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Research article

Multiple entire solutions of fractional Laplacian Schrödinger equations

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Abstract: We consider the semi-linear fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases}$$

where both $V(x)$ and $f(x,u)$ are periodic in x, 0 belongs to a spectral gap of the operator $(-\Delta)^s + V$ and $f(x,u)$ is subcritical in u. We obtain the existence of nontrivial solutions by using a generalized linking theorem, and based on this existence we further establish infinitely many geometrically distinct solutions. We weaken the super-quadratic condition of f, which is usually assumed even in the standard Laplacian case so as to obtain the existence of solutions.

Keywords: fractional Schrödinger equation; Cerami sequence; infinitely many geometrically distinct solutions
Mathematics Subject Classification: 35A01, 35B08, 35J61

1. Introduction

We consider the following semi-linear fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $(-\Delta)^s$, $s \in (0,1)$, denotes the usual fractional Laplace operator, a Fourier multiplier of symbol $|\xi|^{2s}$. Here $H^s(\mathbb{R}^N)$ is the fractional Sobolev space

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty \right\}.$$

Suppose that $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions
(V_s) : $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each component $x_1, x_2, ..., x_N$ of x and

$$\sup\{\sigma[(-\Delta)^s + V] \cap (-\infty, 0)\} < 0 < \inf\{\sigma[(-\Delta)^s + V] \cap (0, \infty)\},$$

where $\sigma[(-\Delta)^s + V]$ denotes the spectrum of $(-\Delta)^s + V$.
(F_1): $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is 1-periodic in each of $x_1, x_2, ..., x_N$ and $|f(x,t)| \leq c_1(1 + |t|^{p-1})$ for some $c_1 > 0$ and $p \in (2, 2_s^*)$, where $2_s^* = \frac{2N}{N-2s}$ if $N > 2s$, $2_s^* = +\infty$ if $N \leq 2s$.
(F_2): $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in \mathbb{R}^N$.

Denote $\Lambda_s := \inf\{\sigma[(-\Delta)^s + V] \cap (0, \infty)\}$. By (V_s), one has $\Lambda_s > 0$.

Two simple examples of function satisfying the conditions (F_1) and (F_2) are the following:

$$f(x,t) = P(x)t\ln(1 + |t|),\ f(x,t) = P(x)t|t|^{p-2},$$

where the function $P(x)$ is 1-periodic in each of $x_1, x_2, ..., x_N$.
The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. It was introduced by Laskin [16] and [17] as a result of expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, where the Feynman path integral leads to the classical Schrödinger equation, and the path integral over Lévy trajectories leads to the fractional Schrödinger equation.

The fractional Laplacian operator is defined as

$$(-\Delta)^s u(x) = C(N,s)\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy.$$

Here P.V. stands for the Cauchy principal value and the positive constant $C(N,s)$ depends only on N and s, which is not essential in our problem and we will omit it for simplicity of notation. For fractional Laplacian operators and fractional spaces, the reader can refer to [4] and [8]. The authors in [1] raise the following assumption (AR) of the nonlinear term to study a semi-linear elliptic boundary value problem
(AR): There exists $\mu > 2$ such that $0 < \mu F(x,t) \leq tf(x,t)$, for $x \in \mathbb{R}^N$, $t \neq 0$,
where $F(x,t) := \int_0^t f(x,\tau)d\tau$. By a direct integration of (AR), one can deduce the existence of positive constants A, B such that $F(x,t) \geq A|t|^\mu - B$ for any $t \in \mathbb{R}$. We first recall some main results of the particular case $s = 1$, namely the standard Laplacian case of (1.1). The existence of a nontrivial solution to (1.1) has been obtained in [2, 3, 7, 15, 23, 29, 31] under (AR) and some other standard assumptions of f. The authors of [21] introduce the following more natural super-quadratic condition to replace (AR)
(SQ) : $\lim_{|t|\to\infty} \frac{F(x,t)}{t^2} = \infty$ uniformly in $x \in \mathbb{R}^N$,
and obtain the existence of nontrivial solutions of (1.1) under (SQ) and some other standard assumptions of f by imposing some compact conditions on the potential function V.After that, condition (SQ) is also used in many papers, see [5, 9, 19, 20, 25, 30, 32, 33]. In the definite cases where $\sigma(-\Delta + V) \subset (0, \infty)$, [20] obtains a ground state solution via a Nehari type argument for (1.1). The corresponding energy functional of (1.1) in the case $s = 1$ is

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

Let $E = H^1(\mathbb{R}^N)$. Recall that $E = E^- \oplus E^+$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum, and $u = u^- + u^+ \in E^- \oplus E^+$. (See Section 2 for more details.) The following set has been introduced in [22]

$$\mathcal{M} = \{u \in E \setminus E^- : \langle \Phi'(u), u \rangle = \langle \Phi'(u), v \rangle = 0 \text{ for all } v \in E^-\}.$$

By definition, $\mathcal{M}$ contains all nontrivial critical points of I. The authors of [25] develop an ingenious approach to find ground state solutions of (1.1). Their approach transforms, by a direct and simple reduction, the indefinite variational problem to a definite one, resulting in a new minimax characterization of the corresponding critical value. More precisely, they establish the following two propositions by introducing the strictly monotonicity assumption (Mo)

(Mo): $t \to \frac{f(x,t)}{|t|}$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

Proposition 1.1. ([25]) *Assume* $(V_1), (F_1)$, (F_2), *(Mo), (SQ) are satisfied and let* $m = \inf\limits_{u \in \mathcal{M}} \Phi(u)$. *Then* m *is attained,* $m > 0$ *and if* $u_0 \in \mathcal{M}$ *satisfies* $\Phi(u_0) = m$, *then* u_0 *is a solution of* (1.1).

Proposition 1.2. ([25]) *Assume* $(V_1), (F_1), (F_2)$, *(Mo), (SQ) are satisfied and* $f(x,t)$ *is odd in* t. *Then* (1.1) *admits infinitely many pairs geometrically distinct solutions* $\pm u$.

