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Research article

Positive radial solutions of p-Laplace equations on exterior domains

Yongxiang Li* **and Mei Wei**

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: liyxnwnu@163.com; Tel: +8609317971111.

Abstract: This paper deals with the existence of positive radial solutions of the p-Laplace equation

$$\begin{cases} -\Delta_p u = K(|x|)\, f(u)\,, & x \in \Omega\,, \\ \frac{\partial u}{\partial n} = 0\,, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0\,, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, $N \geq 2$, $1 < p < N$, $K : [r_0, \infty) \to \mathbb{R}^+$ is continuous and $0 < \int_{r_0}^{\infty} r^{N-1} K(r)\, dr < \infty$, $f \in C(\mathbb{R}^+, \mathbb{R}^+)$. Under the inequality conditions related to the asymptotic behaviour of $f(u)/u^{p-1}$ at 0 and infinity, the existence results of positive radial solutions are obtained. The discussion is based on the fixed point index theory in cones.

Keywords: p-Laplace operator; positive radial solution; exterior domain; cone; fixed point index
Mathematics Subject Classification: 35J25, 35J60, 47H11, 47N20

1. Introduction

Boundary value problems with p-Laplace operator $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ arise in many different areas of applied mathematics and physics, such as non-Newtonian fluids, reaction-diffusion problems, non-linear elasticity, etc. But little is known about the p-Laplace operator cases ($p \neq 2$) compared to the vast amount of knowledge for the Laplace operator ($p = 2$). In this paper, we discuss the existence of positive radial solution for the p-Laplace boundary value problem (BVP)

$$\begin{cases} -\Delta_p u = K(|x|)\, f(u)\,, & x \in \Omega\,, \\ \frac{\partial u}{\partial n} = 0\,, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0\,, \end{cases} \tag{1.1}$$

in the exterior domain $\Omega = \{x \in \mathbb{R}^N : \ |x| > r_0\}$, where $N \geq 2$, $r_0 > 0$, $1 < p < N$, $\frac{\partial u}{\partial n}$ is the outward normal derivative of u on $\partial\Omega$, $K : [r_0, \infty) \to \mathbb{R}^+$ is a coefficient function, $f : \mathbb{R}^+ \to \mathbb{R}$ is a nonlinear function. Throughout this paper, we assume that the following conditions hold:

(A1) $K \in C([r_0, \infty), \mathbb{R}^+)$ and $0 < \int_{r_0}^{\infty} r^{N-1} K(r)\, dr < \infty$;

(A2) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$;

For the special case of $p = 2$, namely the Laplace boundary value problem

$$\begin{cases} -\Delta u = K(|x|) f(u), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.2}$$

the existence of positive radial solutions has been discussed by many authors, see [1–7]. The authors of references[1–6] obtained some existence results by using upper and lower solutions method, priori estimates technique and fixed point index theory. In [7], the present author built an eigenvalue criteria of existing positive radial solutions. The eigenvalue criterion is related to the principle eigenvalue λ_1 of the corresponding radially symmetric Laplace eigenvalue problem (EVP)

$$\begin{cases} -\Delta u = \lambda\, K(|x|)\, u, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ u = u(|x|), \quad \lim_{|x|\to\infty} u(|x|) = 0. \end{cases} \tag{1.3}$$

Specifically, if f satisfies one of the following eigenvalue conditions:

(H1) $f^0 < \lambda_1, \ f_\infty > \lambda_1$;

(H2) $f^\infty < \lambda_1, \ f_0 > \lambda_1$,

the BVP(1.2) has a classical positive radial solution, where

$$f_0 = \liminf_{u\to 0^+} \frac{f(u)}{u}, \qquad f^0 = \limsup_{u\to 0^+} \frac{f(u)}{u},$$
$$f_\infty = \liminf_{u\to\infty} \frac{f(u)}{u}, \qquad f^\infty = \limsup_{u\to\infty} \frac{f(u)}{u}.$$

See [7, Theorem 1.1]. This criterion first appeared in a boundary value problem of second-order ordinary differential equations, and built by Zhaoli Liu and Fuyi Li in [8]. Then it was extended to general boundary value problems of ordinary differential equations, See [9,10]. In [11,12], the radially symmetric solutions of the more general Hessian equations are discussed.

