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*Research article*

# Existence of solutions for impulsive hybrid boundary value problems to fractional differential systems

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**Abstract:** In this paper, the existence of solutions for impulsive mixed boundary value problems involving Caputo fractional derivatives is obtained. And then, our conclusions are based on Krasnoselskii's fixed point theorem and Arzela-Ascoli theorem. Finally, some examples are given to illustrate the main results.

**Keywords:** impulsive fractional differential equations; mixed boundary value; existence and uniqueness of solutions; fixed point theorem
**Mathematics Subject Classification:** 34A08, 34B15

## 1. Introduction

Fractional differential equations have used in many engineering and scientific disciplines, such as physics, chemistry, aerodynamics, electrodynamics of complex media, polymer rheology, and other fields [1–3]. Many authors obtained solutions for fractional differential equations boundary value problems (BVP) by a fixed point theorem [4–8]. Many established mathmatics methods were applied to the existence solutions of BVP. For example, the numerical method [9–11], the Mawhin continuation method [12–14], the upper and lower solution method [15–17], the critical point theory [18–20].

In the past, impulsive differential and partial differential equations have become more and more crucial in mathematical models of real phenomena, especially in the fields of control, biological and medical [21–23].

In 2016, Bai and Dong [4] studied the existence of solutions for a class of hybrid BVP for fractional impulsive differential equation:

$$\begin{cases} {}^{C}D^{q}_{0^+}u(t) = f(t, u(t)), \qquad 1 < q < 2, t \in J \setminus \{t_1, t_2, \cdots, t_p\}, \\ \Delta u\mid_{t=t_k} = I_k(u(t_k)), \Delta u'\mid_{t=t_k} = J_k(u(t_k)), \quad k = 1, 2, \cdots, p, \\ u(0) + u'(1) = 0, u'(0) + u(1) = 0, \end{cases}$$

where ${}^{C}D_{0^+}^{q}$ is the Caputo fractional derivative of order $q \in (1,2)$, $J = [0,1]$, $f : [0,1] \times R \to R$ is a given function, $t_k$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1$, the right and left limits of $u(t)$ at $t = t_k^+$ are represented by $u(t_k^+)$ and $u(t_k^-)$.

In 2017, Mahmudov *et al.* [5] using the same way to investigate the existence and uniqueness of solutions for the following mixed impulsive BVP:

$$\begin{cases} {}^{C}D_{0^+}^{q}u(t) = f(t,u(t)), \qquad 1 < q < 2, J = [0,1], t \in J \setminus \{t_1, t_2, \cdots, t_p\}, \\ \Delta u \mid_{t=t_k} = I_k(u(t_k)), \Delta u' \mid_{t=t_k} = J_k(u(t_k)), \quad k = 1,2,\cdots,p, \\ u(0) + \mu_1 u'(1) = \sigma_1, u'(0) + \mu_2 u(1) = \sigma_2, \end{cases}$$

where ${}^{C}D_{0^+}^{q}$ is the Caputo fractional derivative of order $q \in (1,2)$.

Motivated by the above works, in this paper, we will apply Arzela-Ascoli theorem, Krasnoselskii's fixed point theorem and contraction mapping principle to study the existence of solution for a class of hybrid boundary value problem under impulse conditions. Precisely, we consider the existence and uniqueness of solutions for an impulsive mixed BVP of fractional differential equation:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}u(t) = f(t,u(t)), \qquad 1 < \alpha \le 2, t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \Delta {}^{C}D_{0^+}^{\beta}u(t_k) = J_k(u(t_k)), \quad 0 < \beta \le 1, \\ u(0) + \kappa_1 {}^{C}D_{0^+}^{\beta}u(1) = \theta_1, {}^{C}D_{0^+}^{\beta}u(0) + \kappa_2 u(1) = \theta_2, \end{cases} \tag{1.1}$$

where ${}^{C}D_{0^+}^{\alpha}$ and ${}^{C}D_{0^+}^{\beta}$ are Caputo fractional derivatives of order $\alpha$ ($1 < \alpha \le 2$) and $\beta$ ($0 < \beta \le 1$) respectively; $f : J \times R \to R, I_k, J_k : R \to R$ are continuous functions; $J = [0,1], J' = J \setminus \{t_1, t_2, \cdots, t_p\}$ and $t_k$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1, \Delta u(t_k) = u(t_k^+) - u(t_k^-), \Delta {}^{C}D_{0^+}^{\beta}u(t_k) = {}^{C}D_{0^+}^{\beta}u(t_k^+) - {}^{C}D_{0^+}^{\beta}u(t_k^-)$. Here, respectively, the left and the right limits of $u(t)$ at $t = t_k$ ($k = 1,2,\cdots,p$) are represented by $u(t_k^-)$ and $u(t_k^+)$; $\kappa_1, \kappa_2, \theta_1, \theta_2$ are constants and $\kappa_1, \kappa_2$ are different from zero.

When $\beta = 1$, the results of (1.1) will be degenerate to Lemma 6 in [5], and $\kappa_1 = \kappa_2 = 1$, the results degenerate to Lemma 2.4 in [4]. Therefore, this conclusion further expands the research results of [4] and [5].

The structure of this article is as follows. In Sect 2, The definitions and theorems related to Caputo's fractional integral and derivative are given. In Sect 3, the existence and uniqueness of solutions to mixed impulsive boundary value problems are proved by using Arzela-Ascoli theorem and Krasnoselskii's fixed point theorem. In Sect 4, some examples are provided to illustrate the main research results.

## 2. Preliminaries

In this section, we mainly introduce related definitions, theorems, lemmas and necessary symbol descriptions.

Let $PC(J) = \{u : [0,1] \to R \big| u \in C(J'), u(t_k^+), u(t_k^-)$ exist, and $u(t_k^-) = u(t_k), 1 \le k \le p\}$. Thus, $PC(J)$ is a Banach space with the norm $\|u\|_{PC} = \sup\limits_{0 \le t \le 1} |u(t)|$.

**Definition 2.1.** ( [5]) *The fractional integral of order $\alpha$ of a function $f : [0,+\infty) \to R$ is defined as*

$$I_{0^+}^{\alpha} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds, \quad t > 0, \alpha > 0, \tag{2.1}$$

*provided that the right hand side of the integral is point-wise defined on* $(0, \infty)$.