In [24], the author obtains nontrivial and ground solutions of Schrödinger equation (1.1) under weaker conditions than those of [25]. Via deformation arguments jointed with the notion of Cerami sequence(See Section 2 for concrete definition), [9] establishes the following proposition.

Proposition 1.3. ([9]) *Assume that* V *and* f *satisfy* $(V_1), (F_1), (F_2), (SQ)$ *and the following condition* (DL) : $F(x,t) \geq 0, G(x,t) = \frac{1}{2}f(x,t)t - F(x,t) > 0$ *if* $t \neq 0, G(x,t) \to +\infty$ *as* $|t| \to \infty$ *uniformly in* $x \in \mathbb{R}^N$, *and there exists* $c_2, r_0 > 0$ *and* $\nu > max\{1, \frac{N}{2}\}$ *such that*

$$\left|\frac{f(x,t)}{t}\right|^\nu \leq c_2 G(x,t) \textit{ for all } |t| \geq r_0 \textit{ and } x \in \mathbb{R}^N.$$

Then (1.1) *has a nontrivial solution. If, in addition,* $f(x,t)$ *is odd in* t, *then* (1.1) *admits infinitely many pairs geometrically distinct solutions* $\pm u$.

In [26], the author obtains the existence of ground state solutions by non-Nehari manifold method for (1.1) with periodic and asymptotically periodic potential function V, under (SQ) and some other standard assumptions of f. Recently, under the weaker super-quadratic condition (SQ)$'$ and some other standard assumptions of f, the authors in [28] obtain the existence of nontrivial solution for (1.1) with periodic and non-periodic potential function V. The authors of [27] further obtain the existence of ground state solutions and infinitely many geometrically distinct solutions under (SQ)$'$ and non-strictly monotonicity condition (Mo)$'$, and as a compensation, additional condition (F_0) or $(F_0)'$ is necessary. These conditions are defined as follows.

(Mo)$'$: $u \to \frac{f(x,t)}{|t|}$ is nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$.

(SQ)$'$: There exists a domain $\Omega \subseteq \mathbb{R}^N$, such that $\lim_{|t|\to\infty} \frac{F(x,t)}{t^2} = \infty$, a.e $x \in \Omega$.

(F_0): $G(x,t) := \frac{1}{2}f(x,t)t - F(x,t) \geq 0$, there exists $c_0 > 0, R_0 > 0$ and $\alpha \in (0,1)$, such that

$$\left[\frac{|f(x,t)|}{|t|^{\alpha}}\right]^{\frac{2N}{2N-(1+\alpha)(N-2)}} \leq c_0 G(x,t), \forall\ |t| \geq R_0,\ \text{if } N \geq 3,$$

and for some $k \in (1, \frac{2}{1-\alpha}]$,

$$\left[\frac{|f(x,t)|}{|t|^{\alpha}}\right]^{k} \leq c_0 G(x,t),\ \forall\ |t| \geq R_0,\ \text{if } N = 1,2;$$

$(F_0)'$: $G(x,t) = \frac{1}{2}f(x,t)t - F(x,t) \geq 0$, $F(x,t) \geq 0$, and there exists $c_0 > 0, \delta_0 \in (0,\Lambda_1)$ and $\alpha \in (0,1)$, such that

$$\frac{f(x,t)}{t} \geq \Lambda_1 - \delta_0 \text{ implies } \left[\frac{|f(x,t)|}{|t|^{\alpha}}\right]^{\frac{2N}{2N-(1+\alpha)(N-2)}} \leq c_0 G(x,t),\ \text{if } N \geq 3,$$

and for some $k \in (1, \frac{2}{1-\alpha}]$,

$$\frac{f(x,t)}{t} \geq \Lambda_1 - \delta_0 \text{ implies } \left[\frac{|f(x,t)|}{|t|^{\alpha}}\right]^{k} \leq c_0 G(x,t),\ \text{if } N = 1,2.$$

The following propositions are established in [27].

Proposition 1.4. *Assume that V and f satisfy $(V_1), (F_0)', (F_1), (F_2)$ and $(SQ)'$. Then* (1.1) *has a solution $u_0 \in E\backslash\{0\}$ such that $\Phi(u_0) = \inf_{u\in K} \Phi(u) > 0$, where $K := \{u \in E\backslash\{0\} : \Phi'(u) = 0\}$. If, in addition, $f(x,t)$ is odd in t, then* (1.1) *admits infinitely many pairs geometrically distinct solutions $\pm u$.*

Proposition 1.5. *Assume that V and f satisfy $(V_1), (F_0), (F_1), (F_2)$, (Mo)$'$ and $(SQ)'$. Then* (1.1) *has a solution $u_0 \in E$ such that $\Phi(u_0) = \inf_{u\in\mathcal{M}} \Phi(u) > 0$. If, in addition, $f(x,t)$ is odd in t, then* (1.1) *admits infinitely many pairs geometrically distinct solutions $\pm u$.*

Existence of nontrivial solutions to a strongly indefinite Choquard equation with critical exponent is obtained in [13]. We also want to mention that the existence and some quantitative properties of periodic solutions of fractional equation with double well potential in one-dimensional case are established in [10, 12, 14].

In this paper we will generalize the existence of nontrivial solutions in [9] by replacing (SQ), (DL) by the weaker conditions (F_3) and (F_4), and generalize the existence of infinitely many geometrically different solutions in [27] by replacing (Mo)$'$, (SQ)$'$ by the weaker conditions (F_3) and (F_4).