The purpose of this paper is to establish a similar existence result of positive radial solution of BVP (1.1). Our results are related to the principle eigenvalue $\lambda_{p,1}$ of the radially symmetric p-Laplce eigenvalue problem (EVP)

$$\begin{cases} -\Delta_p\, u = \lambda\, K(|x|)\, |u|^{p-2} u, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ u = u(|x|), \quad \lim_{|x|\to\infty} u(|x|) = 0. \end{cases} \tag{1.4}$$

Different from EVP (1.3), EVP (1.4) is a nonlinear eigenvalue problem, and the spectral theory of linear operators is not applicable to it. In Section 2 we will prove that EVP (1.4) has a minimum positive real eigenvalue $\lambda_{p,1}$, see Lemma 2.3. For BVP (1.1), we conjecture that eigenvalue criteria is valid if f_0, f^0, f_∞ and f^∞ is replaced respectively by

$$\begin{aligned} f_{p\,0} &= \liminf_{u\to 0^+} \frac{f(u)}{u^{p-1}}, \qquad {f_p}^{0} = \limsup_{u\to 0^+} \frac{f(u)}{u^{p-1}}, \\ f_{p\,\infty} &= \liminf_{u\to \infty} \frac{f(u)}{u^{p-1}}, \qquad {f_p}^{\infty} = \limsup_{u\to \infty} \frac{f(u)}{u^{p-1}}. \end{aligned} \tag{1.5}$$

But now we can only prove a weaker version of it: In second inequality of (H1) and (H2), $\lambda_{p,1}$ needs to be replaced by the larger number

$$B = \Big[\int_0^1 \Psi\Big(\int_s^1 t^{p-1}a(t)\,dt\Big)ds\Big]^{-(p-1)}, \tag{1.6}$$

where $a \in C^+(0,1]$ is given by (2.4) and $\Psi \in C(\mathbb{R})$ is given by (2.7). Our result is as follows:

Theorem 1.1. *Suppose that Assumptions* (A1) *and* (A2) *hold. If the nonlinear function f satisfies one of the the following conditions:*

(H1)* ${f_p}^{0} < \lambda_{p,1}$, $f_{p\,\infty} > B$;

(H2)* ${f_p}^{\infty} < \lambda_{p,1}$, $f_{p\,0} > B$,

then BVP (1.1) has at least one classical positive radial solution.

As an example of the application of Theorem 1.1, we consider the following p-Laplace boundary value problem

$$\begin{cases} -\Delta_p u = K(|x|)\,|u|^\gamma, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.7}$$

Corresponding to BVP (1.1), $f(u) = |u|^\gamma$. If $\gamma > p-1$, by (1.5) ${f_p}^{0} = 0$, $f_{p\,\infty} = +\infty$, and (H1) holds. If $0 < \gamma < p-1$, then ${f_p}^{\infty} = 0$, $f_{p\,0} = +\infty$, and (H2) holds. Hence, by Theorem 1.1 we have

Corollary 1.1. *Let* $K : [r_0, \infty) \to \mathbb{R}^+$ *satisfy Assumption* (A1), $\gamma > 0$ *and* $\gamma \neq p-1$. *Then BVP (1.7) has a positive radial solution.*

The proof of Theorem 1.1 is based on the fixed point index theory in cones, which will be given in Section 3. Some preliminaries to discuss BVP (1.1) are presented in Section 2.

2. Preliminaries

For the radially symmetric solution $u = u(|x|)$ of BVP (1.1), setting $r = |x|$, since

$$-\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) = -\Big(|u'(r)|^{p-2}u'(r)\Big)' - \frac{N-1}{r}\,|u'(r)|^{p-2}u'(r),$$

BVP (1.1) becomes the ordinary differential equation BVP in $[r_0, \infty)$

$$\begin{cases} -\left(|u'(r)|^{p-2}u'(r)\right)' - \frac{N-1}{r}\,|u'(r)|^{p-2}u'(r) = K(r)\,f(u(r)), \quad r \in [r_0, \infty), \\ u'(r_0) = 0, \quad u(\infty) = 0, \end{cases} \tag{2.1}$$

where $u(\infty) = \lim_{r\to\infty} u(r)$.

