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Research article

Classification of Möbius minimal and Möbius isotropic hypersurfaces in $\mathbb{S}^5$

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Abstract: In this paper, we will prove that a closed Möbius minimal and Möbius isotropic hypersurface without umbilic points in the unit sphere $\mathbb{S}^5$ is Möbius equivalent to either the torus $\mathbb{S}^2(\frac{1}{\sqrt{2}})\times\mathbb{S}^2(\frac{1}{\sqrt{2}}) \to \mathbb{S}^5$ or the Cartan minimal hypersurface in $\mathbb{S}^5$ with four distinct principal curvatures.

Keywords: Möbius minimal; Möbius isotropic; isoparametric hypersurface
Mathematics Subject Classification: 53A30, 53B25, 53C40

1. Introduction

For a connected smooth n-dimensional hypersurface $x : M^n \to \mathbb{S}^{n+1}$ in the unit sphere without umbilical points, Wang [18] introduced four basic Möbius invariants g, $\mathbf{B}$, $\mathbf{A}$ and $\mathbf{\Phi}$, which are called the Möbius metric, the Möbius second fundamental form, the Blaschke tensor and the Möbius form respectively (for their definitions see Section 2 below), he also gave the fundamental theorem and basic formulas. Since then, the Möbius differential geometry of submanifolds for general dimension and codimension in a sphere has been well investigated and significant progress has been made in this area [6–10, 13].

One of the important aspects of Möbius geometry of hypersurfaces in a sphere is Möbius minimal hypersurfaces (also known as Willmore hypersurfaces), which is the critical point of the Möbius volume functional (volume functional of Möbius metric g) (see [18] for details):

$$\mathbb{W}(M) = \int_M dv_g = \int_M \rho^n dv_{g_0} = \frac{n}{n-1}\int_M (S - nH^2)^{\frac{n}{2}} dv_{g_0}, \tag{1.1}$$

where S is the square of the length of the second fundamental form, H is the mean curvature, $g = \rho^2 dx \cdot dx$ is the Möbius metric, $g_0 = dx \cdot dx$ is the induced metric, $\rho^2 = \frac{n}{n-1}(S - nH^2)$. In [18] (or [11]), the authors computed the Euler-Lagrange equation of the above Möbius volume function for any n-dimensional submanifold. Guo-Li-Wang [5] gave an important example of Möbius minimal hypersurfaces:

$$W_{n,m} := \mathbb{S}^m\left(\sqrt{\frac{n-m}{n}}\right)\times \mathbb{S}^{n-m}\left(\sqrt{\frac{m}{n}}\right) \to \mathbb{S}^{n+1},$$

which is called Willmore tori, it is minimal if and only if $n = 2m$ for some m.

Hypersurface $x : M^n \to \mathbb{S}^{n+1}$ is called Möbius isotropic if $\mathbf{\Phi} \equiv 0$ and $\mathbf{A} = \mu g$, where μ is the Blaschke eigenvalue (i.e. the eigenvalue of Blaschke tensor $\mathbf{A}$ with respect to g) of x. According to Liu-Wang-Zhao's main result in [14], we know that the Blaschke eigenvalue μ has to be constant, and x is Möbius equivalent to either a minimal hypersurface with constant scalar curvature in $\mathbb{S}^{n+1}$ (if $\mu > 0$), or the preimage of a stereographic projection of a minimal hypersurface with constant scalar curvature in $\mathbb{R}^{n+1}$ (if $\mu = 0$), or the image of the standard conformal map $\tau : \mathbb{H}^{n+1} \to \mathbb{S}^{n+1}_+$ of a minimal hypersurface with constant scalar curvature in hyperbolic space $\mathbb{H}^{n+1}$ (if $\mu < 0$), because such hypersurfaces are closely related to Chern's conjecture, it has received extensive attention, and many meaningful results [1–3, 16] have been obtained about minimal hypersurfaces of $\mathbb{S}^{n+1}$ with constant scalar curvature.

Recently, Deng-Gu-Wei [4] proved that a minimal Willmore hypersurface M^4 of $\mathbb{S}^5$ with constant scalar curvature is isoparametric. Motivated by Deng-Gu-Wei's paper, we investigated Möbius minimal hypersurface M^4 in the context of Möbius geometry, and obtained the following classification theorem:

Theorem 1.1. *Let $x : M^4 \to \mathbb{S}^5$ be a closed Möbius minimal and Möbius isotropic hypersurface without umbilic points, then x is Möbius equivalent to one of the following hypersurfaces:*

(1) *the torus $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \times \mathbb{S}^2(\frac{1}{\sqrt{2}}) \to \mathbb{S}^5$;*

(2) *the Cartan minimal hypersurface in $\mathbb{S}^5$ with four distinct principal curvatures.*

We organize the paper as follows. In Section 2, we review the Möbius invariants and its integrability conditions for hypersurfaces in the unit sphere, and give some basic formulas for Möbius minimal and Möbius isotropic hypersurfaces in $\mathbb{S}^5$. In Section 3, we consider the case that there are two distinct principal curvatures at one point and prove that its Möbius second fundamental form is parallel. In Section 4, we show that there do not exist Möbius minimal and Möbius isotropic hypersurfaces with three distinct principal curvatures at the minimum point of f_4. In Section 5, we will discuss the case that there are four distinct principal curvatures at the minimum point of f_4, and prove that the Blaschke eigenvalue $\mu \geq \frac{1}{32}$. Finally, we complete the proof of Theorem 1.1 in Section 6.

2. Möbius invariants and some basic formulas

In this section, we first review Möbius invariants and the structure equations for hypersurfaces in $\mathbb{S}^{n+1}$ (see [18] for details), and then we give some basic formulas for Möbius minimal and Möbius isotropic hypersurface in $\mathbb{S}^5$.

Let $\mathbb{R}^{n+3}_1$ be the Lorentz space with standard inner product $\langle\cdot,\cdot\rangle$ given by

$$\langle X, W\rangle_1 = -x^0w^0 + x^1w^1 + \cdots + x^{n+2}w^{n+2},$$

for $X = (x^0, x^1, \cdots, x^{n+2})$, $W = (w^0, w^1, \cdots, w^{n+2}) \in \mathbb{R}^{n+3}$. The half cone in $\mathbb{R}^{n+3}$ is defined as

$$C^{n+2}_+ = \{X \in \mathbb{R}^{n+3}_1 \mid \langle X, X\rangle = 0,\ x_0 > 0\}.$$

For an immersed umbilic-free hypersurface $x : M^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, we define its Möbius position vector $Y : M^n \to C_+^{n+2}$ by

$$Y = \rho(1, x), \qquad \rho^2 = \frac{n}{n-1}(S - nH^2) > 0.$$

Theorem 2.1 ([18]). *Two hypersurfaces $x, \widetilde{x} : M^n \to \mathbb{S}^{n+1}$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(n+2; 1)$ acting on $\mathbb{R}_1^{n+3}$, such that $Y = \widetilde{Y}T$.*

It follows immediately from Theorem 2.1 that $g = \langle dY, dY\rangle = \rho^2 dx \cdot dx$ is a Möbius invariant, which is called the Möbius metric of x.

Let Δ and R denote the Laplacian and the normalized scalar curvature of the Möbius metric g, we define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}(1 + n^2 R)Y. \tag{2.1}$$

Then we have $\langle Y, Y\rangle = \langle N, N\rangle = 0, \quad \langle Y, N\rangle = 1$.

We use the following range of indices: $1 \le i, j, k, \ldots \le n$.

Choosing a local orthonormal basis $\{E_i\}$ with respect to g with dual basis $\{\omega_i\}$, and E is the Möbius normal vector field of x. Putting $E_i(Y) = Y_i$, then we have

$$\langle Y_i, Y_j\rangle = \delta_{ij}, \quad \langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0, \tag{2.2}$$

and $\{Y, N, Y_1, \ldots, Y_n, E\}$ forms a moving frame in $\mathbb{R}_1^{n+3}$ along M. The structure equations are given by

$$dY = \sum_i \omega_i Y_i, \qquad dN = \sum_{i,j} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i E, \tag{2.3}$$

$$dY_i = -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij}\omega_j E, \tag{2.4}$$

$$dE = -\sum_i C_i \omega_i Y - \sum_{i,j} B_{ij}\omega_j Y_i, \tag{2.5}$$

where $\{\omega_{ij}\}$ is the connection form of the Möbius metric g. It is clear that

$$\mathbf{B} = \sum_{i,j} B_{ij}\omega_i \otimes \omega_j, \quad \mathbf{A} = \sum_{i,j} A_{ij}\omega_i \otimes \omega_j, \quad \mathbf{\Phi} = \sum_i C_i \omega_i \tag{2.6}$$

are all Möbius invariants and called the Möbius second fundamental form, the Blaschke tensor and the Möbius form of x, respectively. They can be represented by Euclidean invariants as follows:

$$\begin{aligned}
B_{ij} =& \rho^{-1}(h_{ij} - H\delta_{ij}),\\
A_{ij} =& \rho^{-2}(Hess_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij})\\
&- \frac{1}{2}\rho^{-2}(\|\bar{\nabla}\log\rho\|^2 - 1 + H^2)\delta_{ij},\\
C_i =& -\rho^{-2}(H_{,i} + \sum_j (h_{ij} - H\delta_{ij})e_j(\log\rho)),
\end{aligned}$$

where $Hess_{ij}$ and $\bar{\nabla}$ are the Hessian-matrix and the gradient with respect to $dx \cdot dx$.

The components of the covariant differentiation of $\mathbf{B}$, they are defined by

$$\sum_k B_{ijk}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2.7}$$

Let B_{ijkl} be the second covariant derivative, they satisfy the following Ricci identity

$$B_{ijkl} - B_{ijlk} = \sum_m B_{im}R_{mjkl} + \sum_m B_{mj}R_{mikl}. \tag{2.8}$$

Among the integrability conditions for the structure equations (2.3)-(2.5), we have the following (cf. [18]):

$$B_{ijk} - B_{ikj} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.9}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{2.10}$$

$$\sum_i B_{ii} = 0, \qquad \sum_{i,j} B_{ij}^2 = \frac{n-1}{n}, \tag{2.11}$$

R_{ijkl} denotes the components of the Riemannian curvature tensor of g.

