

Therefore, from Lemmas 2 and 3 we have

$$[1 + \rho(n-1)] n a_n = \varsigma G_1 p_{n-1}$$

and

$$[1 + \rho(n-1)] n c_n = \varsigma G_1 q_{n-1}.$$

Now, from Lemma 4 taking the absolute values of the above relations with $|p_{n-1}| \le 1$ and $|q_{n-1}| \le 1$, (see [22, page 172]), it follows

$$|a_n| = |c_n| \le \frac{|\varsigma|\, G_1}{n\,[1 + \rho(n-1)]} \tag{2.5}$$

Further from Lemma 3, it results in

$$[1 + \rho(2n-2)]\,(2n-1) a_{2n-1} = G_1 p_{2n-2} + G_2 p_{n-1}^2.$$

Using [18, page 10] and [22, page 172] we get

$$|a_{2n-1}| = |c_{2n-1}| \le \frac{|\varsigma|\, G_1}{(2n-1)\,[1 + \rho(2n-2)]}. \tag{2.6}$$

Now, in view of Lemma 5, utilizing the relation (2.6) we conclude that

$$|a_n| \le \sqrt{\frac{|a_{2n-1}| + |c_{2n-1}|}{n}} \le \sqrt{\frac{2\,|\varsigma|\, G_1}{n(2n-1)[1 + \rho(2n-2)]}}. \tag{2.7}$$

From (2.5) and (2.7), we see that the relation (2.4) holds. Further, by (2.6) and applying Lemma 4, we get

$$|n a_n^2 - a_{2n-1}| = |c_{2n-1}| \le \frac{|\varsigma|\, G_1}{(2n-1)\,[1 + \rho(2n-2)]}.$$

This completes the proof. □

In special cases, we get the next corollaries.

Corollary 1. *Let* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \Sigma(\varsigma, \rho, (1 + (1 - 2\beta)z)/(1 - z))\,; (n \geq 2)$. *Then*

$$|a_n| \leq \min\left\{\frac{2|\varsigma|(1-\beta)}{n[1+\rho(n-1)]}, \sqrt{\frac{4|\varsigma|(1-\beta)}{n(2n-1)[1+\rho(2n-2)]}}\right\}$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{2|\varsigma|(1-\beta)}{(2n-1)[1+\rho(2n-2)]}.$$

Proof. For

$$\vartheta(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \ldots \quad (0 \leq \beta < 1,\ z \in \mathbb{U}),$$

where $G_1 = G_2 = 2(1-\beta)$ in Theorem 3, it gives the result mentioned in the corollary. □

Corollary 2. *Let* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \Sigma(\varsigma, \rho, ((1+z)/(1-z))^{\alpha})\,; (n \geq 2)$. *Then*

$$|a_n| \leq \min\left\{\frac{2|\varsigma|\alpha}{n[1+\rho(n-1)]}, \sqrt{\frac{4|\varsigma|\alpha}{n(2n-1)[1+\rho(2n-2)]}}\right\}$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{2|\varsigma|\alpha}{(2n-1)[1+\rho(2n-2)]}.$$

Proof. For

$$\vartheta(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \ldots \quad (0 < \alpha \leq 1;\ z \in \mathbb{U}),$$

where $G_1 = 2\alpha$ and $G_2 = 2\alpha^2$ in Theorem 3, it gives the required result. □

Remark 1. (i) The bound for $|a_n|$ in Theorem 3(*i*) is an improvement of the estimation given in Theorem 1.

(ii) From Corollaries 2 and 1, the bound for $|a_n|$ is smaller than the estimates obtained in [5, Corollary 1] and [5, Corollary 2], respectively.

(iii) Letting $\varsigma = 1$ in Corollary 1, we get an improvement of the estimate obtained by Srivastava *et al.* [33, Theorem 1] for all $0 \leq \rho \leq 1$.

Theorem 4. *Let* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k; (n \geq 2)$ *and its inverse map* $g = f^{-1}$ *be in* $\mathscr{S}(\nu, \rho; \vartheta)$ *with* $|G_2| \leq G_1$. *Then*

(i)

$$|a_n| \leq \min\left\{\frac{|\rho|G_1}{(n-1)[1+\nu(n-1)]}, \sqrt{\frac{2|\rho|G_1}{n(2n-2)[1+\nu(2n-2)]}}\right\}, \tag{2.8}$$

(ii)

$$|na_n^2 - a_{2n-1}| \leq \frac{|\rho|\, G_1}{(2n-2)\,[1+\nu(2n-2)]}.$$

Proof. According to the relations (2.6) and (2.8) in [12], we have

$$[1+\nu(n-1)](n-1)a_n = \rho G_1 p_{n-1}$$

and

$$[1+\nu(n-1)](n-1)c_n = -[1+\nu(n-1)](n-1)a_n = \rho G_1 q_{n-1}.$$

Now, taking the absolute values of the above equalities with $|p_{n-1}| \leq 1,\ |q_{n-1}| \leq 1$, we have

$$|a_n| \leq \frac{|\rho|\, G_1}{(n-1)\,[1+\nu(n-1)]} \tag{2.9}$$

and

$$|c_n| \leq \frac{|\rho|\, G_1}{(n-1)\,[1+\nu(n-1)]}. \tag{2.10}$$

By a similar argument in Theorem 1

$$|a_{2n-1}| = |c_{2n-1}| \leq \frac{|\rho|\, G_1}{(2n-2)\,[1+\nu(2n-2)]} \tag{2.11}$$