**Definition 2.2.** ( [5]) *The Caputo fractional-order derivative of order* $\alpha > 0$ *for a function* $f : [0, +\infty) \to R$ *is defined by*

$$ {}^{C}D^{\alpha}_{0^+} f(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s)ds, \quad t > 0, n = [\alpha] + 1, \tag{2.2} $$

*where* $[\alpha]$ *denotes the integer part of real number* $\alpha$, *and* $\Gamma(\cdot)$ *is the gamma function.*

**Lemma 2.3.** ( [3]) *For* $\alpha > 0$, *the general solution of the fractional differential equation* ${}^{C}D^{\alpha}u(t) = 0$ *is given by*

$$ u(t) = k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1}, k_i \in R, \tag{2.3} $$

*and*

$$ I^{\alpha}_{0^+}(D^{\alpha}_{0^+}u)(t) = u(t) + k_0 + k_1 t + k_2 t^2 + \cdots + k_{n-1} t^{n-1}, \tag{2.4} $$

*where* $n = -[-\alpha], [\alpha]$ *denotes the integer part of the real number* $\alpha$.

Now, we state two known results due to Krasnoselskii and Arzela-Ascoli which are used to prove the existence and uniqueness of solutions of (1.1), respectively.

**Lemma 2.4.** (Krasnoselskiis fixed point Theorem [24]) *Assume C is a closed, convex and non-empty subset of a Banach space H, and the operators A and B be such that:* $Ax + By \in C$, *whenever* $x, y \in C$; *A is compact and continuous; and B is a contraction mapping. Therefore, there exists* $z \in C$ *such that* $z = Az + Bz$.

**Lemma 2.5.** (Arzela-Ascoli Theorem [3]) *The set* $G \subset PC([0, 1], R^n)$ *is relatively compact set if and only if G is bounded, therefore,* $\|x\| \le M$ *for each* $x \in G$ *and some* $M > 0$; *the G is quasi-equicontinuous in* $[0, 1]$, *in other words, for any* $\varepsilon > 0$ *there exists* $\gamma > 0$ *such that if* $x \in G, k \in N$, $\chi_1, \chi_2 \in (t_{k-1}, t_k]$ *and* $|\chi_1 - \chi_2| < \gamma$, *we have* $|x(\chi_1) - x(\chi_2)| < \varepsilon$.

**Lemma 2.6.** *Let* $\alpha \in (1, 2]$, $\beta \in (0, 1]$, *and* $g : J \to R$ *be continuous. A functional u is a solution of the following impulsive hybrid BVP:*

$$ \begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = g(t), \qquad 1 < \alpha \le 2, t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \Delta^{C}D^{\beta}_{0^+}u(t_k) = J_k(u(t_k)), \quad 0 < \beta \le 1, \\ u(0) + \kappa_1 {}^{C}D^{\beta}_{0^+}u(1) = \theta_1, {}^{C}D^{\beta}_{0^+}u(0) + \kappa_2 u(1) = \theta_2, \end{cases} \tag{2.5} $$

*if u is a unique solution of the following impulsive fractional integral equation:*

$$(P^*)\quad u(t)=\begin{cases}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds+\nu_1(t)\theta_1\\ +\nu_2(t)\theta_2-\kappa_2\nu_1(t)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\\ -\kappa_1\nu_1(t)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds, \quad t\in[0,t_1);\\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds+\nu_1(t)\theta_1+\nu_2(t)\theta_2\\ -\kappa_2\nu_1(t)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\\ -\kappa_1\nu_1(t)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds\\ -\Gamma(2-\beta)\nu_1(t)\sum\limits_{j=1}^{p} t_j^{\beta}J_j(u(t_j))+\nu_1(t)\sum\limits_{j=1}^{p} I_j(u(t_j))\\ +\Gamma(2-\beta)\nu_2(t)\sum\limits_{j=1}^{p} t_j^{\beta-1}J_j(u(t_j))\\ +\Gamma(2-\beta)\sum\limits_{j=k+1}^{p} t_j^{\beta-1}(t_j-t)J_j(u(t_j))\\ -\sum\limits_{j=k+1}^{p} I_j(u(t_j)), \quad t\in[t_k,t_{k+1}], \quad k=1,2,\cdots,p-1;\\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds+\nu_1(t)\theta_1+\nu_2(t)\theta_2\\ -\kappa_2\nu_1(t)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\\ -\kappa_1\nu_1(t)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds+\nu_1(t)\sum\limits_{j=1}^{p} I_j(u(t_j))\\ -\Gamma(2-\beta)\nu_1(t)\sum\limits_{j=1}^{p} t_j^{\beta}J_j(u(t_j))\\ +\Gamma(2-\beta)\nu_2(t)\sum\limits_{j=1}^{p} t_j^{\beta-1}J_j(u(t_j)), \quad t\in(t_p,t_{p+1}],\end{cases}$$

*where* $\kappa_2\neq 1+\frac{1}{\kappa_1}$ *and*

$$\nu_1(t)=\frac{\Gamma(2-\beta)(1+\kappa_2-\kappa_2 t)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}, \qquad \nu_2(t)=\frac{\Gamma(2-\beta)t-\kappa_1}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}.$$

*Proof.* With the Lemma 2.3, a general solution $u$ of the equation ${}^{C}D_{0^+}^{\alpha}u(t)=g(t)$ on each interval $(t_k,t_{k+1}]$ $(k=0,1,2,\cdots,p)$ is given by

$$u(t)=I_{0^+}^{\alpha}g(t)+d_k+w_k t=\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds+d_k+w_k t, \quad t\in[0,t_1), \tag{2.6}$$

for some $d_k,w_k\in R$, where $t_0=0$ and $t_{p+1}=1$.
If $0<\beta<1$, we get

$${}^{C}D_{0^+}^{\beta}u(t)=\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds+\frac{t^{1-\beta}}{\Gamma(2-\beta)}w_k, \quad t\in(t_k,t_{k+1}], \tag{2.7}$$

if $\beta=1$, we obtain

$${}^{C}D_{0^+}^{\beta}u(t)=u'(t)=\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}g(s)ds+w_k, \quad t\in(t_k,t_{k+1}], \tag{2.8}$$

where ${}^{C}D_{0^+}^{\beta} d_k = 0 (0 < \beta \le 1), {}^{C}D_{0^+}^{\beta} t = \frac{t^{1-\beta}}{\Gamma(2-\beta)} (0 < \beta < 1)$. When $\beta = 1$, (2.7) and (2.8) are equivalent, thus, $0 < \beta \le 1$, get

$$ {}^{C}D_{0^+}^{\beta} u(t) = \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} g(s)ds + \frac{t^{1-\beta}}{\Gamma(2-\beta)} w_k, \quad t \in (t_k, t_{k+1}]. \tag{2.9} $$