Hypersurface $x : M^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ is Möbius minimal if and only if (see [18])

$$\sum_{i,j} B_{ijij} + \sum_{i,j,k} B_{ik}B_{kj}B_{ji} + \sum_{i,j} A_{ij}B_{ij} = 0. \tag{2.12}$$

If $x : M^n \to \mathbb{S}^{n+1}$ is Möbius isotropic, then we have

$$A_{ij} = \mu\delta_{ij}, \qquad C_i = 0, \qquad B_{ijk} = B_{ikj}, \tag{2.13}$$

$$R_{ijkl} = (B_{ik}B_{jl} - B_{il}B_{jk}) + 2\mu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{2.14}$$

Substituting (2.13) and $\sum_i B_{ii} = 0$ into (2.12), then it reduces to

$$\sum_{i,j,k} B_{ik}B_{kj}B_{ji} = 0. \tag{2.15}$$

We define four functions:

$$f_1 = ||\mathbf{B}||^2 = \sum_{i,j} B_{ij}^2, \quad f_2 = ||\nabla\mathbf{B}||^2 = \sum_{i,j,k} B_{ijk}^2,$$
$$f_3 = \sum_{i,j,k} B_{ik}B_{kj}B_{ji}, \quad f_4 = \sum_{i,j,k,l} B_{ik}B_{kj}B_{jl}B_{li}.$$

Assume that x is a Möbius minimal and Möbius isotropic hypersurface in $\mathbb{S}^5$, then by straightforward computation, we have the following lemma.

Lemma 2.1. *Let $x : M^4 \to \mathbb{S}^5(1)$ be a umbilic-free closed Möbius minimal hypersurface. If x is Möbius isotropic, then*

$$f_2 = ||\nabla\mathbf{B}||^2 = 6(\frac{3}{32} - \mu) = const., \tag{2.16}$$

$$\triangle f_3 = 6 \sum_{i,j,k,l} B_{ikl} B_{kjl} B_{ji} = 0, \tag{2.17}$$

$$\Delta f_4 = 32(\mu - \frac{3}{32}) f_4 + 8\widetilde{A} + 4\widetilde{B}, \tag{2.18}$$

where ∇ *and* Δ *are the gradient and Laplacian of the Möbius metric* g,

$$\widetilde{A} = \sum_{i,j,k,l,m} B_{mlk} B_{mli} B_{kj} B_{ji}, \quad \widetilde{B} = \sum_{i,j,k,l,m} B_{mkl} B_{mi} B_{kj} B_{ijl}.$$

Proof. Using (2.11), (2.14) and the Ricci identity (2.8), we have

$$0 = \frac{1}{2} \triangle f_1 = \frac{1}{2} \sum_{i,j,k} (B_{ij}^2)_{kk} = \|\nabla \mathbf{B}\|^2 - \|\mathbf{B}\|^2 (\|\mathbf{B}\|^2 - 8\mu),$$

where $f_1 = \|\mathbf{B}\|^2 = \frac{3}{4}$, thus (2.16) follows.

Since (2.15) means $f_3 = 0$, then we have

$$\begin{aligned} 0 = (f_3)_{mn} &= (\sum_{i,j,k} B_{ik} B_{kj} B_{ji})_{mn} \\ &= 3 \sum_{i,j,k} B_{ikmn} B_{kj} B_{ji} + 6 \sum_{i,j,k} B_{ikm} B_{kjn} B_{ji}, \end{aligned} \tag{2.19}$$

combining this and (2.8), (2.14), we can have

$$0 = \Delta f_3 = 24(\mu - \frac{3}{32}) f_3 + 6 \sum_{i,j,k,l} B_{ikl} B_{kjl} B_{ji}, \tag{2.20}$$

which gives (2.17) immediately.

Analogously, by (2.11), (2.13) and (2.14), we can have

$$\begin{aligned} \triangle f_4 = \sum_{i,j,k,l,m} (B_{km} B_{mi} B_{ij} B_{jk})_{ll} &= 4 \sum_{i,j,k,l,m} B_{kmll} B_{mi} B_{ij} B_{jk} \\ &+ 4 \sum_{i,j,k,l,m} (2 B_{kml} B_{mil} B_{ij} B_{jk} + B_{kml} B_{mi} B_{ijl} B_{jk}) \\ &= 32(\mu - \frac{3}{32}) f_4 + 4 \sum_{i,j,k,l,m} (2 B_{mlk} B_{mli} B_{kj} B_{ji} + B_{mkl} B_{mi} B_{kj} B_{ijl}), \end{aligned}$$

then we get (2.18). □

For an arbitrary fixed point $p \in M$, we choose orthonormal frame such that $B_{ij} = \lambda_i \delta_{ij}$. We call λ_i the Möbius principal curvatures at p, they satisfy

$$\sum_{i=1}^{4} \lambda_i = 0, \quad \sum_{i=1}^{4} \lambda_i^3 = 0, \quad \sum_{i=1}^{4} \lambda_i^2 = \frac{3}{4}. \tag{2.21}$$

Without loss of generality, we will assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, then we have the following observation.

Lemma 2.2. *Let $x : M^4 \to \mathbb{S}^5$ be an umbilic-free closed Möbius minimal and Möbius isotropic hypersurface, if $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ at p. Then*

$$\lambda_1 + \lambda_4 = 0, \qquad \lambda_2 + \lambda_3 = 0.$$

Proof. Since $\lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 = 0$, then we have

$$(\lambda_1 + \lambda_2)(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2) + (\lambda_3 + \lambda_4)(\lambda_3^2 - \lambda_3\lambda_4 + \lambda_4^2) = 0.$$

Combining $\lambda_1 + \lambda_2 = -(\lambda_3 + \lambda_4)$, we get

$$(\lambda_1 + \lambda_2)[\lambda_1^2 + \lambda_2^2 - (\lambda_3^2 + \lambda_4^2)] = 0.$$

By $\lambda_1 + \lambda_3 = -(\lambda_2 + \lambda_4)$ and $\lambda_1 + \lambda_4 = -(\lambda_2 + \lambda_3)$, we obtain

$$(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4) = 0. \tag{2.22}$$

We claim $\lambda_1+\lambda_2 \neq 0$ under the assumption that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, if otherwise, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, which is a contradiction to $\sum_i \lambda_i^2 = \frac{3}{4}$. Hence

$$(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4) = 0. \tag{2.23}$$

If $\lambda_1 + \lambda_3 \neq 0$, then $\lambda_1 + \lambda_4 = -(\lambda_2 + \lambda_3) = 0$.
If $\lambda_1 + \lambda_3 = 0$, then $\lambda_1 \geq \lambda_2 \geq -\lambda_1 \geq -\lambda_2$, which implies

$$\lambda_1 = \lambda_2 > 0, \quad \lambda_3 = \lambda_4 = -\lambda_1. \tag{2.24}$$

These complete the proof of Lemma 2.2. □

According to Lemma 2.2, at a fixed point, there are three cases:

$$\begin{array}{ll} \textbf{Case I}, & \lambda_1 = \lambda_2 > 0 > \lambda_3 = \lambda_4 = -\lambda_1; \\ \textbf{Case II}, & \lambda_1 > \lambda_2 = \lambda_3 > \lambda_4, \ \lambda_2 = \lambda_3 = 0, \ \lambda_4 = -\lambda_1; \\ \textbf{Case III}, & \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4, \ \lambda_3 = -\lambda_2, \ \lambda_4 = -\lambda_1. \end{array} \tag{2.25}$$

3. Two distinct Möbius principal curvatures at one point

In this section, we deal with Case I and prove the following result.

Theorem 3.1. *Let $x : M^4 \to \mathbb{S}^5$ be a closed Möbius minimal and Möbius isotropic hypersurface without umbilic points. If there are two distinct principal curvatures at a fixed point, then $\|\nabla \mathbf{B}\| = 0$.*

We first prove a simple and useful lemma.

Lemma 3.1. *For any $k, l \in \{1, 2, 3, 4\}$, we have*

$$B_{11k} + B_{22k} = 0, \qquad B_{33k} + B_{44k} = 0; \tag{3.1}$$

$$B_{11kl} + B_{22kl} = -\frac{1}{2\lambda}\sum_{i,j} B_{ijk}B_{ijl} = -(B_{33kl} + B_{44kl}). \tag{3.2}$$

Proof. Since $\sum_i B_{ii} = 0$, taking the first and second covariant derivative, we have

$$B_{11k} + B_{22k} + B_{33k} + B_{44k} = 0, \tag{3.3}$$

$$B_{11kl} + B_{22kl} + B_{33kl} + B_{44kl} = 0. \tag{3.4}$$

Similarly, by $f_1 = \|\mathbf{B}\|^2 = \sum_{i,j}(B_{ij})^2 = \frac{3}{4}$, we can get

$$\sum_{i,j} B_{ij}B_{ijk} = 0, \quad \sum_{i,j}(B_{ijk}B_{ijl} + B_{ij}B_{ijkl}) = 0. \tag{3.5}$$

In case I, $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = \lambda \neq 0$, so we get

$$\lambda(B_{11k} + B_{22k} - B_{33k} - B_{44k}) = 0, \tag{3.6}$$

$$\sum_{i,j} B_{ijk}B_{ijl} + \lambda(B_{11kl} + B_{22kl} - B_{33kl} - B_{44kl}) = 0. \tag{3.7}$$

Hence, by (3.3), (3.6), (3.4) and (3.7), we obtain (3.1), (3.2) hold. □

The following lemma describes the relations between the components of $\nabla\mathbf{B}$, which are crucial to the proof of Theorem 3.1.

Lemma 3.2. *The elements of* $\{B_{ijk}\}$ *satisfy the following equations:*

$$B_{111}^2 + B_{222}^2 = B_{133}^2 + B_{134}^2 = B_{233}^2 + B_{234}^2, \tag{3.8}$$

$$B_{333}^2 + B_{444}^2 = B_{113}^2 + B_{123}^2 = B_{114}^2 + B_{124}^2, \tag{3.9}$$

$$B_{133}B_{233} + B_{134}B_{234} = 0, \tag{3.10}$$

$$B_{113}B_{114} + B_{123}B_{124} = 0, \tag{3.11}$$

$$-B_{111}B_{113} + B_{222}B_{123} + B_{133}B_{333} - B_{134}B_{444} = 0, \tag{3.12}$$

$$B_{111}B_{114} - B_{222}B_{124} + B_{134}B_{333} + B_{133}B_{444} = 0, \tag{3.13}$$

$$B_{111}B_{123} + B_{222}B_{113} + B_{233}B_{333} - B_{234}B_{444} = 0, \tag{3.14}$$

$$B_{111}B_{124} + B_{222}B_{114} - B_{234}B_{333} - B_{233}B_{444} = 0. \tag{3.15}$$

Proof. From (2.19) and the fact that $\lambda_1 = \lambda_2 = \lambda = -\lambda_3 = -\lambda_4 \neq 0$, we have

$$\begin{aligned} 0 &= \sum_{i,j,k} B_{ikmn}B_{kj}B_{ji} + 2\sum_{i,j,k} B_{ikm}B_{kjn}B_{ji} = 2\sum_{i,k} B_{ikm}B_{ikn}\lambda_i \\ &= 2\lambda\sum_k (B_{1km}B_{1kn} + B_{2km}B_{2kn} - B_{3km}B_{3kn} - B_{4km}B_{4kn}). \end{aligned}$$

Thus

$$\sum_k (B_{1km}B_{1kn} + B_{2km}B_{2kn} - B_{3km}B_{3kn} - B_{4km}B_{4kn}) = 0. \tag{3.16}$$

Using (3.1) and setting $(m,n) = (1,1),\ (2,2),\ (3,3),\ (4,4)$ in (3.16), then (3.8) and (3.9) can be obtained immediately.

