Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(8): 8453–8465.
DOI:10.3934/math.2021490
Received: 05 March 2021
Accepted: 19 May 2021
Published: 02 June 2021

Research article

Linear maps on von Neumann algebras acting as Lie type derivation via local actions

Mohd Arif Raza[1,*], **Aisha Jabeen**[2], **Abdul Nadim Khan**[1] **and Husain Alhazmi**[3]

[1] Department of Mathematics, Faculty of science and Arts-Rabigh, King Abdulaziz University, KSA
[2] Department of Applied Sciences and Humanities, Jamia Millia Islamia, New Delhi-110025, India
[3] Department of Mathematics, Faculty of science, King Abdulaziz University, Jeddah, KSA

* **Correspondence:** Email: mreda@kau.edu.sa; arifraza03@gmail.com.

Abstract: Let $\aleph$ be a factor von Neumann algebra with $dim > 1$ that operates on a Hilbert space. Within the manuscript, we let out the characterization of Lie type derivation on factor von Neumann algebra of zero product as well as at projection product and notice that it has standard form.

Keywords: derivation; Lie derivation; von Neumann algebra
Mathematics Subject Classification: 47B47, 47L10

1. Introduction

Let $\aleph$ be an algebra over complex field $\mathbb{C}$. A mapping (linear) $\mho : \aleph \to \aleph$ is considered as a derivation (respectively Lie derivation) on $\aleph$ if $\mho(\varpi_1\varpi_2) = \mho(\varpi_1)\varpi_2 + \varpi_1\mho(\varpi_2)$ (resp. $\mho([\varpi_1, \varpi_2]) = [\mho(\varpi_1), \varpi_2] + [\varpi_1, \mho(\varpi_2)]$) holds for all $\varpi_1, \varpi_2 \in \aleph$. Right away we explore a popular family of maps. Characterize the arrangement of polynomials:

$$\begin{aligned}
&\mathscr{P}_1(\zeta_1) = \zeta_1\\
&\mathscr{P}_2(\zeta_1, \zeta_2) = [\mathscr{P}_1(\zeta_1), \zeta_2] = [\zeta_1, \zeta_2]\\
&\vdots\\
&\mathscr{P}_n(\zeta_1, \zeta_2, \cdots, \zeta_n) = [\mathscr{P}_{n-1}(\zeta_1, \zeta_2, \cdots, \zeta_{n-1}), \zeta_n].
\end{aligned}$$

For $n \geq 2$, the polynomial $\mathscr{P}_n(\zeta_1, \zeta_2, \cdots, \zeta_n)$ is known as $(n-1)$-th commutator. A Lie n-derivation on $\aleph$ is defined as

$$\mho(\mathscr{P}_n(\zeta_1, \zeta_2, \cdots, \zeta_n)) = \sum_{i=1}^{i=n} \mathscr{P}_n(\zeta_1, \zeta_2, , \cdots, \zeta_{i-1}, \mho(\zeta_i), \zeta_{i+1}, \cdots, \zeta_n)$$

for all $\zeta_1, \zeta_2, \cdots, \zeta_{\mathtt{n}} \in \aleph$, where $\mho$ is a linear map $\mho : \aleph \to \aleph$. Along these lines, Abdullaev [1] initiated and conceived the idea of Lie $\mathtt{n}$-derivation on von Neumann algebras. Notice that any Lie 2-derivation is known as Lie derivation and Lie 3-derivation is said to be Lie triple derivation. Therefore Lie, Lie triple, Lie $\mathtt{n}$-derivation are comprehensively recognized as Lie type derivations on $\aleph$.

In the recent past assessment of the conditions under which a linear map becomes a derivation (Lie derivation) fascinate the courtesy of many algebraists (see [2–6] and in their bibliographic content). Commonly, the object of the above studies was to attain the stipulations under which derivations (Lie derivations) can be absolutely determined by way of the action on some subsets of the algebras. On the analysis of local actions of Lie derivations on operator algebras, although there numerous research articles have been published. In 2010, Lu and Jing [2] initiated the study of local actions of Lie derivations of operator algebras and they characterized the action of Lie derivation on $\mathscr{B}(\zeta)$. Exactly, they established that if ζ is Banach space of dimension greater then two and a linear map $\mho : \mathscr{B}(\zeta) \to \mathscr{B}(\zeta)$ such that $\mho([\varpi_1, \varpi_2]) = [\mho(\varpi_1), \varpi_2] + [\varpi_1, \mho(\varpi_2)]$ for all $\varpi_1, \varpi_2 \in \mathscr{B}(\zeta)$ with $\varpi_1\varpi_2 = 0$ (resp. $\varpi_1\varpi_2 = \mathscr{Y}$, where $\mathscr{Y}$ is a fixed nontrivial idempotent), then there exists an operator $r \in \mathscr{B}(\zeta)$ and a linear map $\phi : \mathscr{B}(\zeta) \to \mathbb{C}I$ vanishes at all the commutators $[\varpi_1, \varpi_2]$ with $\varpi_1\varpi_2 = 0$ (resp. $\varpi_1\varpi_2 = \mathscr{Y}$) such that $\mho(\varpi_1) = r\varpi_1 - \varpi_1 r + \phi(\varpi_1)$ for all $\varpi_1 \in \mathscr{B}(\zeta)$. Motivated by the work of Lu and Jing [2], Ji and Qi in [4] studied the conditions under which Lie derivations can be completely determined by their actions on the triangular algebras. Namely, they proved that under certain restrictions on triangular algebra $\mathscr{T}$ over commutative ring $\mathscr{R}$, if $\mho : \mathscr{T} \to \mathscr{T}$ is an $\mathscr{R}$-linear map such that $\mho([\varpi_1, \varpi_2]) = [\mho(\varpi_1), \varpi_2] + [\varpi_1, \mho(\varpi_2)]$ for all $\varpi_1, \varpi_2 \in \mathscr{T}$ with $\varpi_1\varpi_2 = 0$ (resp. $\varpi_1\varpi_2 = p$, where p is the standard idempotent of $\mathscr{T}$), then there exists a derivation $\delta : \mathscr{T} \to \mathscr{T}$ and an $\mathscr{R}$-linear map $\phi : \mathscr{T} \to Z(\mathscr{T})$ vanishes at all the commutators $[\varpi_1, \varpi_2]$ with $\varpi_1\varpi_2 = 0$ (resp. $\varpi_1\varpi_2 = p$) such that $\mho = \delta + \phi$. In 2013, Ji et al. [3] characterized Lie derivations on factor von Neumann algebra with dimension greater than 4 and obtained the similar conclusion. Furthermore, Qi [5] characterized Lie derivation on $\mathscr{J}$-subspace lattice algebras and proved the same result due to Lu and Jing [2] on $\mathscr{J}$-subspace lattice algebra $Alg\mathscr{L}$, where $\mathscr{L}$ is $\mathscr{J}$-subspace lattice on a Banach space ζ over the real or complex field with dimension greater than 2.

Apart from these, Liu [6] investigated the Lie triple derivation on factor von Neumann algebra with $dim > 1$ and stated that a linear map $\mho : \aleph \to \aleph$ satisfying $\mho([[\varpi_1, \varpi_2], \varpi_3]) = [[\mho(\varpi_1), \varpi_2], \varpi_3] + [[\varpi_1, \mho(\varpi_2)], \varpi_3] + [[\varpi_1, \varpi_2], \mho(\varpi_3)]$ for all $\varpi_1, \varpi_2, \varpi_3 \in \aleph$ with $\varpi_1\varpi_2 = 0$ (resp. $\varpi_1\varpi_2 = \mathscr{Y}$, where $\mathscr{Y}$ is a fixed nontrivial projection of $\aleph$). Then there exist an operator $T \in \aleph$ and a linear map $\gamma : \aleph \to \mathbb{C}I$ annihilates each 2-commutator $\gamma([[\varpi_1, \varpi_2], \varpi_3]) = 0$ with $\varpi_1\varpi_2 = 0$ (resp. $\varpi_1\varpi_2 = \mathscr{Y}$) such that $\mho(\zeta) = \zeta T - T\zeta + \gamma(\zeta)$ for all $\zeta \in \aleph$. Recently, many authors examined Lie $\mathtt{n}$-derivation on various kind of algebras (see [7–10] and references therein). However, so far, there is no known study about of the local actions of Lie type derivations on operator algebras, it needs to be analyzed further. A linear map $\mho : \aleph \to \aleph$ is said to be Lie $\mathtt{n}$-derivable at a given point $Z \in \aleph$ if

$$\mho(\mathscr{P}_{\mathtt{n}}(\zeta_1, \zeta_2, \cdots, \zeta_{\mathtt{n}})) = \sum_{\mathtt{i}=1}^{\mathtt{i}=\mathtt{n}} \mathscr{P}_{\mathtt{n}}(\zeta_1, \zeta_2, \cdots, \zeta_{\mathtt{i}-1}, \mho(\zeta_{\mathtt{i}}), \zeta_{\mathtt{i}+1}, \cdots, \zeta_{\mathtt{n}})$$

for all $\zeta_1, \zeta_2, \cdots, \zeta_{\mathtt{n}} \in \aleph$ with $\zeta_1\zeta_2 = Z$. The condition of being a Lie $\mathtt{n}$-derivable map at some point can easily be seen to be much weaker than the condition of being a Lie $\mathtt{n}$-derivation.