Let $q > 1$ be the constant satisfying $\frac{1}{p} + \frac{1}{q} = 1$. To solve BVP (2.1), make the variable transformations

$$t = \left(\frac{r_0}{r}\right)^{(q-1)(N-p)}, \quad r = r_0 t^{-1/(q-1)(N-p)}, \quad v(t) = u(r(t)), \tag{2.2}$$

Then BVP (2.1) is converted to the ordinary differential equation BVP in $(0, 1]$

$$\begin{cases} -\left(|v'(t)|^{p-2}v'(t)\right)' = a(t)\,f(v(t)), \qquad t \in (0, 1], \\ v(0) = 0, \quad v'(1) = 0, \end{cases} \tag{2.3}$$

where

$$a(t) = \frac{r^{q(N-1)}(t)}{(q-1)^p (N-p)^p\, {r_0}^{q(N-p)}}\, K(r(t)), \quad t \in (0, 1]. \tag{2.4}$$

BVP (2.3) is a quasilinear ordinary differential equation boundary value problem with singularity at $t = 0$. A solution v of BVP (2.3) means that $v \in C^1[0, 1]$ such that $|v'|^{p-2}v' \in C^1(0, 1]$ and it satisfies the Eq (2.3). Clearly, if v is a solution of BVP (2.3), then $u(r) = v(t(r))$ is a solution of BVP (2.1) and $u(|x|)$ is a classical radial solution of BVP (1.1). We discuss BVP (2.3) to obtain positive radial solutions of BVP (1.1).

Let $I = [0, 1]$ and $R^+ = [0, +\infty)$. Let $C(I)$ denote the Banach space of all continuous function $v(t)$ on I with norm $\|v\|_C = \max_{t\in I} |v(t)|$, $C^1(I)$ denote the Banach space of all continuous differentiable function on I. Let $C^+(I)$ be the cone of all nonnegative functions in $C(I)$.

To discuss BVP (2.3), we first consider the corresponding simple boundary value problem

$$\begin{cases} -\left(|v'(t)|^{p-2}v'(t)\right)' = a(t)\,h(t)\,, \quad t \in (0, 1], \\ v(0) = 0, \quad v'(1) = 0, \end{cases} \tag{2.5}$$

where $h \in C^+(I)$ is a given function. Let

$$\Phi(v) = |v|^{p-2}v = |v|^{p-1}\mathrm{sgn}\, v, \qquad v \in \mathbb{R}, \tag{2.6}$$

then $w = \Phi(v)$ is a strictly monotone increasing continuous function on $\mathbb{R}$ and its inverse function

$$\Phi^{-1}(w) := \Psi(w) = |w|^{q-1}\mathrm{sgn}\, w, \qquad w \in \mathbb{R}, \tag{2.7}$$

is also a strictly monotone increasing continuous function.

Lemma 2.1. *For every $h \in C(I)$, BVP (2.5) has a unique solution $v := Sh \in C^1(I)$. Moreover, the solution operator $S : C(I) \to C(I)$ is completely continuous and has the homogeneity*

$$S(\nu h) = \nu^{q-1} S h, \qquad h \in C(I), \quad \nu \geq 0. \tag{2.8}$$

Proof. By (2.4) and Assumption (A1), the coefficient $a(t) \in C^+(0, 1]$ and satisfies

$$\int_0^1 a(t)dt = \frac{1}{[(q-1)(N-p)]^{p-1} {r_0}^{N-p}} \int_{r_0}^{\infty} r^{N-1} K(r)\, dr < \infty. \tag{2.9}$$

Hence $a \in L(I)$.

For every $h \in C(I)$, we verify that

$$v(t) = \int_0^t \Psi\Big(\int_s^1 a(\tau)h(\tau)d\tau\Big)ds := Sh(t), \quad t \in I \tag{2.10}$$

is a unique solution of BVP (2.5). Since the function $G(s) := \int_s^1 a(\tau)h(\tau)d\tau \in C(I)$, from (2.10) it follows that $v \in C^1(I)$ and

$$v'(t) = \Psi\Big(\int_t^1 a(\tau)h(\tau)d\tau\Big), \qquad t \in I. \tag{2.11}$$

Hence,

$$|v'(t)|^{p-2}v'(t) = \Phi(v'(t)) = \int_t^1 a(\tau)h(\tau)d\tau, \quad t \in I.$$

This means that $(|v'(t)|^{p-2}v'(t) \in C^1(0, 1]$ and

$$(|v'(t)|^{p-2}v'(t))' = -a(t)h(t), \quad t \in (0, 1],$$

that is, v is a solution of BVP (2.5).