Analogously, taking $(m,n) = (1,2),\ (3,4),\ (1,3),\ (1,4),\ (2,3),\ (2,4)$ in (3.16), respectively, we can get the remaining equations in Lemma 3.2. □

As a direct application of Lemma 3.2, we have

Lemma 3.3. *For any $k \in \{1,2,3,4\}$, the following equation holds*

$$\sum_{i,j} B_{ijk}^2 = 4(B_{111}^2 + B_{222}^2 + B_{333}^2 + B_{444}^2) = \frac{1}{4}\|\nabla \mathbf{B}\|^2. \tag{3.17}$$

Proof. Using (3.1), (3.8) and (3.9), we have

$$\begin{aligned}\sum_{i,j} B_{ij1}^2 =& B_{111}^2 + B_{221}^2 + B_{331}^2 + B_{441}^2 + 2B_{121}^2 \\ &+ 2B_{131}^2 + 2B_{141}^2 + 2B_{231}^2 + 2B_{241}^2 + 2B_{341}^2 \\ =& 4(B_{111}^2 + B_{222}^2 + B_{333}^2 + B_{444}^2).\end{aligned}$$

Similarly, we get

$$\sum_{i,j} B_{ij2}^2 = \sum_{i,j} B_{ij3}^2 = \sum_{i,j} B_{ij4}^2 = 4(B_{111}^2 + B_{222}^2 + B_{333}^2 + B_{444}^2),$$

thus (3.17) follows. □

By making use of Lemma 3.1-3.3, we get some relations about $\{B_{ijkl}\}$.

Lemma 3.4.

$$\begin{aligned} &B_{1122} = B_{2211}, \quad && B_{1111} = B_{2222}, \quad && B_{1112} = -B_{2212}, \\ &B_{1134} = -B_{2234}, \quad && B_{3334} = -B_{4434}, \quad && B_{3312} = -B_{4412}. \end{aligned}$$

Proof. By (2.14) and the Ricci identity (2.8), we obtain

$$B_{1122} = B_{1212} = B_{2211} + (2\mu + \lambda_1\lambda_2)(\lambda_1 - \lambda_2) = B_{2211}.$$

On the other hand, from (3.2) and (3.17), we know that

$$B_{1111} + B_{2211} = B_{1122} + B_{2222},$$

thus we have $B_{1111} = B_{2222}$.

By making use of Lemma 3.1 and (3.10), (3.11), we have

$$\begin{aligned}\sum_{i,j} B_{ij1}B_{ij2} &= 2(B_{133}B_{233} + B_{134}B_{234}) = 0, \\ \sum_{i,j} B_{ij3}B_{ij4} &= 2(B_{113}B_{114} + B_{123}B_{124}) = 0.\end{aligned}$$

From the above equations and (3.2), we get

$$\begin{aligned} B_{3312} + B_{4412} &= B_{1112} + B_{2212} = 0, \\ B_{1134} + B_{2234} &= B_{3334} + B_{4434} = 0. \end{aligned}$$

Hence, we have completed Lemma 3.4. □

From (3.17), we can see that $(B_{111}^2 + B_{222}^2) + (B_{333}^2 + B_{444}^2) = \frac{1}{16}\|\nabla \mathbf{B}\|^2$ is independent of the choice of the basis $\{E_i\}_{i=1}^4$, thus for any fixed E_3 and E_4, $B_{111}^2 + B_{222}^2$ is invariant under the rotation of $\{E_1, E_2\}$. Similarly, using (3.17), we can also see that $B_{333}^2 + B_{444}^2$ is invariant under the rotation of $\{E_3, E_4\}$. For convenience, we introduce the following notation:

$$d_1 = B_{111}^2 + B_{222}^2, \qquad d_2 = B_{333}^2 + B_{444}^2. \tag{3.18}$$

In order to get further relations between the components of $\nabla \mathbf{B}$, we set

$$G_1 = \begin{pmatrix} B_{133} & B_{134} \\ B_{233} & B_{234} \end{pmatrix}, \quad G_2 = \begin{pmatrix} B_{113} & B_{123} \\ B_{114} & B_{124} \end{pmatrix}. \tag{3.19}$$

Then (3.8)-(3.11) can be rewritten as

$$G_1 {G_1}^T = d_1 I, \quad G_2 {G_2}^T = d_2 I, \tag{3.20}$$

where I is the identity matrix. Using Lemma 3.2, we have the following:

Lemma 3.5. *If $d_1 d_2 \neq 0$, then $|G_1||G_2| > 0$. Furthermore, when $|G_1| > 0$, we have*

$$|G_1| = d_1,\ G_1^T = G_1^*, \quad |G_2| = d_2,\ G_2^T = G_2^*. \tag{3.21}$$

where $|G_i|$, G_i^T and G_i^* denote the determinant, transpose matrix and adjoint matrix of G_i respectively.

Proof. From (3.8), (3.9), we know that $G_i = 0$ if $d_i = 0$ $(i = 1, 2)$, this and (3.20) imply that

$$|G_i| = \pm d_i,\ G_i^T = \pm G_i^*, \quad i = 1, 2. \tag{3.22}$$

By (3.12)-(3.15), we know that $(B_{111}, B_{222}, B_{333}, B_{444})$ satisfies the following homogeneous system of linear equations:

$$\begin{pmatrix} -B_{113} & B_{123} & B_{133} & -B_{134} \\ B_{114} & -B_{124} & B_{134} & B_{133} \\ B_{123} & B_{113} & B_{233} & -B_{234} \\ B_{124} & B_{114} & -B_{234} & -B_{233} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.23}$$

Denote the coefficient matrix by F, then by (3.8)-(3.11), (3.19) and (3.22) we deduce that

$$\begin{aligned} |F|^2 = |FF^T| &= \{(d_1 + d_2)^2 - (|G_1| + |G_2|)^2\}^2 \\ &= \{(d_1 + d_2)^2 - (d_1 \pm d_2)^2\}^2. \end{aligned} \tag{3.24}$$

Since (3.23) has nonzero solution if and only if $|F| = 0$. Hence, if $d_1 d_2 \neq 0$, we have $|F|^2 = 0$, thus by (3.22), (3.24) we get $|G_1||G_2| > 0$. Assume $|G_1| > 0$, then (3.22) gives (3.21). We have completed the proof of Lemma 3.5. □

To prove Theorem 3.1, we assume on the contrary that $\|\nabla \mathbf{B}\|^2 = 16(d_1 + d_2) \neq 0$, then at least one of $\{d_1,\ d_2\}$ is nonzero. Without loss of generality, suppose $d_1 \neq 0$. Thus, we may divide the discussion into two cases.

Case I-(i): $d_1 \neq 0, d_2 = 0$; **Case I-(ii)**: $d_1 \neq 0,\ d_2 \neq 0$.

3.1. Case I-(i) does not occur.

In this case, from (3.9) and (3.18), we immediately get

$$B_{333} = B_{444} = B_{123} = B_{113} = B_{124} = B_{114} = 0. \tag{3.25}$$

For fixed directions E_3 and E_4, we can reselect $\{E_1, E_2\}$ if necessary such that $B_{222} = 0$, then fix E_1 and rotate the directions $\{E_3, E_4\}$, such that $B_{134} = 0$. Since $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, so the above transformations of the frame preserve $B_{ij} = \lambda_i \delta_{ij}$ at the point p. More precisely we have

$$d_1 = B_{111}^2 = B_{133}^2 = B_{234}^2 \neq 0, \quad B_{233} = B_{134} = B_{222} = 0. \tag{3.26}$$

Lemma 3.6. *If $d_1 \neq 0$, $d_2 = 0$, then there exist local othonomal frame $\{E_i\}$ such that* (3.25) *and* (3.26) *hold. Furthermore, we have*

$$B_{1111} = 3B_{1122} = B_{2222}, \tag{3.27}$$

$$(B_{3311} - B_{4411})B_{133} + 2B_{1234}B_{234} = 0. \tag{3.28}$$

Proof. From Lemma 2.1, we know that $f_2 = \|\nabla \mathbf{B}\|^2 = const.$, taking the covariant derivatives of f_2 by direction E_1, we get:

$$\begin{aligned} 0 = \frac{1}{2}(f_2)_1 &= B_{1111}B_{111} + 3B_{2211}B_{122} \\ &\quad + 3(B_{3311}B_{133} + B_{4411}B_{144} + 2B_{1234}B_{234}) \\ &= B_{111}(B_{1111} - 3B_{2211}) \\ &\quad + 3(B_{3311}B_{133} - B_{4411}B_{133} + 2B_{1234}B_{234}). \end{aligned} \tag{3.29}$$

Taking the covariant derivatives of $\triangle f_3$, by (2.17), gives us

$$\begin{aligned} 0 = \frac{1}{6}(\triangle f_3)_1 &= \sum_{i,j,k,l}(B_{ikl}B_{kjl}B_{ji})_1 \\ &= 2\sum_{i,j,k,l} B_{ikl1}B_{kjl}B_{ij} + \sum_{i,j,k,l} B_{ikl}B_{kjl}B_{ij1} \\ &= 2\sum_{i,j,k} B_{ijk1}B_{ijk}\lambda_i + \sum_{i,j,k,l} B_{ijk}B_{ijl}B_{kl1}. \end{aligned} \tag{3.30}$$

Using (3.25), (3.26) and (3.17), we conclude the second term

$$\sum_{i,j,k,l} B_{ijk}B_{ijl}B_{kl1} = \sum_{i,j,k} B_{ijk}^2 B_{ii1} = \frac{1}{4}\|\nabla \mathbf{B}\|^2 \sum_i B_{ii1} = 0.$$

So by making use of (3.1), (3.25), (3.26), we have

$$B_{111}(B_{1111} - 3B_{2211}) - (B_{3311}B_{133} - B_{4411}B_{133} + 2B_{2341}B_{234}) = 0. \tag{3.31}$$

Combining (3.31) and (3.29), we get

$$\begin{aligned} &(B_{3311} - B_{4411})B_{133} + 2B_{1234}B_{234} = 0, \\ &B_{111}(B_{1111} - 3B_{2211}) = 0. \end{aligned}$$

Which together with (3.26) imply that $B_{1111} = 3B_{2211}$.