Spurred by using the above cited references, it is very natural to examine Lie type derivation on factor von Neumann algebra of $dim > 1$. In this manuscript, we characterize Lie type derivation on factor von Neumann algebra which has standard form at zero product as well as at projection product.

2. Preliminaries

Across the whole manuscript, let $\mathscr{B}(\mathscr{H})$ be an algebra of all bound linear operators on $\mathscr{H}$, where $\mathscr{H}$ be a complex Hilbert space. Recognize that a von Neumann algebra $\aleph$ acting on $\mathscr{H}$ is a self-adjoint, weakly closed algebra of operators containing an identity operator. A factor von Neumann algebra is a von Neumann algebra whose center contains only scalar operators. The factor-von-Neumann algebra (i.e., the center of $\aleph$ is $\mathbb{C}I$, where I is the identity of $\aleph$) is referred by $\aleph \subseteq \mathscr{B}(\mathscr{H})$. Let $\mathscr{Q}_1$ and $\mathscr{Q}_2$ be two projections in $\aleph$ satisfying $\mathscr{Q}_1 + \mathscr{Q}_2 = I$ and let $\aleph_{ij} = \mathscr{Q}_i \aleph \mathscr{Q}_j,\ 1 \leq i, j \leq 2$. Then $\aleph = \sum_{1\leq i,j\leq 2} \aleph_{ij}$. This signifies $\zeta_{ij} \in \aleph_{ij}, 1 \leq i, j \leq 2$ according to what accepts, whenever we start reading ζ_{ij}. The factor von Neumann algebra $\aleph$ is widely known to be prime (i.e., a lack of nontrivial tensor product decomposition for $\aleph$). Towards the concept of von Neumann algebras, we conclude with recommendations to [11]. We often use the following observation while proving the key result of this manuscript.

Lemma 2.1. *Let $\zeta_{ii} \in \aleph_{ii}, i = 1, 2$. If $\zeta_{11}Y_{12} = Y_{12}\zeta_{22}$ for all $Y_{12} \in \aleph_{12}$, then $\zeta_{11} + \zeta_{22} \in \mathbb{C}I$.*

Proof. As $\aleph$ is prime, then for any $\zeta_{11} \in \aleph_{11},\ Y_{12} \in \aleph_{12}$, we find that $\zeta_{11}Y_{11}Y_{12} = Y_{11}Y_{12}\zeta_{22} = Y_{11}\zeta_{11}Y_{12}$. This leads to $\zeta_{11}Y_{11} = Y_{11}\zeta_{11}$. Clearly, $\aleph_{11}$ is a factor von Neumann algebra on $\mathscr{Q}_1\mathscr{H}$ and hence $\zeta_{11} = \lambda_1\mathscr{Q}_1,\ \lambda_1 \in \mathbb{C}$. In the similar manner, $\zeta_{22} = \lambda_2\mathscr{Q}_2,\ \lambda_2 \in \mathbb{C}$. This implies that $\lambda_1 = \lambda_2$ and then $\zeta_{11} + \zeta_{22} \in \mathbb{C}I$. □

3. Characterization at zero product

During this segment, the characterization of Lie $\mathfrak{n}$-derivation on factor von Neumann algebras at zero product is considered as follows:

Theorem 3.1. *Let $\aleph$ be a factor von Neumann algebra with* dim > 1 *acting on a Hilbert space and a linear map $\mho : \aleph \to \aleph$ satisfying*

$$\mho(\mathscr{P}_n(\zeta_1, \zeta_2, \cdots, \zeta_n)) = \sum_{i=1}^{i=n} \mathscr{P}_n(\zeta_1, \zeta_2, \cdots, \zeta_{i-1}, \mho(\zeta_i), \zeta_{i+1}, \cdots, \zeta_n)$$

for all $\zeta_1, \zeta_2, \cdots, \zeta_n \in \aleph$ with $\zeta_1\zeta_2 = 0$. Then there exist an operator $T \in \aleph$ and a linear map $\gamma : \aleph \to \mathbb{C}I$ that annihilates every $(\mathfrak{n}-1)$-th commutator $\mathscr{P}_n(\zeta_1, \zeta_2, \cdots, \zeta_n)$ with $\zeta_1\zeta_2 = 0$ such that $\mho(\zeta) = \zeta T - T\zeta + \gamma(\zeta)$ for all $\zeta \in \aleph$.

Let $\mathscr{Q}_0 = \mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_2 - \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_1$ and let us define a map $\delta : \aleph \to \aleph$ as an inner derivation $\delta(\zeta) = [\zeta, \mathscr{Q}_0]$ for all $x \in \aleph$. Clearly $\mho' = \mho - \delta$ is also a Lie $\mathfrak{n}$-derivation. Since

$$\begin{aligned}
\mho'(\mathscr{Q}_1) &= \mho(\mathscr{Q}_1) - [\mathscr{Q}_1, \mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_2 - \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_1] \\
&= \mho(\mathscr{Q}_1) - \mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_2 - \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_1 \\
&= \mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_1 + \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_2,
\end{aligned}$$

we get $\mathscr{Q}_1\mho'(\mathscr{Q}_1)\mathscr{Q}_2 = \mathscr{Q}_2\mho'(\mathscr{Q}_1)\mathscr{Q}_1 = 0$. One need only consider these Lie $\mathfrak{n}$-derivation $\mho : \aleph \to \aleph$ that satisfy $\mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_2 = \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_1 = 0$.

Lemma 3.1. $\mho(\mathscr{Q}_1), \mho(\mathscr{Q}_2) \in \mathbb{C}I$.

Proof. Now $\zeta_{12}\mathscr{Q}_1 = 0$ for all $\zeta_{12} \in \aleph_{12}$, then

$$\begin{aligned} \mho(\mathscr{P}_{\mathtt{n}}(\zeta_{12}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1)) &= \mathscr{P}_{\mathtt{n}}(\mho(\zeta_{12}), \mathscr{Q}_1, \cdots, \mathscr{Q}_1) + \sum_{\mathtt{k}=2}^{\mathtt{n}} \mathscr{P}_{\mathtt{n}}(\zeta_{12}, \mathscr{Q}_1, \cdots, \underbrace{\mho(\mathscr{Q}_1)}_{\mathtt{k}th-place}, \cdots, \mathscr{Q}_1) \\ \mho((-1)^{\mathtt{n}-1}\zeta_{12}) &= (-1)^{\mathtt{n}-1}\mathscr{Q}_1\mho(\zeta_{12})\mathscr{Q}_2 + \mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1 + (-1)^{(\mathtt{n}-2)}(\mathtt{n}-1)[\zeta_{12}, \mho(\mathscr{Q}_1)]. \end{aligned} \tag{3.1}$$

Multiplying from the left side $\mathscr{Q}_1$ and from the right side of the aforementioned equation $\mathscr{Q}_2$, we find that $\mathscr{Q}_1\mho(\mathscr{Q}_1)\zeta_{12} = \zeta_{12}\mho(\mathscr{Q}_1)\mathscr{Q}_2$ and by Lemma 2.1, we have $\mho(\mathscr{Q}_1) \in \mathbb{C}I$.

Now using $\mathscr{Q}_2\mathscr{Q}_1 = 0$, it follows that

$$\begin{aligned} 0 &= \mho(\mathscr{P}_{\mathtt{n}}(\mathscr{Q}_2, \mathscr{Q}_1, \cdots, \mathscr{Q}_1)) \\ &= \mathscr{P}_{\mathtt{n}}(\mho(\mathscr{Q}_2), \mathscr{Q}_1, \cdots, \mathscr{Q}_1) + \sum_{\mathtt{k}=2}^{\mathtt{n}} \mathscr{P}_{\mathtt{n}}(\mathscr{Q}_2, \mathscr{Q}_1, \ldots, \underbrace{\mho(\mathscr{Q}_1)}_{\mathtt{k}th-place}, \ldots, \mathscr{Q}_1) \\ &= (-1)^{\mathtt{n}-1}\mathscr{Q}_1\mho(\mathscr{Q}_2)\mathscr{Q}_2 + \mathscr{Q}_2\mho(\mathscr{Q}_2)\mathscr{Q}_1. \end{aligned}$$

This implies that $\mathscr{Q}_1\mho(\mathscr{Q}_2)\mathscr{Q}_2 = \mathscr{Q}_2\mho(\mathscr{Q}_2)\mathscr{Q}_1 = 0$. Also, on using $\mathscr{P}_{\mathtt{n}}(\mathscr{Q}_2, \zeta_{12}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1) = 0$ and applying the similar calculation as above, we get $\mho(\mathscr{Q}_2) \in \mathbb{C}I$. □

Lemma 3.2. $\mho(\aleph_{ij}) \subseteq \aleph_{ij}, 1 \leq i \neq j \leq 2$.