Conversely, if v is a solution of BVP (2.5), by the definition of the solution of BVP (2.5), it is easy to show that v can be expressed by (2.10). Hence, BVP (2.5) has a unique solution $v = Sh$.

By (2.10) and the continuity of Ψ, the solution operator $S : C(I) \to C(I)$ is continuous. Let $D \subset C(I)$ be bounded. By (2.10) and (2.11) we can show that $S(D)$ and its derivative set $\{v' \mid v \in S(D)\}$ are bounded sets in $C(I)$. By the Ascoli-Arzéla theorem, $S(D)$ is a precompact subset of $C(I)$. Thus, $S : C(I) \to C(I)$ is completely continuous.

By the uniqueness of solution of BVP (2.5), we easily verify that the solution operator S satisfies (2.8). □

Lemma 2.2. *If $h \in C^+(I)$, then the solution $v = Sh$ of LBVP (2.5) satisfies: $\|v\|_c = v(1)$, $v(t) \geq t\,\|v\|_C$ for every $t \in I$.*

Proof. Let $h \in C^+(I)$ and $v = Sh$. By (2.10) and (2.11), for every $t \in I$ $v(t) \geq 0$ and $v'(t) \geq 0$. Hence, $v(t)$ is a nonnegative monotone increasing function and $\|v\|_C = \max_{t\in I} v(t) = v(1)$. From (2.11) and the monotonicity of Ψ, we notice that $v'(t)$ is a monotone decreasing function on I. For every $t \in (0, 1)$, by Lagrange's mean value theorem, there exist $\xi_1 \in (0, t)$ and $\xi_2 \in (t, 1)$, such that

$$(1-t)v(t) = (1-t)(v(t) - v(0)) = v'(\xi_1)t(1-t) \geq v'(t)t(1-t),$$

$$tv(t) = tv(1) - t(v(1) - v(t)) = tv(1) - tv'(\xi_2)(1-t) \geq tv(1) - v'(t)t(1-t).$$

Hence

$$v(t) = tv(t) + (1-t)v(t) \geq tv(1) = t\|v\|_C.$$

Obviously, when $t = 0$ or 1, this inequality also holds. The proof is completed. □

Consider the radially symmetric p-Laplace eigenvalue problem EVP (1.3). We have

Lemma 2.3. *EVP (1.4) has a minimum positive real eigenvalue $\lambda_{p,1}$, and $\lambda_{p,1}$ has a radially symmetric positive eigenfunction.*

Proof. For the radially symmetric eigenvalue problem EVP (1.4), writing $r = |x|$ and making the variable transformations of (2.2), it is converted to the one-dimensional weighted p-Laplace eigenvalue problem (EVP)

$$\begin{cases} -\left(|v'(t)|^{p-2}v'(t)\right)' = \lambda\, a(t)\, |v(t)|^{p-2}v(t), \qquad t \in (0,\, 1], \\ v(0) = 0, \quad v'(1) = 0, \end{cases} \tag{2.12}$$

where $v(t) = u(r(t))$. Clearly, $\lambda \in \mathbb{R}$ is an eigenvalue of EVP (1.4) if and only if it is an eigenvalue of EVP (2.12). By (2.4) and (2.9), $a \in C^+(0, 1] \cap L(I)$ and $\int_0^1 a(s)ds > 0$. This guarantees that EVP (2.12) has a minimum positive real eigenvalue $\lambda_{p,1}$, which given by

$$\lambda_{p,1} = \inf\left\{ \frac{\int_0^1 |w'(t)|^p dt}{\int_0^1 a(t)w^p(t)dt} \;\middle|\; \begin{array}{c} w \in C^1(I), w(0) = 0,\ w'(1) = 0, \\ \int_0^1 a(t)w^p(t)dt \neq 0 \end{array} \right\}. \tag{2.13}$$