The corresponding energy functional of (1.1) is

$$\Phi_s(u) := \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2(x)\,dx - \int_{\mathbb{R}^N} F(x,u(x))\,dx.$$

It is easy to verify that Φ_s is $C^1(H^s(\mathbb{R}^N),\mathbb{R})$ and

$$\langle \Phi_s'(u), v\rangle = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy$$

$$+\int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx-\int_{\mathbb{R}^N} f(x,u(x))v(x)\,dx.$$

From (F_1) and (F_2), for any given $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$|f(x,t)|\le \epsilon|t|+C_\epsilon|t|^{p-1},\ \forall (x,t)\in\mathbb{R}^N\times\mathbb{R}, \tag{1.2}$$

which yields

$$|F(x,t)|\le \epsilon|t|^2+C_\epsilon|t|^{p},\ \forall (x,t)\in\mathbb{R}^N\times\mathbb{R}. \tag{1.3}$$

The following conditions are required to arrive at our results.
(F_3): $F(x,t)\ge 0$ for any $(x,t)\in\mathbb{R}^N\times\mathbb{R}$, and there exists $r_1>0$ such that $F(x,t)\ge\Lambda_s t^2$ for any $|t|\ge r_1$ and $x\in\mathbb{R}^N$.
(F_4): $G(x,t)=\frac{1}{2}f(x,t)t-F(x,t)>0$ if $|t|\ne 0$, $G(x,t)\to+\infty$ as $|t|\to\infty$ uniformly in x, and there exists $c_3,r_2>0$ and $\sigma>\max\{1,\frac{N}{2s}\}$ such that $\left|\frac{f(x,t)}{t}\right|^\sigma\le c_3G(x,t)$ for $|t|\ge r_2$ and $x\in\mathbb{R}^N$.

An example that satisfies the conditions (F_1)-(F_4), but does not satisfy (SQ) is

$$f(x,t)=h(x)t\ln\frac{1+e|t|}{1+|t|},$$

where $h(x)$ is 1-periodic in each of $x_1,x_2,...,x_N$ and $\inf h(x)\ge 4\Lambda_s$.

The followings are our main results.

Theorem 1.1. *Assume* (V_s) *and* (F_1)-(F_4) *are satisfied. Then* (1.1) *has a nontrivial solution.*

Theorem 1.2. *Assume* (V_s) *and* (F_1)-(F_4) *are satisfied and* $f(x,t)$ *is odd in* t. *Then* (1.1) *admits infinitely many pairs geometrically distinct solutions* $\pm u$.

We note that if u_0 is a solution of (1.1), then so are all elements of the orbit of u_0 under the action of $\mathbb{Z}^N$, $O(u)=\{k*u: k\in\mathbb{Z}^N\}$, where $k*u(x):=u(x+k)$. Two solutions u_1 and u_2 are said to be geometrically distinct if $O(u_1)$ and $O(u_2)$ are disjoint.

2. Preliminaries

Denote $A_s=(-\Delta)^s+V$. Plainly A_s is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathcal{D}(A_s)=H^{2s}(\mathbb{R}^N)$. Let $\{\Upsilon_s(\lambda):-\infty\le\lambda\le+\infty\}$ and $|A_s|$ be the spectral family and the absolute value of A_s respectively, and $|A_s|^{\frac12}$ be the square root of $|A_s|$. Let $E_s=\mathcal{D}(|A_s|^{\frac12})$ and

$$E_s^-=\Upsilon_s(0^-)E_s,\quad E_s^+=[\mathrm{id}-\Upsilon_s(0)]E_s. \tag{2.1}$$

For any $u\in E_s$, it is easy to see that $u=u^-+u^+$ and

$$A_su^-=-|A_s|u^-,\quad A_su^+=|A_s|u^+ \text{ for any } u\in E_s\cap\mathcal{D}(A_s), \tag{2.2}$$

where

$$u^-=\Upsilon_s(0^-)u\in E_s^-,\quad u^+=[\mathrm{id}-\Upsilon_s(0)]u\in E_s^+. \tag{2.3}$$

Under assumption (V_s), we can define an inner product

$$(u,v)=(|A_s|^{\frac12}u,|A_s|^{\frac12}v)_{L^2},\quad u,v\in E_s \tag{2.4}$$

and the corresponding norm

$$\|u\| = \||A_s|^{\frac{1}{2}}u\|_{L^2}.$$

By $(V_s), E_s = H^s(\mathbb{R}^N)$ with equivalent norms. Therefore E_s embeds continuously in $L^p(\mathbb{R}^N)$ for all $2 \le p \le 2_s^*$. Hence, there exists constant $\gamma_p > 0$ such that $\|u\|_{L^p} \le \gamma_p\|u\|$. By the definitions of Λ_s and E_s^+ we also have

$$\|u\|^2 \ge \Lambda_s\|u\|_{L^2}^2 \quad \text{for any } u \in E_s^+. \tag{2.5}$$

From $(2, 2)$–(2.4), one has

$$\begin{aligned} B(u, v) &:= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)u(x)v(x)\,dx \\ &= (A_s u, v)_{L^2} \\ &= (|A_s|u^+, v)_{L^2} - (|A_s|u^-, v)_{L^2} \\ &= (|A_s|u^+, v^+)_{L^2} - (|A_s|u^-, v^-)_{L^2} \\ &= (|A_s|^{\frac{1}{2}}u^+, |A_s|^{\frac{1}{2}}v^+)_{L^2} - (|A_s|^{\frac{1}{2}}u^-, |A_s|^{\frac{1}{2}}v^-)_{L^2} \\ &= (u^+, v^+) - (u^-, v^-). \end{aligned}$$

Then

$$\begin{aligned} \Phi_s(u) &= \frac{1}{2}B(u, u) - \int_{\mathbb{R}^N} F(x, u)\,dx \\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^N} F(x, u)\,dx \ \text{ for any } u \in E_s. \end{aligned} \tag{2.6}$$

Let X be a real Hilbert space. Recall that a functional $\psi \in C^1(X, \mathbb{R})$ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in X one has $\psi(u) \le \liminf_{n\to\infty}\psi(u_n)$, and ψ' is said to be weakly sequentially continuous if $\lim_{n\to\infty}\langle\psi'(u_n), v\rangle = \langle\psi'(u), v\rangle$ for each $v \in X$. Let $\Psi(u) = \int_{\mathbb{R}^N} F(x, u)\,dx$. By (F_1)–(F_3), one can easily get that Ψ is weakly sequentially lower semi-continuous and Ψ' is weakly sequentially continuous.