Proof. Now consider the case for $i = 1$ and $j = 2$. On using (3.1) and $\mho(\mathscr{Q}_1) \in \mathbb{C}I$, we have

$$\mho(\zeta_{12}) = \mathscr{Q}_1\mho(\zeta_{12})\mathscr{Q}_2 + (-1)^{\mathtt{n}-1}\mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1.$$

It follows that $\mathscr{Q}_1\mho(\zeta_{12})\mathscr{Q}_1 = \mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_2 = 0$. Also, if $\mathtt{n}$ is even, then $2\mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1 = 0$. But when $\mathtt{n}$ is odd, then for any $\zeta_{12}, Y_{12} \in \aleph_{12}$, we calculate that

$$\begin{aligned} 0 &= \mho(\mathscr{P}_{\mathtt{n}}(\zeta_{12}, Y_{12}, Z_{12}, -\mathscr{Q}_1, \cdots, -\mathscr{Q}_1)) \\ &= \mathscr{P}_{\mathtt{n}}(\mho(\zeta_{12}), Y_{12}, Z_{12}, -\mathscr{Q}_1, \cdots, -\mathscr{Q}_1) + \mathscr{P}_{\mathtt{n}}(\zeta_{12}, \mho(Y_{12}), Z_{12}, -\mathscr{Q}_1, \cdots, -\mathscr{Q}_1) \\ &\quad + \mathscr{P}_{\mathtt{n}}(\zeta_{12}, Y_{12}, \mho(Z_{12}), -\mathscr{Q}_1, \cdots, -\mathscr{Q}_1) \\ &\quad + \sum_{\mathtt{k}=4}^{\mathtt{n}} \mathscr{P}_{\mathtt{n}}(\zeta_{12}, Y_{12}, Z_{12}, -\mathscr{Q}_1, \cdots, \underbrace{\mho(-\mathscr{Q}_1)}_{\mathtt{k}th-place}, \cdots, -\mathscr{Q}_1) \\ &= [[\mho(\zeta_{12}), Y_{12}], Z_{12}] + [[\zeta_{12}, \mho(Y_{12}], Z_{12}]. \end{aligned}$$

This leads to $[\mho(\zeta_{12}), Y_{12}] + [\zeta_{12}, \mho(Y_{12})] = \lambda I \in \mathbb{C}I$. Then

$$\begin{aligned} [\mho(\zeta_{12}), Y_{12}] &= \lambda I - [\zeta_{12}, \mho(Y_{12})] \\ &= \lambda I - \mathscr{P}_{\mathtt{n}}(\zeta_{12}, -\mathscr{Q}_1, \cdots, -\mathscr{Q}_1, \mho(Y_{12})) \\ &= \lambda I + \mho(\mathscr{P}_{\mathtt{n}}(\zeta_{12}, -\mathscr{Q}_1, \cdots, -\mathscr{Q}_1, Y_{12})) - \mathscr{P}_{\mathtt{n}}(\mho(\zeta_{12}), -\mathscr{Q}_1, \cdots, -\mathscr{Q}_1, Y_{12}) \\ &= \lambda I - \mathscr{P}_{\mathtt{n}}(\mho(\zeta_{12}), -\mathscr{Q}_1, \cdots, -\mathscr{Q}_1, Y_{12}) \end{aligned}$$

$$= \lambda I - [\mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1, Y_{12}].$$

This gives $[\mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1, Y_{12}] \in \mathbb{C}I$ and hence $\mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1 Y_{12} = 0$. Since $\aleph$ is prime, we have $\mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1 = 0$. Therefore, $\mho(\aleph_{12}) \subseteq \aleph_{12}$. In the similar manner, we can show that $\mho(\aleph_{21}) \subseteq \aleph_{21}$. □

Lemma 3.3. *There exist linear functionals γ_i on $\aleph_{ii}$ such that $\mho(\zeta_{ii}) - \gamma_i(\zeta_{ii})I \in \aleph_{ii}$ for any $\zeta_{ii} \in \aleph_{ii}$, $i = 1, 2$.*

Proof. Since $\zeta_{11}\mathscr{Q}_2 = 0$ and from Lemma 3.1, we have

$$\begin{aligned} 0 &= \mho(\mathscr{P}_{\mathrm{n}}(\zeta_{11}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2)) \\ &= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}), \mathscr{Q}_2, \cdots, \mathscr{Q}_2) + \sum_{\mathrm{k}=2}^{\mathrm{n}} \mathscr{P}_{\mathrm{n}}(\zeta_{11}, \mathscr{Q}_2, \cdots, \underbrace{\mho(\mathscr{Q}_2)}_{\mathrm{k}th-place}, \cdots, \mathscr{Q}_2) \\ &= \mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_2 + (-1)^{\mathrm{n}-1}\mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_1. \end{aligned}$$

Then $\mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_2 = \mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_1 = 0$. Now for any $\zeta_{22} \in \aleph_{22}$ and $Y_{12} \in \aleph_{12}$, we arrive at

$$\begin{aligned} 0 &= \mho(\mathscr{P}_{\mathrm{n}}(\zeta_{11}, \zeta_{22}, Y_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2)) \\ &= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}), \zeta_{22}, Y_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) + \mathscr{P}_{\mathrm{n}}(\zeta_{11}, \mho(\zeta_{22}), Y_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) \\ &\quad + \mathscr{P}_{\mathrm{n}}(\zeta_{11}, \zeta_{22}, \mho(Y_{12}), \mathscr{Q}_2, \cdots, \mathscr{Q}_2) + \sum_{\mathrm{k}=4}^{\mathrm{n}} \mathscr{P}_{\mathrm{n}}(\zeta_{11}, \zeta_{22}, Y_{12}, \mathscr{Q}_2, \cdots, \underbrace{\mho(\mathscr{Q}_2)}_{\mathrm{k}th-place}, \ldots, \mathscr{Q}_2) \\ &= [[\mho(\zeta_{11}), \zeta_{22}], Y_{12}] + [[\zeta_{11}, \mho(\zeta_{22}], Y_{12}]. \end{aligned}$$

This leads to $[\mho(\zeta_{11}), \zeta_{22}] + [\zeta_{11}, \mho(\zeta_{22})] = \lambda I \in \mathbb{C}I$. By multiplying the above equation by $\mathscr{Q}_2$, on both ends, we conclude that $[\mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2, \zeta_{22}] = \lambda\mathscr{Q}_2$ which leads to $[\mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2, \zeta_{22}] = 0$. Then there exists $\overline{\lambda} \in \mathbb{C}$ such that $\mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2 = \overline{\lambda}\mathscr{Q}_2$ and hence

$$\begin{aligned} \mho(\zeta_{11}) &= \mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_1 + \mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2 \\ &= \mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_1 - \overline{\lambda}\mathscr{Q}_1 + \overline{\lambda}I. \end{aligned}$$

A linear functional one can describe as γ_1 on $\aleph_{11}$ by $\gamma_1(\zeta_{11}) = \overline{\lambda} \in \mathbb{C}$ and combining with the above equation, we have $\mho(\zeta_{11}) - \gamma_1(\zeta_{11})I = \mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_1 - \overline{\lambda}\mathscr{Q}_1 \in \aleph_{11}$ for all $\zeta_{11} \in \aleph_{11}$.

With the similar arguments, we can get a linear functional γ_2 on $\aleph_{22}$ such that $\gamma_2(\zeta_{22}) = \overline{\lambda} \in \mathbb{C}$ and $\mho(\zeta_{22}) - \gamma_2(\zeta_{22})I \in \aleph_{22}$ for all $\zeta_{22} \in \aleph_{22}$. □

Now, we define a linear map $\chi : \aleph \to \aleph$ by $\chi(\zeta) = \mho(\zeta) - \gamma_1(\mathscr{Q}_1\zeta\mathscr{Q}_1)I - \gamma_2(\mathscr{Q}_2\zeta\mathscr{Q}_2)I$ for all $\zeta \in \aleph$. It can be easily seen that $\chi(\mathscr{Q}_i) = 0$, $\chi(\aleph_{ij}) \subseteq \aleph_{ij}, i, j = 1, 2$ and $\chi(\zeta_{ij}) = \mho(\zeta_{ij})$ for all $\zeta_{ij} \in \aleph_{ij}, 1 \le i \ne j \le 2$.