Moreover, $\lambda_{p,1}$ is simple and has a positive eigenfunction $\phi \in C^+(I) \cap C^1(I)$. See [13, Theorem 5], [14, Theorem 1.1] or [15, Theorem 1.2]. Hence, $\lambda_{p,1}$ is also the minimum positive real eigenvalue of EVP (1.4), and $\phi((r_0/|x|)^{(q-1)(N-p)})$ is corresponding positive eigenfunction. □

Now we consider BVP (2.3). Define a closed convex cone K of $C(I)$ by

$$K = \{\, v \in C(I) \mid v(t) \geq t\,\|v\|_C, \;\; t \in I \,\}. \tag{2.14}$$

By Lemma 2.2, $S(C^+(I)) \subset K$. Let $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, and define a mapping $F : K \to C^+(I)$ by

$$F(v)(t) := f(v(t)), \qquad t \in I. \tag{2.15}$$

Then $F : K \to C^+(I)$ is continuous and it maps every bounded subset of K into a bounded subset of $C^+(I)$. Define the composite mapping by

$$A = S \circ F. \tag{2.16}$$

Then $A : K \to K$ is completely continuous by the complete continuity of the operator $S : C^+(I) \to K$. By the definitions of S and K, the positive solution of BVP (2.3) is equivalent to the nonzero fixed point of A.

Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume D is a bounded open subset of E with boundary ∂D, and $K \cap D \neq \emptyset$. Let $A : K \cap \overline{D} \to K$ be a completely continuous mapping. If $Av \neq v$ for every $v \in K \cap \partial D$, then the fixed point index $i(A, K \cap D, K)$ is well defined. One important fact is that if $i(A, K \cap D, K) \neq 0$, then A has a fixed point in $K \cap D$. In next section, we will use the following two lemmas in [16, 17] to find the nonzero fixed point of the mapping A defined by (2.16).

Lemma 2.4. *Let D be a bounded open subset of E with $\mathbf{0} \in D$, and $A : K \cap \overline{D} \to K$ a completely continuous mapping. If $\mu Av \neq v$ for every $v \in K \cap \partial D$ and $0 < \mu \leq 1$, then $i(A, K \cap D, K) = 1$.*

Lemma 2.5. *Let D be a bounded open subset of E with $\mathbf{0} \in D$, and $A : K \cap \overline{D} \to K$ a completely continuous mapping. If $\|Av\| \geq \|v\|$ and $Av \neq v$ for every $v \in K \cap \partial D$, then $i(A, K \cap D, K) = 0$.*

3. Proof of the main result

Proof of Theorem 1.1. We only consider the case that (H1)* holds, and the case that (H2)* holds can be proved by a similar way.

Let $K \subset C(I)$ be the closed convex cone defined by (2.14) and $A : K \to K$ be the completely continuous mapping defined by (2.16). If $v \in K$ is a nontrivial fixed point of A, then by the definitions of S and A, $v(t)$ is a positive solution of BVP (2.3) and $u = v(r_0{}^{N-2}/|x|^{N-2})$ is a classical positive radial solution of BVP (1.1). Let $0 < R_1 < R_2 < +\infty$ and set

$$D_1 = \{v \in C(I) : \ \|v\|_C < R_1\}, \quad D_2 = \{v \in C(I) : \ \|v\|_C < R_2\}. \tag{3.1}$$

We prove that A has a fixed point in $K \cap (\overline{D}_2 \setminus D_1)$ when R_1 is small enough and R_2 large enough.

Since $f_p{}^0 < \lambda_{p,1}$, by the definition of $f_p{}^0$, there exist $\varepsilon \in (0, \lambda_{p,1})$ and $\delta > 0$, such that

$$f(u) \le (\lambda_{p,1} - \varepsilon) u^{p-1}, \quad 0 \le u \le \delta. \tag{3.2}$$

Choosing $R_1 \in (0, \delta)$, we prove that A satisfies the condition of Lemma 2.4 in $K \cap \partial D_1$, namely

$$\mu A v \neq v, \qquad \forall\, v \in K \cap \partial D_1, \ \ 0 < \mu \le 1. \tag{3.3}$$