We introduce the following generalized linking theorem.

Lemma 2.1. ([15, 18]) *Let X be a real Hilbert space, $\varphi \in C^1(X, \mathbb{R}), \varphi(0) = 0$ and*

$$\varphi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^- + u^- \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied

i) : $\psi \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;

ii) : ψ' is weakly sequentially continuous;

iii) : there exists $\tau > 0$ such that

$$m_\tau := \inf_{u\in X^+, \|u\|=\tau} \varphi(u) > 0;$$

iv) : there exists $r > \tau > 0$, $e \in X^+$and $\|e\| = 1$, such that

$$m_\tau > sup\ \varphi(\partial Q_{e,r}),$$

where

$$Q_{e,r} := \{v + ze : v \in X^-, z \geq 0, \|v + ze\| \leq r\}.$$

Then there exists a constant $C_0 \in [m_\tau, sup(Q_{e,r})]$ and a sequence $\{u_n\} \subseteq X$, such that $\varphi(u_n) \to C_0$, $\|\varphi'(u_n)\|(1 + \|u_n\|) \to 0$.

A sequence $\{u_n\}$ is called Cerami sequence (denoted also as $(C_e)_c$-sequence) of the energy functional φ, if there exists constant c such that $\varphi(u_n) \to c$ and $\|\varphi'(u_n)\|(1 + \|u_n\|) \to 0$.

3. The existence of nontrivial solutions

In this section, we will prove Theorem 1.1 by applying Lemma 2.1.

Lemma 3.1. *Under the assumptions $(V_s), (F_1)$ and (F_2), there exists $\rho > 0$ such that*

$$m_\rho = inf\{\Phi_s(u) : u \in E_s^+, \|u\| = \rho\} > 0$$

Proof. By (2.6), for $u \in E_s^+$, we have $\Phi_s(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u)\, dx$. Inequality (1.3) shows that for any given $\epsilon > 0$ the inequality $|F(x, u)| \leq \epsilon |u|^2$ holds for small $|u|$. So $|\int_{\mathbb{R}^N} F(x, u)\, dx| \leq \epsilon \|u\|^2$, and the conclusion follows if ρ is sufficiently small. □

Lemma 3.2. ([11] ,*Theorem 1.1) The fractional Schrödinger operator $A_s = (-\Delta)^s + V$ has purely continuous spectrum, which is bounded below and consists of closed disjoint intervals.*

Since $\Upsilon_s(A_s)$ is purely continuous, for any given $\mu > \Lambda_s$, the space $Y_\mu := ((\Upsilon_s)_\mu - (\Upsilon_s)_0)_{L^2}$ is infinitely dimensional, where $((\Upsilon_s)_\lambda)_{\lambda \in \mathbb{R}}$ denotes spectrum family of A_s. By (2.5), for $\Lambda_s < \mu < 2\Lambda_s$ we have

$$Y_\mu \subseteq E_s^+, \text{ and } \Lambda_s \|v\|_{L^2}^2 \leq \|v\|^2 \leq \mu \|v\|_{L^2}^2 \text{ for all } v \in Y_\mu. \tag{3.1}$$

Lemma 3.3. *Suppose that (V_s) and (F_3) are satisfied. Then for any $e \in Y_\mu$, $sup\ \Phi_s\ (E_s^- \oplus \mathbb{R}^+ e) < \infty$ and there is $r_e > 0$ such that*

$$\Phi_s(u) \leq 0 \textit{ for any } u \in E_s^- \oplus \mathbb{R}^+ e,\ \|u\| \geq r_e.$$

Proof. Arguing indirectly, assume that for some sequence $\{u_n\} \subseteq E_s^- \oplus \mathbb{R}^+ e$, $e \in Y_\mu$ with $\|u_n\| \to \infty$ and $\Phi_s(u_n) > 0$. Setting $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Hence there exists $v = v^+ + v^-$ such that $v_n \rightharpoonup v$, $v_n^- \rightharpoonup v^-$, $v_n^+ \to v^+ \in \mathbb{R}^+ e$. Here the strong convergence of $\{v_n^+\}$ is due to the reason that $\mathbb{R}^+ e$ is finite dimensional. We have

$$0 < \frac{\Phi_s(u_n)}{\|u_n\|^2} = \frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2}\, dx.$$

We claim that $v \neq 0$. Suppose not, then

$$0 \leq \frac{1}{2}\|v_n^-\|^2 + \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2}\, dx < \frac{1}{2}\|v_n^+\|^2 \to 0,$$

where the first inequality use the fact that $F \geq 0$. The above relation gives $\|v_n^-\| \to 0$, hence $1 = \|v_n\|^2 = \|v_n^+\|^2 + \|v_n^-\|^2 \to 0$, which is a contradiction, and the claim is true. By (3.1)

$$\begin{aligned}\|v^+\|^2 - \|v^-\|^2 - 2\Lambda_s \int_{\mathbb{R}^N} v^2\,dx &\leq \mu\|v^+\|_{L^2}^2 - \|v^-\|^2 - 2(\Lambda_s\|v^+\|_{L^2}^2 + \Lambda_s\|v^-\|_{L^2}^2)\\ &\leq -((2\Lambda_s - \mu)\|v^+\|_{L^2}^2 + \|v^-\|^2) < 0.\end{aligned}$$