From this and Lemma 3.4, the proof of Lemma 3.6 is completed. □

In order to get more information about the second order covariant differential of tensor **B**, we take the third covariant derivative of f_3 and get the following:

Lemma 3.7. *In Case I-(i), the following equations hold*

$$B_{1122}B_{111} + B_{1234}B_{234} = 0, \tag{3.32}$$

$$B_{1234}B_{111} + B_{3344}B_{234} = 0, \tag{3.33}$$

$$(B_{1122} - B_{2222})B_{111} - (B_{3322} - B_{4422})B_{133} = 0, \tag{3.34}$$

$$(B_{1144} - B_{2244})B_{111} - (B_{3344} - B_{4444})B_{133} = 0. \tag{3.35}$$

Proof. Taking the third covariant derivative of f_3, we have

$$\begin{aligned} 0 = \sum_{i,j,k}(B_{ik}B_{kj}B_{ji})_{pqm} &= 3\sum_{i,j,k} B_{ijpqm}B_{jk}B_{ki} + 6\sum_{i,j,k} B_{ijp}B_{jkq}B_{kim} \\ &+ 6\sum_{i,j,k}(B_{ijpq}B_{jkm}B_{ki} + B_{ijpm}B_{jkq}B_{ki} + B_{ijqm}B_{jkp}B_{ki}) \\ &= 3\sum_{i} B_{iipqm}\lambda^2 + 6\sum_{i,j,k} B_{ijp}B_{jkq}B_{kim} \\ &+ 6\sum_{i,j}\lambda_i(B_{ijpq}B_{ijm} + B_{ijpm}B_{ijq} + B_{ijqm}B_{ijp}). \end{aligned}$$

Thus we have

$$\sum_{i,j,k} B_{ijp}B_{jkq}B_{kim} + \sum_{i,j}\lambda_i(B_{ijpq}B_{ijm} + B_{ijpm}B_{ijq} + B_{ijqm}B_{ijp}) = 0. \tag{3.36}$$

We will rewrite (3.36) in case of $(p, q, m) = (1, 1, 1),\ (2, 3, 4),\ (2, 2, 1),\ (4, 4, 1)$ respectively.

If $(p, q, m) = (1, 1, 1)$**,** we have

$$\sum_{i,j,k} B_{ij1}B_{jk1}B_{ki1} + 3\sum_{i,j}\lambda_i B_{ij11}B_{ij1} = 0. \tag{3.37}$$

Using (3.25), (3.26) and (3.1), the first term of (3.37) reduces to

$$\begin{aligned} \sum_{i,j,k} B_{ij1}B_{jk1}B_{ki1} &= B_{111}^3 + B_{122}^3 + B_{133}^3 + B_{144}^3 \\ &= B_{111}^3 + (-B_{111})^3 + B_{133}^3 + (-B_{133})^3 = 0. \end{aligned}$$

Thus we have $\sum_{i,j} B_{ij11}B_{ij1}\lambda_i = 0$, hence

$$\lambda\sum_{j}(B_{1j11}B_{1j1} + B_{2j11}B_{2j1} - B_{3j11}B_{3j1} - B_{4j11}B_{4j1}) = 0,$$

which together with (3.1) yields

$$(B_{1111} - B_{2211})B_{111} - (B_{3311} - B_{4411})B_{133} = 0. \tag{3.38}$$

Substitute (3.27), (3.28) into (3.38), we can get (3.32).

If $(p,q,m)=(2,3,4)$, we have

$$\sum_{i,j,k} B_{ij2}B_{jk3}B_{ki4}+\sum_{i,j}\lambda_i(B_{ij23}B_{ij4}+B_{ij24}B_{ij3}+B_{ij34}B_{ij2})=0. \tag{3.39}$$

It is easy to check that the first term $\sum_{i,j,k}B_{ij2}B_{jk3}B_{ki4}=0$, thus (3.39) can be simplified as

$$\begin{aligned}
0=&\sum_{i,j}\lambda_i(B_{ij23}B_{ij4}+B_{ij24}B_{ij3}+B_{ij34}B_{ij2})\\
=&\lambda(B_{1423}B_{144}+B_{2323}B_{234}-B_{3223}B_{324}-B_{4123}B_{414})\\
&+\lambda(B_{1324}B_{133}+B_{2424}B_{243}-B_{3124}B_{313}-B_{4224}B_{423})\\
&+\lambda(B_{1234}B_{122}+B_{2134}B_{212}-B_{3434}B_{342}-B_{4334}B_{432}),
\end{aligned}$$

this and (3.25), (3.26), (3.1) imply that

$$B_{1234}B_{111}+B_{3344}B_{234}=0.$$

Thus (3.33) is obtained.

If $(p,q,m)=(2,2,1)$, we rewrite (3.36) as

$$0=\sum_{i,j,k}B_{ij2}B_{jk2}B_{ki1}+\sum_{i,j}\lambda_i(B_{ij22}B_{ij1}+2B_{ij21}B_{ij2}). \tag{3.40}$$

For the first term, by direct computation, we find it is vanished, then (3.40) can be simplified as

$$\begin{aligned}
0=&\sum_{i,j}\lambda_i(B_{ij22}B_{ij1}+2B_{ij21}B_{ij1})\\
=&\lambda(B_{1122}B_{111}+B_{2222}B_{221}-B_{3322}B_{331}-B_{4422}B_{441})\\
&+2\lambda(B_{1221}B_{122}+B_{2121}B_{212}-B_{3421}B_{342}-B_{4321}B_{432})\\
=&\lambda(B_{1122}B_{111}-B_{2222}B_{111}-B_{3322}B_{331}+B_{4422}B_{133}),
\end{aligned}$$

in the last equality, we used (3.32) and (3.1). Thus we get (3.34).

If $(p,q,m)=(4,4,1)$, we have

$$0=\sum_{i,j,k}B_{ij4}B_{jk4}B_{ki1}+\sum_{i,j}\lambda_i(B_{ij44}B_{ij1}+2B_{ij41}B_{ij4}). \tag{3.41}$$

The first term vanished, (3.41) can be reduced to

$$\begin{aligned}
0=&\sum_{i,j}\lambda_i(B_{ij44}B_{ij1}+2B_{ij41}B_{ij4})\\
=&\lambda(B_{1144}B_{111}+B_{2244}B_{221}-B_{3344}B_{331}-B_{4444}B_{441})\\
&+2\lambda(B_{1441}B_{144}+B_{2341}B_{234}-B_{3241}B_{324}-B_{4141}B_{414}),
\end{aligned}$$

which together with (3.1) give that

$$(B_{1144}-B_{2244})B_{111}-(B_{3344}-B_{4444})B_{133}=0.$$

We have completed the proof of Lemma 3.7. □

Corresponding to subcase I-(i): $d_1 \neq 0$, $d_2 = 0$, we have the following result:

Proposition 3.1. *Let $x : M^4 \to \mathbb{S}^5(1)$ be a closed Möbius minimal and Möbius isotropic hypersurface without umbilic points. If there are two distinct principal curvatures at a fixed point, then subcase I-(i) does not occur.*

Proof. Since $B_{111}^2 = B_{234}^2 \neq 0$, then from (3.32) and (3.33) we conclude that

$$B_{3344} = B_{1122}. \tag{3.42}$$

From $B_{111}^2 = B_{133}^2$, we have either $B_{111} = B_{133}$ or $B_{111} = -B_{133}$.

If $B_{111} = B_{133}$, then (3.34) and (3.35) reduce to

$$B_{1122} - B_{2222} - B_{3322} + B_{4422} = 0,$$
$$B_{1144} - B_{2244} - B_{3344} + B_{4444} = 0.$$

On the other hand, $\sum_i B_{ii} = 0$ gives

$$B_{1122} + B_{2222} + B_{3322} + B_{4422} = 0, \tag{3.43}$$
$$B_{1144} + B_{2244} + B_{3344} + B_{4444} = 0. \tag{3.44}$$

The above four equations imply that

$$B_{4422} = -B_{1122}, \quad B_{3344} = -B_{2244}. \tag{3.45}$$

Using (3.45) and (3.42), we get $B_{4422} = B_{2244}$. By means of Ricci identity, we have

$$0 = B_{2244} - B_{4422} = B_{2424} - B_{2442} = 2\lambda(2\mu - \lambda^2). \tag{3.46}$$

Thus $\mu = \frac{1}{2}\lambda^2 = \frac{3}{32}$, by (2.16) we have $\|\nabla\mathbf{B}\|^2 = 0$, this is a contradiction.

If $B_{111} = -B_{133} \neq 0$, similar to the case $B_{111} = B_{133}$, from (3.34), (3.35) and (3.42), we can get

$$B_{4422} = -B_{2222}, \quad B_{1144} = -B_{3344} = -B_{1122}. \tag{3.47}$$

By using (3.47), (3.2) and (3.17) we have

$$\begin{aligned} B_{2244} - B_{4422} &= B_{2244} - B_{1122} - B_{4422} + B_{1122} \\ &= (B_{2244} + B_{1144}) + (B_{2222} + B_{1122}) \\ &= -\frac{2}{2\lambda}\sum_{i,j} B_{ij4}^2 = -\frac{1}{\lambda}(\frac{1}{4}\|\nabla\mathbf{B}\|^2). \end{aligned} \tag{3.48}$$

On the other hand, by Ricci identity, (2.16) and $\sum_i \lambda_i^2 = \frac{3}{4}$, we have

$$B_{2244} - B_{4422} = 2\lambda(2\mu - \lambda^2) = 4\lambda(\mu - \frac{3}{32}) = -\frac{2\lambda}{3}\|\nabla\mathbf{B}\|^2. \tag{3.49}$$

Combining (3.49), (3.48) and $\lambda^2 = \frac{3}{16}$, we have $\|\nabla\mathbf{B}\|^2 = 0$, which is a contradiction.

In conclusion, we have completed the proof of Proposition 3.1. □

3.2. Case I-(ii) does not occur.

We will deal with Case I-(ii) and prove it does not occur.

For any fixed direction E_3, we can rotate the directions $\{E_1, E_2\}$ if necessary such that $B_{123} = 0$, then fix E_1 and rotate $\{E_3, E_4\}$, such that $B_{333} = 0$. Since $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, the above transformations of the frame preserve $B_{ij} = \lambda_i \delta_{ij}$ at the point p. We now assume that $B_{123} = B_{333} = 0$, then by (3.19), (3.21) we get

$$\begin{aligned} &B_{113} = B_{124}, \quad B_{133} = B_{234}, \quad B_{233} = -B_{134}, \\ &d_2 = B_{444}^2 = B_{113}^2 \neq 0, \quad B_{123} = B_{114} = B_{333} = 0. \end{aligned} \tag{3.50}$$

Hence, there are two subcases: (**a**): $B_{444} = B_{113}$, (**b**): $B_{444} = -B_{113}$.