By the (2.6) and (2.9), we obtain $u(0) = d_0, {}^{C}D_{0^+}^{\beta} u(0) = w_0$, and

$$ \begin{aligned} u(1) &= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds + d_p + w_p, \\ {}^{C}D_{0^+}^{\beta} u(1) &= \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} g(s)ds + \frac{w_p}{\Gamma(2-\beta)}. \end{aligned} $$

Using the boundary conditions in (2.5) to get

$$ d_0 + \kappa_1 \Big[ \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} g(s)ds + \frac{w_p}{\Gamma(2-\beta)} \Big] = \theta_1, \tag{2.10} $$

and

$$ w_0 + \kappa_2 \Big[ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds + d_p + w_p \Big] = \theta_2. \tag{2.11} $$

Next, using the condition of $\Delta^{C}D_{0^+}^{\beta} u(t_k) = {}^{C}D_{0^+}^{\beta} u(t_k^+) - {}^{C}D_{0^+}^{\beta} u(t_k^-) = J_k(u(t_k))$, we get

$$ \begin{aligned} \frac{t_k^{1-\beta}}{\Gamma(2-\beta)} w_k &= \frac{t_k^{1-\beta}}{\Gamma(2-\beta)} w_{k-1} + J_k(u(t_k)), \\ w_k &= w_{k-1} + \Gamma(2-\beta) t_k^{\beta-1} J_k(u(t_k)), \\ w_{k+1} &= w_k + \Gamma(2-\beta) t_{k+1}^{\beta-1} J_{k+1}(u(t_k)), \\ w_{k+2} &= w_{k+1} + \Gamma(2-\beta) t_{k+2}^{\beta-1} J_{k+2}(u(t_k)), \\ w_p &= w_{k-1} + \Gamma(2-\beta) t_p^{\beta-1} J_p(u(t_p)), \\ w_p &= w_{k-1} + \Gamma(2-\beta) \sum_{j=k}^{p} t_j^{\beta-1} J_j(u(t_j)), \\ w_{k-1} &= w_p - \Gamma(2-\beta) \sum_{j=k}^{p} t_j^{\beta-1} J_j(u(t_j)), \\ w_k &= w_p - \Gamma(2-\beta) \sum_{j=k+1}^{p} t_j^{\beta-1} J_j(u(t_j)). \end{aligned} \tag{2.12} $$

In the same way, using the condition of $\Delta u \mid_{t=t_k} = I_k(u(t_k)) = u(t_k^+) - u(t_k^-)$, we obtain

$$ d_k + w_k t_k = d_{k-1} + w_{k-1} t_k + I_k(u(t_k)), $$

which by (2.12) implies that

$$ d_k + (w_{k-1} + \Gamma(2-\beta) t_k^{\beta-1} J_k(u(t_k))) t_k = d_{k-1} + w_{k-1} t_k + I_k(u(t_k)), $$

$$
\begin{aligned}
&d_k + w_{k-1}t_k + \Gamma(2-\beta)t_k^{\beta-1}J_k(u(t_k))t_k = d_{k-1} + w_{k-1}t_k + I_k(u(t_k)),\\
&d_k + \Gamma(2-\beta)t_k^{\beta}J_k(u(t_k)) = d_{k-1} + I_k(u(t_k)),\\
&d_k = d_{k-1} - \Gamma(2-\beta)t_k^{\beta}J_k(u(t_k)) + I_k(u(t_k)),\\
&d_k = d_p + \Gamma(2-\beta)\sum_{j=k+1}^{p} t_j^{\beta}J_j(u(t_j)) - \sum_{j=k+1}^{p} I_j(u(t_j)).
\end{aligned} \tag{2.13}
$$

By combining (2.10), (2.11), (2.12) and (2.13), we have

$$
\begin{aligned}
&d_p + \Gamma(2-\beta)\sum_{j=1}^{p} t_j^{\beta}J_j(u(t_j)) - \sum_{j=1}^{p} I_j(u(t_j))\\
&\quad + \kappa_1\frac{w_p}{\Gamma(2-\beta)} + \kappa_1\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds = \theta_1,\\
&w_p - \Gamma(2-\beta)\sum_{j=1}^{p} t_j^{\beta-1}J_j(u(t_j)) + \kappa_2 w_p + \kappa_2 d_p\\
&\quad + \kappa_2\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds = \theta_2.
\end{aligned}
$$

Then

$$
\begin{aligned}
d_p =&\Big(\frac{\Gamma(2-\beta)(\kappa_2+1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\theta_1 - \Big(\frac{\kappa_1}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\theta_2\\
&+\Big(\frac{\kappa_1\kappa_2}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\\
&-\Big(\frac{\Gamma(2-\beta)(\kappa_1\kappa_2+\kappa_1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds\\
&+\Big(\frac{\Gamma(2-\beta)(\kappa_2+1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} I_j(u(t_j))\\
&-\Big(\frac{(\Gamma(2-\beta))^2(\kappa_2+1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} t_j^{\beta}J_j(u(t_j))\\
&-\Big(\frac{\Gamma(2-\beta)\kappa_1}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} t_j^{\beta-1}J_j(u(t_j)),
\end{aligned} \tag{2.14}
$$

and

$$
\begin{aligned}
w_p =& -\Big(\frac{\Gamma(2-\beta)(\kappa_2)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\theta_1 + \Big(\frac{\Gamma(2-\beta)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\theta_2\\
&-\Big(\frac{\kappa_2\Gamma(2-\beta)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\\
&+\Big(\frac{\Gamma(2-\beta)(\kappa_1\kappa_2)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&-\Big(\frac{\Gamma(2-\beta)\kappa_2}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} I_j(u(t_j))\\
&+\Big(\frac{(\Gamma(2-\beta))^2\kappa_2}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} t_j^{\beta} J_j(u(t_j))\\
&+\Big(\frac{(\Gamma(2-\beta))^2}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} t_j^{\beta-1} J_j(u(t_j)).
\end{aligned}
\tag{2.15}
$$