Lemma 3.4. (1) *$\chi(\zeta_{ii}Y_{ij}) = \chi(\zeta_{ii})Y_{ij} + \zeta_{ii}\chi(Y_{ij})$ for any $\zeta_{ii} \in \aleph_{ii}, Y_{ij} \in \aleph_{ij}, 1 \le i \ne j \le 2$.*
(2) *$\chi(\zeta_{ij}Y_{jj}) = \chi(\zeta_{ij})Y_{jj} + \zeta_{ij}\chi(Y_{jj})$ for any $\zeta_{ij} \in \aleph_{ij}, Y_{jj} \in \aleph_{jj}, 1 \le i \ne j \le 2$.*

Proof. (1) Since $Y_{ij}\zeta_{ii} = 0$, $i \ne j$, it follows that

$$\chi(\zeta_{ii}Y_{ij}) = \mho(\zeta_{ii}Y_{ij})$$

$$
\begin{aligned}
&= \mho(\mathscr{P}_{\mathrm{n}}(Y_{ij}, \zeta_{ii}, -\mathscr{Q}_i, \cdots, -\mathscr{Q}_i)) \\
&= \mathscr{P}_{\mathrm{n}}(\mho(Y_{ij}), \zeta_{ii}, -\mathscr{Q}_i, \cdots, -\mathscr{Q}_i) + \mathscr{P}_{\mathrm{n}}(Y_{ij}, \mho(\zeta_{ii}), -\mathscr{Q}_i, \cdots, -\mathscr{Q}_i) \\
&\quad + \sum_{\mathrm{k}=3}^{\mathrm{n}} \mathscr{P}_{\mathrm{n}}(Y_{ij}, \zeta_{ii}, -\mathscr{Q}_i, \cdots, -\underbrace{\mho(\mathscr{Q}_i)}_{\mathrm{k}th-place}, \ldots, -\mathscr{Q}_i) \\
&= \chi(\zeta_{ii})Y_{ij} + \zeta_{ii}\chi(Y_{ij}).
\end{aligned}
$$

(2) Similar to (1). □

Lemma 3.5. *$\chi(\zeta_{ii}Y_{ii}) = \chi(\zeta_{ii})Y_{ii} + \zeta_{ii}\chi(Y_{ii})$ for all $\zeta_{ii}, Y_{ii} \in \aleph_{ii},\ i = 1, 2$.*

Proof. For any $Y_{ij} \in \aleph_{ij}$, we have

$$
\begin{aligned}
\zeta_{ii}Y_{ii}\chi(Y_{ij}) + \chi(\zeta_{ii}Y_{ii})Y_{ij} &= \chi(\zeta_{ii}Y_{ii}Y_{ij}) \\
&= \zeta_{ii}\chi(Y_{ii}Y_{ij}) + \chi(\zeta_{ii}Y_{ii})Y_{ij} \\
&= \zeta_{ii}Y_{ii}\chi(Y_{ij}) + \zeta_{ii}\chi(Y_{ii})Y_{ij} + \chi(\zeta_{ii}Y_{ii})Y_{ij}.
\end{aligned}
$$

It follows that $\chi(\zeta_{ii}Y_{ii})Y_{ij} = \zeta_{ii}\chi(Y_{ii})Y_{ij} + \chi(\zeta_{ii}Y_{ii})Y_{ij}$. Since $\aleph$ is prime, we find that $\chi(\zeta_{ii}Y_{ii}) = \chi(\zeta_{ii})Y_{ii} + \zeta_{ii}\chi(Y_{ii})$ for all $\zeta_{ii}, Y_{ii} \in \aleph_{ii},\ i = 1, 2$. □

Lemma 3.6. *$\chi(\zeta_{ij}Y_{ji}) = \chi(\zeta_{ij})Y_{ji} + \zeta_{ij}\chi(Y_{ji})$ for any $\zeta_{ij} \in \aleph_{ij}, Y_{ji} \in \aleph_{ji}, 1 \le i \ne j \le 2$.*

Proof. For any $\zeta_{12} \in \aleph_{12},\ \zeta_{12}\mathscr{Q}_1 = 0$, then

$$
\begin{aligned}
\mho(\mathscr{P}_{\mathrm{n}}(\zeta_{12}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1, Y_{21})) &= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{12}), \mathscr{Q}_1, \cdots, \mathscr{Q}_1, Y_{21}) + \mathscr{P}_{\mathrm{n}}(Y_{12}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1, \mho(Y_{21})) \\
&\quad + \sum_{\mathrm{k}=2}^{\mathrm{n}-1} \mathscr{P}_{\mathrm{n}}(\zeta_{12}, \mathscr{Q}_1, \cdots, \underbrace{\mho(\mathscr{Q}_1)}_{\mathrm{k}th-place}, \ldots, \mathscr{Q}_1, Y_{21}) \\
&= \mathscr{P}_{\mathrm{n}}(\chi(\zeta_{12}), \mathscr{Q}_1, \cdots, \mathscr{Q}_1, Y_{21}) + \mathscr{P}_{\mathrm{n}}(\zeta_{12}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1, \chi(Y_{21})) \\
\mho(\zeta_{12}Y_{21} - Y_{21}\zeta_{12}) &= \chi(\zeta_{12})Y_{21} + \zeta_{12}\chi(Y_{21}) - \chi(Y_{21})\zeta_{12} - Y_{21}\chi(\zeta_{12}).
\end{aligned}
$$

As $\chi(\zeta) = \mho(\zeta) - \gamma_1(\mathscr{Q}_1\zeta\mathscr{Q}_1)I - \gamma_2(\mathscr{Q}_2\zeta\mathscr{Q}_2)I$ for all $\zeta \in \aleph$. This implies that

$$
\begin{aligned}
&\chi(\zeta_{12}Y_{21} - Y_{21}\zeta_{12}) + \gamma_1(\zeta_{12}Y_{21})I - \gamma_2(Y_{21}\zeta_{12})I \\
&\quad = \chi(\zeta_{12})Y_{21} + \zeta_{12}\chi(Y_{21}) - \chi(Y_{21})\zeta_{12} - Y_{21}\chi(\zeta_{12}).
\end{aligned}
$$

Multiplying the aforementioned equation to the left and right side by ζ_{12} respectively, we get some of that

$$
\zeta_{12}\chi(Y_{21}\zeta_{12}) - \zeta_{12}\gamma_1(\zeta_{12}Y_{21}) + \zeta_{12}\gamma_2(Y_{21}\zeta_{12}) = \zeta_{12}\chi(Y_{21})\zeta_{12} + \zeta_{12}Y_{21}\chi(\zeta_{12}) \tag{3.2}
$$

$$
\chi(\zeta_{12}Y_{21})\zeta_{12} + \gamma_1(\zeta_{12}Y_{21})\zeta_{12} - \gamma_2(Y_{21}\zeta_{12})\zeta_{12} = \chi(\zeta_{12})Y_{21}\zeta_{12} + \zeta_{12}\chi(Y_{21})\zeta_{12}. \tag{3.3}
$$

Even before we comparing these two above expressions, we notice that

$$
\begin{aligned}
&\zeta_{12}\chi(Y_{21}\zeta_{12}) - \zeta_{12}\gamma_1(\zeta_{12}Y_{21}) + \zeta_{12}\gamma_2(Y_{21}\zeta_{12}) - \zeta_{12}Y_{21}\chi(\zeta_{12}) \\
&\quad = \chi(\zeta_{12}Y_{21})\zeta_{12} + \zeta_{12}\gamma_1(\zeta_{12}Y_{21}) - \zeta_{12}\gamma_2(Y_{21}\zeta_{12}) - \chi(\zeta_{12})Y_{21}\zeta_{12}.
\end{aligned} \tag{3.4}
$$

On application of Lemma 3.4, we get

$$\begin{aligned}\zeta_{12}\chi(Y_{21}\zeta_{12})+\chi(\zeta_{12})Y_{21}\zeta_{12} &= \chi(\zeta_{12}Y_{21}\zeta_{12})\\ &= \chi(\zeta_{12}Y_{21})\zeta_{12}+\zeta_{12}\chi(\zeta_{12}Y_{21}).\end{aligned}$$

From (3.4) it follows that $\zeta_{12}\gamma_1(\zeta_{12}Y_{21})-\zeta_{12}\gamma_2(Y_{21}\zeta_{12})=0$ and hence $\gamma_1(\zeta_{12}Y_{21})I-\gamma_2(Y_{21}\zeta_{12})I=0$. This imply to $\chi(\zeta_{12}Y_{21})=\chi(\zeta_{12})Y_{21}+\zeta_{12}\chi(Y_{21})$ and $\chi(Y_{21}\zeta_{12})=\chi(Y_{21})\zeta_{12}+Y_{21}\chi(\zeta_{12})$ for all $\zeta_{12}\in\aleph_{12},\ Y_{21}\in\aleph_{21}$. □

Proof of Theorem 3.1. In view of Lemma 3.4-3.6, it can be easily seen that χ is an additive derivation. Now in order to complete the proof, we define a map $\gamma(\zeta)=\mho(\zeta)-\chi(\zeta)$ for all $\zeta\in\aleph$. It is easy to observe that $\gamma(\zeta_{ii})\in\mathbb{C}I$, for $i=1,2$ and $\gamma(\zeta_{ij})=0$ for $i\neq j$. Clearly, γ is map from $\aleph$ to $\mathbb{C}I$. Also, by [11] we know that every derivation is an inner derivation, then there exists an operator $T\in\aleph$ such that $\chi(\zeta)=\zeta T-T\zeta$ for all $\zeta\in\aleph$.