Here we will only discuss the Case (a), the other case follows similarly.
Substituting $B_{444} = B_{113} = B_{124}$ into (3.12) and (3.13), we get

$$-(B_{111} + B_{134})B_{444} = 0, \quad (B_{133} - B_{222})B_{444} = 0,$$

which implies

$$B_{111} = -B_{134}, \quad B_{222} = B_{133}. \tag{3.51}$$

Summarizing the above discussion, under case (a), we have

$$\begin{aligned} &B_{111} = -B_{134} = B_{233}, \quad B_{222} = B_{133} = B_{234}, \\ &B_{444} = B_{113} = B_{124}, \qquad B_{123} = B_{114} = B_{333} = 0. \end{aligned} \tag{3.52}$$

Lemma 3.8. *In Case I-(ii), we have the following equations*

$$B_{1111} - B_{2211} + 2B_{3411} = 0, \tag{3.53}$$

$$B_{3344} - B_{4444} + 2B_{1244} = 0, \tag{3.54}$$

$$3B_{1122} + B_{1111} - 4B_{4412} + 2B_{3411} = 0, \tag{3.55}$$

$$B_{3411} - B_{3422} + 2B_{3344} = 0. \tag{3.56}$$

Proof. In the proof of Lemma 3.7, we calculated $(f_3)_{pqm}$ and got

$$\sum_{i,j,k} B_{ijp}B_{jkq}B_{kim} + \sum_{i,j} \lambda_i (B_{ijpq}B_{ijm} + B_{ijpm}B_{ijq} + B_{ijqm}B_{ijp}) = 0. \tag{3.57}$$

For convenience, we denote matrix $B_k = (B_{ijk})$, then the first term of the above equation can be written as $\mathbf{tr}(B_pB_qB_m)$. By direct computation, we have

$$\begin{aligned} &tr(B_1^3) = tr(B_2^3) = tr(B_4^3) = 0, \\ &tr(B_2^2B_1) = tr(B_1^2B_2) = tr(B_3^2B_4) = 0. \end{aligned} \tag{3.58}$$

To prove this lemma, we will rewrite (3.57) in case of $(p, q, m) = (1, 1, 1)$, $(2, 2, 2)$, $(2, 2, 1)$, $(1, 1, 2)$, $(4, 4, 4)$, $(3, 3, 4)$ respectively.

If $(p, q, m) = (1, 1, 1)$, by (3.57) and (3.58), we have

$$0 = \lambda \sum_j (B_{1j11}B_{1j1} + B_{2j11}B_{2j1} - B_{3j11}B_{3j1} - B_{4j11}B_{4j1}), \tag{3.59}$$

which yields the following equation by (3.1) and (3.52):

$$B_{111}(B_{1111} - B_{2211} + 2B_{3411}) - B_{222}(B_{3311} - B_{4411} + 2B_{2111}) = 0. \tag{3.60}$$

If $(p, q, m) = (2, 2, 2)$, similar to the proof of (3.60), we can obtain

$$B_{222}(B_{2222} - B_{1122} - 2B_{3422}) + B_{111}(-B_{3322} + B_{4422} - 2B_{1222}) = 0. \tag{3.61}$$

From Lemma 3.4, we get

$$\begin{aligned} &B_{2222} - B_{1122} - 2B_{3422} = B_{1111} - B_{2211} + 2B_{3411}, \\ &B_{2111} = B_{1112} = -B_{2212} = -B_{1222}. \end{aligned} \tag{3.62}$$

From Lemma 3.3, we have

$$B_{1133} + B_{2233} = B_{1144} + B_{2244}, \tag{3.63}$$

which implies

$$B_{3311} - B_{4411} = -B_{3322} + B_{4422}. \tag{3.64}$$

Substituting (3.62), (3.64) into (3.61), we have

$$B_{222}(B_{1111} - B_{2211} + 2B_{3411}) + B_{111}(B_{3311} - B_{4411} + 2B_{2111}) = 0. \tag{3.65}$$

By using (3.65), (3.60) and our assumption that $d_1 = B_{111}^2 + B_{222}^2 \neq 0$, we get

$$B_{1111} - B_{2211} + 2B_{3411} = 0, \quad B_{3311} - B_{4411} + 2B_{1112} = 0. \tag{3.66}$$

Hence we obtain the Eq (3.53).

If $(p, q, m) = (4, 4, 4)$, from (3.58), we know the first term of (3.57) is equal to zero, then rewrite the second term, we have

$$\sum_{i,j} B_{ij44}B_{ij4}\lambda_i = \lambda(-B_{3344}B_{334} - B_{4444}B_{444}) + 2\lambda B_{1244}B_{124}.$$

Combining (3.1) and (3.52), we obtain that:

$$B_{3344} - B_{4444} + 2B_{1244} = 0.$$

Thus we get (3.54).

If $(p, q, m) = (2, 2, 1)$, the first term of (3.57) is also equal to zero, from the second term, we have

$$0 = \sum_{i,j} B_{ij22}B_{ij1}\lambda_i + 2\sum_{i,j} B_{ij21}B_{ij2}\lambda_i. \tag{3.67}$$

Using (3.1) and (3.52), we can obtain

$$\begin{aligned} &B_{111}(3B_{1122} + B_{2222} - 2B_{4412} + 2B_{3312} - 2B_{3422}) \\ &- B_{222}(2B_{2221} - B_{3322} + B_{4422} - 4B_{1234}) = 0. \end{aligned} \tag{3.68}$$

If $(p,q,m)=(1,1,2)$, analogously, by (3.58), (3.52), (3.1) and (3.57), we can get

$$\begin{aligned} &B_{222}(3B_{2211}+B_{1111}+2B_{3312}-2B_{4412}+2B_{3411})\\ &+B_{111}(2B_{2221}+B_{3311}-B_{4411}-4B_{1234})=0. \end{aligned} \tag{3.69}$$

By using Lemma 3.4, (3.64), (3.69) and (3.68), we get:

$$\begin{aligned} 3B_{2211}+B_{1111}-4B_{4412}+2B_{3411}&=0,\\ 2B_{2221}+B_{3311}-B_{4411}-4B_{1234}&=0. \end{aligned}$$

Thus we get (3.55).

If $(p,q,m)=(3,3,4)$, from (3.57), we get

$$(B_{3333}-B_{3344}+2B_{3312})+2(B_{3411}-B_{3422}+B_{3344})=0. \tag{3.70}$$

By using Lemma 3.1, (3.54) and (3.70), we get the following equalities:

$$B_{3411}-B_{3422}+2B_{3344}=0.$$

Thus we get (3.56).

In summing up, we have completed the proof of Lemma 3.8. □

Proposition 3.2. *Let $x: M^4\to\mathbb{S}^5(1)$ be a closed Möbius minimal and Möbius isotropic hypersurface without umbilic points. If there are two distinct principal curvatures at a fixed point, then Case I-(ii) does not occur.*

Proof. By Ricci identity and Gauss equation, we have

$$B_{1134}=B_{3411},\ B_{2234}=B_{3422},\ B_{4412}=B_{1244}.$$

Lemma 3.4 and Lemma 3.8 yield

$$B_{1111}+2B_{3411}=B_{2211}, \tag{3.71}$$

$$B_{3344}+2B_{4412}=B_{4444}, \tag{3.72}$$

$$3B_{1122}+B_{1111}+2B_{3411}-4B_{4412}=0, \tag{3.73}$$

$$B_{3411}+B_{3344}=0. \tag{3.74}$$

Combining (3.71) and (3.73), we have

$$B_{1122}=-B_{3312}=B_{4412}. \tag{3.75}$$

Substituting (3.75), (3.74) into (3.71), (3.72), we have

$$B_{1111}+2B_{3411}=B_{1111}-2B_{3344}=B_{2211}, \tag{3.76}$$

$$B_{3344}+2B_{4412}=B_{3344}+2B_{1122}=B_{4444}. \tag{3.77}$$

Plus (3.76) and (3.77), we can get

$$B_{1111}+B_{2211}=B_{3344}+B_{4444}. \tag{3.78}$$

On the other hand, from (3.2) and (3.17), we have

$$B_{1111} + B_{2211} = -(B_{3344} + B_{4444}) = -\frac{1}{2\lambda}(\frac{1}{4}\|\nabla\mathbf{B}\|^2). \tag{3.79}$$

Then (3.78) and (3.79) imply that $\|\nabla\mathbf{B}\| = 0$, which contradicts our hypothesis.

Thus we have completed the proof of Proposition 3.2. □

Combining Proposition 3.1 and Proposition 3.2, we obtain that the assumption $\|\nabla\mathbf{B}\|^2 = 16(d_1 + d_2) \neq 0$ does not occur, therefore we have completed the proof of Theorem 3.1.

4. Three distinct principal curvatures at the minimum point of f_4

In this section, we will deal with Case II, and prove that there do not exist Möbius minimal and Möbius isotropic hypersurfaces with three distinct principal curvatures in $\mathbb{S}^5$.

Suppose the function $f_4 = \sum_{i,j,k,l} B_{ik}B_{kl}B_{lj}B_{ji}$ attained the minimum value at point p and we assume there are three distinct Möbius principal curvatures at the point p. Then the Möbius principal curvatures at the point p are

$$\lambda_1(p) = -\lambda_4(p) = \frac{\sqrt{6}}{4}, \quad \lambda_2(p) = \lambda_3(p) = 0.$$

For any point $q \in M$, we have

$$\begin{aligned}
\frac{9}{32} &= f_4(p) \leq f_4(q) = \lambda_1^4(q) + \lambda_2^4(q) + \lambda_3^4(q) + \lambda_4^4(q)\\
&= [(\lambda_1^2(q) + \lambda_4^2(q))^2 - 2\lambda_1^2(q)\lambda_4^2(q)] + [(\lambda_2^2(q) + \lambda_3^2(q))^2 - 2\lambda_2^2(q)\lambda_3^2(q)]\\
&= 2\lambda_1^2(q)\lambda_4^2(q) + 2\lambda_2^2(q)\lambda_3^2(q) = 2\lambda_1^4(q) + 2\lambda_2^4(q)\\
&= 2[(\lambda_1^2(q) + \lambda_2^2(q))^2 - 2\lambda_1^2(q)\lambda_2^2(q)]\\
&= \frac{9}{32} - 4\lambda_1^2(q)\lambda_2^2(q) \leq \frac{9}{32} = f_4(p).
\end{aligned}$$

Hence $f_4 \equiv \frac{9}{32}$ is a constant, which implies that M is a Möbius isoparametric hypersurface with Möbius principal curvatures $\lambda_1 = -\lambda_4 = \frac{\sqrt{6}}{4}, \quad \lambda_2 = \lambda_3 = 0$. However the hypersurface with Möbius principal curvatures above is not Möbius isotropic (refer to [8]). Thus this case does not occur.