Hence, there exists a bounded set $\Omega \subseteq \mathbb{R}^N$ such that

$$\|v^+\|^2 - \|v^-\|^2 - 2\Lambda_s \int_\Omega v^2\,dx < 0.$$

Note that

$$\begin{aligned}\frac{\Phi_s(u_n)}{\|u_n\|^2} &\leq \frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2}\,dx\\ &= \frac{1}{2}\left(\|v_n^+\|^2 - \|v_n^-\|^2 - 2\Lambda_s \int_\Omega v_n^2\,dx\right) + \int_\Omega \frac{\Lambda_s|u_n|^2 - F(x,u_n)}{\|u_n\|^2}\,dx\\ &\leq \frac{1}{2}\left(\|v_n^+\|^2 - \|v_n^-\|^2 - 2\Lambda_s \int_\Omega v_n^2\,dx\right) + \frac{\Lambda_s r_0^2|\Omega|}{\|u_n\|^2},\end{aligned}$$

where the last inequality use the assumption (F_3). Here $|\Omega|$ denotes Lebesgue's measure of Ω. By the weak lower-semi continuity of the norm, we have $\|v^-\|^2 \leq \liminf_{n\to\infty} \|v_n^-\|^2$. Thus

$$\begin{aligned}0 \leq \liminf_{n\to\infty} \frac{\Phi_s(u_n)}{\|u_n\|^2} &\leq \liminf_{n\to\infty} \frac{1}{2}\left(\|v_n^+\|^2 - \|v_n^-\|^2 - 2\Lambda_s \int_\Omega v_n^2\,dx\right)\\ &\leq \frac{1}{2}\left(\|v^+\|^2 - \|v^-\|^2 - 2\Lambda_s \int_\Omega v^2\,dx\right) < 0,\end{aligned}$$

a contradiction follows. □

Lemma 3.4. *Under the assumptions of* (V_s), (F_2) *and* (F_4), *any* $(C_e)_c$*-sequence is bounded.*

Proof. Let $\{u_n\} \subseteq E_s$ be a $(C_e)_c$-sequence. Suppose that u_n is unbounded, define $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Passing to subsequence, we may assume that $v_n \rightharpoonup v$ in E_s, $v_n \to v$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \leq p < 2_s^*$, and $v_n \to v$ a.e. in $\mathbb{R}^N$. Moreover $\|v_n\|_{L^p} \leq \gamma_p\|v_n\| = \gamma_p$. Note that

$$\Phi_s'(u_n)(u_n^+ - u_n^-) = \|u_n\|^2\left(1 - \int_{\mathbb{R}^N} \frac{f(x,u_n)(v_n^+ - v_n^-)}{\|u_n\|}\right),$$

hence

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)(v_n^+ - v_n^-)}{\|u_n\|} \to 1, \tag{3.2}$$

since $\{u_n\} \subseteq E_s$ is a $(C_e)_c$-sequence. For $R > 0, 0 < a < b$, we define

$$\Omega_n(a,b) := \{x \in \mathbb{R}^N,\ a \leq |u_n(x)| < b\},$$

$$G_R := \inf\{G(x,u) : x \in \mathbb{R}^N,\ |u| \geq R\}$$

and
$$G_a^b = \inf\left\{\frac{G(x,u)}{u^2} x \in \mathbb{R}^N,\ a \le |u| \le b\right\}.$$
By (F_4), $G_R > 0$ for $R > 0$ and $G_R \to +\infty$ as $R \to +\infty$. Since $G(x,u) > 0$ for $u \neq 0$ and depends periodically on x, we have
$$G(x,u_n) \ge G_a^b |u_n|^2, \quad x \in \Omega_n(a,b). \tag{3.3}$$
Observe that for n large
$$\begin{aligned} 1 + C_0 \ge \Phi_s(u_n) - \frac{1}{2}\Phi_s'(u_n)u_n &= \int_{\mathbb{R}^N} G(x,u_n)\,dx \\ &= \int_{\Omega_n(0,a)} G(x,u_n)\,dx + \int_{\Omega_n(a,b)} G(x,u_n)\,dx + \int_{\Omega_n(b,\infty)} G(x,u_n)\,dx \\ &\ge \int_{\Omega_n(0,a)} G(x,u_n)\,dx + G_a^b \int_{\Omega_n(a,b)} |u_n|^2\,dx + G_b|\Omega_n(b,\infty)|. \end{aligned} \tag{3.4}$$
Denote σ' as the conjugate number of σ, namely $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$. Set $\tau_\sigma = \frac{2\sigma}{\sigma-1}$, then $\tau_\sigma = 2\sigma' \in (2, 2_s^*)$, since $\sigma > \max\{1, \frac{N}{2s}\}$. Fix a $\tau \in (\tau_\sigma, 2_s^*)$. By (3.4), we can see
$$|\Omega_n(b,\infty)| \le \frac{1 + C_0}{G_b} \to 0$$
uniformly in n as $b \to +\infty$. By using Hölder inequality we have
$$\int_{\Omega_n(b,\infty)} |v_n|^{\tau_\sigma} \le \gamma_\tau^{\tau_\sigma} |\Omega_n(b,\infty)|^{1-\frac{\tau_\sigma}{\tau}} \to 0 \tag{3.5}$$
uniformly in n as $b \to \infty$. For any given $0 < \delta < \frac{1}{3}$, let $b_\delta \ge r_2$. From (3.4), (3.5) and (F_4), we have
$$\begin{aligned} &\int_{\Omega_n(b_\delta,\infty)} \frac{f(x,u_n)}{|u_n|}(v_n^+ - v_n^-)|v_n| \\ &\le \left(\int_{\Omega_n(b_\delta,\infty)} \left|\frac{f(x,u_n)}{u_n}\right|^\sigma\right)^{\frac{1}{\sigma}} \left(\int_{\Omega_n(b_\delta,\infty)} (|v_n^+ - v_n^-||v_n|)^{\sigma'}\right)^{\frac{1}{\sigma'}} \\ &\le \left(\int_{\mathbb{R}^N} c_3 G(x,u_n)\right)^{\frac{1}{\sigma}} \left(\int_{\Omega_n(b_\delta,\infty)} |v_n^+ - v_n^-|^{\tau_\sigma}\right)^{\frac{1}{\tau_\sigma}} \left(\int_{\Omega_n(b_\delta,\infty)} |v_n|^{\tau_\sigma}\right)^{\frac{1}{\tau_\sigma}} \\ &\le \delta, \end{aligned} \tag{3.6}$$
where we use the relation $\tau_\sigma = 2\sigma'$. By (F_2), there exist $a_\delta > 0$ such that $|f(x,t)| < \frac{\delta}{(\gamma_2)^2}|t|$ for any $|t| \le a_\delta$ and $x \in \mathbb{R}^N$, hence
$$\begin{aligned} &\int_{\Omega_n(0,a_\delta)} \frac{f(x,u_n)}{|u_n|}(v_n^+ - v_n^-)|v_n| \\ &\qquad \le \int_{\Omega_n(0,a_\delta)} \frac{\delta}{(\gamma_2)^2}|v_n^+ - v_n^-||v_n| \le \frac{\delta}{(\gamma_2)^2}\|v_n\|_{L^2}^2 \le \delta. \end{aligned} \tag{3.7}$$