Now we show that $\gamma(\mathscr{P}_{\mathrm{n}}(\zeta_1,\zeta_2,\ldots,\zeta_{\mathrm{n}}))=0$ for all $\zeta_1,\zeta_2,\ldots,\zeta_{\mathrm{n}}\in\aleph$.

$$\begin{aligned}\gamma(\mathscr{P}_{\mathrm{n}}(\zeta_1,\zeta_2,\ldots,\zeta_{\mathrm{n}})) &= \mho(\mathscr{P}_{\mathrm{n}}(\zeta_1,\zeta_2,\ldots,\zeta_{\mathrm{n}}))-\chi(\mathscr{P}_{\mathrm{n}}(\zeta_1,\zeta_2,\ldots,\zeta_{\mathrm{n}}))\\ &= \sum_{\mathrm{k}=1}^{\mathrm{n}}\mathscr{P}_{\mathrm{n}}(\zeta_1,\ldots,\mho(\zeta_{\mathrm{k}}),\ldots,\zeta_{\mathrm{n}}))-\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathscr{P}_{\mathrm{n}}(\zeta_1,\ldots,\chi(\zeta_{\mathrm{k}}),\ldots,\zeta_{\mathrm{n}}))\\ &= \sum_{\mathrm{k}=1}^{\mathrm{n}}\mathscr{P}_{\mathrm{n}}(\zeta_1,\ldots,\chi(\zeta_{\mathrm{k}}),\ldots,\zeta_{\mathrm{n}})-\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathscr{P}_{\mathrm{n}}(\zeta_1,\ldots,\chi(\zeta_{\mathrm{k}}),\ldots,\zeta_{\mathrm{n}})\\ &= 0.\end{aligned}$$

We can draw the conclusion according to the above observations if $\mho:\aleph\to\aleph$ is a Lie $\mathtt{n}$-derivation, then there exists an additive derivation χ of $\aleph$ and a map $\gamma:\aleph\to\mathbb{C}I$ vanishing at $\mathscr{P}_{\mathrm{n}}(\zeta_1,\zeta_2,\ldots,\zeta_{\mathrm{n}})$ with $\zeta_1\zeta_2=0$ for all $\zeta_1,\zeta_2,\ldots,\zeta_{\mathrm{n}}\in\aleph$ such that $\mho=\chi+\gamma$. □

4. Characterization at projection product

This segment is devoted to the analysis of a characterization by action of the nontrivial projection product of Lie $\mathtt{n}$-derivations on factor von Neumann algebras and demonstrates the following observations:

Theorem 4.1. *Let $\mho:\aleph\to\aleph$ be a linear map such that*

$$\mho(\mathscr{P}_n(\zeta_1,\zeta_2,\cdots,\zeta_n))=\sum_{i=1}^{i=n}\mathscr{P}_n(\zeta_1,\zeta_2,\cdots,\zeta_{i-1},\mho(\zeta_i),\zeta_{i+1},\cdots,\zeta_n),$$

where $\aleph$ is a factor von Neumann algebra with dim >1 *acting on a Hilbert space and for every $\zeta_1,\zeta_2,\cdots,\zeta_n\in\aleph$ with $\zeta_1\zeta_2=\mathscr{Q}_1$, $\mathscr{Q}_1\in\aleph$ a fixed nontrivial projection. Then there exist an operator $T\in\aleph$ and a linear map $\gamma:\aleph\to\mathbb{C}I$ that annihilates every $(\mathtt{n}-1)$-th commutator $\mathscr{P}_n(\zeta_1,\zeta_2,\cdots,\zeta_n)$ with $\zeta_1\zeta_2=\mathscr{Q}_1$ such that $\mho(\zeta)=\zeta T-T\zeta+\gamma(\zeta)$ for every $\zeta\in\aleph$.*

Let $\mathscr{Q}_0 = \mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_2 - \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_1$ and let us define $\delta : \aleph \to \aleph$ as an inner derivation $\delta(\zeta) = [\zeta, \mathscr{Q}_0]$ for all $\zeta \in \aleph$. Clearly $\mho' = \mho - \delta$ is also a Lie $\mathtt{n}$-derivation. Therefore we've got

$$\begin{aligned} \mho'(\mathscr{Q}_1) &= \mho(\mathscr{Q}_1) - [\mathscr{Q}_1, \mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_2 - \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_1] \\ &= \mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_1 - \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_2 \end{aligned}$$

to get $\mathscr{Q}_1\mho'(\mathscr{Q}_1)\mathscr{Q}_2 = \mathscr{Q}_2\mho'(\mathscr{Q}_1)\mathscr{Q}_1 = 0$. It's indeed reasonable, therefore, to recognize only Lie $\mathtt{n}$-derivation $\mho : \aleph \to \aleph$ satisfy $\mathscr{Q}_1\mho(\mathscr{Q}_1)\mathscr{Q}_2 = \mathscr{Q}_2\mho(\mathscr{Q}_1)\mathscr{Q}_1 = 0$.

Lemma 4.1. $\mho(\mathscr{Q}_1), \mho(\mathscr{Q}_2) \in \mathbb{C}I$.

Proof. Now we know that $(\zeta_{12} + \mathscr{Q}_1)\mathscr{Q}_1 = \mathscr{Q}_1$ for all $\zeta_{12} \in \aleph_{12}$, then

$$\begin{aligned} &\mho(\mathscr{P}_{\mathtt{n}}(\zeta_{12} + \mathscr{Q}_1, \mathscr{Q}_1, \cdots, \mathscr{Q}_1)) \\ &= \mathscr{P}_{\mathtt{n}}(\mho(\zeta_{12} + \mathscr{Q}_1), \mathscr{Q}_1, \cdots, \mathscr{Q}_1) + \sum_{\mathtt{k}=2}^{\mathtt{n}} \mathscr{P}_{\mathtt{n}}(\zeta_{12} + \mathscr{Q}_1, \mathscr{Q}_1, \cdots, \underbrace{\mho(\mathscr{Q}_1)}_{\mathtt{k}th-place}, \cdots, \mathscr{Q}_1) \\ \mho((-1)^{\mathtt{n}-1}\zeta_{12}) &= (-1)^{\mathtt{n}-1}\mathscr{Q}_1\mho(\zeta_{12})\mathscr{Q}_2 + \mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1 + (-1)^{(\mathtt{n}-2)}(\mathtt{n}-1)[\zeta_{12}, \mho(\mathscr{Q}_1)]. \end{aligned} \tag{4.1}$$

We achieve $\mathscr{Q}_1\mho(\mathscr{Q}_1)\zeta_{12} = \zeta_{12}\mho(\mathscr{Q}_1)\mathscr{Q}_2$ upon multiplying the Eq (4.1) by $\mathscr{Q}_1$ from the left side and $\mathscr{Q}_2$ from the right side. Also, by Lemma 2.1, we get $\mho(\mathscr{Q}_1) \in \mathbb{C}I$. Further, by using $(\mathscr{Q}_2 + \mathscr{Q}_1)\mathscr{Q}_1 = \mathscr{Q}_1$ we obtain

$$\begin{aligned} 0 &= \mho(\mathscr{P}_{\mathtt{n}}(\mathscr{Q}_2 + \mathscr{Q}_1, \mathscr{Q}_1, \cdots, \mathscr{Q}_1)) \\ &= \mathscr{P}_{\mathtt{n}}(\mho(\mathscr{Q}_2 + \mathscr{Q}_1), \mathscr{Q}_1, \cdots, \mathscr{Q}_1) + \sum_{\mathtt{k}=2}^{\mathtt{n}} \mathscr{P}_{\mathtt{n}}(\mathscr{Q}_2 + \mathscr{Q}_1, \mathscr{Q}_1, \cdots, \underbrace{\mho(\mathscr{Q}_1)}_{\mathtt{k}th-place}, \cdots, \mathscr{Q}_1) \\ &= (-1)^{\mathtt{n}-1}\mathscr{Q}_1\mho(\mathscr{Q}_2)\mathscr{Q}_2 + \mathscr{Q}_2\mho(\mathscr{Q}_2)\mathscr{Q}_1. \end{aligned}$$

This implies that $\mathscr{Q}_1\mho(\mathscr{Q}_2)\mathscr{Q}_2 = \mathscr{Q}_2\mho(\mathscr{Q}_2)\mathscr{Q}_1 = 0$. Also, using

$$\mathscr{P}_{\mathtt{n}}(\mathscr{Q}_1 + \zeta_{12}, \mathscr{Q}_2 + \mathscr{Q}_1 - \zeta_{12}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1) = 0$$

and applying the similar calculation as above, we get $\mho(\mathscr{Q}_2) \in \mathbb{C}I$. □

Lemma 4.2. $\mho(\aleph_{ij}) \subseteq \aleph_{ij}, 1 \leq i \neq j \leq 2$.

Proof. Taking into account the situation for $i = 1$ and $j = 2$, applying (4.1) and $\mho(\mathscr{Q}_1) \in \mathbb{C}I$, we have

$$\mho(\zeta_{12}) = \mathscr{Q}_1\mho(\zeta_{12})\mathscr{Q}_2 + (-1)^{\mathtt{n}-1}\mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1.$$

It follows that $\mathscr{Q}_1\mho(\zeta_{12})\mathscr{Q}_1 = \mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_2 = 0$. Also, if $\mathtt{n}$ is even, then $2\mathscr{Q}_2\mho(\zeta_{12})\mathscr{Q}_1 = 0$. But when $\mathtt{n}$ is odd, then for any $\zeta_{12} \in \aleph_{12}$, we calculate that

$$\begin{aligned} 0 &= \mho(\mathscr{P}_{\mathtt{n}}(\mathscr{Q}_1 + \zeta_{12}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1, \zeta_{12})) \\ &= \mathscr{P}_{\mathtt{n}}(\mho(\mathscr{Q}_1 + \zeta_{12}), \mathscr{Q}_1, \cdots, \mathscr{Q}_1, \zeta_{12}) + \sum_{\mathtt{k}=2}^{\mathtt{n}-1} \mathscr{P}_{\mathtt{n}}(\mathscr{Q}_1 + \zeta_{12}, \mathscr{Q}_1, \cdots, \underbrace{\mho(\mathscr{Q}_1)}_{\mathtt{k}th-place}, \cdots, \mathscr{Q}_1, \zeta_{12}) \end{aligned}$$

$$
\begin{aligned}
&+\mathscr{P}_{\mathrm{n}}(\mathscr{Q}_1+\zeta_{12},\mathscr{Q}_1,\cdots,\mathscr{Q}_1,\mho(\zeta_{12}))\\
=\;&\mathscr{Q}_2\mho(\zeta_{12})\zeta_{12}-\zeta_{12}\mho(\zeta_{12})\mathscr{Q}_1+\mho(\zeta_{12})\zeta_{12}-\zeta_{12}\mho(\zeta_{12}).
\end{aligned}
$$