Combining (2.12), (2.13), (2.14) and (2.15), we obtain

$$
\begin{aligned}
d_k =& d_p + \Gamma(2-\beta)\sum_{j=k+1}^{p} t_j^{\beta} J_j(u(t_j)) - \sum_{j=k+1}^{p} I_j(u(t_j))\\
=&\Big(\frac{\Gamma(2-\beta)(\kappa_2+1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\theta_1 - \Big(\frac{\kappa_1}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\theta_2\\
&+\Big(\frac{\kappa_1\kappa_2}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds\\
&-\Big(\frac{\Gamma(2-\beta)(\kappa_1\kappa_2+\kappa_1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} g(s)ds\\
&+\Big(\frac{\Gamma(2-\beta)(\kappa_2+1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} I_j(u(t_j))\\
&-\Big(\frac{(\Gamma(2-\beta))^2(\kappa_2+1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} t_j^{\beta} J_j(u(t_j))\\
&-\Big(\frac{\Gamma(2-\beta)\kappa_1}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} t_j^{\beta-1} J_j(u(t_j))\\
&+\Gamma(2-\beta)\sum_{j=k+1}^{p} t_j^{\beta} J_j(u(t_j)) - \sum_{j=k+1}^{p} I_j(u(t_j)),
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
w_k =& w_p - \Gamma(2-\beta)\sum_{j=k+1}^{p} t_j^{\beta-1} J_j(u(t_j))\\
=&-\Big(\frac{\Gamma(2-\beta)(\kappa_2)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\theta_1 + \Big(\frac{\Gamma(2-\beta)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\theta_2\\
&-\Big(\frac{\kappa_2\Gamma(2-\beta)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds\\
&+\Big(\frac{\Gamma(2-\beta)(\kappa_1\kappa_2)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} g(s)ds\\
&-\Big(\frac{\Gamma(2-\beta)\kappa_2}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\Big)\sum_{j=1}^{p} I_j(u(t_j))
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{(\Gamma(2-\beta))^2\kappa_2}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\sum_{j=1}^{p} t_j^{\beta}J_j(u(t_j))\\
&+\left(\frac{(\Gamma(2-\beta))^2}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\sum_{j=1}^{p} t_j^{\beta-1}J_j(u(t_j))\\
&-\Gamma(2-\beta)\sum_{j=k+1}^{p} t_j^{\beta-1}J_j(u(t_j)),
\end{aligned}
\tag{2.17}
$$

for $k = 0, 1, \cdots, p-1$. By using (2.16) and (2.17), we get

$$
\begin{aligned}
d_k + w_k t =&\left(\frac{\Gamma(2-\beta)(1+\kappa_2-\kappa_2 t)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\theta_1+\left(\frac{\Gamma(2-\beta)t-\kappa_1}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\theta_2\\
&+\left(\frac{-\kappa_2(\Gamma(2-\beta)t-\kappa_1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\\
&+\left(\frac{-\kappa_1\Gamma(2-\beta)(1+\kappa_2-\kappa_2 t)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds\\
&+\left(\frac{\Gamma(2-\beta)(1+\kappa_2-\kappa_2 t)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\sum_{j=1}^{p} I_j(u(t_j))\\
&+\left(\frac{(\Gamma(2-\beta))^2(\kappa_2 t-\kappa_2-1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\sum_{j=1}^{p} t_j^{\beta}J_j(u(t_j))\\
&+\left(\frac{\Gamma(2-\beta)(\Gamma(2-\beta)t-\kappa_1)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}\right)\sum_{j=1}^{p} t_j^{\beta-1}J_j(u(t_j))\\
&+\Gamma(2-\beta)\sum_{j=k+1}^{p} t_j^{\beta-1}(t_j-t)J_j(u(t_j))-\sum_{j=k+1}^{p} I_j(u(t_j)).
\end{aligned}
$$

Therefore, by the (2.6), we get

$$
\begin{aligned}
u(t) =& \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds+\nu_1(t)\theta_1+\nu_2(t)\theta_2\\
&-\kappa_2\nu_2(t)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g(s)ds\\
&-\kappa_1\nu_1(t)\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}g(s)ds+\nu_1(t)\sum_{j=1}^{p} I_j(u(t_j))\\
&-\Gamma(2-\beta)\nu_1(t)\sum_{j=1}^{p} t_j^{\beta}J_j(u(t_j))+\Gamma(2-\beta)\nu_2(t)\sum_{j=1}^{p} t_j^{\beta-1}J_j(u(t_j))\\
&+\Gamma(2-\beta)\sum_{j=k+1}^{p} t_j^{\beta-1}(t_j-t)J_j(u(t_j))-\sum_{j=k+1}^{p} I_j(u(t_j)),
\end{aligned}
$$

where

$$
\nu_1(t)=\frac{\Gamma(2-\beta)(1+\kappa_2-\kappa_2 t)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2},\qquad \nu_2(t)=\frac{\Gamma(2-\beta)t-\kappa_1}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2}.
$$

Thus, we obtain $(P^*)$ for solution of (2.5). Conversely, assume that $u$ is a solution of the impulsive fractional integral equation (2.5), then by a direct computation, it follows that the solution given by $(P^*)$ satisfies (2.5). This completes the proof. □

## 3. Existence and uniqueness results

In this section, we state and prove the existence and uniqueness results of the fractional boundary value of (1.1) by using the fixed point theorem. We use the following notations throughout this paper:

$$\nu_1(t) = \frac{\Gamma(2-\beta)(1+\kappa_2-\kappa_2 t)}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2} \leq \nu_1 := \frac{\Gamma(2-\beta)(1+2|\kappa_2|)}{|\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2|},$$

$$\nu_2(t) = \frac{\Gamma(2-\beta)t-\kappa_1}{\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2} \leq \nu_2 := \frac{\Gamma(2-\beta)+|\kappa_1|}{|\Gamma(2-\beta)(\kappa_2+1)-\kappa_1\kappa_2|}.$$

In this paper, we following conditions of $(H_1)$ and $(H_2)$, and then we state and prove our first result.
$(H_1)$ The function $f : J \times R \to R$ is continuous.
$(H_2)$ There exist positive constants $M_1, M_2, M_3, L_1, L_2$ such that

$$|f(t,u) - f(t,v)| \leq M_1|u-v|, t \in [0,1], u, v \in R,$$

$$|I_k(u) - I_k(v)| \leq M_2|u-v|, \qquad |J_k(u) - J_k(v)| \leq M_3|u-v|,$$

$$|I_k(u)| \leq L_1, \qquad |J_k(u)| \leq L_2.$$

Also it is clear that

$$\begin{aligned} |f(t,u)| &\leq |f(t,u) - f(t,0)| + |f(t,0)| \\ &\leq M_1|u| + L, \end{aligned}$$

where $\sup_{t\in[0,1]} |f(t,0)| = L$.