From (3.4) we have

$$\int_{\Omega_n(a,b)} |v_n|^2\,dx = \frac{1}{\|u_n\|^2}\int_{\Omega_n(a,b)} |u_n|^2\,dx \le \frac{1+C_0}{G_a^b\|u_n\|^2} \to 0 \text{ as } n\to\infty. \tag{3.8}$$

Note that there exists $\gamma = \gamma(\delta) > 0$ such as $|f(x,u_n)| \le \gamma|u_n|$ for $x \in \Omega_n(a_\delta, b_\delta)$. By (3.8), there exists $n_0 > 0$, for $n \ge n_0$ we have

$$\begin{aligned}\int_{\Omega_n(a_\delta,b_\delta)} \frac{f(x,u_n)}{|u_n|}(v_n^+ - v_n^-)|v_n| &\le \int_{\Omega_n(a_\delta,b_\delta)} \gamma|v_n^+ - v_n^-||v_n| \\ &\le \gamma\|v_n\|_{L^2}\left(\int_{\Omega_n(a_\delta,b_\delta)} |v_n|^2\right)^{\frac{1}{2}} \le \delta.\end{aligned} \tag{3.9}$$

Combining (3.6), (3.7) and (3.9) we have

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)(v_n^+ - v_n^-)}{\|u_n\|} < 3\delta < 1,$$

which contradicts with (3.2). $\square$

Proof of Theorem 1.1 From Lemmas 3.1 and 3.3, we verify that all the conditions of Lemma 2.1 hold true. Hence there exist a Cerami sequence such that $\Phi_s(u_n) \to C_0$, $\|\Phi_s'(u_n)\|(1+\|u_n\|) \to 0$. By Lemma 3.4 and Sobolev imbedding theorem, there exists $C \ge 0$ such that $\|u_n\|_{L^2} \le C$. If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)} |u_n|^2\,dx = 0,$$

then by Lions' concentration compactness principle, $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2_s^*$. For $\epsilon = \frac{C_0}{4C^2}$, from (1.2) and (1.3) it follows that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx \le \frac{3}{2}\epsilon C^2 + C_\epsilon \lim_{n\to\infty}\|u_n\|_{L^p}^p = \frac{3}{8}C_0.$$

We obtain

$$\begin{aligned}C_0 + o(1) &= \Phi_s(u_n) - \frac{1}{2}\langle\Phi_s'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^N}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx \le \frac{3}{8}C_0 + o(1),\end{aligned}$$

a contradiction follows, and so $\delta > 0$.

Passing to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{n}}(k_n)} |u_n|^2 dx \ge \frac{\delta}{2}.$$

Let us define $v_n(x) = u_n(x + k_n)$, then

$$\int_{B_{1+\sqrt{n}}(0)} |v_n|^2 dx \ge \frac{\delta}{2}. \tag{3.10}$$

Since $V(x)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, then $\|u_n\| = \|v_n\|$ and

$$\Phi_s(v_n) \to C_0, \quad \|\Phi_s'(v_n)\|(1+\|v_n\|) \to 0.$$

Passing to a subsequence, we have $v_n \rightharpoonup \bar{v}$ in E_s. Obviously, (3.10) implies that $\bar{v} \ne 0$. By a standard argument, one has $\Phi_s'(\bar{v}) = 0$. We complete the proof of Theorem 1.1. $\square$

4. The existence of infinite many solutions

In this section, we give the proof of Theorem 1.2.

We need to introduce some notations. For $d_2 \geq d_1 > -\infty$, we set

$$\Phi_s^{d_2} = \{u \in E_s : \Phi_s(u) \leq d_2\}, \ \ I_{s,d_1} = \{u \in E_s : \Phi_s(u) \geq d_1\}, \quad \Phi_{s,d_1}^{d_2} = I_s^{d_2} \cap \Phi_{s,d_1},$$

$$K = \{u \in E_s \setminus \{0\} : \Phi_s'(u) = 0\}, \ \ K_d = \{u \in K : \Phi_s(u) = d\}.$$

Lemma 4.1. *Assume that* $(V_s),\ (F_1),\ (F_2),\ (F_4)$ *hold, then*

$$\begin{aligned} &i)\ b_1 := inf\{\|u\| : u \in K\} > 0;\\ &ii)\ b_2 := inf\{\Phi_s(u) : u \in K\} > 0. \end{aligned} \tag{4.1}$$

Proof. i) Assume $b_1 = 0$, then there is a sequence $\{u_n\} \subset K$ with $\|u_n\| \to 0$, and

$$0 = \|u_n\|^2 - \int_{\mathbb{R}^N} f(x, u_n)(u_n^+ - u_n^-).$$

This and (1.2) yield that

$$\|u_n\|^2 \leq \epsilon \|u_n\|_{L^2}^2 + C_\epsilon \|u_n\|_{L^p}^p. \tag{4.2}$$