5. Four distinct principal curvatures at the minimum point of f_4

In this section, we will deal with Case III: there are four distinct Möbius principal curvatures at the minimum point of f_4. Suppose $\lambda_1 > \lambda_2 > 0 > \lambda_3 > \lambda_4$ and $\lambda_1 = -\lambda_4, \quad \lambda_2 = -\lambda_3$. We assume the function $f_4 = \sum_{i,j,k,l} B_{ik}B_{kl}B_{lj}B_{ji}$ attained the minimum value at point p. If not specified, all the computations in this section are considered at this point p.

Since the condition that M is Willmore minimal and Möbius isotropic mean $f_3 = \sum_{i,j,k} B_{ik}B_{kj}B_{ji} = 0$, while the function f_4 attained the minimum value at point p means the covariant derivative of f_4 equal to zero at the point p. From these and $\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = \frac{3}{4}$, we obtain that

$$\begin{aligned}
&B_{11k} + B_{22k} + B_{33k} + B_{44k} = 0,\\
&\lambda_1 B_{11k} + \lambda_2 B_{22k} + \lambda_3 B_{33k} + \lambda_4 B_{44k} = 0,\\
&\lambda_1^2 B_{11k} + \lambda_2^2 B_{22k} + \lambda_3^2 B_{33k} + \lambda_4^2 B_{44k} = 0,\\
&\lambda_1^3 B_{11k} + \lambda_2^3 B_{22k} + \lambda_3^3 B_{33k} + \lambda_4^3 B_{44k} = 0.
\end{aligned}$$

Because of the coefficient determinant satisfies $\prod_{1\le i<j\le 4}(\lambda_j - \lambda_i) \neq 0$, then we deduce that

$$B_{iik} = 0, \ \forall \ i, \ k.$$

According to the result of [7], we know that the Möbius second fundamental form is nonparallel if $x : M^n \to \mathbb{S}^{n+1}$ has four distinct principal curvatures, hence

$$\|\nabla \mathbf{B}\|^2 = 6(B_{123}^2 + B_{124}^2 + B_{134}^2 + B_{234}^2) \neq 0. \tag{5.1}$$

From Lemma 2.1, we know that $\|\nabla \mathbf{B}\|^2 = constant$, thus for any $l = 1, 2, 3, 4$, we get $\sum_{i,j,k} B_{ijk} B_{ijkl} = 0$, more precisely,

$$\begin{aligned}
&B_{1123}B_{123} + B_{1124}B_{124} + B_{1134}B_{134} + B_{1234}B_{234} = 0,\\
&B_{2213}B_{123} + B_{2214}B_{124} + B_{1234}B_{134} + B_{2234}B_{234} = 0,\\
&B_{3312}B_{123} + B_{1234}B_{124} + B_{3314}B_{134} + B_{3324}B_{234} = 0,\\
&B_{1234}B_{123} + B_{4412}B_{124} + B_{4413}B_{134} + B_{4423}B_{234} = 0.
\end{aligned} \tag{5.2}$$

Secondly, by using (2.17), we take the derivative of $\Delta f_3 = 0$ and get

$$2\sum_{i,j,k} B_{ijkm} B_{ijk} \lambda_i + \sum_{i,j,k,l} B_{ijk} B_{ijl} B_{klm} = 0, \quad \forall \ m. \tag{5.3}$$

In Lemma 2.1, we have defined

$$\widetilde{A} = \sum_{i,j,k,l,m} B_{mkl} B_{mil} B_{kj} B_{ji}, \quad \widetilde{B} = \sum_{i,j,k,l,m} B_{mkl} B_{mi} B_{kj} B_{ijl}.$$

By a direct computation, we can deduce

$$\widetilde{A} - 2\widetilde{B} = \frac{3}{4} f_4 - \frac{9}{8}\mu. \tag{5.4}$$

Since

$$\begin{aligned}
0 &= \frac{1}{3}\sum_{m,n} B_{mn}(f_3)_{mn} = \frac{1}{3}\sum_{m,n,i,j,k} B_{mn}(B_{ik}B_{kj}B_{ji})_{mn}\\
&= 2\widetilde{B} + \sum_{m,n,i,j,k} B_{mn} B_{mnik} B_{kj} B_{ji} + \|\mathbf{B}\|^2 f_4 - 2\mu\|\mathbf{B}\|^4 - (f_3)^2.
\end{aligned} \tag{5.5}$$

On the other hand, from $\|\mathbf{B}\|^2 = \sum_{m,n} B_{mn}^2 = \frac{3}{4}$, we can get

$$\sum_{m,n} B_{mn} B_{mnik} + \sum_{m,n} B_{mnk} B_{mni} = 0. \tag{5.6}$$

Thus (5.5) and (5.6) imply (5.4).

Taking the covariant derivative of (5.4) at the minimum point of f_4, we have

$$\sum_{m,n,i,j,k}(B_{mnk}B_{mni}B_{kj}B_{ji})_s = 2\sum_{m,n,i,j,k}(B_{mn}B_{ikm}B_{kjn}B_{ji})_s. \tag{5.7}$$

Making use of the Eqs (5.2)-(5.7), we can get the following two lemmas:

Lemma 5.1. $B_{1234} = 0$.

Proof. By differentiating $f_3 = 0$ twice, we get

$$\sum_i \lambda_i^2 B_{iimn} + 2\sum_{i,j}\lambda_i B_{ijm}B_{ijn} = 0 \tag{5.8}$$

for all $m, n = 1, 2, 3, 4$.

Specially, for $m = 2,\ n = 3$, we get

$$\lambda_1^2(B_{1123} + B_{4423}) + \lambda_2^2(B_{3323} + B_{2223}) = 0.$$

Note that $B_{1123} + B_{2223} + B_{3323} + B_{4423} = 0$ and $\lambda_1 \neq \lambda_2$ together indicate

$$B_{1123} + B_{4423} = 0. \tag{5.9}$$

On the other hand, for $s = 1, 4$, by rewriting (5.7) and using (5.2), we get

$$\begin{aligned}(\lambda_2^2 - \lambda_1^2)(B_{1123}B_{123} + B_{1234}B_{234}) &= 2B_{123}B_{124}B_{134}\lambda_1,\\ (\lambda_2^2 - \lambda_1^2)(B_{1234}B_{123} + B_{4423}B_{234}) &= 2B_{124}B_{134}B_{234}\lambda_4.\end{aligned} \tag{5.10}$$

Combining (5.9), the first equation of (5.10) times B_{234} plus the second equation times B_{123} implies

$$(\lambda_2^2 - \lambda_1^2)(B_{123}^2 + B_{234}^2)B_{1234} = 0.$$

Analogously, it is easy to see $B_{2214} + B_{3314} = 0$ from (5.8), then using (5.7) and (5.2), we can deduce that

$$(\lambda_1^2 - \lambda_2^2)(B_{124}^2 + B_{134}^2)B_{1234} = 0.$$

Since $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ means $\lambda_1^2 - \lambda_2^2 \neq 0$, so from (5.1) and the above equations, we get $B_{1234} = 0$. □

According to Lemma 5.1, we have $B_{1234} = 0$. then (5.2) can be reduced to

$$\begin{aligned}B_{1123}B_{123} + B_{1124}B_{124} + B_{1134}B_{134} &= 0,\\ B_{2213}B_{123} + B_{2214}B_{124} + B_{2234}B_{234} &= 0,\\ B_{3312}B_{123} + B_{3314}B_{134} + B_{3324}B_{234} &= 0,\\ B_{4412}B_{124} + B_{4413}B_{134} + B_{4423}B_{234} &= 0.\end{aligned} \tag{5.11}$$

From (5.3), we have

$$\begin{aligned}B_{1123}B_{123}\lambda_1 + B_{1124}B_{124}\lambda_2 + B_{1134}B_{134}\lambda_3 &= -3B_{123}B_{124}B_{134},\\ B_{2213}B_{123}\lambda_1 + B_{2214}B_{124}\lambda_2 + B_{2234}B_{234}\lambda_4 &= -3B_{123}B_{124}B_{234},\\ B_{3312}B_{123}\lambda_1 + B_{3314}B_{134}\lambda_3 + B_{3324}B_{234}\lambda_4 &= -3B_{123}B_{134}B_{234},\\ B_{4412}B_{124}\lambda_2 + B_{4413}B_{134}\lambda_3 + B_{4423}B_{234}\lambda_4 &= -3B_{124}B_{134}B_{234}.\end{aligned} \tag{5.12}$$

From (5.7), we have

$$\begin{aligned}
B_{1123}B_{123}\lambda_2^2 + B_{1124}B_{124}\lambda_1^2 + B_{1134}B_{134}\lambda_1^2 &= 2B_{123}B_{124}B_{134}\lambda_1,\\
B_{2213}B_{123}\lambda_2^2 + B_{2214}B_{124}\lambda_1^2 + B_{2234}B_{234}\lambda_2^2 &= 2B_{123}B_{124}B_{234}\lambda_2,\\
B_{3312}B_{123}\lambda_2^2 + B_{3314}B_{134}\lambda_1^2 + B_{3324}B_{234}\lambda_2^2 &= 2B_{123}B_{134}B_{234}\lambda_3,\\
B_{4412}B_{124}\lambda_1^2 + B_{4413}B_{134}\lambda_1^2 + B_{4423}B_{234}\lambda_2^2 &= 2B_{124}B_{134}B_{234}\lambda_4.
\end{aligned} \tag{5.13}$$

By making use of these equations, we can get the following lemma:

Lemma 5.2. $B_{123}B_{124}B_{134}B_{234} = 0.$

Proof. Suppose on the contrary that $B_{123}B_{124}B_{134}B_{234} \neq 0$.