In fact, if (3.3) does not hold, there exist $v_0 \in K \cap \partial D_1$ and $0 < \mu_0 \le 1$ such that $\mu_0 A v_0 = v_0$. By the homogeneity of S, $v_0 = \mu_0 S(F(v_0)) = S(\mu_0{}^{p-1} F(v_0))$. By the definition of S, v_0 is the unique solution of BVP (2.5) for $h = \mu_0{}^{p-1} F(v_0) \in C^+(I)$. Hence, $v_0 \in C^1(I)$ satisfies the differential equation

$$\begin{cases} -\left(|v_0'(t)|^{p-2} v_0{}'(t)\right)' = \mu_0{}^{p-1}\, a(t)\, f(v_0(t)), & t \in (0, 1], \\ v_0(0) = 0, \quad v_0{}'(1) = 0. \end{cases} \tag{3.4}$$

Since $v_0 \in K \cap \partial D_1$, by the definitions of K and D_1,

$$0 \le v_0(t) \le \|v_0\|_C = R_1 < \delta, \quad t \in I.$$

Hence by (3.2),

$$f(v_0(t)) \le (\lambda_{p,1} - \varepsilon)\, v_0{}^{p-1}(t), \quad t \in I.$$

By this inequality and Eq (3.4), we have

$$-\left(|v_0'(t)|^{p-2} v_0{}'(t)\right)' \le \mu_0{}^{p-1} (\lambda_{p,1} - \varepsilon)\, a(t)\, v_0{}^{p-1}(t), \qquad t \in (0,\ 1].$$

Multiplying this inequality by $v_0(t)$ and integrating on $(0,\ 1]$, then using integration by parts for the left side, we have

$$\begin{aligned} \int_0^1 |v_0'(t)|^p dt &\le \mu_0{}^{p-1} (\lambda_{p,1} - \varepsilon) \int_0^1 a(t)\, v_0{}^p(t) dt \\ &\le (\lambda_{p,1} - \varepsilon) \int_0^1 a(t)\, v_0{}^p(t) dt. \end{aligned} \tag{3.5}$$

Since $v_0 \in K \cap \partial D$, by the definition of K,

$$\int_0^1 a(t)\, {v_0}^p(t)dt \geq \|v_0\|_C{}^p \int_0^1 t^p a(t)dt = R_1{}^p \int_0^1 t^p a(t)dt > 0.$$

Hence, by (2.13) and (3.5) we obtain that

$$\lambda_{p,1} \leq \frac{\int_0^1 |v_0'(t)|^p dt}{\int_0^1 a(t)\, {v_0}^p(t)dt} \leq \lambda_{p,1} - \varepsilon,$$

which is a contradiction. This means that (3.3) holds, namely A satisfies the condition of Lemma 2.4 in $K \cap \partial D_1$. By Lemma 2.4, we have

$$i\,(A,\ K \cap D_1,\ K) = 1. \tag{3.6}$$

On the other hand, by the definition (1.6) of B, we have

$$B < \Big[\int_\sigma^1 \Psi\Big(\int_s^1 t^{p-1} a(t)\, dt\Big) ds\Big]^{-(p-1)} \to B\ (\sigma \to 0^+), \quad \sigma \in (0,\ 1). \tag{3.7}$$

Since $f_{p\,\infty} > B$, by (3.7) there exists $\sigma_0 \in (0,\ 1)$, such that

$$B_0 := \Big[\int_{\sigma_0}^1 \Psi\Big(\int_s^1 t^{p-1} a(t)\, dt\Big) ds\Big]^{-(p-1)} < f_{p\,\infty}. \tag{3.8}$$

By this inequality and the definition of $f_{p\,\infty}$, there exists $H > 0$ such that

$$f(u) > B_0 u^{p-1}, \qquad u > H. \tag{3.9}$$

Choosing $R_2 > \max\{\delta,\ H/\sigma_0\}$, we show that

$$\|Av\|_C \geq \|v\|_C, \qquad v \in K \cap \partial D_2. \tag{3.10}$$

For $\forall\ v \in K \cap \partial D_2$ and $t \in [\sigma_0,\ 1]$, by the definitions of K and D_2

$$v(t) \geq t\|v\|_C \geq \sigma_0 R_2 > H.$$

By this inequality and (3.9),

$$f(v(t)) > B_0 v^{p-1}(t) \geq B_0 \|v\|_C^{p-1} t^{p-1}, \qquad t \in [\sigma_0,\ 1]. \tag{3.11}$$