Multiplying both sides by $\mathscr{Q}_2$, we obtain that $\mathscr{Q}_2\mho(\zeta_{12})\zeta_{12}=0$. Moreover, we have

$$
\begin{aligned}
0 &= \mho(\mathscr{P}_{\mathrm{n}}(\mathscr{Q}_1+\zeta_{12},\mathscr{Q}_1,\cdots,\mathscr{Q}_1,Y_{12}))\\
&= \mathscr{P}_{\mathrm{n}}(\mho(\mathscr{Q}_1+\zeta_{12}),\mathscr{Q}_1,\cdots,\mathscr{Q}_1,Y_{12})+\sum_{\mathrm{k}=2}^{\mathrm{n}-1}\mathscr{P}_{\mathrm{n}}(\mathscr{Q}_1+\zeta_{12},\mathscr{Q}_1,\cdots,\underbrace{\mho(\mathscr{Q}_1)}_{\mathrm{k}th-place},\cdots,\mathscr{Q}_1,Y_{12})\\
&\quad+\mathscr{P}_{\mathrm{n}}(\mathscr{Q}_1+\zeta_{12},\mathscr{Q}_1,\cdots,\mathscr{Q}_1,\mho(Y_{12}))\\
&= \mathscr{Q}_2\mho(\zeta_{12})Y_{12}-Y_{12}\mho(\zeta_{12})\mathscr{Q}_1+\mho(Y_{12})\zeta_{12}-\zeta_{12}\mho(Y_{12}).
\end{aligned}
$$

Multiplying by ζ_{12} from right side and using the fact $\mathscr{Q}_2\mho(\zeta_{12})\zeta_{12}=0$, we obtain that $\zeta_{12}\mho(Y_{12})\zeta_{12}=0$. On linearization we find $\zeta_{12}\mho(Y_{12})\varsigma_{12}+\varsigma_{12}\mho(Y_{12})\zeta_{12}=0$ for all $\varsigma_{12},\zeta_{12}\in\aleph$. It is easy to observe that

$$
\begin{aligned}
0 &= \mathscr{Q}_2\mho(Y_{12})\zeta_{12}\mho(Y_{12})[\zeta_{12}\mho(Y_{12})\varsigma_{12}]\mho(Y_{12})\mathscr{Q}_1\\
&\quad+\mathscr{Q}_2\mho(Y_{12})\zeta_{12}\mho(Y_{12})[\varsigma_{12}\mho(Y_{12})\zeta_{12}]\mho(Y_{12})\mathscr{Q}_1\\
&= \mathscr{Q}_2\mho(Y_{12})\zeta_{12}\mho(Y_{12})\varsigma_{12}\mho(Y_{12})\zeta_{12}\mho(Y_{12})\mathscr{Q}_1.
\end{aligned}
$$

Since $\aleph$ is prime, we have $\mathscr{Q}_2\mho(Y_{12})\zeta_{12}\mho(Y_{12})\mathscr{Q}_1=0$, and hence $\mathscr{Q}_2\mho(Y_{12})\mathscr{Q}_1=0$ for all $Y_{12}\in\aleph$. Therefore, $\mho(\aleph_{12})\subseteq\aleph_{12}$. In the similar manner, we can show that $\mho(\aleph_{21})\subseteq\aleph_{21}$. □

Lemma 4.3. *There exist linear functionals γ_i on $\aleph_{ii}$ such that $\mho(\zeta_{ii})-\gamma_i(\zeta_{ii})I\in\aleph_{ii}$ for any $\zeta_{ii}\in\aleph_{ii}$, $i=1,2$.*

Proof. Consider for $i=1$. Suppose that ζ_{11} is invertible in $\aleph_{11}$, then there exists $\zeta_{11}^{-1}\in\aleph_{11}$ such that $\zeta_{11}\zeta_{11}^{-1}=\zeta_{11}^{-1}\zeta_{11}=\mathscr{Q}_1$. Now we have

$$
\begin{aligned}
0 &= \mho(\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1},\zeta_{11},\mathscr{Q}_1,\cdots,\mathscr{Q}_1))\\
&= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}^{-1}),\zeta_{11},\mathscr{Q}_1,\cdots,\mathscr{Q}_1)+\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1},\mho(\zeta_{11}),\mathscr{Q}_1,\cdots,\mathscr{Q}_1)\\
&\quad+\sum_{\mathrm{k}=3}^{\mathrm{n}}\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1},\zeta_{11},\mathscr{Q}_1,\cdots,\underbrace{\mho(\mathscr{Q}_1)}_{\mathrm{k}th-place},\cdots,\mathscr{Q}_1)\\
&= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}^{-1}),\zeta_{11},\mathscr{Q}_1,\cdots,\mathscr{Q}_1)+\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1},\mho(\zeta_{11}),\mathscr{Q}_1,\cdots,\mathscr{Q}_1).
\end{aligned}
$$

It follows from $(\zeta_{11}^{-1}+\mathscr{Q}_2)\zeta_{11}=\mathscr{Q}_1$

$$
\begin{aligned}
0 &= \mho(\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}+\mathscr{Q}_2,\zeta_{11},\mathscr{Q}_1,\cdots,\mathscr{Q}_1))\\
&= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}^{-1}+\mathscr{Q}_2),\zeta_{11},\mathscr{Q}_1,\cdots,\mathscr{Q}_1)+\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}+\mathscr{Q}_2,\mho(\zeta_{11}),\mathscr{Q}_1,\cdots,\mathscr{Q}_1)\\
&\quad+\sum_{\mathrm{k}=3}^{\mathrm{n}}\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}+\mathscr{Q}_2,\zeta_{11},\mathscr{Q}_1,\cdots,\underbrace{\mho(\mathscr{Q}_1)}_{\mathrm{k}th-place},\cdots,\mathscr{Q}_1)\\
&= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}^{-1}+\mathscr{Q}_2),\zeta_{11},\mathscr{Q}_1,\cdots,\mathscr{Q}_1)+\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}+\mathscr{Q}_2,\mho(\zeta_{11}),\mathscr{Q}_1,\cdots,\mathscr{Q}_1)\\
&= \mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_1+(-1)^{\mathrm{n}-1}\mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_2.
\end{aligned}
$$

For any $Y_{22} \in \aleph_{22}$ and $Z_{12} \in \aleph_{12}$, since $(\zeta_{11}^{-1} + Y_{22})\zeta_{11} = \mathscr{Q}_1$, it is easy to observe that

$$\begin{aligned}
0 &= \mho(\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1} + Y_{22}, \zeta_{11}, Z_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2)) \\
&= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}^{-1} + Y_{22}), \zeta_{11}, Z_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) + \mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1} + Y_{22}, \mho(\zeta_{11}), Z_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) \\
&\quad + \sum_{\mathrm{k}=3}^{\mathrm{n}} \mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1} + Y_{22}, \zeta_{11}, Z_{12}, \mathscr{Q}_2, \cdots, \underbrace{\mho(\mathscr{Q}_2)}_{\mathrm{k}th-place}, \cdots, \mathscr{Q}_2) \\
&= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}^{-1} + Y_{22}), \zeta_{11}, Z_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) + \mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1} + Y_{22}, \mho(\zeta_{11}), Z_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) \\
&= \mathscr{P}_{\mathrm{n\text{-}1}}([\mho(Y_{22}), \zeta_{11}], Z_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) + \mathscr{P}_{\mathrm{n\text{-}1}}([Y_{22}, \mho(\zeta_{11})], Z_{12}, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) \\
&= [[\mho(Y_{22}), \zeta_{11}], Z_{12}] + [[Y_{22}, \mho(\zeta_{11})], Z_{12}].
\end{aligned}$$

This leads to $[\mathscr{Q}_1\mho(Y_{22})\mathscr{Q}_1, \zeta_{11}] + [Y_{22}, \mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2] = \lambda I \in \mathbb{C}I$. We achieve the following upon multiplying the equation drive above by $\mathscr{Q}_2$ on both sides, $[Y_{22}, \mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2] = \lambda\mathscr{Q}_2$ and hence $[Y_{22}, \mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2] = 0$. Then there exists $\overline{\lambda} \in \mathbb{C}$ such that $\mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2 = \overline{\lambda}\mathscr{Q}_2$.