**Theorem 3.1.** *Assume $(H_1)$ and $(H_2)$ holds. If*

$$\begin{aligned} &M_1\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{|\kappa_2|\nu_2}{\Gamma(\alpha+1)} + \frac{|\kappa_1|\nu_1}{\Gamma(\alpha-\beta+1)}\Big) \\ &\qquad + \Gamma(2-\beta)p(\nu_1+\nu_2+2)M_3 + p(\nu_1+1)M_2 < 1, \end{aligned} \tag{3.1}$$

*then BVP of (1.1) has a unique solution on [0,1].*

*Proof.* By using (3.1), $r$ can be chosen as follows:

$$\begin{aligned} r > &\Big\{1 - \frac{M_1}{\Gamma(\alpha+1)}(1+\nu_2|\kappa_2|) - \frac{M_1}{\Gamma(\alpha-\beta+1)}\nu_1|\kappa_1|\Big\}^{-1}\Big(\frac{L}{\Gamma(\alpha+1)} \\ &+ \nu_1|\theta_1| + \nu_2|\theta_2| + \nu_1|\kappa_2|\frac{L}{\Gamma(\alpha+1)} + \nu_2|\kappa_1|\frac{L}{\Gamma(\alpha-\beta+1)} \\ &+ \Gamma(2-\beta)p(\nu_1+\nu_2+2)L_2 + p(\nu_1+1)L_1\Big). \end{aligned}$$

Define an operator $F : PC\left([0,1],R\right) \to PC\left([0,1],R\right)$ to transform (1.1) into the fixed point problem

$$
\begin{aligned}
(Fu)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds + \nu_1(t)\theta_1 + \nu_2(t)\theta_2 \\
& - \kappa_2 \nu_2(t) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))ds \\
& - \kappa_1 \nu_1(t) \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s,u(s))ds + \nu_1(t) \sum_{j=1}^{p} I_j(u(t_j)) \\
& - \Gamma(2-\beta)\nu_1(t) \sum_{j=1}^{p} t_j^{\beta} J_j(u(t_j)) + \Gamma(2-\beta)\nu_2(t) \sum_{j=1}^{p} t_j^{\beta-1} J_j(u(t_j)) \\
& + \Gamma(2-\beta) \sum_{j=k+1}^{p} t_j^{\beta-1}(t_j - t) J_j(u(t_j)) - \sum_{j=k+1}^{p} I_j(u(t_j)),
\end{aligned}
$$

where $t_k < t < t_{k+1}, k = 0, \cdots, p$. Then

$$
\begin{aligned}
\left|Fu(t)\right| \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s,u(s))\right| ds + \left|\nu_1(t)\right| |\theta_1| + \left|\nu_2(t)\right| |\theta_2| \\
& + |\kappa_2| \left|\nu_2(t)\right| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s,u(s))\right| ds \\
& + |\kappa_1| \left|\nu_1(t)\right| \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \left|f(s,u(s))\right| ds + \left|\nu_1(t)\right| \sum_{j=1}^{p} \left|I_j(u(t_j))\right| \\
& + \Gamma(2-\beta) \left|\nu_1(t)\right| \sum_{j=1}^{p} \left|J_j(u(t_j))\right| + \Gamma(2-\beta) \left|\nu_2(t)\right| \sum_{j=1}^{p} \left|J_j(u(t_j))\right| \\
& + 2\Gamma(2-\beta) \sum_{j=k+1}^{p} \left|J_j(u(t_j))\right| + \sum_{j=k+1}^{p} \left|I_j(u(t_j))\right|,
\end{aligned}
$$

and then

$$
\begin{aligned}
|Fu|(t) \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s,u(s)) - f(s,0)\right| ds \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s,0)\right| ds + \left|\nu_1(t)\right| |\theta_1| + \left|\nu_2(t)\right| |\theta_2| \\
& + |\kappa_2| \left|\nu_2(t)\right| \Big[ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s,u(s)) - f(s,0)\right| ds \\
& + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f(s,0)\right| ds \Big] \\
& + |\kappa_1| \left|\nu_1(t)\right| \Big[ \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \left|f(s,u(s)) - f(s,0)\right| ds \\
& + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\beta)} \left|f(s,0)\right| ds \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \left|\nu_1(t)\right| \sum_{j=1}^{p} \left|I_j(u(t_j))\right| + \Gamma(2-\beta)\left|\nu_1(t)\right| \sum_{j=1}^{p} \left|J_j(u(t_j))\right| \\
&+ \Gamma(2-\beta)\left|\nu_2(t)\right| \sum_{j=1}^{p} \left|J_j(u(t_j))\right| \\
&+ 2\Gamma(2-\beta) \sum_{j=k+1}^{p} \left|J_j(u(t_j))\right| + \sum_{j=k+1}^{p} \left|I_j(u(t_j))\right|.
\end{aligned}
$$

Using the upper bound $I_k(u) \leq L_1$, we have

$$
\begin{aligned}
\sum_{j=1}^{p} \left|I_j(u(t_j))\right| &= \left|I_1(u(t_1))\right| + \left|I_2(u(t_2))\right| + \cdots + \left|I_{p-1}(u(t_{p-1}))\right| + \left|I_p(u(t_p))\right| \\
&\leq \overbrace{L_1 + L_1 + \cdots + L_1 + L_1}^{p} = pL_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=1}^{p} \left|J_j(u(t_j))\right| &= \left|J_1(u(t_1))\right| + \left|J_2(u(t_2))\right| + \cdots + \left|J_{p-1}(u(t_{p-1}))\right| + \left|J_p(u(t_p))\right| \\
&\leq \overbrace{L_2 + L_2 + \cdots + L_2 + L_2}^{p} = pL_2.
\end{aligned}
$$