Also, in view of Lemma 5 and using the inequality (2.11), we obtain

$$|a_n| \leq \sqrt{\frac{|a_{2n-1}| + |c_{2n-1}|}{n}} \leq \sqrt{\frac{2\,|\rho|\, G_1}{n(2n-2)[1+\nu(2n-2)]}}. \tag{2.12}$$

From (2.9) and (2.12), we conclude that the inequality (2.8) holds. In addition, by (2.11) and applying Lemma 4, we get

$$|na_n^2 - a_{2n-1}| = |c_{2n-1}| \leq \frac{|\rho|\, G_1}{(2n-2)\,[1+\nu(2n-2)]}.$$

This completes the proof. □

For different values of ν and ρ and well-known ϑ, the above theorem yields the following interesting corollaries.

Corollary 3. *Let* $f(z) = z + \sum\limits_{k=n}^{\infty} a_k z^k;\ (n \geq 2)$ *and its inverse map* $g = f^{-1}$ *be in* $\mathscr{S}(1, \rho; \vartheta)$*. Then*

$$|a_n| \leq \min\left\{ \frac{|\rho|\, G_1}{n(n-1)},\ \sqrt{\frac{2\,|\rho|\, G_1}{n(2n-1)(2n-2)}} \right\}$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{|\rho|\, G_1}{(2n-1)(2n-2)}.$$

Corollary 4. *Let* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k; (n \geq 2)$ *and its inverse map* $g = f^{-1}$ *be in* $\mathscr{S}(\nu, \rho; (1 + Az)/(1 + Bz))$ *where* $-1 \leq B < A \leq 1$. *Then*

$$|a_n| \leq \min\left\{ \frac{|\rho|(A - B)}{(n-1)[1 + \nu(n-1)]}, \sqrt{\frac{2|\rho|(A - B)}{n(2n-2)[1 + \nu(2n-2)]}} \right\}$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{|\rho|(A - B)}{(2n-2)[1 + \nu(2n-2)]}.$$

Proof. For

$$\vartheta(z) = \frac{1 + Az}{1 + Bz} = 1 + (A - B)z - B(A - B)z^2 + \ldots \quad (-1 \leq B < A \leq 1;\ z \in \mathbb{U}),$$

where $G_1 = A - B$ and $G_2 = -B(A - B)$ in Theorem 4, it gives the required result. □

Corollary 5. *Let* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k; (n \geq 2)$ *and its inverse map* $g = f^{-1}$ *be in* $\mathscr{S}(0, 1; (1 + Az)/(1 + Bz))$ *where* $-1 \leq B < A \leq 1$. *Then*

$$|a_n| \leq \min\left\{ \frac{A - B}{(n-1)}, \sqrt{\frac{2(A - B)}{n(2n-2)}} \right\}$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{A - B}{2n - 2}.$$

Corollary 6. *Let* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k; (n \geq 2)$ *and its inverse map* $g = f^{-1}$ *be in* $\mathscr{S}(\nu, \rho; ((1 + z)/(1 - z))^{\alpha})$. *Then*

$$|a_n| \leq \min\left\{ \frac{2|\rho|\alpha}{(n-1)[1 + \nu(n-1)]}, \sqrt{\frac{4|\rho|\alpha}{n(2n-2)[1 + \nu(2n-2)]}} \right\}$$

and

$$|na_n^2 - a_{2n-1}| \leq \frac{2|\rho|\alpha}{(2n-2)[1 + \nu(2n-2)]}.$$

Remark 2. (i) The bound for $|a_n|$ in Theorem 4(*i*) is an improvement of the estimation given in Theorem 2.

(ii) From Corollary 5, the bound for $|a_n|$ is smaller than the estimate obtained by Hamidi and Jahangiri in [13, Theorem 2.1].

(iii) From Corollary 3, the bound for $|a_n|$ is smaller than the result obtained in [12, Corolary 2.4].

(iv) Letting $\rho = (1 - \beta)e^{i\delta}\cos\delta$ $(|\delta| \leq \pi/2;\ 0 \leq \beta < 1)$, $\nu = 0$, $A = 1$, $B = -1$ and $\rho = (1 - \beta)e^{i\delta}\cos\delta$ $(|\delta| \leq \pi/2;\ 0 \leq \beta < 1)$, $\nu = 1$, $A = 1$, $B = -1$ in Corollary 4, respectively, we get an improvement of the estimates obtained in [12, Corolary 2.5].

3. Conclusions

In our present study, we have applied the Faber polynomial expansion method to estimate the coefficient bounds of analytic and bi-univalent functions in the certain two subclasses, which are defined by subordination. Also, we have obtained some corollaries and consequences of the main results. Moreover, the given bounds improve and generalize some of the pervious results.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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