The third equation of (5.11), (5.12), (5.13) constitute a linear equation system of B_{3312}, B_{3314}, B_{3324}. We denote

$$D^{(3)} = \begin{vmatrix} B_{123} & B_{134} & B_{234} \\ B_{123}\lambda_1 & B_{134}\lambda_3 & B_{234}\lambda_4 \\ B_{123}\lambda_2^2 & B_{134}\lambda_1^2 & B_{234}\lambda_2^2 \end{vmatrix} = 2B_{123}B_{134}B_{234}\lambda_1(\lambda_1^2 - \lambda_2^2),$$

$$\begin{aligned} D_1^{(3)} &= \begin{vmatrix} 0 & B_{134} & B_{234} \\ -3B_{123}B_{134}B_{234} & B_{134}\lambda_3 & B_{234}\lambda_4 \\ 2B_{123}B_{134}B_{234}\lambda_3 & B_{134}\lambda_1^2 & B_{234}\lambda_2^2 \end{vmatrix} \\ &= -(\lambda_1 - \lambda_2)(3\lambda_1 + \lambda_2)B_{123}B_{134}^2B_{234}^2. \end{aligned}$$

By Cramer's rule we can have

$$B_{3312} = \frac{D_1^{(3)}}{D^{(3)}} = -\frac{(3\lambda_1 + \lambda_2)B_{134}B_{234}}{2\lambda_1(\lambda_1 + \lambda_2)}.$$

Similarly, from the last equation of (5.11), (5.12), (5.13), we can get

$$B_{4412} = \frac{D_1^{(4)}}{D^{(4)}} = -\frac{(\lambda_1 + 3\lambda_2)B_{134}B_{234}}{2\lambda_2(\lambda_1 + \lambda_2)}.$$

Thus

$$B_{3312} + B_{4412} = -\frac{B_{134}B_{234}}{2(\lambda_1 + \lambda_2)}\Big(4 + \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1\lambda_2}\Big). \tag{5.14}$$

On the other hand, taking $m = 1, n = 2$ in (5.8), we have

$$B_{1112} = \frac{2B_{134}B_{234}}{\lambda_1 - \lambda_2} - B_{4412}, \qquad B_{2212} = -\frac{2B_{134}B_{234}}{\lambda_1 - \lambda_2} - B_{3312}. \tag{5.15}$$

From $\|\mathbf{B}\|^2 = \frac{3}{4}$, we have

$$\sum_i \lambda_i B_{iimn} + \sum_{i,j} B_{ijm}B_{ijn} = 0, \forall \quad m, n. \tag{5.16}$$

In particular, for $m = 1, n = 2$, we get

$$\lambda_1(B_{1112} - B_{4412}) + \lambda_2(B_{2212} - B_{3312}) + 2B_{134}B_{234} = 0. \tag{5.17}$$

Substitute (5.15) into (5.17), we get the following equality:

$$-2(\lambda_1+\lambda_2)(B_{3312}+B_{4412})=0. \tag{5.18}$$

Since $\lambda_1>\lambda_2>0$, so (5.18) and (5.14) yield $B_{134}B_{234}=0$, which contradicts our assumption that $B_{123}B_{124}B_{134}B_{234}\neq 0$.

Hence we complete the proof of this lemma. □

Lemma 5.2 says that at least one of $\{B_{123},B_{124},B_{134},B_{234}\}$ is equal to zero.

From (2.16), (2.17), (5.4) and (5.1), we have the following linear equation system of $B_{123}^2,\quad B_{124}^2,\quad B_{134}^2,\quad B_{234}^2$:

$$\begin{cases} B_{123}^2+B_{124}^2+B_{134}^2+B_{234}^2=e, \\ \lambda_1B_{123}^2+\lambda_2B_{124}^2-\lambda_2B_{134}^2-\lambda_1B_{234}^2=0, \\ (4\lambda_1^2+\lambda_2^2)(B_{124}^2+B_{134}^2)+(\lambda_1^2+4\lambda_2^2)(B_{123}^2+B_{234}^2)=y. \end{cases} \tag{5.19}$$

where $e=\frac{3}{32}-\mu,\quad y=\frac{3}{8}f_4-\frac{9}{16}\mu.$

By (5.1), we know that $B_{123}^2+B_{124}^2+B_{134}^2+B_{234}^2=e\neq 0$, thus the second equation of (5.19) and our assumption $\lambda_1>\lambda_2>0$ imply that

$$B_{123}^2+B_{124}^2\neq 0,\quad B_{134}^2+B_{234}^2\neq 0. \tag{5.20}$$

According to (5.20) and Lemma 5.2, we can divide the discussions into four cases:

Case III − (i) : $B_{123}=0$, or $B_{234}=0$;
Case III − (ii) : $B_{134}=0$, $B_{124}=0$;
Case III − (iii) : $B_{134}=0$, $B_{123}B_{124}B_{234}\neq 0$;
Case III − (iv) : $B_{124}=0$, $B_{123}B_{134}B_{234}\neq 0$.

By discussing these cases one by one, we will prove that Blaschke eigenvalue μ is greater than or equal to $\frac{1}{32}$, precisely, we have the following lemmas.

Lemma 5.3. *If $B_{123}=0$ or $B_{234}=0$, then $\mu\geq\frac{1}{32}$.*

Proof. If $B_{123}=0$, from (5.19), we have

$$0\leq B_{134}^2=-\frac{(4\lambda_1^2+4\lambda_2^2-3\lambda_1\lambda_2)e-y}{6\lambda_2(\lambda_1-\lambda_2)}.$$

Since $\lambda_1>\lambda_2>0$ and $\lambda_1=-\lambda_4,\quad \lambda_2=-\lambda_3$, then

$$2(\lambda_1^2+\lambda_2^2)=\|\mathbf{B}\|^2=\frac{3}{4},\quad f_4=2(\lambda_1^4+\lambda_2^4)=\frac{9}{32}-4\lambda_1^2\lambda_2^2.$$

Thus

$$\begin{aligned} 0&\geq(4\lambda_1^2+4\lambda_2^2-3\lambda_1\lambda_2)e-y \\ &=(\frac{3}{32}-\mu)(\frac{3}{2}-3\lambda_1\lambda_2)-\frac{3}{8}(\frac{9}{32}-4\lambda_1^2\lambda_2^2)+\frac{9}{16}\mu \\ &=\frac{3}{2}\Big(\lambda_1^2\lambda_2^2+2(\mu-\frac{3}{32})\lambda_1\lambda_2-\frac{5}{8}\mu+\frac{3}{64\times 2}\Big) \\ &=\frac{3}{2}\Big[(\lambda_1\lambda_2+\mu-\frac{3}{32})^2-\mu^2-\frac{7}{16}\mu+\frac{15}{32\times 32}\Big] \\ &\geq-\frac{3}{2}(\mu^2+\frac{7}{16}\mu-\frac{15}{32\times 32}). \end{aligned}$$

The above inequality indicates

$$\mu^2 + \frac{7}{16}\mu - \frac{15}{32\times 32} = (\mu - \frac{1}{32})(\mu + \frac{15}{32}) \geq 0. \tag{5.21}$$

On the other hand, from

$$\begin{aligned}
0 &\geq (4\lambda_1^2 + 4\lambda_2^2 - 3\lambda_1\lambda_2)e - y \\
&= (-\frac{15}{16} + 3\lambda_1\lambda_2)\mu + \frac{3}{2}\lambda_1^2\lambda_2^2 - \frac{9}{32}\lambda_1\lambda_2 + \frac{9}{256} \\
&= (-\frac{15}{16} + 3\lambda_1\lambda_2)\mu + \frac{3}{2}(\lambda_1\lambda_2 - \frac{3}{32})^2 + \frac{45}{32\times 64} \\
&> (-\frac{15}{16} + 3\lambda_1\lambda_2)\mu,
\end{aligned}$$

and

$$-\frac{15}{16} + 3\lambda_1\lambda_2 \leq -\frac{15}{16} + \frac{3}{2}(\lambda_1^2 + \lambda_2^2) = -\frac{6}{16} < 0,$$

we know that $\mu > 0$. Hence, (5.21) immediately implies $\mu \geq \frac{1}{32}$.

If $B_{234} = 0$, from (5.19), we have

$$B_{124}^2 = -\frac{(4\lambda_1^2 + 4\lambda_2^2 - 3\lambda_1\lambda_2)e - y}{6\lambda_2(\lambda_1 - \lambda_2)}.$$

By similar arguments as the case $B_{123} = 0$, we also obtain $\mu \geq \frac{1}{32}$.

We have completed the proof of Lemma 5.3. □

Lemma 5.4. *Case III-(ii):* $B_{134} = 0,\ \ B_{124} = 0$ *does not occur.*

Proof. By differentiating f_3 and $\|\mathbf{B}\|^2$ twice, we get (5.8) and (5.16). Setting $(m, n) = (1, 1),\ \ (4, 4)$ in (5.8) and (5.16), we get the following equations:

$$\begin{aligned}
\lambda_1^2(B_{1111} + B_{4411}) + \lambda_2^2(B_{2211} + B_{3311}) &= 0, \\
\lambda_1^2(B_{1144} + B_{4444}) + \lambda_2^2(B_{2244} + B_{3344}) &= 0, \\
\lambda_1(B_{1111} - B_{4411}) + \lambda_2(B_{2211} - B_{3311}) + 2B_{123}^2 &= 0, \\
\lambda_1(B_{1144} - B_{4444}) + \lambda_2(B_{2244} - B_{3344}) + 2B_{234}^2 &= 0.
\end{aligned} \tag{5.22}$$

We note that

$$\begin{aligned}
(B_{1111} + B_{4411}) + (B_{2211} + B_{3311}) &= 0, \\
(B_{1144} + B_{4444}) + (B_{2244} + B_{3344}) &= 0.
\end{aligned} \tag{5.23}$$

Combining (5.22) and (5.23), we get

$$B_{1111} + B_{4411} = 0, \qquad B_{1144} + B_{4444} = 0, \tag{5.24}$$

and

$$\begin{aligned}
-2\lambda_1 B_{4411} + 2\lambda_2 B_{2211} + 2B_{123}^2 &= 0, \\
2\lambda_1 B_{1144} + 2\lambda_2 B_{2244} + 2B_{234}^2 &= 0.
\end{aligned} \tag{5.25}$$

Plus the above two equations, we can have

$$2\lambda_1(B_{1144} - B_{4411}) + 2\lambda_2(B_{2211} + B_{2244}) + 2(B_{123}^2 + B_{234}^2) = 0. \tag{5.26}$$

On the other hand, by differentiating f_4 twice, we get

$$\frac{1}{4}(f_4)_{mn} = \sum_i \lambda_i^3 B_{iimn} + 2\sum_{i,j} \lambda_i^2 B_{ijm} B_{ijn} + \sum_{i,j} \lambda_i \lambda_j B_{ijm} B_{ijn}, \tag{5.27}$$

for all $m, n = 1, 2, 3, 4$.