By the same way, we get

$$
\begin{aligned}
\sum_{j=k+1}^{p} \left|I_j(u(t_j))\right| =& \left|I_{k+1}(u(t_{k+1}))\right| + \left|I_{k+2}(u(t_{k+2}))\right| + \cdots + \left|I_{p-k-1}(u(t_{p-k-1}))\right| \\
&+ \left|I_{p-k}(u(t_{p-k}))\right| \\
\leq& \overbrace{L_1 + L_1 + \cdots + L_1 + L_1}^{p-k} = (p-k)L_1 \leq pL_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=k+1}^{p} \left|J_j(u(t_j))\right| =& \left|J_{k+1}(u(t_{k+1}))\right| + \left|J_{k+2}(u(t_{k+2}))\right| + \cdots + \left|J_{p-k-1}(u(t_{p-k-1}))\right| \\
&+ \left|J_{p-k}(u(t_{p-k}))\right| \\
\leq& \overbrace{L_2 + L_2 + \cdots + L_2 + L_2}^{p-k} = (p-k)L_2 \leq pL_2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|Fu|(t) \leq& \frac{M_1 r}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha+1)} + \nu_1|\theta_1| + \nu_2|\theta_2| \\
&+ \nu_2|\kappa_2|\left[\frac{M_1 r}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha+1)}\right] \\
&+ \nu_1|\kappa_1|\left[\frac{M_1 r}{\Gamma(\alpha-\beta+1)} + \frac{L}{\Gamma(\alpha-\beta+1)}\right] \\
&+ \Gamma(2-\beta)p(\nu_1+\nu_2+2)L_2 + p(\nu_1+1)L_1 < r.
\end{aligned}
$$

Therefore, for all $t \in [0, 1]$, the fixed point of the operator $F$ is the solution of our BVP (1.1). Next, the fixed point theorem is used and then it is shown that $F$ is a contraction mapping and we have

$$
\begin{aligned}
&|(Fu)(t) - (Fv)(t)| \\
&\quad \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big| f(s, u(s)) - f(s, v(s)) \big| ds \\
&\quad + |\kappa_2| \big| \nu_2(t) \big| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \big| f(s, u(s)) - f(s, v(s)) \big| ds \\
&\quad + |\kappa_1| \big| \nu_1(t) \big| \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \big| f(s, u(s)) - f(s, v(s)) \big| ds \\
&\quad + \big| \nu_1(t) \big| \sum_{j=1}^{p} \big| I_j(u(t_j)) - I_j(v(t_j)) \big| \\
&\quad + \Gamma(2-\beta) \big| \nu_1(t) \big| \sum_{j=1}^{p} \big| J_j(u(t_j)) - J_j(v(t_j)) \big| \\
&\quad + \Gamma(2-\beta) \big| \nu_2(t) \big| \sum_{j=1}^{p} \big| J_j(u(t_j)) - J_j(v(t_j)) \big| \\
&\quad + 2\Gamma(2-\beta) \sum_{j=k+1}^{p} \big| J_j(u(t_j)) - J_j(v(t_j))| \\
&\quad + \sum_{j=k+1}^{p} \big| I_j(u(t_j)) - I_j(v(t_j)) \big|,
\end{aligned}
$$

Thus

$$
\begin{aligned}
|(Fu)(t) - (Fv)(t)| \le \Bigg[ & M_1 \Big( \frac{1}{\Gamma(\alpha+1)} + \frac{|\kappa_2| \big|\nu_2(t)\big|}{\Gamma(\alpha+1)} + \frac{|\kappa_1| \big|\nu_1(t)\big|}{\Gamma(\alpha-\beta+1)} \Big) \\
& + \Gamma(2-\beta) p \Big( \big|\nu_1(t)\big| + \big|\nu_2(t)\big| + 2 \Big) M_3 \\
& + p \Big( \big|\nu_1(t)\big| + 1 \Big) M_2 \Bigg] \|u - v\|,
\end{aligned}
$$

$F$ is contraction mapping. By condition (3.1), we have

$$
\begin{aligned}
|Fu - Fv| \le \Bigg[ & M_1 \Big( \frac{1}{\Gamma(\alpha+1)} + \frac{|\kappa_2| \nu_2}{\Gamma(\alpha+1)} + \frac{|\kappa_1| \nu_1}{\Gamma(\alpha-\beta+1)} \Big) \\
& + \Gamma(2-\beta) p(\nu_1 + \nu_2 + 2) M_3 + p(\nu_1 + 1) M_2 \Bigg] \|u - v\|
\end{aligned}
$$

Therefore, $F$ is a contraction mapping. The conclusion follows the principle of contraction mapping, that is the unique solution of impulsive mixed BVP can be obtained by using the fixed point theorem. □

**Theorem 3.2.** *Assume that* $|f(t, u)| \le \zeta(t)$ *for all* $(t, u) \in J \times R$ *where* $\zeta \in L^{\frac{1}{\mu}}(J \times R), \mu \in (0, \alpha - 1)$ *and the* $(H_2)$ *holds. If*

$$
\Gamma(2-\beta) p(\nu_1 + \nu_2 + 2) M_3 + p(\nu_1 + 1) M_2 < 1. \tag{3.2}
$$

*Then problem (1.1) has at least one solution on J.*

*Proof.* Let

$$
\begin{aligned}
r \geq & \|\zeta\|_{L^{\frac{1}{\mu}}(J)}\Big(\frac{(1+|\kappa_2|\nu_2)}{\Gamma(\alpha)(\frac{\alpha-\mu}{1-\mu})^{1-\mu}}+\frac{|\kappa_1|\nu_1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\mu}{1-\mu})^{1-\mu}}\Big)\\
&+\Gamma(2-\beta)p(\nu_1+\nu_2+2)L_2+p(\nu_1+1)L_1,
\end{aligned}
$$

and denote $S_r=\{u\in PC(J,R)\Big|\|u\|_{PC}\leq r\}$.