Suppose if ζ_{11} is not invertible in $\aleph_{11}$. Let r be a real constant satisfying $r > \|\zeta_{11}\|$. Then $r\mathscr{Q}_1 - \zeta_{11}$ is invertible in $\aleph_{11}$. Following the preceding case $\mathscr{Q}_1\mho(r\mathscr{Q}_1 - \zeta_{11})\mathscr{Q}_2 + \mathscr{Q}_2\mho(r\mathscr{Q}_1 - \zeta_{11})\mathscr{Q}_1 = 0$ and $\mathscr{Q}_2\mho(r\mathscr{Q}_1 - \zeta_{11})\mathscr{Q}_2 = \tilde{\lambda}\mathscr{Q}_2$. Since $\mho(\mathscr{Q}_1) = \overline{\mu}I$, we also have $\mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_2 + \mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_1 = 0$ and $\mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2 = \overline{\lambda}\mathscr{Q}_2$, where $\overline{\lambda} = r\mu - \tilde{\lambda}$. Without loss of generality, we denote $\mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2 = \tilde{\lambda}\mathscr{Q}_2$. Thus for any $\zeta_{11} \in \aleph_{11}$, we have $\mho(\zeta_{11}) = \mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_1 + \mathscr{Q}_2\mho(\zeta_{11})\mathscr{Q}_2 = \mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_1 - \tilde{\lambda}\mathscr{Q}_1 + \tilde{\lambda}I$.

We define a linear functional γ_1 on $\aleph_{11}$ by $\gamma_1(\zeta_{11}) = \tilde{\lambda}$. Then combining with the above equation, we get $\mho(\zeta_{11}) - \gamma_1(\zeta_{11})I = \mathscr{Q}_1\mho(\zeta_{11})\mathscr{Q}_1 - \tilde{\lambda}\mathscr{Q}_1 \in \aleph_{11}$ for any $\zeta_{11} \in \aleph_{11}$.

For $i = 2$, we consider $(\mathscr{Q}_1 + Y_{22})\mathscr{Q}_1 = \mathscr{Q}_1$ to get $\mathscr{Q}_2\mho(Y_{22})\mathscr{Q}_1 + (-1)^{\mathrm{n}-1}\mathscr{Q}_1\mho(Y_{22})\mathscr{Q}_2 = 0$ and then follow the similar steps as for $i = 1$. Hence $\mho(Y_{22}) - \gamma_2(Y_{22})I = \mathscr{Q}_2\mho(Y_{22})\mathscr{Q}_2 - \tilde{\lambda}\mathscr{Q}_2 \in \aleph_{22}$ for any $Y_{22} \in \aleph_{22}$. □

Now, we define a linear map $\chi : \aleph \to \aleph$ by $\chi(\zeta) = \mho(\zeta) - \gamma_1(\mathscr{Q}_1\zeta\mathscr{Q}_1)I - \gamma_2(\mathscr{Q}_2\zeta\mathscr{Q}_2)I$ for all $\zeta \in \aleph$. This would easily observed that $\chi(\mathscr{Q}_i) = 0$, $\chi(\aleph_{ij}) \subseteq \aleph_{ij}, i, j = 1, 2$, and $\chi(\zeta_{ij}) = \mho(\zeta_{ij})$ for any $\zeta_{ij} \in \aleph_{ij}, 1 \leq i \neq j \leq 2$.

Lemma 4.4. (1) $\chi(\zeta_{ii}Y_{ij}) = \chi(\zeta_{ii})Y_{ij} + \zeta_{ii}\chi(Y_{ij})$ *for any* $\zeta_{ii} \in \aleph_{ii}, Y_{ij} \in \aleph_{ij}, 1 \leq i \neq j \leq 2$.
(2) $\chi(\zeta_{ij}Y_{jj}) = \chi(\zeta_{ij})Y_{jj} + \zeta_{ij}\chi(Y_{jj})$ *for any* $\zeta_{ij} \in \aleph_{ij}, Y_{jj} \in \aleph_{jj}, 1 \leq i \neq j \leq 2$.

Proof. (1) Firstly, we discuss for $i = 1, j = 2$. If ζ_{11} is invertible in $\aleph_{11}$, then for any $Z_{12} \in \aleph_{12}$, we have $(\zeta_{11}^{-1}Z_{12} + \zeta_{11}^{-1})\zeta_{11} = \mathscr{Q}_1$. It follows that

$$\begin{aligned}
\chi(Z_{12}) &= \mho(\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}Z_{12} + \zeta_{11}^{-1}, \zeta_{11}, \mathscr{Q}_1, \mathscr{Q}_2, \cdots, \mathscr{Q}_2)) \\
&= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{11}^{-1}Z_{12} + \zeta_{11}^{-1}), \zeta_{11}, \mathscr{Q}_1, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) \\
&\quad + \mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}Z_{12} + \zeta_{11}^{-1}, \mho(\zeta_{11}), \mathscr{Q}_1, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) \\
&\quad + \mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}Z_{12} + \zeta_{11}^{-1}, \zeta_{11}, \mho(\mathscr{Q}_1), \mathscr{Q}_2, \cdots, \mathscr{Q}_2) \\
&\quad + \sum_{\mathrm{k}=4}^{\mathrm{n}} \mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}Z_{12} + \zeta_{11}^{-1}, \zeta_{11}, \mathscr{Q}_1, \mathscr{Q}_2, \cdots, \underbrace{\mho(\mathscr{Q}_2)}_{\mathrm{k}th-place}, \cdots, \mathscr{Q}_2) \\
&= \mathscr{P}_{\mathrm{n}}(\chi(\zeta_{11}^{-1}Z_{12} + \zeta_{11}^{-1}), \zeta_{11}, \mathscr{Q}_1, \mathscr{Q}_2, \cdots, \mathscr{Q}_2)
\end{aligned}$$

$$
\begin{aligned}
&+\mathscr{P}_{\mathrm{n}}(\zeta_{11}^{-1}Z_{12} + \zeta_{11}^{-1}, \chi(\zeta_{11}), \mathscr{Q}_1, \mathscr{Q}_2, \cdots, \mathscr{Q}_2) \\
= \ & \chi(\zeta_{11})\zeta_{11}^{-1}Z_{12} + \zeta_{11}\chi(\zeta_{11}^{-1}Z_{12}).
\end{aligned}
$$

Replacing Y_{12} with $\zeta_{11}^{-1}Z_{12}$, we have $\chi(\zeta_{11}Y_{12}) = \chi(\zeta_{11})Y_{12} + \zeta_{11}\chi(Y_{12})$. Suppose if ζ_{11} is not invertible in $\aleph_{11}$. Let r be a real constant satisfying $r > \|\zeta_{11}\|$. Then $r\mathscr{Q}_1 - \zeta_{11}$ is invertible in $\aleph_{11}$. Then $\chi((r\mathscr{Q}_1 - \zeta_{11})Y_{12}) = (r\mathscr{Q}_1 - \zeta_{11})\chi(Y_{12}) + \chi(r\mathscr{Q}_1 - \zeta_{11})Y_{12}$. Clearly, $\mathscr{Q}_1$ is invertible in $\aleph_{11}$, so we get $\chi(\zeta_{11}Y_{12}) = \chi(\zeta_{11})Y_{12} + \zeta_{11}\chi(Y_{12})$ from the above equation.

For $i = 2,\ j = 1$, consider $(\mathscr{Q}_1 + \zeta_{22} - \zeta_{22}Z_{21})(\mathscr{Q}_1 + Z_{21}) = \mathscr{Q}_1$, we have

$$
\begin{aligned}
-\chi(Y_{21}) &= \mho(\mathscr{P}_{\mathrm{n}}(\mathscr{Q}_1 + \zeta_{22} - \zeta_{22}Y_{21}, \mathscr{Q}_1 + Y_{21}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1)) \\
&= \mathscr{P}_{\mathrm{n}}(\mho(\mathscr{Q}_1 + \zeta_{22} - \zeta_{22}Y_{21}), \mathscr{Q}_1 + Y_{21}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1) \\
&\quad + \mathscr{P}_{\mathrm{n}}(\mathscr{Q}_1 + \zeta_{22} - \zeta_{22}Y_{21}, \mho(\mathscr{Q}_1 + Y_{21}), \mathscr{Q}_1, \cdots, \mathscr{Q}_1) \\
&\quad + \sum_{\mathrm{k}=3}^{\mathrm{n}} \mathscr{P}_{\mathrm{n}}(\mathscr{Q}_1 + \zeta_{22} - \zeta_{22}Y_{21}, \mathscr{Q}_1 + Y_{21}, \mathscr{Q}_1, \cdots, \underbrace{\mho(\mathscr{Q}_1)}_{\mathrm{k}th-place}, \cdots, \mathscr{Q}_1) \\
&= \mathscr{P}_{\mathrm{n}}(\chi(\mathscr{Q}_1 + \zeta_{22} - \zeta_{22}Y_{21}), \mathscr{Q}_1 + Y_{21}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1) \\
&\quad + \mathscr{P}_{\mathrm{n}}(\mathscr{Q}_1 + \zeta_{22} - \zeta_{22}Y_{21}, \chi(\mathscr{Q}_1 + Y_{21}), \mathscr{Q}_1, \cdots, \mathscr{Q}_1) \\
&= -\chi(\zeta_{22}Y_{21}) + \chi(\zeta_{22})Y_{21} + \zeta_{22}\chi(Y_{21}) - \chi(Y_{21}).
\end{aligned}
$$

This implies that $\chi(\zeta_{22}Y_{21}) = \chi(\zeta_{22})Y_{21} + \zeta_{22}\chi(Y_{21})$ for all $\zeta_{22} \in \aleph_{22}$ and $Y_{21} \in \aleph_{21}$.
(2) For $i = 1,\ j = 2$. Considering $(\mathscr{Q}_1 + \zeta_{12})(\mathscr{Q}_1 - Y_{22} + \zeta_{12}Y_{22}) = \mathscr{Q}_1$ and using the same approach as above, we obtain that $\chi(\zeta_{12}Y_{22}) = \chi(\zeta_{12})Y_{22} + \zeta_{12}\chi(Y_{22})$ for all $\zeta_{12} \in \aleph_{12}$ and $Y_{22} \in \aleph_{22}$.