Using (5.27) and (5.26), we obtain

$$\begin{aligned}\frac{1}{4}[(f_4)_{11} + (f_4)_{44}] &= 2\lambda_1^3(B_{1144} - B_{4411}) + 2\lambda_2^3(B_{2211} + B_{2244}) + 2\lambda_2^2(B_{123}^2 + B_{234}^2)\\ &= 2\lambda_1(\lambda_1^2 - \lambda_2^2)(B_{1144} - B_{4411}) = 4\lambda_1^2(\lambda_1^2 - \lambda_2^2)(2\mu - \lambda_1^2),\end{aligned}$$

in the last equality, we have used Ricci identity. Since f_4 take the minimum value at p, so the Hessian matrix of f_4 is positive, which implies

$$0 \le (f_4)_{11} + (f_4)_{44} = 16\lambda_1^2(\lambda_1^2 - \lambda_2^2)(2\mu - \lambda_1^2). \tag{5.28}$$

Thus we have $\mu \ge \frac{1}{2}\lambda_1^2 > \frac{3}{32}$, which contradicts $\|\nabla \mathbf{B}\|^2 \ge 0$. □

Lemma 5.5. *If $B_{134} = 0$, $B_{123}B_{124}B_{234} \neq 0$, then $\mu \ge \frac{1}{32}$.*

Proof. From (5.19), we get

$$\begin{aligned}B_{234}^2 &= \frac{(4\lambda_1^2 + 4\lambda_2^2 + 3\lambda_1\lambda_2)e - y}{6\lambda_1(\lambda_1 + \lambda_2)},\\ B_{123}^2 &= \frac{(4\lambda_1^2 + 4\lambda_2^2 - 3\lambda_1\lambda_2)e - y}{6\lambda_1(\lambda_1 - \lambda_2)},\\ B_{124}^2 &= -\frac{(4\lambda_2^2 + \lambda_1^2)e - y}{3(\lambda_1^2 - \lambda_2^2)},\end{aligned}$$

where $e = \frac{3}{32} - \mu$, $\quad y = \frac{3}{8}f_4 - \frac{9}{16}\mu$.

Taking $(m, n) = (1, 4)$ in (5.27) and (5.16) respectively, we get

$$\frac{1}{4}(f_4)_{14} = \lambda_1^3(B_{1114} - B_{4414}) + \lambda_2^3(B_{2214} - B_{3314}) + 2\lambda_2^2 B_{134}B_{234}, \tag{5.29}$$

$$\lambda_1(B_{1114} - B_{4414}) + \lambda_2(B_{2214} - B_{3314}) + 2B_{123}B_{234} = 0. \tag{5.30}$$

From (5.24), we know that $B_{1114} = -B_{4414}$, $\quad B_{2214} = -B_{3314}$, thus

$$2\lambda_1 B_{1114} + 2\lambda_2 B_{2214} + 2B_{123}B_{234} = 0, \tag{5.31}$$

$$\frac{1}{4}(f_4)_{14} = 2\lambda_1^3 B_{1114} + 2\lambda_2^3 B_{2214} + 2\lambda_2^2 B_{134}B_{234}. \tag{5.32}$$

We take the value B_{2214} of Lemma 5.1 into (5.32), make it reduce to

$$\frac{1}{4}(f_4)_{14} = -2(\lambda_1^2 + \lambda_2^2)B_{123}B_{234}. \tag{5.33}$$

Since

$$\begin{aligned}\frac{1}{4}\Delta f_4 &= 2\widetilde{A} + \widetilde{B} + 8(\mu - \frac{3}{32})f_4\\ &= \frac{5\|B\|^4(\|\mathbf{B}\|^2 - 8\mu)}{18} + \frac{\|\mathbf{B}\|^2 f_4 - 2\mu\|B\|^4}{3} + 8(\mu - \frac{3}{32})f_4.\end{aligned} \tag{5.34}$$

Since f_4 take the minimum at p, we conclude the Hessian matrix of f_4 is positive at p. Thus we have the following inequality:

$$(\Delta f_4)^2 \geq \left((f_4)_{11} + (f_4)_{44}\right)^2 \geq 4(f_4)_{11}(f_4)_{44} \geq 4[(f_4)_{14}]^2,$$

combining (5.34) and (5.33), this inequality indicates that

$$\begin{aligned}&\left(\frac{5\|B\|^4(\|\mathbf{B}\|^2 - 8\mu)}{18} + \frac{\|\mathbf{B}\|^2 f_4 - 2\mu\|\mathbf{B}\|^4}{3} + 8(\mu - \frac{3}{32})f_4\right)^2\\ &\geq 16(\lambda_1^2 + \lambda_2^2)^2 B_{123}^2 B_{234}^2.\end{aligned} \tag{5.35}$$

Substituting B_{234}^2 and B_{123}^2 into inequality (5.35). Let $\lambda_1^2 = t\lambda_2^2$ $(t > 1)$, then $\|\mathbf{B}\|^2 = 2(1+t)\lambda_2^2 = \frac{3}{4}$ and $f_4 = 2(1+t^2)\lambda_2^4$, thus from (5.35) we can get

$$a\mu^2 + 2b\mu + c \geq 0, \tag{5.36}$$

where

$$\begin{aligned}a &= -4(399t^5 - 755t^4 + 1270t^3 - 54t^2 + 139t + 25)\\ &= -4(7t+1)(57t^4 - 116t^3 + 198t^2 - 36t + 25)\\ &= -4(7t+1)\{57(t - \frac{58}{57})^2 + [198 - 57(1 + \frac{1}{57})^2 - 36]t^2 + (6t-3)^2 + 16\},\\ b &= 2(1+t)\lambda_2{}^2(123^5 - 223t^4 + 2\times 259t^3 + 42t^2 + 47t + 5)\\ &= \frac{3}{4}[123t^3(t-1)^2 + 23t^4 + 395t^3 + 42t^2 + 47t + 5],\\ c &= -(t+1)^2\lambda_2^4(30t^5 - 59t^4 + 232t^3 + 42t^2 + 10t + 1)\\ &= -\frac{9}{64}[30t^3(t-1)^2 + t^4 + 202t^3 + 42t^2 + 10t + 1].\end{aligned}$$

It is easy to check that $a < 0, \quad c < 0, \quad b > 0$, thus (5.36) implies

$$\frac{b - \sqrt{b^2 - ac}}{-a} \leq \mu \leq \frac{b + \sqrt{b^2 - ac}}{-a} \tag{5.37}$$

In order to prove $\mu \geq \frac{1}{32}$, we will prove that $\frac{1}{32} \leq \frac{b - \sqrt{b^2 - ac}}{-a}$, which is equivalent to

$$a + 64b + 1024c \leq 0. \tag{5.38}$$

By direct computation, we have

$$\begin{aligned} &a + 64b + 1024c = -4\{3t^5 - 203t^4 + 3406t^3 + 954t^2 - 65t + 1\} \\ &= -4(3t + 1)(t^2 - 34t + 1)^2 \leq 0. \end{aligned}$$

Hence, we get $\mu \geq \frac{1}{32}$ in case III-(iii).

By the equivalent condition (5.38), we get $\mu \geq \frac{1}{32}$ under the case $B^2_{134} = 0$. □

For the **case III-(iv)**: $B_{124} = 0, B_{123}B_{134}B_{234} \neq 0$, by totally similar arguments as that in Lemma 5.5, we can get $\mu \geq \frac{1}{32}$.

Theorem 5.1. *Let $x : M^4 \to \mathbb{S}^5$ be a closed Möbius minimal and Möbius isotropic hypersurface without umbilic points. If there are four distinct Möbius principal curvatures at the minimum point of f_4, then M^4 is Möbius equivalent to an isoparametric hypersurface.*

Proof. Denote the normalized scalar curvature of the Möbius metric g and the induced metric $dx \cdot dx$ by R and κ respectively.

From (2.14), it is easy to get

$$12R = 24\mu - \sum_{i,j} B^2_{ij} = 24\mu - \frac{3}{4} = 24(\mu - \frac{1}{32}). \tag{5.39}$$

Combining Lemma 5.3, Lemma 5.5 and (5.39), we obtain $R \geq 0$.

In addition, $x : M^4 \to \mathbb{S}^5$ is Möbius isotropic ($\mu \geq \frac{1}{32}$) means that it is Möbius equivalent to a minimal hypersurface with constant scalar curvature in $\mathbb{S}^5$, more precisely, $H = 0$ and $\kappa = constant$, then by Gauss equation

$$\bar{R}_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

we know that

$$\rho^2 = \frac{4}{3}(S - 4\|H\|^2) = \frac{4}{3}S = 16(1 - \kappa) = const.$$

Thus by Eq (1.8) in [18], we can get $\kappa = \rho^2 R \geq 0$.

By using the result of section 5 in [4] or section 4 in [15], we conclude that x is Euclidean isoparametric, and the Blaschke eigenvalue $\mu = \frac{1}{32}$.

□

Remark 5.1. We will give another interpretation of the proof of Theorem 5.1. (5.39) and (2.21) mean that the following conditions are satisfied:

(1) $R \geq 0$; (2) $\sum_{i=1}^4 \lambda_i^k (k = 1, 2, 3)$ are constants; (3) M^4 has four distinct principal curvatures somewhere.

By the Corollary 1.1 of [17], we conclude that all the Möbius principal curvatures are constant and $R = 2(\mu - \frac{1}{32}) \equiv 0$, which together with $\rho = const$, $H = 0$ and $B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij})$ imply that M^4 is an isoparametric hypersurface.

6. Proof of Theorem 1.1

Proof of Theorem 1.1.

In Section 3, 4 and 5 we discussed three cases to prove Theorem 1.1.

Case 1: If there are two distinct principal curvatures at the minimum point of f_4, by Theorem 3.1 and (2.16), we have $\nabla\mathbf{B} \equiv 0$ and $\mu > 0$. According to Proposition 5.1 of [7], we conclude that $x : M^4 \to \mathbb{S}^5$ is Möbius isoparametric. Then using the results of [12] and [14], we know that x is Möbius equivalent to the minimal Willmore torus in $\mathbb{S}^5$, to be precise, it is $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \times \mathbb{S}^2(\frac{1}{\sqrt{2}})$.

Case 2: According to the arguments in section 4, we know that there do not exist any Möbius minimal and Möbius isotropic hypersurfaces with three distinct principal curvatures.

Case 3: If there are four distinct principal curvatures at the minimum point of f_4, according to the proof of Theorem 5.1, x is Möbius equivalent to an Euclidean isoparametric hypersurface in $\mathbb{S}^5$. More precisely, x is Möbius equivalent to the Cartan minimal hypersurface in $\mathbb{S}^5$ with four distinct principal curvatures (see [1] or [15] for details).

Therefore, we have completed the proof of Theorem 1.1.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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