By this and Sobolev imbedding theorem we deduce $\|u_n\|^{2-p} \leq \bar{C}_\epsilon$, which contradicts with the assumption $\|u_n\| \to 0$.

ii) By

$$\Phi_s(u_n) = \Phi_s(u_n) - \frac{1}{2}\Phi_s'(u_n)u_n = \int_{\mathbb{R}^N} G(x, u_n) \geq 0, \ \ u_n \in K, \tag{4.3}$$

we have $b_2 \geq 0$. Assume $b_2 = 0$, then there is a sequence $\{u_n\} \subset K$ such that $\Phi_s(u_n) \to 0$. Since $\{u_n\}$ is a $(C_e)_{c=0}$ sequence, by lemma 3.4, u_n is bounded. By Sobolev imbedding theorem, there exists $C \geq 0$ such that $\|u_n\|_{L^2}^2 \leq C$. Note that

$$\|u_n\|^2 = \int_{\mathbb{R}^N} f(x, u_n)(u_n^+ - u_n^-) \tag{4.4}$$

By (4.3), for any $0 < a < b$, we have

$$\begin{aligned} \mathrm{o}(1) &= \int_{\mathbb{R}^N} G(x, u_n)\, dx \\ &= \int_{\Omega_n(0,a)} G(x, u_n)\, dx + \int_{\Omega_n(a,b)} G(x, u_n)\, dx + \int_{\Omega_n(b,\infty)} G(x, u_n)\, dx \\ &\geq \int_{\Omega_n(0,a)} G(x, u_n)\, dx + G_a^b \int_{\Omega_n(a,b)} |u_n|^2\, dx + G_b |\Omega_n(b, \infty)|, \end{aligned} \tag{4.5}$$

which gives

$$\int_{\Omega_n(a,b)} |u_n|^2\, dx = \mathrm{o}(1), \ \ |\Omega_n(b, \infty)| \leq \frac{\mathrm{o}(1)}{G_b} = \mathrm{o}(1)$$

as $n \to \infty$. Similar as the derivation of (3.5), for any $p \in (2, 2_s^*)$ we have

$$\int_{\Omega_n(b,\infty)} |u_n|^p dx \to 0 \text{ as } n \to \infty. \tag{4.6}$$

Next, we prove $\|u_n\| \to 0$. For any given $\epsilon > 0$, by (F_2), there exist $a_\epsilon > 0$ such that $|f(x,t)| < \frac{\epsilon}{3C}|t|$ for any $|t| \le a_\epsilon$ and $x \in \mathbb{R}^N$. Hence,

$$\int_{\Omega_n(0,a_\epsilon)} |f(x,u_n)||u_n^+ - u_n^-|dx \le \frac{\epsilon}{3C}\left(\int_{\Omega_n(0,a_\epsilon)} |u_n|^2\right) \le \frac{\epsilon}{3}. \tag{4.7}$$

There exists $b_\epsilon > a_\epsilon > 0$ such as $|f(x,t)| \le \bar{C}_\epsilon |t|^{p-1}$ for $|t| \ge b_\epsilon$ and $x \in \mathbb{R}^N$, and for large n we have

$$\begin{aligned}&\int_{\Omega_n(b_\epsilon,\infty)} |f(x,u_n)||u_n^+ - u_n^-|dx \\ &\le \bar{C}_\epsilon \left(\int_{\Omega_n(b_\epsilon,\infty)} |u_n|^p dx\right)^{\frac{p-1}{p}} \times \left(\int_{\Omega_n(b_\epsilon,\infty)} |u_n^+ - u_n^-|^p dx\right)^{\frac{1}{p}} \le \frac{\epsilon}{3},\end{aligned} \tag{4.8}$$

where the last inequality follows from (4.6). Note that there exists $\bar{\gamma} = \bar{\gamma}(\epsilon) > 0$ such as $|f(x,t)| \le \bar{\gamma}|t|$ for $|t| \in (a_\epsilon, b_\epsilon)$ and $x \in \mathbb{R}^N$. So for large n

$$\begin{aligned}\int_{\Omega_n(a_\epsilon,b_\epsilon)} |f(x,u_n)||u_n^+ - u_n^-|dx &\le \int_{\Omega_n(a_\epsilon,b_\epsilon)} \bar{\gamma}|u_n^+ - u_n^-||u_n|dx \\ &\le \bar{\gamma}\|u_n\|_{L^2}\left(\int_{\Omega_n(a_\epsilon,b_\epsilon)} |u_n|^2 dx\right)^{\frac{1}{2}} \le \frac{\epsilon}{3}.\end{aligned} \tag{4.9}$$

Therefore, it follows from (4.7)–(4.9) and (4.4), we have that

$$\limsup_{n\to\infty} \|u_n\|^2 \le \epsilon,$$

which contradicts with the result $\|u_n\| \ge b_1 > 0$ of i). □

Lemma 4.2. *Assume that (F_1) and (F_2) hold. If $u_n \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^N)$, then along a subsequence of $\{u_n\}$,*

$$\lim_{n\to\infty} \sup_{\phi \in H^s(\mathbb{R}^N), \|\phi\| \le 1} \left| \int_{\mathbb{R}^N} [f(x,u_n) - f(x,u_n - \bar{u}) - f(x,\bar{u})]\phi dx \right| = 0.$$

Proof. We can refer to Lemma 4.1 of [27]. The only difference is that the space $H^1(\mathbb{R}^N)$ there is replaced by $H^s(\mathbb{R}^N)$ in this lemma, the rest argument is almost similar as the proof of Lemma 4.1 in [27] and we omit it. □

Applying Lemma 4.2, we can obtain the next lemma.