Since $Av = S(F(v))$, by the expression (2.10) of the solution operator S and (3.11), noticing $(p-1)(q-1) = 1$, we have

$$\begin{aligned} \|Av\|_C \geq Av(1) &= \int_0^1 \Psi\Big(\int_s^1 a(t)\, f(v(t))dt\Big) ds \\ &\geq \int_{\sigma_0}^1 \Psi\Big(\int_s^1 a(t)\, f(v(t))dt\Big) ds \end{aligned}$$

$$\geq \int_{\sigma_0}^{1} \Psi\Big(\int_{s}^{1} a(t) B_0 \|v\|_C^{p-1} t^{p-1} dt \Big) ds$$

$$= B_0^{q-1} \|v\|_C \int_{\sigma_0}^{1} \Psi\Big(\int_{s}^{1} t^{p-1} a(t) dt \Big) ds$$

$$= \|v\|_C.$$

Namely, (3.10) holds. Suppose A has no fixed point on ∂D_2. Then by (3.10), A satisfies the condition of Lemma 2.5 in $K \cap \partial D_2$. By Lemma 2.5, we have

$$i\,(A,\ K \cap D_2,\ K) = 0. \tag{3.11}$$

By the additivity of fixed point index, (3.6) and (3.11), we have

$$i\,(A,\ K \cap (D_2 \setminus \overline{D}_1),\ K) = i\,(A,\ K \cap D_2,\ K) - i\,(A,\ K \cap D_1,\ K) = -1.$$

Hence A has a fixed point in $K \cap (D_2 \setminus \overline{D}_1)$.

The proof of Theorem 1.1 is complete. □

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Conflict of interest

The authors declare that they have no competing interests.

References

1. J. Santanilla, Existence and nonexistence of positive radial solutions of an elliptic Dirichlet problem in an exterior domain, *Nonlinear Anal.*, **25** (1995), 1391–1399.
2. Y. H. Lee, Eigenvalues of singular boundary value problems and existence results for positive radial solutions of semilinear elliptic problems in exterior domains, *Differ. Integral Equ*, **13** (2000), 631–648.
3. Y. H. Lee, A multiplicity result ofpositive radial solutions for a multiparameter elliptic system on an exterior domain, *Nonlinear Anal.*, **45** (2000), 597–611.
4. R. Stanczy, Decaying solutions for sublinear elliptic equations in exterior domains, *Topol. Method. Nonlinear Anal.*, **14** (1999), 363–370.
5. R. Stanczy, Positive solutions for superlinear elliptic equations, *J. Math. Anal. Appl.*, **283** (2003), 159–166.

6. R. Precup, Existence, localization and multiplicity results for positive radial solutions of semilinear elliptic systems, *J. Math. Anal. Appl.*, **352** (2009), 48–56.
7. Y. Li, H. Zhang, Existence of positive radial solutions for the elliptic equations on an exterior domain, *Ann. Polon. Math.*, **116** (2016), 67–78.
8. Z. Liu, F. Li, Multiple positive solutions of nonlinear boundary problems, *J. Math. Anal. Appl.,* **203** (1996), 610–625.
9. L. Erbe, Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems, *Math. Comput. Model.*, **32** (2000), 529–539.
10. Y. Li, Abstract existence theorems of positive solutions for nonlinear boundary value problems, *Nonlinear Anal.*, **57** (2004), 211–227.
11. L. M. Dai, Existence and nonexistence of subsolutions for augmented Hessian equations, *DCDS*, **40** (2020), 579–596.
12. L. M. Dai, J. G. Bao, On uniqueness and existence of viscosity solutions to Hessian equations in exterior domains, *Front. Math. China*, **6** (2011), 221–230.
13. W. Walter, Sturm-Liouville theory for the radial Δ_p-operator, *Math. Z.*, **227** (1998), 175–185.
14. J. F. Bonder, J. P. Pinasco, Asymptotic behavior of the eigenvalues of the one-dimensional weighted p-Laplace operator, *Ark. Mat.*, **41** (2003), 267–280.
15. M. Lucia, S. Prashanth, Simplicity of principal eigenvalue for p-Laplace operator with singular indefinite weight, *Arch. Math.*, **86** (2006), 79–89.
16. K. Deimling, *Nonlinear functional analysis*, New York: Springer-Verlag, 1985.
17. D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, New York: Academic Press, 1988.

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