Define the operators $B$ and $N$ on $S_r$ as

$$
\begin{aligned}
(Bu)(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds-\kappa_2\nu_2(t)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds\\
&-\kappa_1\nu_1(t)\int_0^1\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s,u(s))ds,\\
(Nu)(t)=&\nu_1(t)\sum_{j=1}^{p}I_j(u(t_j))-\Gamma(2-\beta)\nu_1(t)\sum_{j=1}^{p}t_j^{\beta}J_j(u(t_j))\\
&+\Gamma(2-\beta)\nu_2(t)\sum_{j=1}^{p}t_j^{\beta-1}J_j(u(t_j))\\
&+\Gamma(2-\beta)\sum_{j=k+1}^{p}t_j^{\beta-1}(t_j-t)J_j(u(t_j))-\sum_{j=k+1}^{p}I_j(u(t_j)).
\end{aligned}
$$

For any $u,v\in S_r(t\in J)$, using the condition that $|f(t,u)\leq\zeta(t)|$ and the Hölder inequality, we get

$$
\begin{aligned}
\int_0^t\Big|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\Big|ds\leq&\frac{\Big(\int_0^t(t-s)^{\frac{\alpha-1}{1-\mu}}ds\Big)^{1-\mu}\Big(\int_0^t(\zeta(s))^{\frac{1}{\mu}}ds\Big)^{\mu}}{\Gamma(\alpha)}\\
\leq&\frac{\|\zeta\|_{L^{\frac{1}{\mu}}(J)}}{\Gamma(\alpha)(\frac{\alpha-\mu}{1-\mu})^{1-\mu}},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1\Big|\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\Big|ds\leq&\frac{\Big(\int_0^1(1-s)^{\frac{\alpha-1}{1-\mu}}ds\Big)^{1-\mu}\Big(\int_0^1(\zeta(s))^{\frac{1}{\mu}}ds\Big)^{\mu}}{\Gamma(\alpha)}\\
\leq&\frac{\|\zeta\|_{L^{\frac{1}{\mu}}(J)}}{\Gamma(\alpha)(\frac{\alpha-\mu}{1-\mu})^{1-\mu}},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1\Big|\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(t,u(s))\Big|ds\leq&\frac{\Big(\int_0^1(1-s)^{\frac{\alpha-\beta-1}{1-\mu}}ds\Big)^{1-\mu}\Big(\int_0^1(\zeta(s))^{\frac{1}{\mu}}ds\Big)^{\mu}}{\Gamma(\alpha-\beta)}\\
\leq&\frac{\|\zeta\|_{L^{\frac{1}{\mu}}(J)}}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\mu}{1-\mu})^{1-\mu}}.
\end{aligned}
$$

Thus

$$\begin{aligned}\|(Bu)+(Nv)\| \le & \frac{(1+|\kappa_2|\nu_2)\|\zeta\|_{L^{\frac{1}{\mu}}(J)}}{\Gamma(\alpha)(\frac{\alpha-\mu}{1-\mu})^{1-\mu}} + \frac{(|\kappa_1|\nu_1)\|\zeta\|_{L^{\frac{1}{\mu}}(J)}}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\mu}{1-\mu})^{1-\mu}} \\ & + \Gamma(2-\beta)p(\nu_1+\nu_2+2)L_2 + p(\nu_1+1)L_1 \\ = & \|\zeta\|_{L^{\frac{1}{\mu}}(J)}\Big(\frac{(1+|\kappa_2|\nu_2)}{\Gamma(\alpha)(\frac{\alpha-\mu}{1-\mu})^{1-\mu}} + \frac{|\kappa_1|\nu_1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\mu}{1-\mu})^{1-\mu}}\Big) \\ & + \Gamma(2-\beta)p(\nu_1+\nu_2+2)L_2 + p(\nu_1+1)L_1.\end{aligned}$$

Therefore, $Bu+Nv \in S_r$. By the (3.2), it is obvious that $N$ is a contraction mapping. And the continuity of $f$ implies that the operator $B$ is continuous. Thus, $B$ is uniformly bounded on $S_r$ where

$$(Bu)\| \le \|\zeta\|_{L^{\frac{1}{\mu}}(J)}\Big(\frac{(1+|\kappa_2|\nu_2)}{\Gamma(\alpha)(\frac{\alpha-\mu}{1-\mu})^{1-\mu}} + \frac{|\kappa_1|\nu_1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\mu}{1-\mu})^{1-\mu}}\Big) \le r.$$

Next the quasi-equicontinuity of the operator $B$ is proved. Let $\Omega = J\times S_r$, $f_{\sup} = \sup\limits_{(t,u)\in\Omega} |f(t,u)|$. For any $t_k < \chi_2 < \chi_1 \le t_{k+1}$, we have

$$\begin{aligned}&\|(Bu)(\chi_2)-(Bu)(\chi_1)\| \\ =&\Big|\int_0^{\chi_2}\frac{(\chi_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds - \int_0^1\frac{\kappa_2\nu_2(\chi_2)(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds \\ &- \int_0^1\frac{\kappa_1\nu_1(\chi_2)(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s,u(s))ds - \int_0^{\chi_1}\frac{(\chi_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds \\ &+ \int_0^1\frac{\kappa_2\nu_2(\chi_1)(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))ds \\ &+ \int_0^1\frac{\kappa_1\nu_1(\chi_1)(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s,u(s))ds\Big| \\ \le&\frac{f_{\sup}}{\Gamma(\alpha)}\Big|\int_0^{\chi_2}(\chi_2-s)^{\alpha-1}-(\chi_1-s)^{\alpha-1}ds + \int_{\chi_2}^{\chi_1}(\chi_1-s)^{\alpha-1}ds\Big| \\ &+\Big|\frac{\kappa_2\nu_2(\chi_2-\chi_1)f_{\sup}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds\Big| \\ &+\Big|\frac{\kappa_1\nu_1(\chi_1-\chi_2)f_{\sup}}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}ds\Big| \\ \le& f_{\sup}\Big[\frac{(\chi_2-\chi_1)^\alpha+\chi_1^\alpha-\chi_2^\alpha}{\Gamma(\alpha+1)} + |\kappa_2|\frac{(\chi_1^\alpha-\chi_2^\alpha)}{\Gamma(\alpha+1)} + |\kappa_1|\frac{(\chi_2^\alpha-\chi_1^\alpha)}{\Gamma(\alpha-\beta+1)}\Big],\end{aligned}$$

which tends to zero as $\chi_1 \to \chi_2$. This proves that $B$ is quasi-equicontinuous on the $(t_k, t_{k+1}]$. So that $B$ is compact by Lemma 2.5, and $B$ is relatively compact on $S_r$.

Thus all the assumptions of Lemma 2.4 are satisfied and problem (1.1) has at least one solution on $J$. □

## 4. Examples

In this section, two examples are given to verify the feasibility of the results.