For $i = 2,\ j = 1$. Considering $\zeta_{11}(Z_{21}\zeta_{11}^{-1} + \zeta_{11}^{-1}) = \mathscr{Q}_1$, we can prove that $\chi(\zeta_{21}Y_{11}) = \chi(\zeta_{21})Y_{11} + \zeta_{21}\chi(Y_{11})$ for all $\zeta_{21} \in \aleph_{21}$ and $Y_{11} \in \aleph_{11}$. □

Lemma 4.5. $\chi(\zeta_{ii}Y_{ii}) = \chi(\zeta_{ii})Y_{ii} + \zeta_{ii}\chi(Y_{ii})$ *for any* $\zeta_{ii}, Y_{ii} \in \aleph_{ii},\ i = 1, 2.$

Proof. Same as proof of Lemma 3.5. □

Lemma 4.6. $\chi(\zeta_{ij}Y_{ji}) = \chi(\zeta_{ij})Y_{ji} + \zeta_{ij}\chi(Y_{ji})$ *for any* $\zeta_{ij} \in \aleph_{ij}, Y_{ji} \in \aleph_{ji}, 1 \le i \ne j \le 2.$

Proof. For any $\zeta_{12} \in \aleph_{12},\ (\zeta_{12} + \mathscr{Q}_1)\mathscr{Q}_1 = \mathscr{Q}_1$, then

$$
\begin{aligned}
\mho(\mathscr{P}_{\mathrm{n}}(\zeta_{12} + \mathscr{Q}_1, \mathscr{Q}_1, \cdots, \mathscr{Q}_1, Y_{21})) &= \mathscr{P}_{\mathrm{n}}(\mho(\zeta_{12} + \mathscr{Q}_1), \mathscr{Q}_1, \cdots, \mathscr{Q}_1, Y_{21}) \\
&\quad + \mathscr{P}_{\mathrm{n}}(\zeta_{12} + \mathscr{Q}_1, \mathscr{Q}_1, \cdots, \mathscr{Q}_1, \mho(Y_{21})) \\
&\quad + \sum_{\mathrm{k}=2}^{\mathrm{n-1}} \mathscr{P}_{\mathrm{n}}(\zeta_{12} + \mathscr{Q}_1, \mathscr{Q}_1, \cdots, \underbrace{\mho(\mathscr{Q}_1)}_{\mathrm{k}th-place}, \cdots, \mathscr{Q}_1, Y_{21}) \\
&= \mathscr{P}_{\mathrm{n}}(\chi(\zeta_{12}), \mathscr{Q}_1, \cdots, \mathscr{Q}_1, Y_{21}) \\
&\quad + \mathscr{P}_{\mathrm{n}}(\zeta_{12}, \mathscr{Q}_1, \cdots, \mathscr{Q}_1, \chi(Y_{21})) \\
\mho(\zeta_{12}Y_{21} - Y_{21}\zeta_{12}) &= \chi(\zeta_{12})Y_{21} + \zeta_{12}\chi(Y_{21}) - \chi(Y_{21})\zeta_{12} - Y_{21}\chi(\zeta_{12}).
\end{aligned}
$$

As $\chi(\zeta) = \mho(\zeta) - \gamma_1(\mathscr{Q}_1\zeta\mathscr{Q}_1)I - \gamma_2(\mathscr{Q}_2\zeta\mathscr{Q}_2)I$ for all $\zeta \in \aleph$. This implies that

$$
\chi(\zeta_{12}Y_{21} - Y_{21}\zeta_{12}) + \gamma_1(\zeta_{12}Y_{21})I - \gamma_2(Y_{21}\zeta_{12})I
$$

$$= \chi(\zeta_{12})Y_{21} + \zeta_{12}\chi(Y_{21}) - \chi(Y_{21})\zeta_{12} - Y_{21}\chi(\zeta_{12}).$$

Multiply the above mentioned equation by ζ_{12} from both side, we notice that

$$\zeta_{12}\chi(Y_{21}\zeta_{12}) - \zeta_{12}\gamma_1(\zeta_{12}Y_{21}) + \zeta_{12}\gamma_2(Y_{21}\zeta_{12}) = \zeta_{12}\chi(Y_{21})\zeta_{12} + \zeta_{12}Y_{21}\chi(\zeta_{12}), \quad (4.2)$$
$$\chi(\zeta_{12}Y_{21})\zeta_{12} + \gamma_1(\zeta_{12}Y_{21})\zeta_{12} - \gamma_2(Y_{21}\zeta_{12})\zeta_{12} = \chi(\zeta_{12})Y_{21}\zeta_{12} + \zeta_{12}\chi(Y_{21})\zeta_{12}. \quad (4.3)$$

By analyzing the two expressions described above, we notice that

$$\begin{aligned} &\zeta_{12}\chi(Y_{21}\zeta_{12}) - \zeta_{12}\gamma_1(\zeta_{12}Y_{21}) + \zeta_{12}\gamma_2(Y_{21}\zeta_{12}) - \zeta_{12}Y_{21}\chi(\zeta_{12}) \\ &= \chi(\zeta_{12}Y_{21})\zeta_{12} + \zeta_{12}\gamma_1(\zeta_{12}Y_{21}) - \zeta_{12}\gamma_2(Y_{21}\zeta_{12}) - \chi(\zeta_{12})Y_{21}\zeta_{12}. \end{aligned} \quad (4.4)$$

On application of Lemma 4.4, we get

$$\begin{aligned} \zeta_{12}\chi(Y_{21}\zeta_{12}) + \chi(\zeta_{12})Y_{21}\zeta_{12} &= \chi(\zeta_{12}Y_{21}\zeta_{12}) \\ &= \chi(\zeta_{12}Y_{21})\zeta_{12} + \zeta_{12}\chi(\zeta_{12}Y_{21}). \end{aligned}$$

From (4.4) it follows that

$$\zeta_{12}\gamma_1(\zeta_{12}Y_{21}) - \zeta_{12}\gamma_2(Y_{21}\zeta_{12}) = 0$$

and hence $\gamma_1(\zeta_{12}Y_{21})I - \gamma_2(Y_{21}\zeta_{12})I = 0$. This imply to

$$\chi(\zeta_{12}Y_{21}) = \chi(\zeta_{12})Y_{21} + \zeta_{12}\chi(Y_{21})$$

and

$$\chi(Y_{21}\zeta_{12}) = \chi(Y_{21})\zeta_{12} + Y_{21}\chi(\zeta_{12})$$

for all $\zeta_{12} \in \aleph_{12},\ Y_{21} \in \aleph_{21}$. □

Proof of Theorem 4.1. It's just like the Theorem 3.1 claims. □

Acknowledgments

We are very grateful to the referee for his/her appropriate and constructive suggestions which improved the quality of the paper. This work was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant number (G: 270-662-1440). The authors, therefore, gratefully acknowledge with thanks DSR for technical and financial support.

Conflict of interest

No potential conflict of interest in this paper.

References

1. I. Z. Abdullaev, *n*-Lie derivations on von Neumann algebra, *Uzbek. Mat. Zh.,* **5** (1992), 3–9.
2. F. Y. Lu, W. Jing, Characterizations of Lie derivations of $B(X)$, *Linear Algebra Appl.,* **432** (2010), 89–99.

3. P. Ji, W. Qi, X. Sun, Characterizations of Lie derivations of factor von Neumann algebras, *Linear Multilinear A.,* **61** (2013), 417–428.

4. P. Ji, W. Qi, Characterizations of Lie derivations of triangular algebras, *Linear Algebra Appl.,* **435** (2011), 1137–1146.

5. X. Qi, Characterization of (generalized) Lie derivations on $\mathscr{J}$-subspace lattice algebras by local action, *Aequat. Math.,* **87** (2014), 53–69.

6. L. Liu, Lie triple derivations on factor von Neumann algebras, *Bull. Korean Math. Soc.,* **52** (2015), 581–591.

7. Y. Wang, Lie n-derivations of unital algebras with idempotents, *Linear Algebra Appl.,* **458** (2014), 512–525.

8. X. Qi, Characterizing Lie n-derivations for reflexive algebras, *Linear Multilinear A.,* **63** (2015), 1693–1706.

9. Z. Xiao, Y. Yang, Lie n-derivations of incidence algebras, *Comm. Algebra,* **48** (2020), 105–118.

10. M. Khrypchenko, F. Wei, Lie-type derivations of finitary incidence algebras, *Rocky Mountain J. Math.,* **50** (2020), 163–175.

11. R. V. Kadison, J. R. Ringrose, *Fundamentals of the theory of operator algebras*, vol. I, II New York: Academic Press, 1986.