Lemma 4.3. *Assume that $(V_s), (F_1)$ and (F_2) hold. If $u_n \rightharpoonup \bar{u}$ in E_s, then*

$$\Phi_s(u_n) = \Phi_s(\bar{u}) + \Phi_s(u_n - \bar{u}) + o(1), \tag{4.10}$$

$$\Phi_s'(u_n) = \Phi_s'(\bar{u}) + \Phi_s'(u_n - \bar{u}) + o(1). \tag{4.11}$$

Proof. The proof is rather similar as that of lemma 4.2 of [9]. The main differences are that the space E and the energy functional Φ there are replaced by E_s and Φ_s respectively. We omit it here. □

Remark Theorem 1.1 shows that equation (1.1) has a nontrivial solution $\overline{u} \in E_s$, and so $K \neq \emptyset$. We choose a subset Q of K such that $Q = -Q$ (here $-Q := \{w : -w \in Q\}$) and each orbit $O(u) \subseteq K$ has a unique representative in Q. It suffices to show that the set Q is infinite, so from now on we assume by contradiction that

$$Q \text{ is a finite set.} \tag{4.12}$$

Let $[a]$ stands for the largest integer not exceeding a. As a consequence of Lemmas 3.4, 4.1, 4.3, we have the following lemma (see [9] lemma 4.4, [15] proposition 4.2 and [27] lemma 4.4. The only difference is that the space E is replaced by E_s, and the energy functional Φ is replaced by Φ_s.)

Lemma 4.4. *Suppose that (V_s) and (F_1)–(F_4) are satisfied. Let $\{u_n\}$ be a $(C_e)_c$ sequence of I_s in E_s. Then either*
(i) $u_n \to 0$ in E_s (and hence $c = 0$) ; or
(ii) $c \geq b_2$ and there exists a positive integer $\ell \leq [\frac{c}{b_2}]$, points $\bar{u}_1, \bar{u}_2, ..., \bar{u}_\ell \in K$, a subsequence denoted again by $\{u_n\}$ and sequences $\{a_n^i\} \subset \mathbb{Z}^N$, such that

$$\|u_n - \sum_{i=1}^{\ell} a_n^i * \bar{u}_i\| \to 0 \quad as\ n \to \infty,$$

$$|a_n^i - a_n^j| \to \infty \ \ for\ i \neq j \quad as\ n \to \infty$$

and

$$\sum_{i=1}^{\ell} \Phi_s(\bar{u}_i) = c.$$

For any $c \geq b_2$, as in [6, 7, 9, 13, 15], we let

$$Q_c := \left\{ \sum_{i=1}^{j} (a_i * u_i) : 1 \leq j \leq \left[\frac{c}{b_2} \right],\ a_i \in \mathbb{Z}^N, u_i \in Q \right\}.$$

Plainly $Q_{c'} \subseteq Q_c$ for any $c \geq c' \geq b_2$.

Following the argument of Proposition 1.55 in [6], we have the next lemma.

Lemma 4.5. *Let $c \geq b_2$. Then $\kappa_c := inf\{\|u_1 - u_2\| : u_1, u_2 \in Q_c, u_1 \neq u_2\} > 0$.*

To prove Theorem 1.2, we need to establish the following lemmas 4.6–4.10. The proofs of lemmas 4.6–4.10 are rather similar to the the proofs of lemmas 4.6–4.10 in [27]. The main differences are that the space E and the energy functional Φ there are replaced by E_s and Φ_s respectively. We omit them here.

Lemma 4.6. *Let $c \geq b_2$. If $\{u_n^1\}, \{u_n^2\} \subseteq \Phi_{s,b_2}^c$ are two (C_e)-sequence for Φ_s, then either $\lim_{n\to\infty} \|u_n^1 - u_n^2\| = 0$ or $\limsup_{n\to\infty} \|u_n^1 - u_n^2\| \geq \kappa_c$.*

Lemma 4.7. *Let $c > b_2$, $\alpha \in (0, \alpha_0]$ $(\alpha_0 \in (0, (c - b_2)/2]$)and $u \in E \setminus (K \cup \{0\})$ be such that $c - \alpha \leq \Phi_s(\eta(t, u)) \leq c + \alpha$ for all $t \in [0, \infty)$. Then $u_\infty := \lim_{t\to\infty} \eta(t, u)$ exists and $u_\infty \in \Phi_{s,c-\alpha}^{c+\alpha} \cap K$.*

Lemma 4.8. *Let $c > b_2$. If $K_c = \emptyset$, then there exists $\epsilon > 0$ such that* $\lim_{t\to\infty} \Phi_s(\eta(t,u)) < c - \epsilon$ *for* $u \in \Phi_s^{c+\epsilon}$.

Lemma 4.9. *Let $c > b_2$. Then for every $\delta \in (0, \kappa_c/4)$, there exists $\epsilon = \epsilon(c,\delta) > 0$ and an odd and continuous map $\varphi : \Phi_s^{c+\epsilon} \setminus U_\delta(Q_c) \to \Phi_s^{c-\epsilon}$, where $U_\delta(Q_c) := \{v \in E_s : dist(v, Q_c) < \delta\}$.*

Lemma 4.10. *Let $c \geq b_2$. Then for every $\delta \in (0, \kappa_c/4)$, $\gamma(\overline{U_\delta(Q_c)}) = 1$, where $\gamma(\overline{U_\delta(Q_c)})$ denotes the usual Krasnoselskki genus of $\overline{U_\delta(Q_c)}$.*

Proof of Theorem 1.2 We can prove Theorem 1.2 by applying Lemmas 4.6–4.10. Since the proof is rather similar that of the second part of theorem 1.4 and 1.5 in [27], we omit it here. $\square$

5. Conclusions

In this paper we obtained the existence of nontrivial solutions to a semi-linear fractional Schrödinger equation by using a generalized linking theorem. Based on this existence results, infinitely many geometrically distinct solutions are further established under weaken conditions of the nonlinearity of the equation.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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