**Example 4.1.** Let us consider the hybrid fractional BVP:

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2}} u(t) = \frac{\sin^3 u(t)}{(t+20)^2(1+u(t))}, t \in [0,1], t \neq \frac{1}{4}, \\ \Delta u(\frac{1}{4}) = \frac{|u(\frac{1}{4}|)}{100+|u(\frac{1}{4})|}, \Delta {}^C D_{0^+}^{\frac{1}{2}} u(\frac{1}{4}) = \frac{|u(\frac{1}{4}|)}{100+|u(\frac{1}{4})|}, t = \frac{1}{4}, \\ u(0) + {}^C D_{0^+}^{\frac{1}{2}} u(1) = 0, {}^C D_{0^+}^{\frac{1}{2}} u(0) + u(1) = 0, \end{cases} \tag{4.1}$$

where $I_k(u(t)) = J_k(u(t)) = \frac{|u(\frac{1}{4}|)}{100+|u(\frac{1}{4})|}$, and $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $p = 1, f(t,u(t)) = \frac{\sin^3 u(t)}{(t+20)^2(1+u(t))}$. Let $\theta_1 = \theta_2 = 0, M_1 = M_2 = M_3 = L_1 = L_2 = \frac{1}{100}$.

By the $f(t,u(t)) = \frac{\sin^3 u(t)}{(t+20)^2(1+u(t))}$, $(H_1)$ is obviously established. Next, consider the $(H_2)$, we obtain

$$\begin{aligned} \Big|f(t,u) - f(t,v)\Big| &= \Big|\frac{\sin^3 u(t)}{(t+20)^2(1+u(t))} - \frac{\sin^3 v(t)}{(t+12)^2(1+v(t))}\Big| \\ &\leq \frac{1}{400}\Big|\frac{\sin^3 u(t)}{(1+u(t))} - \frac{\sin^3 v(t)}{(1+v(t))}\Big| \\ &\leq \frac{1}{400}\Big(\Big|\sin^3 u(t) - \sin^3 v(t)\Big| + \Big|u(t) - v(t)\Big|\Big) \\ &\leq \frac{1}{400}\Big(3\Big|u(t) - v(t)\Big| + \Big|u(t) - v(t)\Big|\Big) \\ &= \frac{1}{100}|u - v|. \end{aligned}$$

Therefore, the $\Big|f(t,u) - f(t,v)\Big|$ is established. By the same way, we have $|I_k(u) - I_k(v)| \leq M_2|u - v|, |J_k(u) - J_k(v)| \leq M_3|u - v|, |I_k(u)| \leq L_1$ and $|J_k(u)| \leq L_2$ are established. Thus, $(H_2)$ is completed. Last, consider the condition of (3.1). Because the $\theta_1 = \theta_2 = 0, M_1 = M_2 = M_3 = \frac{1}{100}$, we have

$$\nu_1 := \frac{2\Gamma(2-\beta)}{1-\Gamma(2-\beta)} \approx 15.5747, \qquad \nu_2 := \frac{1+\Gamma(2-\beta)}{1-\Gamma(2-\beta)} \approx 16.5747,$$

$$\nu_3 := 2\Gamma(2-\beta) \approx 1.7724.$$

Thus

$$\begin{aligned} &\frac{M_1}{\Gamma(\alpha+1)} + \nu_1\Big(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{\Gamma(\alpha+1)}\Big)M_1 \\ &+ \nu_1 p M_2 + \Gamma(2-\beta)\nu_1 p M_3 + 2pM_2 \\ &+ 2\nu_3 p M_3 + \nu_1\theta_1 + \nu_2\theta_2 \approx 0.6292 < 1. \end{aligned}$$

Therefore, all assumptions in the Theorem 3.1 are satisfied. Hence, the fractional impulsive hybrid BVP of (4.1) has a unique solution on $[0,1]$.

**Example 4.2.** We consider the hybrid fractional BVP of following:

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2}} u(t) = \frac{\sin t\,|u(t)|}{(t+10)^2(1+|u(t)|)}, t \in [0,1], t \neq \frac{1}{3}, \\ \Delta u(\frac{1}{3}) = \frac{|u(\frac{1}{3}|)}{100+|u(\frac{1}{3})|}, \Delta {}^C D_{0^+}^{\frac{1}{2}} u(\frac{1}{5}) = \frac{|u(\frac{1}{3}|)}{100+|u(\frac{1}{3})|}, t = \frac{1}{3} \\ u(0) + {}^C D_{0^+}^{\frac{1}{2}} u(1) = 0, {}^C D_{0^+}^{\frac{1}{2}} u(0) + u(1) = 0, \end{cases} \tag{4.2}$$

where $I_k(u(t)) = J_k(u(t)) = \frac{|u(\frac{1}{3})|}{100+|u(\frac{1}{3})|}$, and $p = 1,\ \alpha = \frac{3}{2},\ \beta = \frac{1}{2},\ f(t, u(t)) = \frac{\sin t\,|u(t)|}{(t+10)^2(1+|u(t)|)}$. The

$$\nu_1 := \frac{2\Gamma(2-\beta)}{1-\Gamma(2-\beta)} \approx 15.5747, \qquad \nu_2 := \frac{1+\Gamma(2-\beta)}{1-\Gamma(2-\beta)} \approx 16.5747,$$

and

$$f(t,u) = \frac{\sin t\,|u|}{(t+10)^2(1+|u|)}, (t,u) \in [0,1] \times [0,+\infty).$$

Clearly, we obtain

$$|f(t,u)| \le \frac{\sin t\,|u|}{(t+10)^2}, \quad \zeta(t) = \frac{\sin t}{(t+10)^2} \in L^3([0,1], R),$$

and

$$\begin{aligned} |f(t,u) - f(t,v)| &= \left| \frac{\sin t\,|u|}{(t+10)^2} - \frac{\sin t\,|v|}{(t+10)^2} \right| \\ &= \left| \frac{\sin t}{(t+10)^2}(|u| - |v|) \right| \\ &\le \frac{\sin t}{(t+10)^2}|u - v|. \end{aligned}$$

Let $M_2 = M_3 = \frac{1}{100}$, we get

$$\Gamma(2-\beta)p(\nu_1 + \nu_2 + 2)M_3 + p(\nu_1 + 1)M_2 \approx 0.4683 < 1.$$

Thus, all assumptions in the Theorem 3.2 are satisfied. Hence, the fractional impulsive mixed BVP of (4.2) has a least one solution on [0, 1].

**Conflict of interest**

The authors declare there is no conflict of interest.

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